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Hawking mass and local rigidity of minimal surfaces in three-manifolds

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The aim of this paper is to generalize some recent local rigidity results for three-dimensional Riemannian manifolds (M^3, g) with a bound on the scalar curvature. More precisely, we study rigidity of strictly stable minimal surfaces $\Sigma \subset M$ which locally maximize the Hawking mass on a Riemannian three-manifold M whose scalar curvature is bounded from below by a negative constant. Moreover, we conclude that the metric of M near Σ must split as $g_a = dr^2 + u_a(r)^2 g_{\widetilde{\Sigma}}$ which is one the Kottler-Schwarzschild metric, where $g_{\widetilde{\Sigma}}$ is a metric of constant gaussian curvature.

1. Introduction and statement of the main results

The study of three-dimensional Riemannian manifolds (M^3, g) for which the scalar curvature R_g has a lower bound has encouraged many geometers to describe the topology and geometry of such manifolds under the assumption of the existence of a compact embedded surface in M with some geometric property. In general, stability as well as area minimizing are assumed. Among seminal works in this matter we mention those due to Schoen and Yau [25] and Meeks, Simon and Yau [21]. In a recent work due to Bray et all [5] the authors studied the phenomena of area minimizing projective planes whereas Bray, Brendle and Neves [6] gave the structure of a three-manifold M under the assumption of the existence of an embedded area minimizing two-sphere, motived by a classical work due to Toponogov concerning closed geodesics in surfaces of positive Gaussian curvature. Next we wish to observe that a rigidity statement as in [5] or [6] fails if we replace the condition of area minimizing by stability, due the counterexamples to Min-Oo's conjecture [9]. A good survey due to Brendle [7] gives an overview of this matter.

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Very recently, Maximo and Nunes [20] have obtained a similar rigidity result by replacing the area minimizing assumption by strictly stability and maximization of the *Hawking Mass* of a surface $\Sigma \subset (M^3, g)$. We recall that the Hawking mass, denote by $\mathfrak{m}_{\mathcal{H}}(\Sigma)$, is defined as follows

$$\mathfrak{m}_{\mathcal{H}}(\Sigma) = \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \left(8\pi\mathcal{X}(\Sigma) - \int_{\Sigma} \left(H^2 + \frac{2}{3}\Lambda \right) d\sigma \right), \tag{1.1}$$

where H, $|\Sigma|$ and $\mathcal{X}(\Sigma)$ stand for the mean curvature, the area and the Euler characteristic of Σ, respectively, while Λ is the infimum of the scalar curvature R_g of M.

Many rigidity results have been inspired by the Positive Mass Theorem which states that an asymptotically flat three-manifold with nonnegative scalar curvature has nonnegative ADM mass (concept developed by Arnowitt, Deser and Misner in [2]) that was settled by Schoen and Yau in [25] and later by Witten [27] using spinors and the Dirac equation. Moreover, the ADM mass is zero if only if M is isometric to $\mathbb{R}^3$ with the standard flat metric.

In another context, Wang [26], Chruściel and Herzlich [13] have proved a Positive Mass Theorem in the asymptotically hyperbolic setting, where now the manifold (M, g) has scalar curvature $R_g \geqslant -6$. Moreover, under certain conditions, it was proved that M must be isometric to the standard hyperbolic space $\mathbb{H}^3$.

Our work gives in some sense a generalization of the one due to Maximo and Nunes [20]. More precisely, they have established a local rigidity result assuming the existence of a locally maximizing two-sphere Σ that is strictly stable in a three-dimensional manifold M with scalar curvature $R_g \geq 2$. Moreover, there exists a neighborhood of M isometric to one of the de Sitter-Schwarzschild metric in $(-\varepsilon, \varepsilon) \times \Sigma$.

In this paper, we will go further. We will prove a local rigidity in the hyperbolic setting. In order to start describing some geometric models, one recall from [19] the following definition of asymptotically locally hyperbolic (ALH) manifolds.

Definition 1. Let $(\widehat{\Sigma}, \widehat{g})$ be a compact surface with constant gaussian curvature $\widehat{k}$. A C^i Riemannian metric g on a smooth manifold M^3 is called C^i asymptotically locally hyperbolic if there exists a compact set $\mathcal{K} \subset M$ such that $M \setminus \mathcal{K}$ is diffeomorphic to $(1, +\infty) \times \widehat{\Sigma}$ with g satisfying

$$g = \frac{d\ell^2}{\widehat{k} + \ell^2} + \ell^2 \widehat{g} + \frac{h}{\ell} + Q,$$

where ℓ is the coordinate on $(1, +\infty)$, h is a C^i symmetric 2-tensor on $\widehat{\Sigma}$ that depends on ℓ in such way that there exists a function μ on $\widehat{\Sigma}$, called the mass aspect function, such that $tr_{\widehat{g}}h$ converges to μ when $\ell \to +\infty$. Moreover, Q is a C^i symmetric 2-tensor on M so that

$$|Q|_b + \ell|\overline{\nabla}Q|_b + \cdots + \ell^i|\overline{\nabla}^i Q|_b = o(\ell^{-3}),$$

where b is the hyperbolic metric $\frac{d\ell^2}{\widehat{k}+\ell^2} + \ell^2\widehat{g}$ and $\overline{\nabla}$ is the derivative taken with respect to b.

Now, fix a real number m and let s_0 be the largest positive solution of $\widehat{k} + s^2 - \frac{2\mathfrak{m}}{s} = 0$. The three-dimensional *Kottler space*, or Kottler-Schwarzschild space is a manifold $[s_0, +\infty) \times \widehat{\Sigma}$ equipped with the following metric

$$g_{ks} = \frac{ds^2}{\widehat{k} + s^2 - \frac{2\mathfrak{m}}{s}} + s^2 g_{\widehat{\Sigma}},$$

where g_{ks} is a metric of constant scalar curvature equal to -6. This space is an analogue to the Schwarzschild space in the context of asymptotically locally hyperbolic manifolds which also corresponds to the static metrics[1] with cosmological constant equal to -3. Standard references for these spaces covered here are [26],[10], [15] and [19].

By change of variable, the Kottler-Schwarzschild metric can be rewritten as

$$g_{ks} = dr^2 + u(r)^2 g_{\widehat{\Sigma}},$$

where $u : [0, +\infty) \to [s_0, +\infty)$ satisfies $u(r(s)) = s$, $u(r) = \mathrm{O}(\mathrm{e}^{\mathrm{r}})$ when $r \to \infty$, $u'(r) = (\widehat{k} + u^2 - \frac{2m}{u})^{1/2}$ and $u''(r) = u(r) + \mathfrak{m}u(r)^{-2} \geq 0$.

After reflection, we define a complete metric on $\mathbb{R} \times \widehat{\Sigma}$ with constant scalar curvature -6. Note that u solves the following second-order nonlinear differential equation

$$u'' - \frac{3}{2}u - \frac{1}{2}\left(\frac{\widehat{k} - (u')^2}{u}\right) = 0. \tag{1.2}$$

We now define a one-parameter family of complete metrics $(g_{ks})_a = dr^2 + u_a(r)^2 g_{\widehat{\Sigma}}$ with constant scalar curvature equal to -6, where u is a smooth positive function satisfying $u_a(0) = a$ and $u'_a(0) = 0$.

[1] A metric g is called static on a manifold N if there exists a nontrivial function V (called of static potential) on N such that $(\Delta V)g + \nabla^2 V - V Ric_g = 0$.

In a special case, if $\widehat{k}=1$, $(\mathbb{R}\times\mathbb{S}^2, g_{adss})$ is the anti-de Sitter-Schwarzschild with $\mathfrak{m}>0$, where $g_{adss}=dr^2+u(r)^2 g_{\mathbb{S}^2}$. We next observe that when $\mathfrak{m}$ tends to 0, then $s_0\to 0$ and g is the hyperbolic metric

$$g=\frac{ds^2}{1+s^2}+s^2 g_{\mathbb{S}^2}.$$

Assuming $\widehat{k}=0$ and that the area of $\widehat{\Sigma}$ is equal to 4π, we have the following model space $(\mathbb{R}\times\widehat{\Sigma}, g')$. We also remark that if we further assume $\mathfrak{m}=0$, after a change of coordinate, the metric g' can be written as $dr^2+e^{2r}\widehat{g}$ on $\mathbb{R}\times\widehat{\Sigma}$.

Finally, when $\widehat{k}=-1$, $\mathfrak{m}$ can be negative, in fact, $\mathfrak{m}\in\big[-\frac{1}{3\sqrt{3}},\infty\big)$. In this case, we define the model space $\mathbb{R}\times\widehat{\Sigma}$ with metric $\overline{g}=dr^2+u(r)^2 g_{\widehat{\Sigma}}$. If $\mathfrak{m}$ attains the critical value $-\frac{1}{3\sqrt{3}}$, then $((0,+\infty)\times\widehat{\Sigma},\overline{g})$ is a two-ended complete Riemannian manifold whose one-end is asymptotically locally hyperbolic and the other is asymptotic to a cylindrical metric $dr^2+\frac{1}{3}g_{\widehat{\Sigma}}$.

For the model of rigidity in the hyperbolic case, let $\big(\widetilde{\Sigma}, g_{\widetilde{\Sigma}}\big)$ denote a compact surface of genus $g(\widetilde{\Sigma})>1$ and constant Gaussian curvature -1 such that $(\mathbb{R}\times\widetilde{\Sigma}, dr^2+u^2(r)g_{\widetilde{\Sigma}})$ has scalar curvature -2. Thus we consider a one-parameter family of metrics

$$g_a=dr^2+u_a(r)^2 g_{\widetilde{\Sigma}}, \tag{1.3}$$

with $u_a(0)=a>1$ and $u_a'(0)=0$.

We now state our main rigidity theorem.

Theorem 1. *Let (M^3,g) be a Riemannian manifold with scalar curvature bounded from below by a constant Λ. Let $\Sigma\subset M$ be a two-sided compact embedded strictly stable minimal surface such that Σ locally maximizes the Hawking mass. Then*

$$\left(\lambda_1+\frac{1}{2}\Lambda\right)|\Sigma|=2\pi\chi(\Sigma), \tag{1.4}$$

where λ_1 is the first eigenvalue of the Jacobi operator. In addition, we assume that $\Lambda=-6$ and one of the following conditions holds:

(i) *If Σ is a 2-sphere, then Σ has constant Gaussian curvature equal to $\frac{1}{a^2}$ and in a neighborhood of Σ, M is isometric to the Anti-de Sitter-Schwarzschild metric $((-\varepsilon,\varepsilon)\times\mathbb{S}^2,(g_{adss})_a)$ for some $\varepsilon>0$.*

(ii) *If Σ has genus $g(\Sigma) \geqslant 2$, then Σ has constant Gauss curvature equal to $-\frac{1}{a^2}$ for $a^2 > \frac{1}{3}$ and there exists a neighborhood which is isometric to $((-\varepsilon,\varepsilon)\times\widehat{\Sigma},\overline{g}_a)$ for some $\varepsilon > 0$, where $\overline{g}_a$ is a metric of constant Gauss curvature -1 in $\widehat{\Sigma}$.*

(iii) *If Σ has genus one, then Σ has constant Gauss curvature equal to zero and there exists a neighborhood which is isometric to $((-\varepsilon,\varepsilon)\times\widehat{\Sigma},g_a')$ for some $\varepsilon > 0$, where g_a' is a flat metric in M.*

Remark 1. We would like to observe that when $\Lambda = -2$, Σ has constant Gaussian curvature equal to $-\frac{1}{a^2}$ for $a > 1$ and in a neighborhood of Σ, (M^3, g) is isometric to $((-\varepsilon,\varepsilon)\times\Sigma, g_a)$ for some $\varepsilon > 0$. This is precisely the hyperbolic case of Theorem 1.4 in [20].

The outline of the paper is as follows: in the next section we give a critical point characterization of the Hawking mass of embedded surfaces in manifolds with scalar curvature bounded from below. After this, guided by the arguments in [20], we use the strictly stability and the second variation formula of the Hawking mass to construct a foliation of M around Σ by CMC embedded compact surfaces $\{\Sigma_t\}_{|t|<\varepsilon}$. Proceeding, we analyze the behavior of the Hawking mass in this family constructed and one conclude that it is nondecreasing along this foliation which implies our local rigidity result. Finally, the appendix is dedicate to deduce the variation formulae of the Hawking mass.

2. Hawking mass and rigidity

2.1. Characterization of critical points

We now choose ν a unit normal field on Σ as well as a variation of Σ along its normal direction ν in M with speed $\varphi \in C^\infty(\Sigma)$, in other words, we consider the following set $\Sigma_t = f(t,\Sigma)$, $|t| < \varepsilon$, where $f : (-\varepsilon,\varepsilon)\times\Sigma \to M$ is a smooth mapping satisfying

$$\frac{\partial}{\partial t} f(0,x) = \varphi(x)\nu(x)$$

and $f(0,x) = x$ for each $x \in \Sigma$, while $f_t = f(t,\cdot) : \Sigma \to M$ is an immersion for all $t \in (-\varepsilon,\varepsilon)$.

Assume that the scalar curvature of M is bounded from below, one can consider the following function defined on any properly embedded surface:

$$\mathcal{Q}(\Sigma) = -\frac{4\pi(g(\Sigma)-1)}{|\Sigma|} - K_\Sigma + \frac{1}{2}(R_g - \Lambda) + \frac{1}{2}\left(|A|^2 - \frac{1}{2|\Sigma|}\int_\Sigma H^2 d\sigma\right), \tag{2.1}$$

where A is the second fundamental form of Σ with respect to the unit normal ν and K_Σ is the Gaussian curvature of Σ.

Now, we prove a result which classifies critical points of the Hawking mass.

Theorem 2. *Let (M^3, g) be a Riemannian manifold with scalar curvature $R_g \geq \Lambda$ and let us consider $\Sigma \subset M$ a compact two-sided surface with non-negative(non positive) mean curvature. In addition, if it is a critical point for the Hawking mass, then*

(i) *either Σ is minimal;*

(ii) *or Σ is umbilic, $R_g = \Lambda$ in Σ and its Gaussian curvature satisfies $K_\Sigma = \frac{2\pi\chi(\Sigma)}{|\Sigma|}$.*

In particular, a closed two-sided surface Σ in $(\mathbb{R}\times\mathbb{S}^2, (g_{adss})_a)$, $(\mathbb{R}\times\widetilde{\Sigma}, g_a)$, $(\mathbb{R}\times\widetilde{\Sigma}, \overline{g}_a)$ or $(\mathbb{R}\times\widehat{\Sigma}, g'_a)$ with nonnegative mean curvature is a critical point for $\mathfrak{m}_{\mathcal{H}}(\Sigma)$, if and only if it is minimal or a slice.

Proof. Using the formula of the first variation of the Hawking mass presented in Proposition 3 of the appendix, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0}\mathfrak{m}_{\mathcal{H}}(\Sigma_t)\\
&= \frac{-2|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma \phi\Delta H d\sigma\\
&\quad + \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma \left[2K_\Sigma + \frac{8\pi(g(\Sigma)-1)}{|\Sigma|} + \frac{1}{2|\Sigma|}\int_\Sigma H^2 d\sigma - |A|^2\right] H\phi d\sigma\\
&\quad - \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma (R_g - \Lambda)H\phi d\sigma\\
&= -\frac{2|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma \big(\Delta H + \mathcal{Q}(\Sigma)H\big)\phi d\sigma.
\end{aligned}$$

Since Σ is a critical point we conclude

$$\int_{\Sigma}\big(\Delta H+\mathcal{Q}(\Sigma)H\big)\phi d\sigma=0. \tag{2.2}$$

By hypothesis, $H\geq 0$. Hence the Maximum Principle guarantees that either $H\equiv 0$ or $H>0$. Supposing Σ not minimal we have

$$\frac{1}{H}\Delta H+\mathcal{Q}(\Sigma)=0. \tag{2.3}$$

Therefore, after integrating this last identity over Σ we infer

$$\int_{\Sigma}\mathcal{Q}(\Sigma)d\sigma=-\int_{\Sigma}\frac{|\nabla H|^2}{H^2}d\sigma.$$

Since $R_g\geq\Lambda$ and $|A|^2\geq\frac{1}{2}H^2$ we use Gauss-Bonnet formula to arrive at $\int_{\Sigma}\mathcal{Q}(\Sigma)d\sigma\geq 0$.

This allow us to conclude that $\int_{\Sigma}\mathcal{Q}(\Sigma)d\sigma=0$. From where we conclude that H is constant. Now, from (2.3) we deduce that $\mathcal{Q}(\Sigma)=0$ which implies that Σ is umbilic, $R_g=\Lambda$ on Σ and $K_{\Sigma}=-\frac{4\pi(g(\Sigma)-1)}{|\Sigma|}$. □

An important remark is that the slices in the Kottler-Schwarzschild space have constant Hawking mass. Indeed, consider the slice Σ_r in $(\mathbb{R}\times\widehat{\Sigma},(g_{ks})_a)$, by a simple computation we have

$$\frac{d}{dr}\mathfrak{m}_{\mathcal{H}}(\Sigma_r)=\frac{1}{2}u_a'(r)(g(\Sigma)-1)(3u(r)^2+\widehat{k}-(u_a(r)')^2-2u_a(r)u_a(r)'').$$

Since u solves (1.2) the Kottler-Schwarzschild space has constant scalar curvature equal to -6, which gives that each slice has constant Hawking mass. Therefore, $\mathfrak{m}_{\mathcal{H}}(\Sigma_r)=\mathfrak{m}_{\mathcal{H}}(\Sigma_0)$, where $\Sigma_0=\{0\}\times\widehat{\Sigma}$.

This slice Σ_0 is a minimal surface that is strictly stable (see next section for a definition) for each Kottler-Schwarzschild space $(\mathbb{R}\times\widehat{\Sigma},(g_{ks})_a)$. Therefore, when $\Sigma_r\hookrightarrow(\mathbb{R}\times\mathbb{S}^2,(g_{adss})_a)$ we obtain $\mathfrak{m}_{\mathcal{H}}(\Sigma_r)=\frac{a}{4}(1+a^2)>0$. If $\Sigma_r\hookrightarrow(\mathbb{R}\times\widetilde{\Sigma},\overline{g}_a)$, then $\mathfrak{m}_{\mathcal{H}}(\Sigma_r)=\frac{-a(g(\Sigma)-1)^{3/2}}{2\sqrt{2}}(1-a^2)$. Finally, if $\Sigma_r\hookrightarrow(\mathbb{R}\times\widehat{\Sigma},g_a')$ we have that $\mathfrak{m}_{\mathcal{H}}(\Sigma_r)=a^2/4$ provided $|\widehat{\Sigma}|=4\pi$ occurs.

2.2. Rigidity result

Before proving the main theorem, a few remarks are necessary.

Let $\Sigma \subset M$ be a surface with associated Jacobi operator given by

$$L = \Delta + Ric(\nu,\nu) + |A|^2, \tag{2.4}$$

where Δ denotes the Laplacian of Σ in the induced metric and Ric is the Ricci curvature of M. For simplicity the first eigenvalue λ_1^L of L will be denoted by λ_1.

The Jacobi operator induces a quadratic form $Q(\varphi) = \int_\Sigma \varphi L\varphi d\sigma$ acting on the space $C^\infty(\Sigma)$ of smooth functions on Σ. An embedded surface Σ is called stable if and only if $-Q(\varphi) \geqslant 0$ for all $\varphi \in C^\infty(\Sigma)$, where the lowest eigenvalue of L is nonnegative. If the lowest eigenvalue is positive we say the surface is *strictly stable*. Equivalently, $-Q(\varphi) \geqslant \lambda_1 \int_\Sigma \varphi^2 d\sigma$, which becomes

$$\lambda_1 \int_\Sigma \varphi^2 d\sigma + \int_\Sigma (Ric(\nu,\nu) + |A|^2)\varphi^2 d\sigma \leq \int_\Sigma |\nabla_\Sigma \varphi|^2 d\sigma, \tag{2.5}$$

for any smooth function φ on Σ.

Proposition 1. *Let (M^3, g) be a Riemannian manifold with scalar curvature $R_g \geq \Lambda$. If $\Sigma \subset M$ is a two-sided compact strictly stable minimal surface which locally maximizes the Hawking mass, then*

$$\left(\lambda_1 + \frac{1}{2}\Lambda\right)|\Sigma| = 2\pi\chi(\Sigma).$$

Moreover, along Σ, we have $A = 0$, $R_g = \Lambda$, $Ric(\nu,\nu) = -\lambda_1$ and its Gaussian curvature $K_\Sigma = -\frac{4\pi(g(\Sigma)-1)}{|\Sigma|}$.

Proof. First of all we use the maximality of Σ in the second variation formula of the Hawking mass, see Proposition 4 in the appendix, to drop off some terms:

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0} \mathfrak{m}_{\mathcal{H}}(\Sigma_t) = {} & \frac{\mathfrak{m}_{\mathcal{H}}(\Sigma)}{2|\Sigma|}\left[\int_\Sigma (|\nabla\varphi|^2 - (Ric(\nu,\nu) + |A|^2))\varphi^2 d\sigma\right] \\ & - \frac{2|\Sigma|^{1/2}}{(16\pi)^{3/2}} \int_\Sigma (L\varphi)^2 d\sigma - \frac{2}{3}\Lambda \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \int_\Sigma |\nabla\varphi|^2 d\sigma \\ & + \frac{2}{3}\Lambda \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \int_\Sigma (Ric(\nu,\nu) + |A|^2)\varphi^2 d\sigma.\end{aligned}$$

Whence we deduce

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathfrak{m}_{\mathcal{H}}(\Sigma_t) = -\frac{8\pi\mathcal{X}(\Sigma)}{2(16\pi)^{3/2}|\Sigma|^{1/2}}\int_\Sigma \varphi L\varphi d\sigma - \frac{2}{3}\Lambda|\Sigma|\int_\Sigma \varphi L\varphi d\sigma$$
$$- 2\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma (L\varphi)^2 d\sigma + \frac{2}{3}\Lambda\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma \varphi L\varphi d\sigma.$$

By assumption that Σ locally maximizes the Hawking mass, we obtain

$$(4\pi\mathcal{X}(\Sigma) - \Lambda|\Sigma|)\int_\Sigma \varphi L\varphi d\sigma \geqslant -2|\Sigma|\int_\Sigma (L\varphi)^2 d\sigma.$$

Choosing an eigenfunction φ associated to λ_1 this last inequality becomes

$$-(4\pi\mathcal{X}(\Sigma) - \Lambda|\Sigma|)\lambda_1 \geqslant -2|\Sigma|\lambda_1^2.$$

Hence, we have

$$\left(\lambda_1 + \frac{1}{2}\Lambda\right)|\Sigma| \geq 2\pi\chi(\Sigma). \tag{2.6}$$

Next we derive the reverse of inequality (2.6) using the fact that Σ is strictly stable. We remark that in [23], this reverse inequality occurs for $\lambda_1 = 0$ when Σ is locally area-minimizing in its homotopy class.

Note that Gauss equation implies

$$R_g - 2Ric(\nu,\nu) = 2K_\Sigma + |A|^2. \tag{2.7}$$

Choosing $\varphi = 1$ in (2.5) and using (2.7) we deduce

$$\lambda_1|\Sigma| + \frac{1}{2}\int_\Sigma (R_g + |A|^2)d\sigma \leq \int_\Sigma K_\Sigma d\sigma.$$

Since $R_g \geq \Lambda$ we use Gauss-Bonnet formula on the last inequality to conclude

$$\lambda_1|\Sigma| + \frac{1}{2}\Lambda|\Sigma| \leq 2\pi\chi(\Sigma).$$

Therefore, we get the following equality

$$\left(\lambda_1 + \frac{1}{2}\Lambda\right)|\Sigma| = 2\pi\chi(\Sigma).$$

This allow us to conclude that Σ is infinitesimally rigid, i.e., along Σ, $R_g = \Lambda$, $Ric(\nu,\nu) = -\lambda_1$, its Gaussian curvature $K_\Sigma = -\frac{4\pi(g(\Sigma)-1)}{|\Sigma|}$ and Σ is totally geodesic, which finishes the proof of the proposition. □

In one of the seminal works concerning stability of hypersurfaces into Riemannian manifolds [3] the authors have chosen a variation of the hypersurface through the exponential mapping of the target space, however such a variation does not in general consist of hypersurfaces of constant mean curvature. But, this is possible to do. In fact, given an embedded, two-sided, infinitesimally rigid minimal surface Σ, one can construct a foliation $\{\Sigma_t\}_{|t|<\varepsilon}$ around $\Sigma = \Sigma_0$ by constant mean curvature surfaces, as an application of the Implicit Function Theorem. A similar approach can be found in [23] as well as in [20]. More specifically, choosing $\mu : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}$ by

$$\Sigma_t := \{\exp_x(\mu(t,x)\nu(x)); x \in \Sigma\},$$

then for each $t \in (-\varepsilon, \varepsilon)$, $\Sigma(t)$ has constant mean curvature as well as the same genus of Σ, where $\Sigma_0 = \Sigma$ and ν is a unit normal field along Σ. Moreover, the assertions of the next proposition also hold. As the proof of the next result is similar to Proposition in [23] we shall omit its proof here.

Proposition 2. *Let (M^3, g) be a Riemannian manifold with scalar curvature $R_g \geq \Lambda$. If $\Sigma \subset M$ is a stable compact embedded minimal surface satisfying*

$$\left(\lambda_1 + \frac{1}{2}\Lambda\right)|\Sigma| = 2\pi\chi(\Sigma),$$

then there exist $\varepsilon > 0$ and a smooth function $\mu : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$\Sigma_t := \{\exp_x(\mu(t,x)\nu(x)); x \in \Sigma\}$$

is a family of compact surfaces with constant mean curvature. Moreover, the following properties hold

$$\mu(0,x) = 0, \quad \frac{\partial \mu}{\partial t}(0,x) = 1 \quad \textit{and} \quad \int_\Sigma (\mu(t,\cdot) - t)d\sigma = 0$$

for each $x \in \Sigma$ and for each $t \in (\varepsilon, \varepsilon)$.

Let us define a mapping $f_t : \Sigma \to M$ given by $f_t(x) = \exp_x\big(\mu(t,x)\nu(x)\big)$ for each $t \in (-\varepsilon, \varepsilon)$. We note that $\Sigma_0 = \Sigma$ for each $x \in \Sigma$. Let $\nu_t(x)$ be a unit normal field along Σ_t such that $\nu_0(x) = \nu(x)$ for all $x \in \Sigma$ and let $d\sigma_t$ denote the area element of Σ_t in the induced metric by f_t.

We also consider the Jacobi operator

$$L(t) = \Delta_t + Ric(\nu_t, \nu_t) + |A_t|^2,$$

where Δ_t is the Laplacian of Σ_t in the induced metric while A_t stands for the second fundamental form of Σ_t with respect to ν_t.

For each $t \in (-\varepsilon, \varepsilon)$, let $H(t)$ denote the mean curvature of Σ_t with respect to ν_t, and define $\rho_t : \Sigma \to \mathbb{R}$ by

$$\rho_t(x) = \left\langle \nu_t(x), \frac{\partial}{\partial t} f_t(x) \right\rangle .$$

We now state the next lemma which corresponds to Theorem 3.2 of [17].

Lemma 1. *The function $\rho_t(x)$ satisfies $H'(t) = L(t)\rho_t$.*

From Proposition 2 it is easy to see that $\rho_0 = 1$. Therefore, using Proposition 1 and the last lemma we deduce that for $t \in (-\varepsilon, \varepsilon)$ the mean curvature $H(t)$ of this CMC foliation $\Sigma_t \subset M$ in a neighborhood of Σ obeys

$$\frac{d}{dt}\Big|_{t=0} H(t) = -\lambda_1 < 0. \tag{2.8}$$

Reducing ε if necessary, we conclude that $H(t) < 0$ for $t \in (0, \varepsilon)$ and $H(t) > 0$ for $t \in (-\varepsilon, 0)$, in other words, H is a decreasing function on t. With these settings we have the next lemma that will be used in order to deduce the monotonicity of the Hawking mass along the foliation Σ_t.

We point out that the next lemma is more general than Lemma 5.2 in [20], where the authors showed that each Σ_t are weakly stable for suitable ε. In this work, we do not need to change the approach depending on the sign of the infimum of the scalar curvature of M as well as in the theory of rigidity of area-minimizing surfaces.

Lemma 2.

$$\begin{aligned}\int_{\Sigma_t} (Ric(\nu_t, \nu_t) + |A_t|^2)\rho_t d\sigma_t = \overline{\rho_t} \int_{\Sigma_t} (Ric(\nu_t, \nu_t) + |A_t|^2) d\sigma_t \\ + H'(t)\theta(t, x) + \overline{\rho_t} \int_{\Sigma_t} \frac{|\nabla \rho_t|^2}{\rho_t^2} d\sigma_t,\end{aligned}$$

where $\theta(t, x)$ is a non positive function, $\overline{\rho_t} = \frac{1}{|\Sigma_t|}\int_{\Sigma_t} \rho_t d\sigma_t$ and $\{\Sigma_t\}_{t \in (-\varepsilon, \varepsilon)}$ is given as in Proposition 2

Proof. It suffices to use Holder inequality to see that $\theta(t,x) = |\Sigma_t| - \overline{\rho_t}\int_{\Sigma_t}\frac{1}{\rho_t}d\sigma_t$ is non positive. Now integrating $H'(t)$ and using Lemma 1 we complete the proof of the lemma. □

2.3. Proof of Theorem 1

In this subsection we present the proof of our main theorem.

Proof. Let (M^3, g) be a Riemannian three-manifold that contains a surface Σ under our assumption. By Proposition 1, the Jacobi operator is given by $L = \Delta - \lambda_1$ and by Proposition 2, we can construct a constant mean curvature foliation $\{\Sigma_t\}_{|t|<\varepsilon}$ around $\Sigma = \Sigma_0$. Applying Lemma 2 in the formula of the first variation of the Hawking mass, we deduce

$$\begin{aligned}\frac{d}{dt}\mathfrak{m}_{\mathcal{H}}(\Sigma_t) &= -\frac{|\Sigma(t)|^{1/2}}{32\pi^{3/2}}H(t)\Big[2\pi\mathcal{X}(\Sigma_t)\overline{\rho_t} - \frac{3}{4}H_t^2\int_{\Sigma(t)}\rho_t d\sigma_t \\ &\qquad + \int_{\Sigma_t}\rho_t d\sigma_t + \int_{\Sigma_t}(Ric(\nu_t,\nu_t) + |A_t|^2)\rho_t d\sigma_t\Big] \\ &= -\frac{|\Sigma(t)|^{1/2}}{32\pi^{3/2}}H(t)\Big[2\pi\mathcal{X}(\Sigma_t)\overline{\rho_t} - \frac{3}{4}H^2(t)\int_{\Sigma(t)}\rho_t d\sigma_t \\ &\qquad + \int_{\Sigma_t}\rho_t d\sigma_t + \overline{\rho_t}\int_{\Sigma_t}(Ric(\nu_t,\nu_t) + |A_t|^2)d\sigma_t \\ &\qquad + H'(t)\theta(x,t) + \overline{\rho_t}\int_{\Sigma_t}\frac{|\nabla\rho_t|^2}{\rho_t^2}d\sigma_t\Big].\end{aligned}$$

Next we use equation (2.7) in the above identity to derive

$$\begin{aligned}\frac{d}{dt}\mathfrak{m}_{\mathcal{H}}(\Sigma_t) &= -\frac{|\Sigma_t|^{1/2}}{32\pi^{3/2}}H(t)\Big[\frac{\overline{\rho_t}}{2}\int_{\Sigma_t}(R_g - \Lambda) + \left(|A_t|^2 - \frac{H^2(t)}{2}\right)d\sigma_t \\ &\qquad + H'(t)\theta(t,x) + \overline{\rho_t}\int_{\Sigma_t}\frac{|\nabla\rho_t|^2}{\rho_t^2}d\sigma_t\Big].\end{aligned} \tag{2.9}$$

Since $\rho_0(x) = 1$, reducing ε if necessary, we still have $\rho_t(x) > 0$ for each $x \in \Sigma_t$ and $t \in (-\varepsilon, \varepsilon)$. So that $\frac{d}{dt}\mathfrak{m}_{\mathcal{H}}(\Sigma_t) \geq 0$ for $t \in [0, \varepsilon)$ and $\frac{d}{dt}\mathfrak{m}_{\mathcal{H}}(\Sigma(t)) \leq 0$ for $t \in (-\varepsilon, 0]$. Therefore, we obtain

$$\mathfrak{m}_{\mathcal{H}}(\Sigma_t) \geq \mathfrak{m}_{\mathcal{H}}(\Sigma)$$

for $t \in (-\varepsilon, \varepsilon)$. Meanwhile, taking into account that Σ locally maximizes the Hawking mass, we conclude that $\mathfrak{m}_{\mathcal{H}}(\Sigma) \geq \mathfrak{m}_{\mathcal{H}}(\Sigma_t)$. Therefore, the equality occurs and $\frac{d}{dt}\mathfrak{m}_{\mathcal{H}}(\Sigma_t) \equiv 0$. Thus we conclude that Σ_t is umbilic and $R_g = \Lambda$ along Σ_t. This implies that ρ_t is constant. In fact, for $t \in (-\varepsilon, \varepsilon)$ we make use of (2.9) to infer

$$H'(t)\theta(t,x) + \overline{\rho_t} \int_{\Sigma_t} \frac{|\nabla \rho_t|^2}{\rho_t^2} = 0.$$

Since $H'(t)\theta(t,x) \geq 0$ we complete our claim, in particular $\rho_t = 1$ in $t \in (-\varepsilon, \varepsilon)$. Thus, up to isometry, the metric in a sufficiently small neighborhood of Σ can be written as $g = dt^2 + g_{\Sigma_t}$.

According to [17] the induced metric on Σ_t evolves as

$$\frac{\partial}{\partial t}(g_{ij})_t = -2\rho_t (A_{ij})_t,$$

where $g_{\Sigma_t} = (g_{ij})_t$ is the induced metric on Σ_t. Since $\rho_t \equiv 1$, Σ_t is umbilic and $H(t)$ is constant we obtain

$$\frac{\partial}{\partial t} g_{\Sigma_t} = -2H(t) g_{\Sigma_t}$$

for all $t \in (-\varepsilon, \varepsilon)$.

Next, under assumption of item i) we arrive at the remarkable equation $g_{\Sigma_t} = u_a(t)^2 g_{\mathbb{S}^2}$ for all $t \in (-\varepsilon, \varepsilon)$, where $u_a(t) = ae^{-\int_0^t H(s)ds}$ and $a^2 = \frac{|\Sigma|}{4\pi}$. Then, by uniqueness of solutions of the corresponding ODE, it follows that M is isometric to a piece of the anti-de Sitter-Schwarzschild space with positive mass and we can conclude that the induced metric by $f(t,x)$ on $(-\varepsilon, \varepsilon) \times \mathbb{S}^2$ is given by $dt^2 + u_a(t)^2 g_{\mathbb{S}^2}$. Analogously in item (ii), we also infer $g_{\Sigma_t} = u_a(t)^2 g_{\widehat{\Sigma}}$ for all $t \in (-\varepsilon, \varepsilon)$, where $u_a(t) = ae^{-\int_0^t H(s)ds}$ with $a^2 = \frac{|\Sigma|}{4\pi(g(\Sigma)-1)}$. It is not difficult to show that $a^2 > \frac{1}{3}$. Finally, in the last item we have that $g_{\Sigma_t} = u_a(t)^2 g_{\widehat{\Sigma}}$ for all $t \in (-\varepsilon, \varepsilon)$, $u_a(t) = ae^{-\int_0^t H(s)ds}$ with $a^2 = \frac{|\Sigma|}{4\pi}$, where we have used that $|\widehat{\Sigma}| = 4\pi$. □

The metric $\{g_a\}_{a>1}$ converges to a product metric when a tends to 1. Thus the rigidity theorem due to [23] can be deduced for stable surfaces that locally maximizes the Hawking mass. We also mention that a similar splitting for three manifolds with non negative scalar curvature which contains an embedded orientable strictly stable two torus that locally maximizes the Hawking mass does not exist (in analogy with [11]) provided there are no such surfaces satisfying this condition.

3. Appendix: Variation formulae

For sake of completeness we present the proofs of the first and the second variation formulae for the Hawking mass.

We start with the following proposition in order to compute the first variation formula. Before, we recall some important basic facts.

The following identities are well known:

$$\frac{d}{dt}\Big|_{t=0} d\sigma_t = -\varphi H d\sigma. \tag{3.1}$$

and

$$Ric(\nu,\nu) + |A|^2 = R_g/2 - K_\Sigma + H^2/2 + |A|^2/2. \tag{3.2}$$

Assertion (3.1) can be found in [17] while (3.2) follows from Gauss equation.

Proposition 3 (First variation of the Hawking mass). *Under the considerations of (1.1) we have*

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathfrak{m}_{\mathcal{H}}(\Sigma_t) = & \frac{-2|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma \varphi\Delta H d\sigma - \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma (R_g - \Lambda)H\varphi d\sigma \\ & + \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma \Bigg[2K_\Sigma + \frac{8\pi(g(\Sigma)-1)}{|\Sigma|} \\ & \qquad + \left(\frac{1}{2|\Sigma|}\int_\Sigma H^2 d\sigma - |A|^2\right)\Bigg]H\varphi d\sigma,\end{aligned}$$

where ν_t is a unit normal vector field along Σ_t with $\nu_0 = \nu$.

Proof. It suffices to take the derivative of the Hawking mass given in (1.1) for the variation f_t of Σ given previously to obtain

$$\begin{aligned}\frac{d}{dt}\mathfrak{m}_{\mathcal{H}}(\Sigma_t) = & \frac{1}{2}\left(8\pi\mathcal{X}(\Sigma) - \int_{\Sigma_t}\left(H^2(t) + \frac{2}{3}\Lambda\right)d\sigma_t\right)\frac{|\Sigma_t|^{-1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}\frac{d}{dt}(d\sigma_t) \\ & - \frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\left(\int_{\Sigma_t} 2H(t)H'(t)d\sigma_t + \int_{\Sigma_t}\left(H^2(t) + \frac{2}{3}\Lambda\right)\frac{d}{dt}(d\sigma_t)\right),\end{aligned}$$

where $H(t)$ and $d\sigma_t$ stand for the mean curvature and the area element of Σ_t, respectively.

Whence, choosing $t = 0$ and using identity (3.1) we deduce the first variation formula for the Hawking mass. □

In order to proceed we need the next lemma.

Lemma 3. *Let ω be the 1-form on Σ given by $\omega(X) = Ric(X, \nu)$ and $Y(x) = \frac{\partial f}{\partial t}(0, x)$, then the following assertions hold*

1) $\frac{d}{dt}\big|_{t=0} K_{\Sigma_t} = -\langle A, Hess\ \varphi\rangle + H\Delta\varphi + 2\omega(\nabla\varphi) + \mathrm{div}_\Sigma(\mathrm{div}_\Sigma\ \omega)\varphi + \mathrm{H}\mathrm{K}_\Sigma\varphi$, *where K_{Σ_t} is the Gaussian curvature of Σ_t with respect to the induced metric;*
2) $\left(\frac{d}{dt}\big|_{t=0}\Delta_{\Sigma_t}\right)\varphi = 2\varphi\langle A, Hess\ \varphi\rangle + 2A(\nabla\varphi, \nabla\varphi) - H|\nabla\varphi|^2 + \varphi\langle\nabla H, \nabla\varphi\rangle - 2\varphi\omega(\nabla\varphi)$;
3) $\frac{d}{dt}\big|_{t=0}|A_{\Sigma_t}|^2 = 2\langle A, Hess\varphi\rangle + 2R_{i\nu\nu j}A_{ij}\varphi + 2A_{ij}A_{ik}A_{jk}\varphi$;
4) $L'(0)\varphi = 2\varphi\langle A, Hess\ \varphi\rangle + 2\varphi\langle A, Hess\varphi\rangle + 2R_{i\nu\nu j}A_{ij}\varphi^2 + 2\varphi^2 A_{ij}A_{ik}A_{jk} - 2\varphi\omega(\nabla\varphi) - \varphi^2\mathrm{div}_\Sigma(\mathrm{div}_\Sigma\ \omega) - \mathrm{H}\mathrm{K}_\Sigma\varphi^2 + 2\mathrm{A}(\nabla\varphi, \nabla\varphi) - \mathrm{H}|\nabla\varphi|^2 + \varphi\langle\nabla\mathrm{H}, \nabla\varphi\rangle - 2\varphi\omega(\nabla\varphi) + \varphi^2\mathrm{HRic}(\nu, \nu) + \varphi^2\mathrm{H}|\mathrm{A}|^2$;
5) $\frac{d^2}{dt^2}\Big|_{t=0} d\sigma_t = [-\varphi L\varphi + H^2\varphi^2 + \mathrm{div}_\Sigma(\nabla_\mathrm{Y}\mathrm{Y})]\mathrm{d}\sigma$.

Proof. The first item is a consequence of Lemma 3.7 in [12] whereas the remainder items can be founded in [18]. □

We are now ready to state the proposition that gives the second variation formula.

Proposition 4. *Let $\Sigma \subset M$ be a critical point for the Hawking mass, then it holds:*

$$\begin{aligned}
&\frac{d^2}{dt^2}\Big|_{t=0}\mathfrak{m}_\mathcal{H}(\Sigma_t) \\
&= -\frac{3}{4}\frac{\mathfrak{m}_\mathcal{H}(\Sigma)}{|\Sigma|^2}\left(\int_\Sigma \varphi H d\sigma\right)^2 + \frac{1}{2}\frac{\mathfrak{m}_\mathcal{H}(\Sigma)}{|\Sigma|}\int_{\Sigma_t}|\nabla\varphi|^2 d\sigma \\
&\quad -\frac{1}{2}\frac{\mathfrak{m}_\mathcal{H}(\Sigma)}{|\Sigma|}\int_{\Sigma_t}(Ric(\nu,\nu) + |A|^2 - H^2)\varphi^2 d\sigma - \frac{2|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma (L\varphi)^2 d\sigma \\
&\quad -\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma\left(H^2 + \frac{2}{3}\Lambda\right)|\nabla\varphi|^2 d\sigma + \frac{4|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma H^2\varphi L\varphi d\sigma \\
&\quad +\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma\left(H^2 + \frac{2}{3}\Lambda\right)(Ric(\nu,\nu) + |A|^2 - H^2)\varphi^2 d\sigma \\
&\quad -\frac{2|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma HL'(0)\varphi d\sigma.
\end{aligned}$$

Before to begin the proof we fix some notations. Letting $S_t = Ric(\nu_t, \nu_t) + |A_t|^2$ and $U_t = 8\pi\mathcal{X}(\Sigma_t) - \int_{\Sigma_t}\Big(H^2(t) + \frac{2}{3}\Lambda\Big)d\sigma_t$ and denoting S_0 and U_0 by S and U, respectively. We infer

$$\begin{aligned}\frac{d}{dt}\mathfrak{m}_{\mathcal{H}}(\Sigma_t) = \frac{1}{2}\frac{|\Sigma_t|^{-1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t} U_t\frac{d}{dt}(d\sigma_t)\\ - \frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\Bigg(\int_{\Sigma_t} 2H(t)(\Delta\varphi_t + S_t\varphi_t)d\sigma_t\\ - \int_{\Sigma_t}\Big(H^2(t) + \frac{2}{3}\Lambda\Big)\frac{d}{dt}(d\sigma_t)\Bigg).\end{aligned}$$

Next we define the following functions:

- $\mathcal{F}_1 = \frac{1}{2}\frac{|\Sigma_t|^{-1/2}}{(16\pi)^{3/2}}\Big(\int_{\Sigma_t}\frac{d}{dt}(d\sigma_t)\Big)U_t$
- $\mathcal{F}_2 = -\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t} 2H(t)\Delta_t\varphi_t d\sigma_t$
- $\mathcal{F}_3 = -\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}(R_g - 2K_{\Sigma_t} + H^2(t) + |A_t|^2)\varphi_t H(t)d\sigma_t$
- $\mathcal{F}_4 = -\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}\Big(H^2(t) + \frac{2}{3}\Lambda\Big)\frac{d}{dt}(d\sigma_t)$

In order to deduce the proof of the second variation formula we shall need a couple of lemmas concerning these functions.

Lemma 4.

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathcal{F}_1 = -\frac{1}{4}\frac{|\Sigma|^{-3/2}}{(16\pi)^{3/2}}U\left(\int_\Sigma \varphi H d\sigma\right)^2\\ + \frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}U\int_\Sigma(|\nabla\varphi|^2 - S\varphi^2 + H^2\varphi^2 + \mathrm{div}_\Sigma(\nabla_{\mathrm{Y}}\mathrm{Y}))\mathrm{d}\sigma\\ + \frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}\Bigg(\int_\Sigma \varphi H d\sigma\Bigg[\int_\Sigma(2H\Delta\varphi + 2SH\varphi)d\sigma\\ - \int_\Sigma\Big(H^2 + \frac{2}{3}\Lambda\Big)(\varphi H)d\sigma\Bigg]\Bigg).\end{aligned}$$

Proof. Taking derivative of $\mathcal{F}_1$ and using Lemma 1, we arrive at

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1 = &-\frac{1}{4}\frac{|\Sigma_t|^{-3/2}}{(16\pi)^{3/2}}U_t\int_{\Sigma_t}\frac{d}{dt}(d\sigma_t)\int_{\Sigma_t}\frac{d}{dt}(d\sigma_t) + \frac{1}{2}\frac{|\Sigma_t|^{-1/2}}{(16\pi)^{3/2}}U\int_{\Sigma_t}\frac{d^2}{dt^2}(d\sigma_t)\\ &+\frac{1}{2}\frac{|\Sigma_t|^{-1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}\frac{d}{dt}(d\sigma_t)\Big[\int_{\Sigma_t}-(2H(t)\Delta_t\varphi_t + 2S_tH(t)\varphi_t)d\sigma_t\\ &\qquad -\int_{\Sigma_t}\left(H^2(t)+\frac{2}{3}\Lambda\right)\frac{d}{dt}(d\sigma_t)\Big].\end{aligned}$$

Now it suffices to apply at $t=0$ and make use of Lemma 3 to finish the proof of the lemma. □

Lemma 5.

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathcal{F}_2 = &\frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}\int_\Sigma \varphi H d\sigma\int_\Sigma 2H\Delta\varphi d\sigma\\ &-\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\Big(\int_\Sigma(2(\Delta\varphi)^2+2\varphi S\Delta\varphi)d\sigma\\ &+\int_\Sigma 2H(2\varphi\langle A, Hess\varphi\rangle + 2A(\nabla\varphi,\nabla\varphi)\\ &-H\langle\nabla\varphi,\nabla\varphi\rangle + \varphi\langle\nabla H,\nabla\varphi\rangle - 2\varphi\omega(\nabla\varphi))d\sigma\\ &+\int_\Sigma 2H\Delta\frac{d}{dt}(\varphi_t)\Big|_{t=0}d\sigma - \int_\Sigma 2H^2(\Delta\varphi)\varphi d\sigma\Big).\end{aligned}$$

Proof. Arguing as before, we derive $\mathcal{F}_2$ to obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2 = &-\frac{1}{2}\frac{|\Sigma_t|^{-1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}\frac{d}{dt}d\sigma_t\int_{\Sigma_t}2H(t)\Delta_t\varphi_t(d\sigma_t)\\ &-\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\Big(\int_{\Sigma_t}2H_t'\Delta_t\varphi_t d\sigma_t + \int_{\Sigma_t}2H(t)\frac{d}{dt}(\Delta_t\varphi_t)d\sigma_t\\ &+\int_{\Sigma_t}2H(t)\Delta_t\varphi_t\frac{d}{dt}(d\sigma_t)\Big).\end{aligned}$$

Now we apply at $t=0$ and using once more Lemma 1 we complete the proof of the lemma. □

Lemma 6.

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\mathcal{F}_3 = &-\frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}\int_\Sigma(-\varphi H)d\sigma\int_\Sigma 2S\varphi H d\sigma \\
&-\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\Bigg[\int_\Sigma \frac{d}{dt}R_g\Big|_{t=0} - 2(-\langle A, Hess\varphi\rangle + H\Delta\varphi \\
&\qquad + 2\omega(\nabla\varphi) + \mathrm{div}_\Sigma(\mathrm{div}_\Sigma\ \omega)\varphi + \mathrm{HK}_\Sigma\varphi) \\
&\qquad + 2H(\Delta\varphi + S\varphi) + (2\langle A, Hess\varphi\rangle \\
&\qquad + 2R_{i\nu\nu j}A_{ij}\varphi + 2A_{ij}A_{ik}A_{jk}\varphi)\Bigg]\varphi H d\sigma_t \\
&-\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma 2S\varphi\Delta\varphi + 2S^2\varphi^2 d\sigma + \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma 2SH^2\varphi^2 d\sigma \\
&-\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma 2SH\frac{d}{dt}(\varphi_t)\Big|_{t=0} d\sigma.
\end{aligned}
$$

Proof. As before we use Lemma 1 after deriving $\mathcal{F}_3$ to obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}_3 = &-\frac{1}{2}\frac{|\Sigma_t|^{-1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}\frac{d}{dt}(d\sigma_t)\int_{\Sigma_t}2S_t\varphi_t H(t)d\sigma_t \\
&-\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\Bigg[\int_{\Sigma_t}\frac{d}{dt}R_g - 2\frac{d}{dt}K_{\Sigma_t} \\
&\qquad + 2H(t)(\Delta_t\varphi_t + S_t\varphi_t) + \frac{d}{dt}|A_t|^2\Bigg]\varphi_t H(t)d\sigma_t \\
&-\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}2S_t(\Delta_t\varphi_t + S_t\varphi_t)\varphi_t d\sigma_t \\
&-\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}2S_tH(t)\varphi_t\frac{d}{dt}(d\sigma_t) \\
&-\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}2S_tH(t)\frac{d}{dt}(\varphi_t)d\sigma_t.
\end{aligned}
$$

Reasoning as before we consider $t=0$ and make use once more of Lemma 3. □

Finally we present the last lemma.

Lemma 7.

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0}\mathcal{F}_4\\
&=\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma(2H^2\varphi\Delta\varphi+2SH^2\varphi^2)d\sigma\\
&\quad-\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma\left(H^2+\frac{2}{3}\Lambda\right)(|\nabla\varphi|^2-S\varphi^2+H^2\varphi^2+\mathrm{div}_\Sigma(\nabla_{\mathrm{Y}}\mathrm{Y}))d\sigma\\
&\quad-\frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}\int_\Sigma\varphi H d\sigma\left[\int_\Sigma\left(H^2+\frac{2}{3}\Lambda\right)\varphi H d\sigma\right].
\end{aligned}
$$

Proof. Following the same steps we derive $\mathcal{F}_4$ and using Lemma 1 we infer

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}_4=&-\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}(2H(t)\Delta_t\varphi_t+2S_tH(t)\varphi_t)\frac{d}{dt}(d\sigma_t)\\
&-\frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}\left(H^2(t)+\frac{2}{3}\Lambda\right)\frac{d^2}{dt^2}(d\sigma_t)\\
&-\frac{1}{2}\frac{|\Sigma_t|^{-1/2}}{(16\pi)^{3/2}}\int_{\Sigma_t}\frac{d}{dt}(d\sigma_t)\left[\int_{\Sigma_t}\left(H^2(t)+\frac{2}{3}\Lambda\right)\frac{d}{d}(d\sigma_t)\right].
\end{aligned}
$$

Proceeding exactly as in the previous lemma we complete the proof of Lemma 7. □

3.1. Proof of Proposition 4

Proof. The proof is a combination of the last four lemmas. Indeed, we can write

$$
\begin{aligned}
&\frac{d^2}{dt^2}\Big|_{t=0}\mathfrak{m}_{\mathcal{H}}(\Sigma)\\
&=-\frac{1}{4}\frac{\mathfrak{m}_{\mathcal{H}}(\Sigma)}{|\Sigma|^2}\left(\int_\Sigma\varphi H d\sigma\right)^2+\frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}S\int_{\Sigma_t}(|\nabla\varphi|^2-S\varphi^2+H^2\varphi^2)d\sigma\\
&\quad+\frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}\left(\int_\Sigma\varphi H d\sigma\left[\int_\Sigma(2H\Delta\varphi+2SH\varphi)-\left(H^2+\frac{2}{3}\Lambda\right)(\varphi H)d\sigma\right]\right)\\
&\quad+\frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}\int_\Sigma\varphi H d\sigma\int_\Sigma 2H\Delta\varphi d\sigma+\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma 2H^2\varphi\Delta\varphi d\sigma\\
&\quad-\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma(2(\Delta\varphi)^2+4S\varphi\Delta\varphi+2S^2\varphi^2)d\sigma
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}\int_\Sigma \varphi H d\sigma \int_\Sigma 2S\varphi H d\sigma - \frac{2|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma HL'(0)\varphi d\sigma\\
&+\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma (2H^2\varphi\Delta\varphi + 4SH^2\varphi^2)d\sigma\\
&-\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma \left(H^2+\frac{2}{3}\Lambda\right)(|\nabla\varphi|^2 - S\varphi^2 + H^2\varphi^2)d\sigma\\
&-\frac{1}{2}\frac{|\Sigma|^{-1/2}}{(16\pi)^{3/2}}\int_\Sigma \varphi H d\sigma\left[\int_\Sigma\left(H^2+\frac{2}{3}\Lambda\right)\varphi H d\sigma\right].
\end{aligned}
$$

On the other hand, since Σ is a critical point for the Hawking mass we have

$$
\int_\Sigma 2(H\Delta\varphi + SH\varphi) - \left(H^2+\frac{2}{3}\Lambda\right)(\varphi H)d\sigma = -\frac{1}{2}|\Sigma|^{-1}\int_\Sigma \varphi H d\sigma.
$$

Therefore we deduce

$$
\begin{aligned}
&\frac{d^2}{dt^2}\Big|_{t=0}\mathfrak{m}_{\mathcal{H}}(\Sigma)\\
&=-\frac{3}{4}\frac{\mathfrak{m}_{\mathcal{H}}(\Sigma)}{|\Sigma|^2}\left(\int_\Sigma \varphi H d\sigma\right)^2 + \frac{1}{2}\frac{\mathfrak{m}_{\mathcal{H}}(\Sigma)}{|\Sigma|}\int_{\Sigma_t}|\nabla\varphi|^2 d\sigma\\
&\quad-\frac{1}{2}\frac{\mathfrak{m}_{\mathcal{H}}(\Sigma)}{|\Sigma|}\int_{\Sigma_t}(S-H^2)\varphi^2 d\sigma - \frac{2|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma (L\varphi)^2 d\sigma\\
&\quad-\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma\left(H^2+\frac{2}{3}\Lambda\right)|\nabla\varphi|^2 d\sigma + \frac{4|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma H^2\varphi L\varphi d\sigma\\
&\quad+\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma\left(H^2+\frac{2}{3}\Lambda\right)(S-H^2)\varphi^2 d\sigma\\
&\quad-\frac{2|\Sigma|^{1/2}}{(16\pi)^{3/2}}\int_\Sigma HL'(0)\varphi d\sigma,
\end{aligned}
$$

that was to be proved. □

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