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Optimal cobordisms between torus knots

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We construct cobordisms of small genus between torus knots and use them to determine the cobordism distance between torus knots of small braid index. In fact, the cobordisms we construct arise as the intersection of a smooth algebraic curve in $\mathbb{C}^2$ with the unit 4-ball from which a 4-ball of smaller radius is removed. Connections to the realization problem of A_n-singularities on algebraic plane curves and the adjacency problem for plane curve singularities are discussed. To obstruct the existence of cobordisms, we use Ozsváth, Stipsicz, and Szabó's Υ-invariant, which we provide explicitly for torus knots of braid index 3 and 4.

1. Introduction

For a knot K—a smooth and oriented embedding of the unit circle S^1 into the unit 3-sphere S^3—the *slice genus* $g_4(K)$ is the minimal genus of smooth, oriented surfaces F in the closed unit 4-ball B^4 with oriented boundary $K \subset \partial B^4 = S^3$. For torus knots, the slice genus is equal to their genus g; i.e., for coprime positive integers p and q, one has

$$g_4(T_{p,q}) = g(T_{p,q}) = \frac{(p-1)(q-1)}{2}. \tag{1}$$

More generally, the slice genus is known for knots K that arise as the transversal intersection of S^3 with a smooth algebraic curve V_f in $\mathbb{C}^2$, by Kronheimer and Mrowka's resolution of the Thom conjecture [KM93, Corollary 1.3]: the surface in $B^4 \subset \mathbb{C}^2$ given as the intersection of B^4 with V_f has genus $g_4(K)$; see also Rudolph's slice-Bennequin inequality [Rud93]. We determine the slice genus for connected sums of torus knots of small braid index; see Corollary 3.

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The *cobordism distance* $d_c(K,T)$ between two knots K and T is the minimal genus of smoothly embedded and oriented surfaces C in $S^3 \times [0,1]$ with boundary $K \times \{0\} \cup T \times \{1\}$ such that the induced orientation agrees with the orientation of T and disagrees with the orientation of K. Such a C is called a *cobordism* between K and T. Equivalently, $d_c(K,T)$ can be defined as the slice genus of the connected sum $K\sharp -T$ of K and $-T$—the mirror image of T with reversed orientation. Cobordism distance satisfies the triangle inequality; in particular, $d_c(K,T) \geq |g_4(T) - g_4(K)|$ for all knots K and T. We call a cobordism between two knots *algebraic* if it arises as the intersection of a smooth algebraic curve in $\mathbb{C}^2$ with $\overline{B_2^4 \backslash B_1^4} \cong S^3 \times [0,1]$, where $B_i^4 \subset \mathbb{C}^2$ are the 4-balls centered at the origin of radius r_i for some $0 < r_1 < r_2$. By the Thom conjecture, algebraic cobordisms between two knots K and T have genus $|g_4(T) - g_4(K)|$. In particular, the existence of an algebraic cobordism between K and T does determine their cobordism distance to be $|g_4(T) - g_4(K)|$. We call a cobordism C between two knots K and T *optimal* if its genus $g(C)$ is $|g_4(T) - g_4(K)|$.

We address the existence of algebraic and optimal cobordisms for torus knots.

Theorem 1. *For positive torus knots $T_{2,n}$ and $T_{3,m}$ of braid index 2 and 3, respectively, the following are equivalent.*

(I) There exists an optimal cobordism between $T_{2,n}$ and $T_{3,m}$; i.e.

$$d_c(T_{2,n}, T_{3,m}) = g_4(T_{2,n}\sharp - T_{3,m}) = |g_4(T_{3,m}) - g_4(T_{2,n})| = \left| m - 1 - \frac{n-1}{2} \right|.$$

(II) $n \leq \frac{5m-1}{3}$.

(III) There exists an algebraic cobordism between $T_{2,n}$ and $T_{3,m}$.

Theorem 2. *For positive torus knots $T_{2,n}$ and $T_{4,m}$ of braid index 2 and 4, respectively, the following are equivalent.*

(I) There exists an optimal cobordism between $T_{2,n}$ and $T_{4,m}$; i.e.

$$d_c(T_{2,n}, T_{4,m}) = g_4(T_{2,n}\sharp - T_{4,m}) = |g_4(T_{4,m}) - g_4(T_{2,n})| = \left| \frac{3m-n-2}{2} \right|.$$

(II) $n \leq \frac{5m-3}{2}$.

(III) There exists an algebraic cobordism between $T_{2,n}$ and $T_{4,m}$.

Theorem 1 and Theorem 2 are established using the following strategy. If (II) holds, we provide an explicit construction of the optimal cobordism using positive braids. In fact, the optimal cobordisms we find, can be seen in S^3 as a sequence of positive destabilizations on the fiber surfaces of the larger-genus knot; see Remark 19. If (II) does not hold, we use the υ-invariant to obstruct the existence of an optimal cobordism. Here, $\upsilon = \Upsilon(1)$ is one of a family $\Upsilon(t)$ of concordance invariants introduced by Ozsváth, Stipsicz, and Szabó [OSS14], which generalize Ozsváth and Szabó's τ-invariant as introduced in [OS03]. Finally, all optimal cobordisms we construct turn out to be algebraic. We establish a more general result, see Lemma 6, for all knots that arise as the transversal intersection of S^3 with a smooth algebraic curve in $\mathbb{C}^2$: the natural way of constructing optimal cobordisms always yields algebraic cobordisms. This brings us to ask: if there exists an optimal cobordism between two positive torus knots, does there exist an algebraic cobordism between them? The proof of Lemma 6 uses realization results of Orevkov and Rudolph. Using deplumbing, we also construct algebraic cobordisms between $T_{2,n}$ and $T_{m,m+1}$; see Section 5.2. This is related to work of Orevkov [Ore12], see Remark 24, and motivated by algebraic geometry questions; see Section 2.

We now turn to the cobordism distance between torus knots. By gluing together the optimal cobordisms given in Theorems 1 and 2, we will obtain the following.

Corollary 3. *Let K and T be torus knots such that the sum of their braid indices is 6 or less. Then we have the following formula for their cobordism distance.*

$$d_c(T, K) = g_4(K \sharp -T) = \max\{|\tau(K) - \tau(T)|, |\upsilon(K) - \upsilon(T)|\}.$$

The values of the concordance invariants τ and υ that arise in Corollary 3 are explicitly calculable: For positive torus knots (for their negative counterparts, which are obtained by taking the mirror image, the sign changes), one has

$$\tau(T_{p,q}) = -g(T_{p,q}) = -\frac{(p-1)(q-1)}{2} \text{ [OS03, Corollary 1.7]},$$

for coprime positive integers p and q, and

$$\begin{aligned} &\upsilon(T_{2,2k+1}) = -k, \quad \upsilon(T_{3,3k+1}) = \upsilon(T_{4,2k+1}) = -2k, \\ &\upsilon(T_{3,3k+2}) = -2k-1, \end{aligned} \tag{2}$$

for positive integers k. The υ-values for torus knots of braid index 2 follow, for example, from the fact that υ equals half the signature for alternating knots [OSS14, Theorem 1.14]. The other υ-values in (2) can be derived from the inductive formula for υ provided by Ozsváth, Stipsticz and Szabó [OSS14, Theorem 1.15]. We do this in Section 6.

Of course, part of the statement of Corollary 3 was known before; for example in the cases covered by the following remark.

Remark 4. If K and T are positive and negative torus knots, respectively, one has

$$d_c(T,K) = g_4(K\sharp -T) = g_4(K) + g_4(T) = g(K) + g(T)$$

by the Thom conjecture. If K and T are both positive (both negative) torus knots of the same braid index, then there exists an optimal cobordism between them; compare Example 20.

Determining the cobordism distance between all torus knots seems out of reach. However, a coarse estimation of the cobordism distance between torus knots was provided by Baader [Baa12].

The study of optimal and algebraic cobordisms between torus knots seems natural from a knot theoretic point of view. We discuss additional motivation from algebraic geometry questions in Section 2. Section 3 recalls the notions of positive and quasi-positive braids. In Section 4 we show that optimal cobordisms given by quasi-positive braid sequences are algebraic. In Section 5 optimal cobordisms are constructed and used to prove Theorem 1, Theorem 2, and Corollary 3. Section 6 provides the Υ-values for torus knots of braid index 3 and 4.

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2. Algebraic motivation: Plane curve singularities over $\mathbb{C}$

In this section, we discuss motivations to study algebraic and optimal cobordisms between torus knots coming from singularity theory. Mathematically, the rest of the paper is independent of this.

We consider isolated singularities on algebraic curves in $\mathbb{C}^2$ and we denote singularities by the function germs that define them. A general question asks, what (topological) type of singularities can occur on an algebraic curve V_f in $\mathbb{C}^2$ given as the zero-set of a square-free polynomial f in $\mathbb{C}[x, y]$ of some fixed degree d; see e.g. Greuel, Lossen, and Shustin's work [GLS98]. Even for simple singularities—those corresponding to Dynkin diagrams [Arn72]—a lot is unknown. For the A_k-singularities—the simple singularities given by $y^2 - x^{k+1}$—the following bounds were provided by Guseĭn-Zade and Nekhoroshev

$$\frac{15}{28}d^2 + O(d) \leq k(d) \leq \frac{3}{4}d^2 + O(d) \quad \text{[GZN00]}, \tag{3}$$

where $k(d)$ denotes the maximal integer k such that A_k occurs on a degree d curve. In fact, Orevkov improved the lower bound to

$$\frac{7}{12}d^2 + O(d) \leq k(d) \quad \text{[Ore12]}. \tag{4}$$

Recall that, for a singularity at $p = (p_1, p_2) \in V_f$, its *link of singularity* is the link obtained as the transversal intersection of V_f with the small 3-sphere

$$S^3_\varepsilon = \{(x, y) \mid |x - p_1|^2 + |y - p_2|^2 = \varepsilon^2\} \subset \mathbb{C}^2,$$

for small enough $\varepsilon > 0$; see Milnor [Mil68]. Similarly, the *link at infinity* of an algebraic curve V_f is defined to be the transversal intersection of V_f with the 3-sphere $S^3_R \subset \mathbb{C}^2$ of radius R for R large enough; see e.g. Neumann and Rudolph [NR87].

Prototypical examples of plane curve singularities are the singularities $f_{p,q} = y^p - x^q$, where p and q are positive integers. They have the torus link $T_{p,q}$ as link of singularity. Up to topological type, singularities are determined by their link of singularity and the links that arise as links of singularities are fully understood: their components are positive torus knots or special cables thereof; see e.g. Brieskorn and Knörrer's book [BK86].

In the algebraic setting it is natural to consider links not only knots. Algebraic cobordisms between links are defined as for knots. Optimal cobordisms are defined via Euler characteristic instead of the genus; see Section 4.

Observation 5. The existence of a singularity on a curve of degree d implies the existence of an optimal cobordism from the link of the singularity to $T_{d,d}$. In particular, there exists an optimal cobordism from $T_{2,k(d)+1}$ to $T_{d,d}$.

Observation 5 motivated our study of optimal cobordisms from $T_{2,n}$ to $T_{d,d}$ and $T_{d,d+1}$. In Section 5.2, we show that there exist algebraic cobordisms between $T_{2,n}$ and $T_{d,d}$ ($T_{d,d+1}$) if $n \leq \frac{2}{3}d^2 + O(d)$. In particular, no obvious topological obstruction exists to having $k(d) \geq \frac{2}{3}d^2 + O(d)$; compare also [Ore12]. Observation 5 allows to give a knot theoretic proof of the upper bound in (3); see Remark 27. To establish Observation 5, we note that, whenever a singularity occurs on an algebraic curve V_f, we get an algebraic cobordism from the link of the singularity K to the link at infinity T of V_f. For this, choose a small sphere S^3_ε and a large sphere S^3_R that intersect V_f transversally in K and T, respectively. Let V_g be another algebraic curve. By transversality, V_f and V_g intersects S^3_ε and S^3_R in the same links as long as g and f are "close". To be precise, this is certainly true if $g = f + t$ and $t \in \mathbb{C}$ is small. For generic t, V_g is smooth; thus, V_g yields an algebraic cobordism between K and L. Furthermore, there is an algebraic cobordism from T to $T_{d,d}$. This follows by using that the link at infinity of $f + s(x^d + y^d)$ is $T_{d,d}$, for generic $s \in \mathbb{C}$, while $S^3_R \pitchfork V_{f+s(x^d+y^d)}$ is T for s small enough; and then arguing as above. Gluing the two algebraic cobordisms together yields an optimal cobordism from K to $T_{d,d}$.

A related question asks about the existence of adjacencies between singularities. Fixing a singularity f, another singularity g is said to be *adjacent* to f if there exists a smooth family of germs f_t such that $f_0 \cong f$ and $g \cong f_t$ for small enough non-zero t. There are different notions of equivalence $\cong$ yielding different notions of adjacencies. However, as long as g defines a simple singularity the notions all agree. See Siersma's work for a discussion of these notions [Sie74] and compare also with Arnol′d's work, who was the first to fully describe adjacency between simple singularities [Arn72, Corollary 8.7]. A modern introduction to singularities and their deformations is provided by Greuel, Lossen and Shustin [GLS07].

If g is adjacent to f, then there exists an algebraic cobordism between their links of the singularity (given by $V_{f_t+\varepsilon}$ for t and ε small as a similar argument as above shows). The adjacency question is mostly unresolved if f is not a simple singularity. A natural first case to consider is $f = f_{p,q}$ for fixed $p > 2$ and to ask: Given a positive integer q, which A_n-singularities are adjacent to $f_{p,q}$? Theorem 1 and Theorem 2 can be seen as answering analogs of this question for p equal to 3 and 4, respectively.

3. Braids and (quasi-)positivity

To set notions, we shortly recall Artin's braid group [Art25]; a nice reference for braids is Birman's book [Bir74]. Let us fixe a positive integer n. The standard group presentation for the *braid group on n strands*, denoted by B_n, is given by generators $a_1, \dots, a_{n-1}$ subject to the *braid relations*

$$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \text{ for } 1 \leq i \leq n-2 \quad \text{and} \quad a_i a_j = a_j a_i \text{ for } |i-j| \geq 2.$$

Elements β of B_n, called *braids* or *n-braids*, have a well-defined (algebraic) length $l(\beta)$, given by the number of generators minus the number of inverses of generators in a word representing β. More geometrically, a n-braid β can be viewed as an isotopy class of an oriented compact 1-submanifold of $[0,1] \times \mathbb{C}$ such that the projection to $[0,1]$ is a n-fold orientation-preserving regular map and β intersects $\{0\} \times \mathbb{C}$ and $\{1\} \times \mathbb{C}$ in $\{0\} \times P$ and $\{1\} \times P$, respectively, where P is a subset of $\mathbb{C}$ consisting of n complex numbers with pairwise different real part. The above standard generators a_i are identified with the braid that exchanges the ith and $i+1$th (with respect to order induced by the real order) point of P by a half-twist parameterized by $[0,1]$ and the group operations is given by stacking braids on top of each other. The *closure* $\overline{\beta}$ of β is the closed 1-submanifold in $S^1 \times \mathbb{C}$ obtained by gluing the top of $\beta \subset [0,1] \times \mathbb{C}$ to its bottom. A closed braid $\overline{\beta}$ yields a link in S^3, also denoted by $\overline{\beta}$, via a fixed standard embedding of the solid torus $S^1 \times \mathbb{C}$ in S^3. The *braid index* of a link is the minimal number n such that L arises as the closure of a n-braid.

Positive braids are the elements of the semi-subgroup $B_{n,+}$ that is generated by all the generators a_i. *Positive torus links* are examples of links that arise as closures of positive braids: For positive integers p and q, the closure of $(a_1 a_2 \cdots a_{p-1})^q$ is $T_{p,q}$, which is a knot, called a *positive torus knot*, if and only if p and q are coprime. The braid index of $T_{p,q}$ is $\min\{p,q\}$.

Rudolph introduced *quasi-positive braids*—the elements of the semi-subgroup of B_n generated by all conjugates of the generators a_i; i.e. the braids given by *quasi-positive braid words* $\prod_{k=1}^{l} \omega_k a_{i_k} \omega_k^{-1}$; compare [Rud83]. A knot or link is called *quasi-positive* if it arises as the closure of a quasi-positive braid. A quasi-positive braid β has an associated canonical ribbon surface F_β embedded in B^4 with the closure of β as boundary, which can be seen in S^3 given by n discs, one for every strand, and $l(\beta)$ ribbon bands between the discs. In particular, the Euler characteristic χ of F_β is $n - l(\beta)$. By the slice-Bennequin inequality [Rud93], $\chi(F_\beta)$ equals $\chi_4(\overline{\beta})$—the maximal Euler characteristic among all oriented and smoothly embedded surfaces F (without closed components) in B^4 such that $\partial F \subset S^3$ is the link $\overline{\beta}$.

Rudolph established that all quasi-positive links arise as the transversal intersection of a smooth algebraic curve in $\mathbb{C}^2$ with S^3 [Rud83]. Boileau and

Orevkov proved that this is a characterization of quasi-positive links [BO01, Theorem 1].

4. Algebraic realization of optimal cobordisms

This section is concerned with establishing the following realization Lemma.

Lemma 6. *Let β_1 and β_2 be quasi-positive n-braid words. If β_2 can be obtained from β_1 by applying a finite number of braid group relations, conjugations, and additions of a conjugate of a generator anywhere in the braid word; then there exists an algebraic cobordism C between the links obtained as the closures of the β_i. In fact, C is given as the zero-set of a polynomial in $\mathbb{C}[x,y]$ of the form*

$$y^n + c_{n-1}(x)y^{n-1} + \cdots + c_0(x),$$

where the $c_i(x)$ are polynomials.

Let β_1 and β_2 be quasi-positive braid words given as in Lemma 6; i.e. there is a sequence of n-braid words $(\alpha_1, \ldots, \alpha_k)$ starting with β_1 ending with β_2 such that α_j and α_{j+1} either define the same conjugacy class in B_n or α_{j+1} is obtained by adding a generator a_i somewhere in α_j. There is an associated cobordism C between $\overline{\beta_1}$ and $\overline{\beta_2}$ given (as a handle decomposition) by 1-handle attachments corresponding to every generator that is added. The cobordism C is *optimal*; i.e. it has Euler characteristic

$$\chi_4(\overline{\beta_2}) - \chi_4(\overline{\beta_1}) = l(\beta_1) - l(\beta_2)$$

and does not have closed components (this is the sensible extension of the notion of optimal cobordisms to links). In fact, although not made explicit, the proof of Lemma 6 given below does show that this C is algebraic. All optimal cobordisms we construct in Section 5 arise as described above. We see Lemma 6 as evidence that the following question might have a positive answer.

Question 7. Are the two necessary conditions for the existence of an algebraic cobordism between two knots—the knots are quasi-positive and there exists an optimal cobordism between them—sufficient?

The proof of Lemma 6 occupies the rest of this section and uses Rudolph diagrams. Only the statement of Lemma 6 is used in the rest of the paper.

4.1. Rudolph diagrams

To set notation and for the reader's convenience, we recall the notion of Rudolph diagrams, following [Rud83] and [Ore96].

For a square-free algebraic function $f\colon \mathbb{C}^2 \to \mathbb{C}$ of the form

$$y^n + c_{n-1}(x)y^{n-1} + \cdots + c_0(x) \in \mathbb{C}[x, y],$$

we study the following subsets of $\mathbb{C}$.

- The finite set B of all x such that some of the n solutions $y_1, \dots, y_n$ of $f(x, y) = 0$ coincide.
- The semi real-analytic set B^+ of all x such that the n solutions of $f(x, y) = 0$ are all different, but do not have n distinct real parts.

Their union $B \cup B^+$ is denoted by $G(f)$.

Example 8. If f is $y^2 + x$, then $B = \{0\}$, $B^+ = (0, \infty)$, and $G(f) = [0, \infty)$.

Let $V_f \subset \mathbb{C}$ denote the zero-set of f. For an oriented simple closed curve γ in $\mathbb{C}\backslash B$, the intersection $V_f \cap (\gamma \times \mathbb{C}) \subset \gamma \times \mathbb{C}$ is a closed n-braid via the identification $\gamma \times \mathbb{C} \cong S^1 \times \mathbb{C}$. Similarly, for every oriented arc α in $\mathbb{C} \setminus B$ with endpoints in $\mathbb{C} \setminus G(f)$ (which guarantees that at endpoints the n-solutions have different real parts), the intersection $V_f \cap (\alpha \times \mathbb{C}) \subset \alpha \times \mathbb{C}$ is a n-braid by identifying $\alpha \times \mathbb{C}$ with $[0, 1] \times \mathbb{C}$. Note that for this to be well-defined, the identification should preserve the order of the real parts in the second factor. An endpoint-fixing isotopy of two arcs and an isotopy of two simple closed curves in $\mathbb{C} \setminus B$ correspond to an isotopy of braids and an isotopy of closed braids, respectively. Any choice of convention not made explicit so far is chosen such that in Example 8 the oriented arc starting at $1 - i$ and ending at $1 + i$ yields the 2-braid a_1.

Let $\pi\colon \mathbb{C}^2 \to \mathbb{C}$ be the projection to the first coordinate. We will only consider f such that $f = 0$ defines a non-singular algebraic curve $V_f \subset \mathbb{C}^2$ and such that for every x in B the intersection $\pi^{-1}(x) \cap V_f$ consists of exactly $n - 1$ points; i.e. fixing an x in B gives a polynomial in y with precisely one repeated root of multiplicity two. Rudolph observed that $G(f) = B \cup B^+$ naturally carries the structure of an oriented, $\{1, \dots, n-1\}$-labeled graph that describes V_f up to π-preserving smooth isotopy in $\mathbb{C}^2$. After a generic small linear coordinate change (to rule out pathologies), the vertices are locally given as in Figure 1. The elements of B are the 1-valent vertices. The edges are the connected components of the semi real-analytic open subset of B^+ given by those x that have precisely $n - 1$ different real parts among the real parts of the solutions $y_1, \dots, y_n$ to $f(x, y) = 0$. Some of the edges

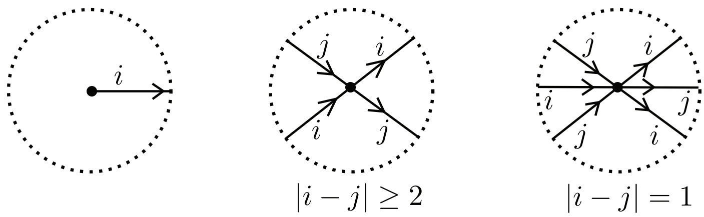


Figure 1: Neighborhoods of vertices of $G(f)$.

tend to infinity instead of ending at a vertex. An edge e is labeled as follows. For x in $e \subset \mathbb{C}$, the n solutions y_i of $f(x, y)$ can be indexed such that the index order agrees with the order given by their real parts; i.e.

$$\mathrm{Re}(\mathrm{y}_1) < \cdots < \mathrm{Re}(\mathrm{y}_{\mathrm{k_e}}) = \mathrm{Re}(\mathrm{y}_{\mathrm{k_e}+1}) < \cdots < \mathrm{Re}(\mathrm{y}_\mathrm{n})$$

for some k_e in $\{1, \ldots, n-1\}$. The edge e is labeled with k_e. The edges are oriented as follows. Pick a small oriented arc $\alpha \subset \mathbb{C}$ that meets e transversally in a point x. The braid associated with α is either a_{k_e} or $a_{k_e}^{-1}$ (the k_eth and $k_e + 1$th solution exchange their real-part order while passing through x). Orient e such that, if the orientation of e followed by the orientation of γ gives the complex orientation of $\mathbb{C}$, then the braid corresponding to the transverse arc is a_{e_k} (rather than for $a_{e_k}^{-1}$). In particular, edges are oriented to point away from the 1-valent vertices.

The oriented, labeled graph $G(f)$ describes V_f up to π-preserving isotopy; in particular, it describes all closed braids given by intersecting V_f with cylinders. For a fixed embedded curve in γ in $\mathbb{C} \setminus B$ with a marked start point $p \notin B^+$, one gets an explicit procedure, how to read off a braid word for the braid β corresponding to the arc starting and ending at p going counter-clockwise around γ: by a small isotopy of γ in $\mathbb{C} \setminus B$, we may assume that γ meets $G(f)$ transversally and in edges only. Starting at p we move counter-clockwise around γ. Whenever we cross an edge e transversally at a point x, we write down the generator a_{k_e} or its inverse $a_{k_e}^{-1}$ depending on whether the orientation at x given by $G(f)$ and γ agrees or disagrees with the complex orientation of $\mathbb{C}$.

The study of the graphs $G(f)$ motivates the following definition. Fix some surface S with boundary. In fact, we will only consider cases where S is either

$$\begin{aligned} &\text{the unit disc } D = \{x \in \mathbb{C} \mid |x| \leq 1\} \\ &\text{or the annulus } A = \{x \in \mathbb{C} \mid 1 \leq |x| \leq 2\}\,. \end{aligned}$$

A *Rudolph diagram* on S is an oriented, $\{1,\ldots,n-1\}$-labeled graph G with smooth edges (we also allow smooth closed cycles) that enters and exits the boundary of S transversely and is locally modelled on graphs $G(f)$ coming from an algebraic function $f \in \mathbb{C}[x,y]$ as above; i.e. locally around a vertex G is given as in Figure 1. We denote the set of 1-valent vertices by $B \subset G$. Of course, a huge source of examples are obtained by embedding (or immersing) S in $\mathbb{C}$ and using this embedding to define G as the pull back of $G(f)$ for some algebraic function f. Any closed curve γ missing 1-valent vertices defines a closed braid $\overline{\beta}$ by isotoping γ to meet G transversally in edges and then reading off a braid word β for that closure as described in the $G(f)$-case. A Rudolph diagram is said to be *smooth* if it contains only 1-valent vertices.

Example 9. Given a quasi-positive braid word $\beta = \prod w_l a_{i_l} w_l^{-1}$, there exists a smooth Rudolph diagram on D such that braid word read off when following $S^1 = \partial D$ is β. Figure 2 illustrates how one factor $\omega a_i \omega^{-1}$ is realized.

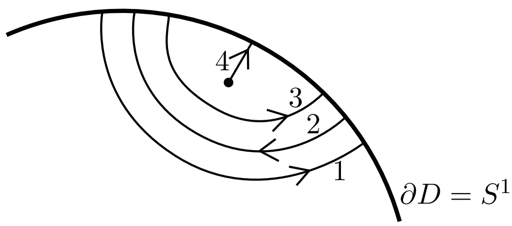


Figure 2: Piece of a Rudolph diagram that yields the braid word $\omega a_i \omega^{-1}$ when following the boundary, for $i = 4$ and $\omega = a_1 a_2^{-1} a_3$.

Orevkov describes which smooth Rudolph diagrams on D arise as $G(f)$.

Proposition 10. *[Ore96, Proposition 2.1] Let G be a smooth Rudolph diagram on D. There exists an algebraic function*

$$f = y^n + c_{n-1}(x)y^{n-1} + \cdots + c_0(x) \in \mathbb{C}[x,y]$$

such that G is isotopic to $G(f) \cap D$ if and only if $G(f)$ contains no closed cycles.

As pointed out by Orevkov, Rudolph (implicitly) proved such a statement while establishing the main theorem of [Rud83].

Remark 11. Given a smooth Rudolph diagram G on D one can remove all closed cycles without changing the closed braids associated with closed curves in D.

4.2. Rudolph diagrams on the annulus and braid word sequences

For a Rudolph diagram G on A, we denote by β_1 and β_2 the two braids defined by G via reading off braid words following the inner and outer boundary of A counter-clockwise starting at 1 and 2, respectively. For β_1 and β_2 to be well-defined, we ask that G does not meet 1 or 2, which from now on is imposed on every Rudolph diagram.

For the proof of Lemma 6, we need the following. If a braid β is obtained from a braid α as described in Lemma 6, then there exists a Rudolph diagram on A such that $\alpha = \beta_1$ and $\beta = \beta_2$. The rest of this subsection provides one way of making this statement precise.

Remark 12. Any Rudolph diagram G on D or A can be isotoped such that it is *outward-oriented*, which is defined as follows: All but a finite number of circles around the origin intersect G transversally in edges. Furthermore, the finite exceptional circle meet G transversally in edges except in one point x, which falls in one of two categories. Either x lies in the interior of an edge and the radial function restricted to that edge has a strict local extremum. Or x is a vertex and locally around x the Rudolph Diagram G and the exceptional circle behave as described in Figure 3. Finally, the positive real

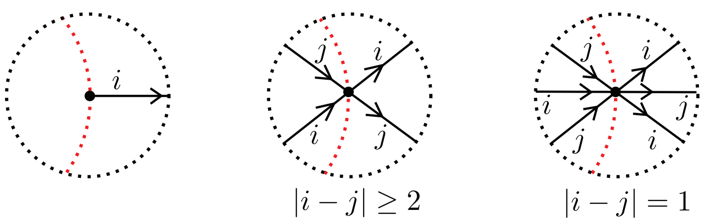


Figure 3: Neighborhoods of vertices of an outward-oriented Rudolph diagram (black) and how they meet their corresponding exceptional circles (red).

ray $[0, \infty)$ meets G transversally in edges and away from the finite number of exceptional circles. Locally around points in $[0, \infty) \cap G$ the radial function increases on G when following the orientation. An example is provided in Figure 4.

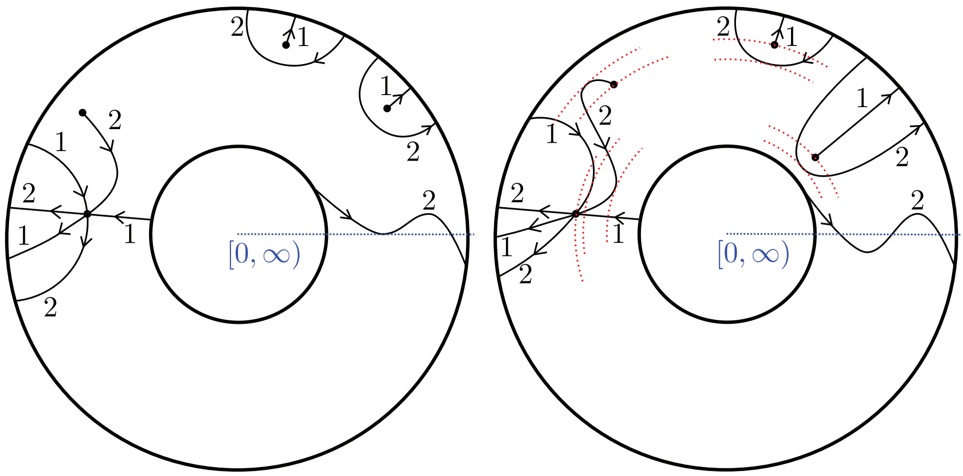


Figure 4: A Rudolph diagram in the annulus (left) is arranged to be outward-oriented (right). The exceptional circles are indicated by circle segments (red).

Given an outward-oriented Rudolph diagram G on A, let $r_1 < \cdots < r_k$ denote the radii corresponding to exceptional circles or points where G meets $[0, \infty)$. For s in $[1, 2] \setminus \{r_1, \ldots, r_k\}$, we denote by β_s the braid read off when following the counter-clockwise oriented circle of radius s with s as marked starting point. We associate to G the following finite sequence of braid words

$$(\beta_{s_0}, \beta_{s_1}, \ldots, \beta_{s_k}), \tag{5}$$

where

$$1 \leq s_0 < r_1 < s_1 < r_2 < s_2 < \cdots < r_k < s_k \leq 2.$$

In particular, the sequence (5) starts and ends with β_1 and β_2, respectively.

Observation 13. For all $0 \leq l < k$, the braid word $\beta_{s_{l+1}}$ is obtained from β_{s_l} by one of the following operations, for some $1 \leq i, j \leq n-1$ with $|i - j| \geq 2$:

(i) adding or removing subwords $a_i a_i^{-1}$ or $a_i^{-1} a_i$;

(ii) performing one braid relation; i.e. replacing $a_i a_{i\pm1} a_i$ by $a_{i\pm1} a_i a_{i\pm1}$ or replacing $a_i a_j$ by $a_j a_i$;

(iii) changing a braid word of the form $a_i \alpha$ to αa_i or vice versa;

(iv) adding a_i somewhere in the braid word.

Remark 14. Note that two braid words can be connected with a sequence using (i) and (ii) if and only if they define the same braid. Two braid words can be connected with a sequence using (i), (ii), and (iii) if and only if they define the same closed braid. And two braid words can be connected with a sequence using (i) through (iv) if and only if they are connected as described in Lemma 6.

Conversely, a sequence of braid words as described in Observation 13 yields a Rudolph diagram on A. This amounts to the following:

Proposition 15. *The assignment given by* (5) *yields an one-to-one correspondence between outward-oriented Rudolph diagrams on A, up to isotopy through outward-oriented Rudolph diagrams, and finite non-empty sequences of braid words $(\beta_0, \ldots, \beta_k)$ such that β_{j+1} is obtained from β_j by one of the operations* (i), (ii), (iii) *and* (iv) *described in Observation 13.*

4.3. Smoothing of Rudolph diagrams and proof of Lemma 6

After this translation of sequences of braid words to Rudolph diagrams we need a final ingredient to prove Lemma 6:

Proposition 16. *Let G be a Rudolph diagram on A. There exists a smooth Rudolph diagram $\widetilde{G}$ on A satisfying the following:*
- *G and $\widetilde{G}$ are identical in a neighborhood of the inner boundary S^1. In particular, the braid words β_1 and $\widetilde{\beta}_1$ corresponding to S^1 are the same.*
- *The braids β_2 and $\widetilde{\beta}_2$ corresponding to the outer boundary S^1_2 have the same closure.*

Proof. Let $B = \{v_1, \ldots, v_k\}$ be the set of 1-valent vertices in G. For every v in B, we choose an embedded arc γ_v in A that connects v to the inner boundary S^1 of A such that γ_v intersects G in the interior of A transversally and outside of vertices (except at v of course). Furthermore, we arrange that all the arcs $\gamma_{v_1}, \ldots, \gamma_{v_k}$ are pairwise disjoint. A neighborhood N in A of the union $S^1 \cup \gamma_{v_1} \cup \cdots \cup \gamma_{v_k}$ defines an annulus on which G is smooth. The boundary of N has two components: S^1 and a curve that is isotopic to S^1_2 in $A \setminus B$. Therefore, we obtain a Rudolph diagram on A as wanted by identifying N with A via a diffeomorphism that is the identity in a neighborhood of S^1. This is illustrated in Figure 5. □

Proof of Lemma 6. Let β_1 and β_2 be quasi-positive braid words satisfying the assumptions in Lemma 6. We find a finite sequence of braid words as

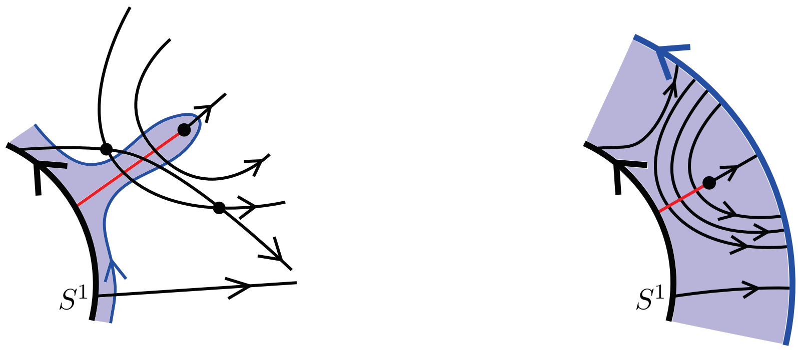


Figure 5: Left: A neighborhood N (blue) of the inner boundary S^1 of A and the embedded arc γ_v (red). Right: Restriction of the Rudolph diagram to N, where N is identified with A.

described in Observation 13 that starts with β_1 and ends with β_2. Let G be the corresponding Rudolph diagram on A provided by Proposition 15. By Proposition 16 we may assume that G is smooth (this may change β_2 but the corresponding closed braid remains the same). Since β_1 is quasi-positive, there is a smooth Rudolph diagram $\widetilde{G}$ on D such that β_1 is the braid word read off when following the boundary of D by Example 9. We glue $(\widetilde{G}, D)$ and (G, A) together along S^1 to get (by rescaling) a smooth Rudolph diagram R on the disk D; see left-hand side of Figure 6. Next, we

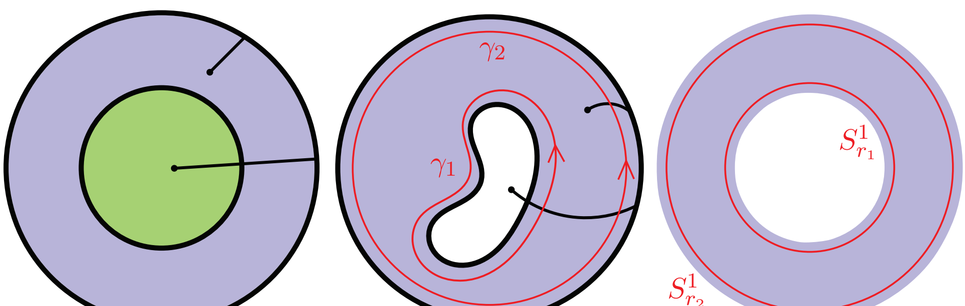


Figure 6: Left: The Rudolph diagram R in D, which is obtained by gluing $(\widetilde{G}, D)$ (green) and (G, A) (blue) together. Middle: Realization of R in D as $G(f)$ with the isotoped annulus A (blue) and the curves γ_1 and γ_2 (red). Right: The annulus $\widetilde{A}$ with $S^1_{r_1}$ and $S^1_{r_1}$ (red).

remove all closed cycles in R. This might change the braid word β_1 but not the closed braid it represents by Remark 11. By Proposition 10, there exists an algebraic function f such that $R = G(f)$ after an isotopy of R. The latter isotopy yields an embedding of A in $\mathbb{C}$ (we denote its image again by A) such that $G(f) \cap A = G$. By the uniformization theorem for open annuli (see e.g. [Ahl78, 6.4. Theorem 10]), there exists a biholomorphic map ϕ from the interior of $A \subset \mathbb{C}$ to an open annulus $\widetilde{A} \subset \mathbb{C}$ with concentric boundary circles centered at the origin. Setting $\widetilde{f}(x, y) = f(\phi^{-1}(x), y)$ defines holomorphic map on $\widetilde{A} \times \mathbb{C}$ of the form

$$y^n + \widetilde{c_{n-1}}(x)y^{n-1} + \cdots + \widetilde{c_0}(x) \text{ with } \widetilde{c_i} \text{ holomorphic on } \widetilde{A}.$$

We choose two concentric circles $S^1_{r_1}$ and $S^1_{r_2}$ in $\widetilde{A}$ such that their preimages under ϕ are curves γ_1 and γ_2 in A for which the closed braids $V_f \pitchfork (\gamma_i \times \mathbb{C})$ are $\overline{\beta_i}$. This is, for example, achieved by choosing r_1 and r_2 close to the radii of the inner and the outer boundary of $\widetilde{A}$, respectively; see Figure 6. Therefore, $V_{\widetilde{f}}$ intersects the cylinders $Z_i = S^1_{r_i} \times \mathbb{C}$ transversely in closed braids and those closed braids are $\overline{\beta_i}$ since $V_{\widetilde{f}} \pitchfork (S^1_{r_i} \times \mathbb{C})$ is the image of $V_f \pitchfork (\gamma_i \times \mathbb{C})$ under the biholomorphic map

$$A \times \mathbb{C} \to \widetilde{A} \times \mathbb{C}, (x, y) \mapsto (\phi(x), y).$$

By polynomial approximation of the holomorphic coefficients $\widetilde{c_i}$ of $\widetilde{f}$, we find a polynomial $g = y^n + c_{n-1}(x)y^{n-1} + \cdots + c_0(x)$ with $c_i \in \mathbb{C}[x]$ such that its zero-set V_g intersects the above cylinders transversally in the same closed braids as $V_{\widetilde{f}}$. We replace the cylinders Z_i with cylinders $Z_{i,R} = \{x, y \in \mathbb{C} \mid |x|^2 + \frac{|y|}{R}^2 = r_i^2, x \neq 0\}$, which for large enough R intersect V_g in the same closed braids as the Z_i. Finally, we set $F(x, y) = \frac{1}{R^n} g(x, Ry)$ and conclude the proof by noticing that the 3-spheres $S^3_{r_i}$ of radius r_i intersect the zero-set V_F in the links that are the closures of β_i since rescaling the y-coordinate by the factor $\frac{1}{R}$ maps $Z_{i,R}$ onto $S^3_{r_i} \setminus (\{0\} \times S^1_{r_i})$. □

5. Construction of optimal and algebraic cobordisms between torus knots via positive braids

In this section, we construct several families of optimal cobordisms between torus knots, which are also algebraic by Lemma 6. It came as a surprise to the author that Ozsváth, Stipsicz, and Szabó's Υ-invariant shows that the constructions for torus knots of braid index 4 or less cannot be improved.

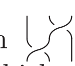
Definition 17. For links K and T that are closures of positive braids, we say K is *subword-adjacent* to T, denoted by $K \leq_s T$, if there are positive

n-braids β_1 and β_2, for some integer n, such that β_1 can be obtained from β_2 by successively deleting generators.

Here, deleting a generator in a positive braid β means removing an a_i in a positive braid word that represents β. We think of subword-adjacency as a combinatorial toy model for adjacency of singularities (as described in Section 2), hence the name.

Remark 18. If a positive n-braid β_1 is obtained from a positive n-braid β_2 by deleting positive generators, then β_2 can be obtained from β_1 as described in Lemma 6. Therefore, if K is subword-adjacent to T, then there exists an algebraic cobordism between them, by Lemma 6.

Remark 19. In what follows we consider positive braids β with non-split closure; in particular, their closures $\overline{\beta}$ are fibered; see Stallings [Sta78]. In this case, the optimal cobordism provided by a subword-adjacency can be understood on the fiber surfaces: Removing a generator in a positive braid β corresponds to deplumbing a positive Hopf band on its fiber surface F_β. In other words, if $K = \overline{\beta_1}$ is subword-adjacent to $T = \overline{\beta_2}$, then the open book of S^3 with binding K is obtained from the open book of S^3 with binding T by $\chi(F_{\beta_1}) - \chi(F_{\beta_2}) = l(\beta_2) - l(\beta_1)$ positive destabilizations.

In this section, we use fence diagrams to represent positive braids. I.e. in a braid diagram, positive crossings  are replaced with horizontal intervals ; see Rudolph [Rud98]. For example, the positive 3-braid a_1a_2 is represented by  instead of the braid diagram .

Simple examples of a subword-adjacency, which yield well-known optimal cobordisms, are the following.

Example 20. Let n, m, a, b be positive integers. If $n \leq a$ and $m \leq b$, then $T_{n,m}$ is subword-adjacent to $T_{a,b}$. This subword-adjacency is obtained by deleting generators in the positive a-braid word $(a_1 \cdots a_{a-1})^b$, which has closure $T_{a,b}$, until one reaches a positive braid word with closure $T_{n,m}$. Figure 7 illustrates this for the adjacency $T(4,5) \leq_s T(7,7)$.

The subword-adjacencies given in Example 20, have analogs in the algebraic adjacency setting since it is easy to write down an adjacency from $y^a - x^b$ to $y^n - x^m$.

A proposition due to Baader provides examples of subword-adjacencies that are more interesting.

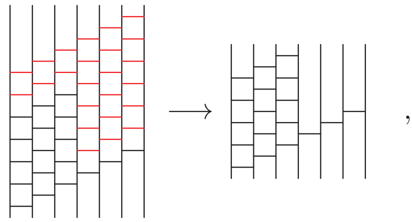


Figure 7: Subword adjacency between $T_{4,5}$ and $T_{7,7}$. The arrow indicates the removal of the generators marked in red.

Proposition 21. *[Baa12, Proposition 1] Let a, b, c be positive integers with $a \leq b$. Then $T_{a,bc}$ is subword-adjacent to $T_{b,ac}$.*

However, again there exists an algebraic adjacency from $y^a - x^{bc}$ to $y^b - x^{ac}$; see [Fel14, Proposition 23], which yields an algebraic cobordisms from $T_{a,bc}$ to $T_{b,ac}$ without appealing to Lemma 6.

5.1. Optimal examples for torus knots of small braid index and proofs of the main results

After these first examples, we proceed with subword adjacencies between torus knots that turn out to be optimal and that, to the author's knowledge, are not known to have algebraic adjacency analogs.

The following propositions provide all optimal cobordisms that are needed to establish Theorem 1, Theorem 2, and Corollary 3.

Proposition 22. *Let n and m be positive integers. If $n \leq \frac{5m-1}{3}$, then the torus link $T_{2,n}$ is subword-adjacent to the torus link $T_{3,m}$.*

Proposition 23. *Let n and m be positive integers. If $n \leq \frac{5m-3}{2}$, then the torus link $T_{2,n}$ is subword-adjacent to the torus link $T_{4,m}$.*

It is part of the statement of Theorems 1 and 2 that Propositions 22 and 23 cannot be improved, at least when the involved links are knots. This is a consequence of the cobordism distance bound

$$d_c(K,T) = g_4(K\sharp -T) \geq \max\{|\tau(K) - \tau(T)|, |\upsilon(K) - \upsilon(T)|\}, \tag{6}$$

provided in [OSS14, Theorem 1.11], generalizing the τ-bound in [OS03]. Before proving Propositions 22 and 23, we use them and (6) to prove Theorem 1, Theorem 2, and Corollary 3.

Proof of Theorems 1 and 2. By Remark 18, the fact that (II) implies (III) is an immediate consequence of Proposition 22 and Proposition 23, respectively. By the Thom conjecture, (III) implies (I). Therefore, it remains to show that (I) implies (II).

Throughout the proof we have $K = T_{2,n}$, for some odd integer $n \geq 3$, and T is a torus knot of braid index 3 or 4. We assume towards a contradiction that (II) does not hold and calculate that

$$|g(T) - g(K)| < |\upsilon(T) - \upsilon(K)|, \tag{7}$$

which contradicts (I) by (6). We do this according to the 3 cases $T = T_{3,3k+1}$, $T = T_{3,3k+2}$ (Theorem 1); and $T = T_{4,2k+1}$ (Theorem 2), where k is a positive integer:

For $T = T_{3,3k+2}$, we have that (II) fails precisely when

$$\begin{aligned} n \geq 5k+4 &\iff 5k+2 < n-1 \\ &\iff 3k+1-\frac{n-1}{2} < \frac{n-1}{2} - (2k+1) \\ &\iff \left|3k+1-\frac{n-1}{2}\right| < \left|\frac{n-1}{2} - (2k+1)\right| \\ &\overset{(1)(2)}{\iff} |g(T) - g(K)| < |-\upsilon(K) + \upsilon(T)|\,. \end{aligned}$$

This shows that, if (II) fails, then (7) holds.

Similarly, for $T = T_{3,3k+1}$ and $T = T_{4,2k+1}$, we have that (II) fails precisely when

$$\begin{aligned} n \geq 5k+2 &\iff 5k < n-1 \\ &\iff 3k-\frac{n-1}{2} < \frac{n-1}{2} - 2k \\ &\iff \left|3k-\frac{n-1}{2}\right| < \left|\frac{n-1}{2} - 2k\right| \\ &\overset{(1)(2)}{\iff} |g(T) - g(K)| < |-\upsilon(K) + \upsilon(T)|\,. \end{aligned}$$

As before this shows that, if (II) fails, then (7) holds. □

Proof of Corollary 3. For torus knots K and T such that the sum of their braid indices is 6 or less, we want to establish

$$d_c(K,T) = g_4(K\sharp\overline{T}) = \max\{|\tau(K)-\tau(T)|, |\upsilon(K)-\upsilon(T)|\}.$$

By (6), it suffices to find a cobordism C between K and T with genus

$$g(C) \leq \max\{|\tau(K)-\tau(T)|, |\upsilon(K)-\upsilon(T)|\}.$$

If K and T are torus knots of opposite sign, then we have that

$$d_c(K,T) = g_4(K\sharp\overline{T}) = g(K)+g(T),$$

where the second equality invokes the Thom Conjecture; compare Remark 4. If K and T are positive torus knots that have the same braid index, then

$$d_c(K,T) = g_4(K\sharp\overline{T}) = |g(K)-g(T)|,$$

where the second equality follows from Example 20; compare Remark 4. Therefore, if K and T are torus knots of opposite sign or torus knots with the same braid index, then

$$d_c(K,T) = g_4(K\sharp\overline{T}) = |\tau(K)-\tau(T)|$$

since $g(K)+g(T) = |\tau(K)-\tau(T)|$ or $|g(K)-g(T)| = |\tau(K)-\tau(T)|$, respectively; compare [OS03, Corollary 1.7]. Thus, after taking mirror images of K and T, if necessary, we may assume that K and T are both positive torus knots such that K has braid index 2 and T has braid index 3 or 4.

Let n and k be the positive integers such that K is $T_{2,n}$ and T is $T_{3,3k+1}$, $T_{3,3k+2}$, or $T_{4,2k+1}$. Furthermore, we do not need to consider the cases covered by Theorems 1 and 2; i.e. when (I), (II), and (III) of Theorems 1 and 2, respectively, are satisfied.

We first consider $T = T_{3,3k+2}$. By Proposition 22 there exist a cobordism C_1 from T to $T_{2,5k+3}$ which is optimal; i.e. C_1 has Euler characteristic

$$\chi_4(T) - \chi_4(T_{2,5k+3}) = (-6k-1)-(-5k-1) = -k.$$

Note that $n > 5k+3$ since we are assuming that there does not exist an optimal cobordism from K to T; compare (II) in Theorem 1. By Example 20, we have a cobordism C_2 from $T_{2,5k+3}$ to K of Euler characteristic

$$\chi_4(K) - \chi_4(T_{2,5k+3}) = -n + 2 - (-5k - 1) = -n + 5k + 3.$$

Gluing C_1 and C_2 together yields a cobordism C of Euler characteristic $-n + 4k + 3$ between T and K. Thus,

$$g(C) = \frac{n - 4k - 3}{2} = \frac{n-1}{2} - 2k - 1 \overset{(2)}{=} -\upsilon(K) + \upsilon(T) = |\upsilon(K) - \upsilon(T)|.$$

Similarly, if $T = T_{3,3k+1}$ or $T = T_{4,2k+1}$, we get a cobordism from T to $T_{2,5k+1}$ with Euler characteristic

$$-6k + 1 - (-5k + 1) = -k$$

by Proposition 22 or Proposition 23, respectively, and a cobordism from $T_{2,5k+1}$ to K with Euler characteristic

$$-n + 2 - (-5k + 1) = -n + 5k + 1.$$

As before, we combine these two cobordisms to a cobordism C from T to K with

$$g(C) = \frac{(n-1) - 4k}{2} = \frac{n-1}{2} - 2k \overset{(2)}{=} -\upsilon(K) + \upsilon(T) = |\upsilon(K) - \upsilon(T)|.$$

□

We now provide proofs of Proposition 22 and Proposition 23. We thank Sebastian Baader for important inputs for these proofs.

Proof of Proposition 22. We denote the 3-strand full twist $(a_1a_2a_1)^2$ by Δ^2. The full twist commutes with every other 3-braid.

Let us first consider the case where $m = 3l$ for some positive integer l. The torus link $T_{3,3l}$ is the closure of 3-braid Δ^{2l}. Note that

$$\Delta^2\Delta^2 = a_1a_2a_1a_1a_2a_1\Delta^2 = a_1a_2a_1a_1a_2\Delta^2a_1.$$

Adding another full twist yields

$$\Delta^2\Delta^2\Delta^2 = a_1a_2a_1a_1a_2\Delta^2\Delta^2a_1 = a_1a_2a_1a_1a_2(a_1a_2a_1a_1a_2\Delta^2a_1)a_1$$

and inductively we get $\Delta^{2l} = (a_1a_2a_1a_1a_2)^l(a_1)^l$. The subword $a_2a_1a_2$ occurs $l-1$ times in $(a_1a_2a_1a_1a_2)^l(a_1)^l$. Applying $l-1$ times the braid relation

$a_2a_1a_2 = a_1a_2a_1$ gives the 3-braid word

$$w = a_1a_2a_1a_1(a_1a_2a_1a_1a_1)^{l-1}a_2(a_1)^l.$$

Deleting all but the first a_2 in w yields $a_1a_2a_1^{5l-2}$, which has $T_{2,5l-1}$ as closure.

If m is $3l+1$ for some positive integer l, we write $T_{3,3l+1}$ as the closure of

$$\begin{aligned} a_1a_2\Delta^{2l} &= a_1a_2w = a_1a_2\left(a_1a_2a_1a_1(a_1a_2a_1a_1a_1)^{l-1}a_2(a_1)^l\right) \\ &= a_1a_1a_2a_1a_1a_1(a_1a_2a_1a_1a_1)^{l-1}a_2(a_1)^l. \end{aligned}$$

Deleting all but the first a_2 yields $a_1a_1a_2(a_1)^{5l-1}$, which has closure $T_{2,5l+1}$.

Finally, if m is $3l+2$ for some positive integer l, view $T_{3,3l+2}$ as the closure of

$$\begin{aligned} a_1a_2a_1a_2\Delta^{2l} &= a_1a_1a_2a_1\Delta^{2l} = a_1(a_1a_2\Delta^{2l})a_1 \\ &= a_1\left(a_1a_1a_2a_1a_1a_1(a_1a_2a_1a_1a_1)^{l-1}a_2(a_1)^l\right)a_1. \end{aligned}$$

Deleting all but the first a_2 yields $a_1a_1a_1a_2(a_1)^{5l}$, which has $T_{2,5l+3}$ as closure. □

Proof of Proposition 23. We view $T_{4,2l+1}$ as the closure of the 4-braid $(a_1a_3a_2)^{2l+1}$. Using the fact that the half twist on 4 strands

$$\Delta = a_1a_3a_2a_1a_3a_2 = a_1a_2a_1a_3a_2a_1$$

anti-commutes with every other 4-braid, i.e. $a_1\Delta = \Delta a_3$, $a_3\Delta = \Delta a_1$, and $a_2\Delta = \Delta a_2$, we have that

$$\begin{aligned} \Delta^k &= a_1a_2a_1a_3a_2a_1\Delta^{k-1} \\ &= a_1a_2a_1a_3a_2\Delta^{k-1}a_{2+(-1)^{k-1}} \\ &= a_1a_2a_1a_3a_2\left(a_1a_2a_1a_3a_2\Delta^{k-2}a_{2+(-1)^{k-2}}\right)a_{2+(-1)^{k-1}} \\ &= (a_1a_2a_1a_3a_2)^2\Delta^{k-2}a_1a_3 \\ &= \cdots = (a_1a_2a_1a_3a_2)^k a_1^{\lfloor\frac{k+1}{2}\rfloor}a_3^{\lfloor\frac{k}{2}\rfloor}. \end{aligned}$$

With this we can write $(a_1a_3a_2)^{2l+1}$ as follows.

$$\begin{aligned}(a_1a_3a_2)^{2l+1} &= (a_1a_3a_2)^2a_1a_3a_2(a_1a_3a_2)^{2(l-1)}\\ &= (a_1a_3a_2)^2a_1a_3a_2(a_1a_2a_1a_3a_2)^{l-1}a_1^{\lfloor\frac{l}{2}\rfloor}a_3^{\lfloor\frac{l-1}{2}\rfloor}\\ &= (a_1a_3a_2)^2a_1a_3(a_2a_1a_2a_1a_3)^{l-1}a_2a_1^{\lfloor\frac{l}{2}\rfloor}a_3^{\lfloor\frac{l-1}{2}\rfloor}\\ &= (a_1a_3a_2)^2a_1a_3(a_1a_2a_1a_1a_3)^{l-1}a_2a_1^{\lfloor\frac{l}{2}\rfloor}a_3^{\lfloor\frac{l-1}{2}\rfloor}.\end{aligned}$$

Deleting the last l occurrences of a_2 in this braid word gives

$$(a_1a_3a_2)^2a_1a_3(a_1a_1a_1a_3)^{l-1}a_1^{\lfloor\frac{l}{2}\rfloor}a_3^{\lfloor\frac{l-1}{2}\rfloor} = (a_1a_3a_2)^2a_1^{3l-2+\lfloor\frac{l}{2}\rfloor}a_3^{l+\lfloor\frac{l-1}{2}\rfloor},$$

which has closure

$$T_{2,4+(3l-2+\lfloor\frac{l}{2}\rfloor)+(l+\lfloor\frac{l-1}{2}\rfloor)} = T_{2,5l+1}.$$

Similarly, the torus link $T_{4,2l+2}$ is the closure of the 4-braid

$$(a_1a_3a_2)^{2l+2} = (a_1a_3a_2)^2a_1a_3a_2a_1a_3(a_1a_2a_1a_1a_3)^{l-1}a_2a_1^{\lfloor\frac{l}{2}\rfloor}a_3^{\lfloor\frac{l-1}{2}\rfloor}.$$

Deleting the last $l+1$ occurrences of a_2 yields a 4-braid that has closure $T_{2,5l+3}$. □

5.2. Subword-adjacency for the torus knot $T_{m,m+1}$

We now study, which $T_{2,n}$ is subword-adjacent to $T_{m,m}$ and $T_{m,m+1}$. Our result is roughly that, whenever $n \leq \frac{2m^2}{3} + O(m)$, then $T_{2,n}$ is subword-adjacent to $T_{m,m}$ and $T_{m,m+1}$. Our interest in this stems from the fact that this is an improvement over what is known in the algebraic setting; compare with (4). In other words, the algebraic cobordism obtained by applying Lemma 6 to the subword-adjacencies provided in the following Proposition is not known to come from an algebraic adjacency between $y^2 - x^n$ and $y^m - x^m$ or a singular algebraic curve of degree m with an A_{n-1}-singularity.

Remark 24. After a first preprint of this article appeared, the author was pointed to work of Orevkov, where the same bound is attained in a very similar setting, and his result was also motivated by questions discussed in Section 2. Indeed, Orevkov's result [Ore12, Theorem 3.13] allows to conclude that there exists an optimal cobordism between $T_{2,n}$ and $T_{m,m}$, whenever $n \leq \frac{2m^2}{3} + O(m)$.

Proposition 25. *Let m and n be positive integers. If $n \leq \frac{2m^2+4}{3} - m$, then $T_{2,n}$ is subword-adjacent to $T_{m,m}$.*

Remark 26. A similar statement holds for the knots $T_{m,m+1}$: let m and n be positive integers. If $n \leq \frac{2m^2-m+5}{3}$, then $T_{2,n}$ is subword-adjacent to $T_{m,m+1}$.

Remark 27. We do not know whether the factor $\frac{2}{3}$ is optimal. If it is, the straight-forward application of Υ does not suffice to show this. In fact, it only gives us that, whenever there is an optimal cobordism between $T_{m,m+1}$ and $T_{2,n}$, then

$$n \leq \frac{3m^2}{4} + O(m), \tag{8}$$

which is the same upper bound that is known for the algebraic setting; see (3). Indeed, let us fix a positive integer m and assume that there exists an optimal cobordism between the $T_{2,n}$ and $T_{m,m+1}$ for some odd $n > 0$; i.e.

$$d_c(T_{2,n}, T_{m,m+1}) = |g(T_{2,n}) - g(T_{m,m+1})|.$$

Using

$$\begin{aligned} \upsilon(T_{2,n}) &\overset{(2)}{=} -\frac{n-1}{2} \quad \text{and} \\ \upsilon(T_{m,m+1}) &= -\left\lfloor \frac{m^2}{4} \right\rfloor \quad \text{[OSS14, Proposition 6.3]}, \end{aligned} \tag{9}$$

the obstruction given in (6) yields

$$\begin{aligned} -\upsilon(T_{2,n}) + \upsilon(T_{m,m+1}) \leq g_4(T_{m,m+1}) - g_4(T_{2,n}) &\quad \overset{(1)(9)}{\Longleftrightarrow} \\ \frac{n-1}{2} - \left\lfloor \frac{m^2}{4} \right\rfloor \leq \frac{m(m-1)}{2} - \frac{n-1}{2} &\quad \Longleftrightarrow \\ n \leq \frac{3m^2-2m+4}{4} &\quad . \end{aligned}$$

A similar calculation using the signature instead of υ also yields (8).

We proceed with the proof of Proposition 25, which can be adapted to yield Remark 26.

Proof of Proposition 25. We denote by Δ_m the half twist on m strands; i.e. the m-braid

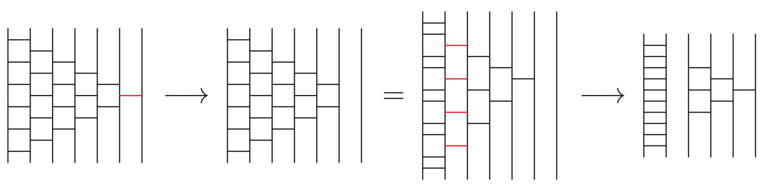
$$(a_1 a_2 \cdots a_{m-1})(a_1 a_2 \cdots a_{m-2}) \cdots (a_1 a_2) a_1.$$

The torus link $T_{m,m}$ is the closure of the full twist on m strands Δ_m^2.

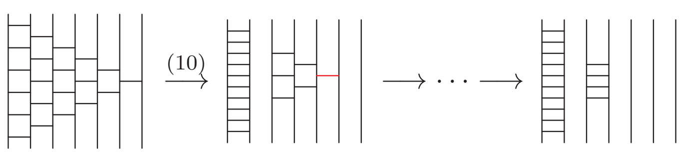
The main step in the proof consists of deleting generators in Δ_m yielding a braid that is a split union of positive 2-braids and which has roughly length $\frac{2}{3} l(\Delta_m)$. More precisely, we delete the generator a_{m-1} in Δ_m and then apply braid relations to get the positive braid word

$$(a_1^2 a_2 \cdots a_{m-2}) \cdots (a_1^2 a_2) a_1^2 \text{ in } B_m.$$

Then, we delete all a_2 yielding a split union of $a_1^{2(m-2)}$ on strands 1 and 2, a half twist on the strands 3 to $m-1$, and strand m. We illustrate this for $m = 7$.

(10) $\Delta_7 =$  ,

where arrows indicate the deletion of the generators marked in red. To the remaining half twist, which we readily identify with Δ_{m-3}, we apply the same procedure. And we do this inductively until the remaining half twist is Δ_3, Δ_2, or Δ_1, where Δ_1 is just the trivial 1-strand braid. Applying the procedure to Δ_3 just yields the split union of a_1^2 and one strand. On $\Delta_2 = a_1^2$ and Δ_1 it does not do anything. This inductive procedure yields a braid β_m, which closes to a split union of torus links of braid index 2. As before we illustrate this for $m = 7$.

(11) $\Delta_7 =$ 

$= \beta_7$

The length $l(\beta_m)$ of β_m is described by the following formula.

$$\begin{aligned} l(\beta_m) &= 2(m-2) + 2(m-5) + 2(m-8) + \cdots \\ &= \begin{cases} (3l-1)l & \text{for } m = 3l \\ (3l+1)l & \text{for } m = 3l+1 \\ (3l+3)l+1 & \text{for } m = 3l+2 \end{cases} \end{aligned}$$

We use the above to obtain a braid γ_m that closes to a $T_{2,n}$ by deleting generators in Δ_m^2, which shows that $T_{2,n}$ is subword-adjacent to $T_{m,m}$. For this we write

$$\Delta_m^2 = \Delta_m\Delta_m = (a_1a_2\cdots a_{m-1})(a_1a_2\cdots a_{m-2})\widetilde{\Delta_{m-2}}\Delta_m,$$

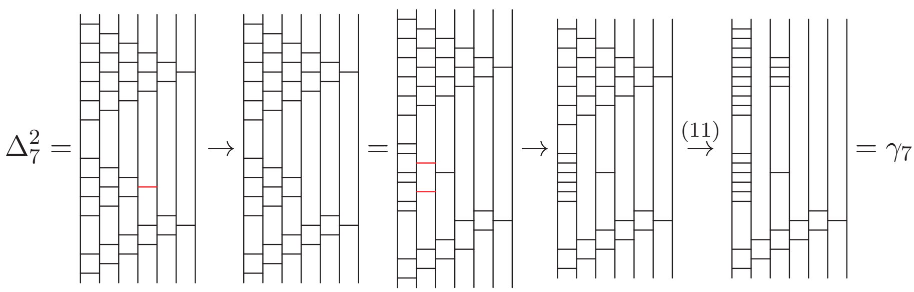
where $\widetilde{\Delta_{m-2}}$ is a half twist on the first $m-2$ strands. Now, we apply the above deleting algorithm to $\widetilde{\Delta_{m-2}}$, which is seen as Δ_{m-2}, and Δ_m yielding

$$\gamma_m = (a_1a_2\cdots a_{m-1})(a_1a_2\cdots a_{m-2})\widetilde{\beta_{m-2}}\beta_m,$$

where $\widetilde{\beta_{m-2}}$ is the m strand braid which is obtained by having β_{m-2} on the first $m-2$ strands. The braid γ_m is of the form

$$\gamma_m = (a_1a_2\cdots a_{m-1})(a_1a_2\cdots a_{m-2})a_1^{\alpha_1}a_3^{\alpha_3}\cdots a_{2k-1}^{\alpha_{2k-1}},$$

where $k = \lfloor \frac{m}{2} \rfloor$ and α_k are positive integers. As above we illustrate this for $m = 7$.



The closure of γ_m is a braid index two torus link $T_{2,n}$. This follows from observing that the closure of $(a_1a_2\cdots a_{m-1})(a_1a_2\cdots a_{m-2})$ is $T_{2,m-1}$. Since

$$l(\gamma_m) - l((a_1a_2\cdots a_{m-1})(a_1a_2\cdots a_{m-2})) = l(\beta_m) + l(\beta_{m-2})$$

we see that $n = m - 1 + l(\beta_{m-2}) + l(\beta_m)$; i.e. the closure of γ_m is $T_{2,m-1+l(\beta_{m-2})+l(\beta_m)}$. With the above calculations for $l(\beta_m)$ we get

$$\begin{aligned}
n &= 3l - 1 + (3l-2)(l-1) + (3l-1)l = 6l^2 - 3l + 1,\\
n &= (3l + 1 - 1) + (3l)(l-1) + 1 + (3l+1)l = 6l^2 + l + 1,\\
n &= (3l + 2 - 1) + (3l-1)l + (3l+3)l + 1 = 6l^2 + 5l + 2,
\end{aligned}$$

for $m = 3l$, $m = 3l+1$, and $m = 3l+2$, respectively. This finishes the proof since n is the largest integer with $n \leq \frac{2m^2+4}{3} - m$. □

6. Calculation of Υ for torus knots of small braid index

For completeness, we provide the calculations that yield the υ-values given in (2).

Proposition 28. *For positive integers n, we have*

$$\upsilon(T_{3,3n+1}) = -2n, \quad \upsilon(T_{3,3n+2}) = -2n-1, \quad \textit{and} \quad \upsilon(T_{4,2n+1}) = -2n.$$

Remark 29. More generally, the calculation we provide below in the proof of Proposition 28 allows to determine the function $\Upsilon_T : [0,2] \to \mathbb{R}$ for torus knots T of braid index 3 and 4: For all positive integers n, we have

$$\Upsilon_{T_{3,3n+1}}(t) = \begin{cases} -3nt & \text{for } 0 \leq t \leq \frac{2}{3} \\ -2n & \text{for } \frac{2}{3} \leq t \leq 1 \end{cases}$$

$$\Upsilon_{T_{3,3n+2}}(t) = \begin{cases} -(3n+1)t & \text{for } 0 \leq t \leq \frac{2}{3} \\ -2n-t & \text{for } \frac{2}{3} \leq t \leq 1 \end{cases}$$

$$\Upsilon_{T_{4,4n+1}}(t) = \begin{cases} -6nt & \text{for } 0 \leq t \leq \frac{1}{2} \\ -2n-2nt & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$\text{and} \quad \Upsilon_{T_{4,4n+3}}(t) = \begin{cases} -(6n+3)t & \text{for } 0 \leq t \leq \frac{1}{2} \\ -2n-(2n+3)t & \text{for } \frac{1}{2} \leq t \leq \frac{2}{3} \\ -2n-2-2nt & \text{for } \frac{2}{3} \leq t \leq 1 \end{cases}$$

This fully describes Υ_T since it is symmetric, i.e. $\Upsilon_T(t) = (2-t)$; see [OSS14].[1]

Our calculations have convinced us that, for a general torus knot $T_{p,q}$, $\Upsilon_{T_{p,q}}(t)$ might look similar to the homogenization of the signature profile of torus knots; i.e. the following function, studied by Gambaudo and Ghys [GG05]:

$$Sign_{T_{p,q}} \colon [0,2] \to \mathbb{R}, \quad t \mapsto \lim_{k \to \infty} \frac{\sigma_{e^{\pi i t}}\left(\overline{((a_1 \cdots a_{p-1})^q)^k}\right)}{k},$$

[1] Added in print: a formula that allows to calculate Υ for torus knots and algebraic knots and, in particular, recovers these calculations has since been described in [FK16].

where σ_ω denotes the Levine-Tristram signature. We hope to explore this further in the future.[2] The only Heegaard-Floer theory input in the proof of Proposition 28 is the following combinatorial procedure to determine Υ for torus knots (or more generally L-space knots) [OSS14]:

Write the Alexander polynomial

$$\Delta(T_{p,q}) = t^{-\frac{(p-1)(q-1)}{2}} \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$$

as $\sum_{k=0}^{l}(-1)^k t^{\alpha_k}$, where $(\alpha_k)_{k=0}^{l}$ is a decreasing sequence of integers. Construct a corresponding decreasing sequence of integers $(m_k)_{k=0}^{l}$ defined by

$$\begin{aligned} &m_0 = 0, \quad m_{2k} = m_{2k-1} - 1, \quad \text{and} \\ &m_{2k+1} = m_{2k} - 2(\alpha_{2k} - \alpha_{2k+1}) + 1. \end{aligned} \tag{12}$$

Then one has

$$\Upsilon_{T_{p,q}}(t) = \max_{0\leq 2k\leq l}\{m_{2k} - t\alpha_{2k}\} \text{ [OSS14, Theorem 1.15].} \tag{13}$$

In particular,

$$\Upsilon_{T_{p,q}}(1) = \upsilon(T_{p,q}) = \max_{0\leq 2k\leq l}\{m_{2k} - \alpha_{2k}\}.$$

In fact, for the calculation one only needs the evenly indexed m_k, for which (12) can be shortened to

$$m_0 = 0 \quad m_{2k} = m_{2k-2} - 2(\alpha_{2k-2} - \alpha_{2k-1}). \tag{14}$$

---

[2] Added in print: the relation between Υ and the signature profil for torus knots has since been described in [FK16].

Proof of Proposition 28. We observe that

$$\begin{aligned}
\Delta(T_{3,3n+1}) &= t^{-3n}\frac{(t^{9n+3}-1)(t-1)}{(t^{3n+1}-1)(t^3-1)} \\
&= t^{-3n}\frac{t^{9n+2}+t^{9n+1}+\cdots+1}{(t^{3n}+t^{3n-1}+\cdots+1)(t^2+t+1)} \\
&= \frac{t^{6n+2}+t^{6n+1}+\cdots+t^{-3n+1}+t^{-3n}}{t^{3n+2}+2t^{3n+1}+3t^{3n}+3t^{3n-1}+\cdots+3t^3+3t^2+2t+1} \\
&= (t^{3n}-t^{3n-1})+(t^{3n-3}-t^{3n-4})+\cdots+(t^3-t^2)+1-t^{-2}+\cdots \\
&= \sum_{i=0}^{n-1}(t^{3n-3i}-t^{3n-3i-1})+1+\sum_{i=1}^{n}(-t^{-3i+1}+t^{-3i}).
\end{aligned}$$

In other words, $\Delta(T_{p,q})=\sum_{k=0}^{l}(-1)^k t^{\alpha_k}$ for

$$l=4n,\quad \alpha_{2k}=3n-3k,\quad \text{and}\quad \alpha_{2k-2}-\alpha_{2k-1}=\begin{cases}1 & \text{for } k\le n\\ 2 & \text{for } k>n\end{cases}$$

Therefore, (14) yields $m_{2k}=-2k$ for $k\le n$ and $m_{2k}=2n-4k$ for $k\ge n$, and so

$$\upsilon(T_{3,3n+1})=\max_{0\le 2k\le l}\{m_{2k}-\alpha_{2k}\}=m_{2n}-\alpha_{2n}=-2n.$$

Similarly, one calculates

$$\begin{aligned}
\Delta(T_{3,3n+2}) &= \frac{t^{6n+4}+t^{6n+3}+\cdots+t^{-3n}+t^{-3n-1}}{t^{3n+3}+2t^{3n+2}+3t^{3n+1}+3t^{3n}+\cdots+3t^3+3t^2+2t+1} \\
&= t^{3n+1}-t^{3n}+t^{3n-2}-t^{3n-3}+\cdots+t^4-t^3+t-1+\cdots \\
&= \sum_{i=0}^{n-1}(t^{3n-3i+1}-t^{3n-3i})+t-1+t^{-1}+\sum_{i=1}^{n}(-t^{-3i}+t^{-3i-1}),
\end{aligned}$$

which means that $\Delta(T_{p,q})=\sum_{k=0}^{l}(-1)^k t^{\alpha_k}$ for $l=4n+2$,

$$\alpha_{2k}=\begin{cases}3n+1-3k & \text{for } k\le n\\ 3n+2-3k & \text{for } k>n\end{cases}\quad\text{and}\quad \alpha_{2k-2}-\alpha_{2k-1}=\begin{cases}1 & \text{for } k\le n+1\\ 2 & \text{for } k>n+1\end{cases}$$

Thus, $m_{2k}=-2k$ for $k\le n+1$ and $m_{2k}=2n+2-4k$ for $k\ge n+1$, which yields

$$\upsilon(T_{3,3n+2})=\max_{0\le 2k\le l}\{m_{2k}-\alpha_{2k}\}=m_{2n}-\alpha_{2n}=m_{2n+2}-\alpha_{2n+2}=-2n-1.$$

For $T_{4,2n+1}$, we calculate first when n is even, i.e. $n = 2s$ for a positive integer s:

$$\Delta(T_{4,4s+1}) = t^{-6s}\frac{(t^{16s+4}-1)(t-1)}{(t^{4s+1}-1)(t^4-1)}$$
$$= \frac{t^{-6s}(t^{16s+3}+t^{16s+2}+\cdots+1)}{(t^{4s}+t^{4s-1}+\cdots+1)(t^3+t^2+t+1)}$$
$$= \frac{t^{10s+3}+t^{10s+2}+\cdots+t^{-6s+1}+t^{-6s}}{t^{4s+3}+2t^{4s+2}+3t^{4s+1}+4t^{4s}+4t^{4s-1}+\cdots+4t^4+4t^3+3t^2+2t+1}$$
$$= \sum_{i=0}^{s-1}(t^{6s-4i}-t^{6s-4i-1}) + \sum_{i=s}^{2s-1}(t^{6s-4i}-t^{6s-4i-2})$$
$$+ \sum_{i=2s}^{3s-1}(t^{6s-4i}-t^{6s-4i-3}) + t^{-6s},$$

which means

$$l = 3n, \quad \alpha_{2k} = 3n-4k, \quad \text{and} \quad \alpha_{2k-2}-\alpha_{2k-1} = \begin{cases} 1 & \text{for } k \leq \frac{n}{2} \\ 2 & \text{for } \frac{n}{2} < k \leq n \\ 3 & \text{for } n < k \end{cases}$$

Therefore, we have

$$m_{2k} = \begin{cases} -2k & \text{for } k \leq \frac{n}{2} \\ n-4k & \text{for } \frac{n}{2} \leq k \leq n \\ 3n-6k & \text{for } n \leq k \end{cases}$$

which yields that $\upsilon(T_{4,2n+1})$ equals

$$\max_{0\leq 2k\leq l}\{m_{2k}-\alpha_{2k}\} = m_n - \alpha_n = m_{n+2}-\alpha_{n+2} = \cdots = m_{2n}-\alpha_{2n} = -2n.$$

Finally, for n odd, a similar calculation yields

$$\Delta(T_{4,2n+1}) = \sum_{i=0}^{\frac{n-1}{2}}(t^{3n-4i}-t^{3n-1-4i}) + \sum_{i=\frac{n+1}{2}}^{n}(t^{3n+1-4i}-t^{3n-1-4i})$$
$$+ \sum_{i=n+1}^{\frac{3n-1}{2}}(t^{3n+2-4i}-t^{3n-1-4i}) + t^{-3n},$$

which means $l = 3n + 1$,

$$\alpha_{2k} = \begin{cases} 3n - 4k & \text{for } k \le \frac{n-1}{2} \\ 3n + 1 - 4k & \text{for } \frac{n-1}{2} < k \le n \\ 3n + 2 - 4k & \text{for } n < k \end{cases} \quad \text{and}$$

$$\alpha_{2k-2} - \alpha_{2k-1} = \begin{cases} 1 & \text{for } k \le \frac{n+1}{2} \\ 2 & \text{for } \frac{n+1}{2} < k \le n+1 \\ 3 & \text{for } n+1 < k \end{cases}$$

Therefore, we have

$$m_{2k} = \begin{cases} -2k & \text{for } k \le \frac{n+1}{2} \\ n + 1 - 4k & \text{for } \frac{n+1}{2} < k \le n+1 \\ 3n + 3 - 6k & \text{for } n+1 < k \end{cases}$$

This yields that $\upsilon(T_{4,2n+1})$ equals

$$\begin{aligned} \max_{0 \le 2k \le l} \{m_{2k} - \alpha_{2k}\} &= m_{n+1} - \alpha_{n+1} \\ &= m_{n+3} - \alpha_{n+3} = \cdots = m_{2n} - \alpha_{2n} = -2n. \end{aligned}$$

□

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