Communications in
Analysis and Geometry
Volume 24, Number 5, 939–967, 2016

Augmentation rank of satellites with braid pattern

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Given a knot K in S^3, a question raised by Cappell and Shaneson asks if the meridional rank of K equals the bridge number of K. Using augmentations in knot contact homology we consider the persistence of equality between these two invariants under satellite operations on K with a braid pattern. In particular, we answer the question in the affirmative for a large class of iterated torus knots.

1. Introduction

Let K be an oriented knot in S^3 and denote by π_K the fundamental group of its complement $\overline{S^3 \setminus n(K)}$, with some basepoint. We call an element of π_K a *meridian* if it is represented by the oriented boundary of a disc, embedded in S^3, whose interior intersects K positively once. The group π_K is generated by meridians; the *meridional rank* of K, written $\mathrm{mr}(K)$, is the minimal size of a generating set containing only meridians.

Choose a height function $h\colon\ S^3 \to \mathbb{R}$. The *bridge number* of K, denoted $b(K)$, is the minimum of the number of local maxima of $h|_{\varphi(S^1)}$ among embeddings $\varphi\colon\ S^1 \to S^3$ which realize K.

By considering Wirtinger's presentation of π_K one can show that $\mathrm{mr}(K) \leq b(K)$ for any $K \subset S^3$. Whether the bound is equality for all knots is an open question attibuted to Cappell and Shaneson [Kir97, Prob. 1.11]. Equality is known to hold for some families of knots due to work of various authors ([BZ85, Cor14b, RZ87]).

Our approach to this problem is to use ideas coming from knot contact homology. We remark, however, that while we will draw from known results in this area, the arguments in the paper are combinatorial in nature and should be quite accessible.

In particular we study *augmentations* of K, which are certain maps associated to the knot contact homology of K. To each augmentation is associated a rank and there is a maximal rank of augmentations of a given K, called the *augmentation rank* $\mathrm{ar}(K)$. For any K the inequality $\mathrm{ar}(K) \leq$

$\mathrm{mr}(K)$ holds (see Section 3.3). We consider the behavior of $\mathrm{ar}(K)$ under satellite operations with a braid pattern.

Denote the group of braids on n strands by B_n and write $\hat{\beta}$ for the *braid closure* of a braid β (see Section 3, Figure 3). We write $\imath_n$ for the identity in B_n. Throughout the paper we let $\alpha \in B_k$ and $\gamma \in B_p$ and set $K = \hat{\alpha}$. We assume our braid closures are (connected) knots. Note that $\mathrm{ar}(K) \leq k$.

Definition 1.1. Let $\imath_p(\alpha)$ be the braid in B_{kp} obtained by replacing each strand of α by p parallel copies (in the blackboard framing). Let $\bar{\gamma}$ be the inclusion of γ into B_{kp} by the map $\sigma_i \mapsto \sigma_i, 1 \leq i \leq p-1$. Set $\gamma(\alpha) = \imath_p(\alpha)\bar{\gamma}$. The *braid satellite* of K associated to α, γ is defined as $K(\alpha, \gamma) = \widehat{\gamma(\alpha)}$.

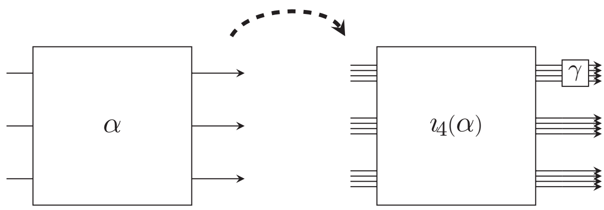


Figure 1: Constructing $\gamma(\alpha)$ from α; case $p = 4$.

As defined $K(\alpha, \gamma)$ depends on the choice of α, rather than just $\hat{\alpha}$. However, this dependence vanishes if one requires the index k of α to be minimal among braid representatives of K (see Section 2).

Note that if $\hat{\alpha}$ and $\hat{\gamma}$ are each a knot, $K(\alpha, \gamma)$ is also. Our principal result is the following.

Theorem 1.2. *If* $\alpha \in B_k$ *and* $\gamma \in B_p$ *are such that* $\mathrm{ar}(\hat{\alpha}) = k$ *and* $\mathrm{ar}(\hat{\gamma}) = p$*, then* $\mathrm{ar}(K(\alpha, \gamma)) = kp$.

A corollary of Theorem 1.2 involves Cappell and Shaneson's question for iterated torus knots. Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be integral vectors. We write $T(\mathbf{p}, \mathbf{q})$ for the $(\mathbf{p}, \mathbf{q})$ *iterated torus knot*, defined as follows.

By convention take $T(\emptyset, \emptyset)$ as the unknot, then define $T(\mathbf{p}, \mathbf{q})$ inductively. Let $\hat{\mathbf{p}}, \hat{\mathbf{q}}$ be the truncated lists obtained from $\mathbf{p}, \mathbf{q}$ by removing the last integer in each. If α is a braid of minimal index such that $T(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = \hat{\alpha}$ then define $T(\mathbf{p}, \mathbf{q}) = K(\alpha, (\sigma_1 \cdots \sigma_{p_n - 1})^{q_n})$.

We remark that $T(\mathbf{p}, \mathbf{q})$ is a cable of $T(\hat{\mathbf{p}}, \hat{\mathbf{q}})$, but not the (p_n, q_n)-cable in the traditional Seifert framing.

Corollary 1.3. *Given integral vectors* $\mathbf{p}$ *and* $\mathbf{q}$*, suppose that* $|p_i| < |q_i|$ *and* $gcd(p_i, q_i) = 1$ *for each* $1 \le i \le n$*. Then*

$$\mathrm{ar}(T(\mathbf{p}, \mathbf{q})) = \mathrm{mr}(T(\mathbf{p}, \mathbf{q})) = b(T(\mathbf{p}, \mathbf{q})) = p_1 p_2 \cdots p_n.$$

The assumption $|p_i| < |q_i|$ is used to apply Theorem 1.2. We note this is truly a requirement; that is, there are cables of $(n, n+1)$ torus knots, with $|p_2| > |q_2|$, which have lower augmentation rank than bridge number.

Theorem 1.4. *Given* $p > 1$ *and* $n > 1$*,* $\mathrm{ar}(T((n, p), (n+1, 1))) < np$*.*

It is natural to wonder if the augmentation rank is multiplicative under weaker assumptions on α, γ than those in Theorem 1.2. The following is a possible generalization.

Conjecture 1.5. *Suppose* $K = \hat{\alpha}$ *for* $\alpha \in B_k$*, and that* α *has minimal index among braids with the same closure. Let* $\gamma \in B_p$*. Then* $\mathrm{ar}(K(\alpha, \gamma)) \ge \mathrm{ar}(\hat{\alpha})\,\mathrm{ar}(\hat{\gamma})$*.*

Remark 1.6. There are examples when the inequality of Conjecture 1.5 is strict (see Section 5).

The paper is organized as follows. Section 2 relates braid satellites to existing conventions on satellite operators. In Section 3 we give the needed background in knot contact homology, specifically Ng's cord algebra, and discuss augmentation rank and the relationship to meridional rank. Section 3.4 reviews techniques used in the proof of Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.2, its requisite supporting lemmas, and Corollary 1.3. Finally, Section 5 considers the sharpness of our results. We prove Theorem 1.4 and briefly discuss the more general case, Conjecture 1.5.

Acknowledgements

We thank the referee for useful comments and suggestions. The first author was supported in part by an AMS-Simons travel grant and is very grateful for this program. The second author was supported through NSF grant DMS-0846346, as a fellow in the PRUV program at Duke University, and thanks David Kraines and the Duke Math Department for organizing the PRUV program. Both authors would like to thank Lenhard Ng for his consultation and helpful comments.

2. Satellite operators and the braid satellite

Definition 1.1 of the braid satellite $K(\alpha, \gamma)$ produces a knot that depends not only on the braid closure $\hat{\alpha}$, but on the chosen braid α as well. We remark here how to avoid this ambiguity.

A tubular neighborhood of an oriented knot J has a standard identification with $S^1 \times D^2$ determined by an oriented Seifert surface that J bounds. Given a knot $P \subset S^1 \times D^2$, per the usual convention, let $P(J)$ be the satellite of J with pattern P obtained with this framing.

Proposition 2.1. *Given a knot J and a braid $\gamma \in B_p$, let ω be the writhe of some minimal index closed braid representing J. Let $P \subset S^1 \times D^2$ be the braid closure of $\Delta^{2\omega}\gamma$, where Δ^2 is the full twist in B_p. Then $K(\alpha, \gamma) = P(J)$ for any minimal index braid α with $J = \hat{\alpha}$.*

Proof. The principal observation is that, since the Jones conjecture holds [DP13, LM14], the writhe of α must be ω. Thus the blackboard framing of the closure of $\imath_p(\alpha)\bar{\Delta}^{-2\omega}$ agrees with the $(p, 0)$-cable of J (with Seifert framing). □

We note, the satellite $T(\mathbf{p}, \mathbf{q})$ corresponds to the $(p_n, p_n\omega_n + q_n)$-cable of $T(\hat{\mathbf{p}}, \hat{\mathbf{q}})$, where ω_n is defined inductively by $\omega_n = p_{n-1}\omega_{n-1} + (p_{n-1} - 1)q_{n-1}$ and $\omega_1 = 0$.

Concerning the bridge number of $K(\alpha, \gamma)$, a result of Schubert [Sch54] (see [Sch03] also) states that if J is not the unknot and $P(J)$ is a satellite such that P has winding number p, then $b(P(J)) \geq p\, b(J)$. Since $K(\alpha, \gamma) = \widehat{\gamma(\alpha)}$, it has bridge number at most kp and thus $b(K(\alpha, \gamma)) = kp$ whenever $b(\hat{\alpha}) = k$. From this we see $b(T(\mathbf{p}, \mathbf{q})) = p_1 p_2 \cdots p_n$, provided $p_1 < q_1$.

3. Background

We review in Section 3.1 the construction of $HC_0(K)$ from the viewpoint of the combinatorial knot DGA, defined in [Ng08]; our conventions are those given in [Ng14]. In Section 3.3 we discuss augmentations in knot contact homology and their rank, which gives a lower bound on the meridional rank of the knot group. Section 3.4 contains a discussion of techniques from [Cor14a] that we use to calculate the augmentation rank.

Throughout the paper we orient n-braids in B_n from left to right, labeling the strands $1, \ldots, n$, with 1 the topmost and n the bottommost strand. We work with Artin's generators $\{\sigma_i^{\pm}, i = 1, \ldots, n-1\}$ of B_n, where in σ_i only the i and $i+1$ strands interact, and they cross once in the manner depicted in Figure 2. Given a braid $\beta \in B_n$, the braid closure $\hat{\beta}$ of β is the

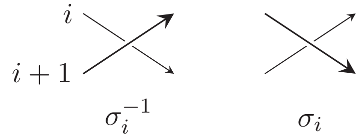


Figure 2: Generators of B_n.

link obtained as shown in Figure 3. The *writhe* (or algebraic length) of β, denoted $\omega(\beta)$, is the sum of exponents of the Artin generators in a word representing β.

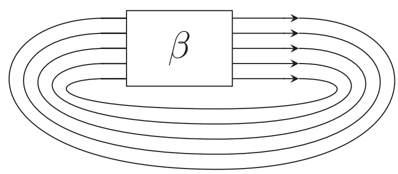


Figure 3: The braid closure of β.

3.1. Knot contact homology

We review the combinatorial knot DGA of Ng (we discuss only the degree zero part as this will suffice for our purposes). This DGA was defined in order to be a calculation of knot contact homology and was shown to be so in [EENS13].

Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by a_{ij}, $1 \le i \ne j \le n$. We define a homomorphism $\phi\colon\ B_n \to \operatorname{Aut} \mathcal{A}_n$ by defining it on the generators of B_n:

$$\phi_{\sigma_k}: \begin{cases} a_{ij} \mapsto a_{ij} & i, j \ne k, k+1 \\ a_{k+1,i} \mapsto a_{ki} & i \ne k, k+1 \\ a_{i,k+1} \mapsto a_{ik} & i \ne k, k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}a_{ki} & i \ne k, k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}a_{k,k+1} & i \ne k, k+1 \end{cases} \tag{1}$$

It can be checked that this definition extends to a well-defined homomorphism $\phi\colon B_n \to \operatorname{Aut}\mathcal{A}_n$. Let $\iota\colon B_n \to B_{n+1}$ be the inclusion $\sigma_i \mapsto \sigma_i$ so that the $(n+1)$ strand does not interact with those from $\beta \in B_n$, and define $\phi^*_\beta \in \operatorname{Aut}\mathcal{A}_{n+1}$ by $\phi^*_\beta = \phi_{\iota(\beta)}$. We then define the $n \times n$ matrices Φ^L_β and Φ^R_β with entries in $\mathcal{A}_n$ by

$$\phi^*_\beta(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi^L_\beta)_{ij} a_{j,n+1}$$

$$\phi^*_\beta(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi^R_\beta)_{ji}$$

Finally, let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm1}, \mu^{\pm1}]$ and define matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ over $\mathcal{A}_n \otimes R_0$ by

$$\mathbf{A}_{ij} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1 - \mu & i = j \end{cases} \tag{2}$$

$$\mathbf{\Lambda} = \operatorname{diag}[\lambda\mu^{\omega(\beta)}, 1, \ldots, 1]. \tag{3}$$

Definition 3.1. Suppose that K is the closure of $\beta \in B_n$. Define $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$ to be the ideal generated by the entries of $\mathbf{A} - \mathbf{\Lambda} \cdot \mathbf{\Phi}^{\mathbf{L}}_\beta \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \mathbf{\Phi}^{\mathbf{R}}_\beta \cdot \mathbf{\Lambda}^{-1}$. The *degree zero homology of the combinatorial knot DGA* is $\mathrm{HC}_0(K) = (\mathcal{A}_n \otimes R_0)/\mathcal{I}$.

3.2. Spanning arcs

The proofs in Sections 4 and 5 require a number of computations of ϕ_β (and of ϕ^*_β) for particular braids. Such computations are benefited by an alternate description of the automorphism, which we now explain.

Definition 3.2. Given $n > 0$, let D_n be a disk in $\mathbb{C}$ containing points $P = \{1, 2, \ldots, n\} \subset \mathbb{R}$ in its interior. A *spanning arc* of D_n is the isotopy class, relative to P, of an oriented embedded path in D_n which begins and ends in P and otherwise does not intersect P. We define $\mathscr{S}_n$ as the associative ring freely generated by spanning arcs of D_n modulo the ideal generated by the relation in Figure 4. Denote by $c_{ij} \in \mathscr{S}_n$ the element represented by a spanning arc contained in the upper half-disk beginning at i and ending at j.

We understand the spanning arcs in Figure 4 to agree outside of a neighborhood of the depicted point, which is in P.

Figure 4: Relation in $\mathscr{S}_n$.

We consider β as a mapping class of (D, P) and denote by $\beta \cdot c$ the image of the spanning arc c. By convention σ_k acts by rotating k and $k+1$ about their midpoint in counter-clockwise fashion. It was shown in [Ng05b, Section 2] that there is a unique, well-defined map χ which sends each spanning arc of D_n to an element of $\mathcal{A}_n$ such that

(i) $\chi(\beta \cdot c) = \phi_\beta(\chi(c))$ for any spanning arc c and $\beta \in B_n$;

(ii) $\chi(c_{ij}) = a_{ij}$ if $i < j$, $\chi(c_{ij}) = -a_{ij}$ if $i > j$.

Furthermore, χ factors through $\mathscr{S}_n$, is injective, and by the relation in Figure 4 the value of $\phi_\beta(a_{ij})$ can be determined from (i) and (ii). This constitutes an essential technique for our calculations of ϕ_β.

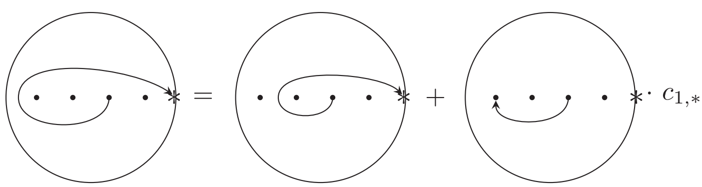


Figure 5: $(\sigma_1\sigma_2\sigma_3)^2 \cdot c_{1*}$, as an element of $\mathscr{S}_{4+1}$.

Computations of Φ^L_β are carried out in likewise manner, including β into B_{n+1} and considering spanning arcs $c_{j,n+1}$, $1 \le j \le n$ of D_{n+1}. We will distinguish this situation by relabeling $n+1$ (and corresponding indices) with the symbol $*$. In figures, we put the point $*$ at the boundary of D. As an example, refer to Figure 5, where an intermediate step is shown in using this method to calculate $\phi^*_{(\sigma_1\sigma_2\sigma_3)^2}(a_{1*}) = a_{3*} - a_{32}a_{2*} - (a_{31} - a_{32}a_{21})a_{1*}$.

It will be convenient for us in Section 4 to consider the free left $\mathcal{A}_n$-module $\mathcal{A}^L_n = \mathcal{A}_n\langle a_{1*}, \dots, a_{n*}\rangle$ and right $\mathcal{A}_n$-module $\mathcal{A}^R_n = \langle a_{*1}, \dots, a_{*n}\rangle\mathcal{A}_n$. By definition, Φ^L_β (respectively Φ^R_β) is the transpose of the matrix in the above basis for the $\mathcal{A}_n$-automorphism of $\mathcal{A}^L_n$ (respectively $\mathcal{A}^R_n$) determined by the image of the basis under ϕ^*_β.

Finally, as we are considering braid satellites $K(\alpha, \gamma)$ with $\gamma \in B_p$ our perspective often considers the points in D_{kp} as k groups of p points each.

We find it convenient in figures of spanning arcs in $\mathscr{S}_{kp}$ to reflect this point of view. To do so, for each $i = 0, \dots, k-1$, we depict the points $\{ip+1, \dots, (i+1)p\}$ by a horizontal segment, and if a spanning arc ends at $ip+s$ for $1 \le s \le p$, it is depicted ending on the $(i+1)^{st}$ segment with a label s (see example in Figure 6).

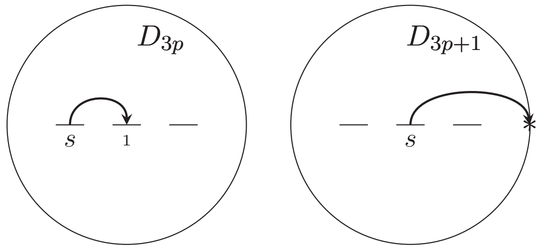


Figure 6: Spanning arcs $c_{s,p+1}$ and $c_{p+s,*}$, $1 \le s \le p$.

Let perm: $B_n \to S_n$ denote the homomorphism from B_n to the symmetric group sending σ_k to the simple transposition interchanging $k, k+1$.

Lemma 3.3. *For $\beta \in B_n$ and $1 \le i \ne j \le n$, consider $(\Phi^L_\beta)_{ij} \in \mathcal{A}_n$ as a polynomial expression in the (non-commuting) variables $\{a_{kl}, 1 \le k \ne l \le n\}$. Writing $i_0 = perm(\beta)(i)$, every monomial in $(\Phi^L_\beta)_{ij}$ is a constant times $a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{l-1},j}$ for some $l \ge 0$, the monomial being a constant if $l = 0$ and only if $i_0 = j$.*

Proof. Consider the spanning arc $\beta \cdot c_{i,*}$ which begins at i_0 and ends at $*$. Applying the relation in Figure 4 to the path equates it with a sum (or difference) of another path with the same endpoints and a product of two paths, the first beginning at i_0 and the other ending at $*$. A finite number of applications of this relation allows one to express the path as a polynomial in the $c_{kl}, 1 \le k \ne l \le n$ where each monomial has the form $c_{i_0 i_1} \cdots c_{i_{l-1},j} c_{j,*}$ for some j (see [Ng05b, Lemma 2.6]).

The result follows from $\phi^*_\beta(a_{i,*}) = \phi^*_\beta(\chi(c_{i,*})) = \chi(\beta \cdot c_{i,*})$. $\square$

3.3. Augmentations and augmentation rank

For our purpose, augmentations of a differential graded algebra $(\mathcal{A}, \partial)$ are graded maps $(\mathcal{A}, \partial) \to (\mathbb{C}, 0)$ that intertwine the differential (here $\mathbb{C}$ has grading zero). If $\beta \in B_n$ is a braid representative of K, such a map corresponds precisely to a homomorphism $\epsilon\colon \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that ϵ sends elements of $\mathcal{I}$ to zero (see Definition 3.1).

Definition 3.4. Suppose that K is the closure of $\beta \in B_n$. An *augmentation* of K is a homomorphism $\epsilon\colon \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that $\epsilon(1) = 1$ and each element of $\mathcal{I}$ is sent by ϵ to zero.

A correspondence between augmentations and certain representations of the knot group π_K was studied in [Cor14a]. Recall that π_K is generated by meridians, which for a knot are all conjugate. Fix some meridian m.

Definition 3.5. For any integer $r \geq 1$, a homomorphism $\rho\colon \pi_K \to \mathrm{GL}_r\mathbb{C}$ is a *KCH representation* if $\rho(m)$ is diagonalizable and has an eigenvalue of 1 with multiplicity $r - 1$. We call ρ a *KCH irrep* if it is irreducible.

In [Ng08], Ng describes an isomorphism between $HC_0(K)$ and an algebra constructed from elements of π_K. As discussed in [Ng14] a KCH representation $\rho\colon \pi_K \to \mathrm{GL}_r\mathbb{C}$ induces an augmentation ϵ_ρ of K. Given an augmentation, the first author showed how to construct a KCH representation that induces it. In fact, we have the following rephrasing of results from [Cor14a].

Theorem 3.6 ([Cor14a]). *Let $\epsilon\colon \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ be an augmentation with $\epsilon(\mu) \neq 1$. There is a KCH irrep $\rho\colon \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$. Furthermore, for any KCH irrep $\rho\colon \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$, the rank of $\epsilon(\mathbf{A})$ equals r.*

The abuse of notation $\epsilon(\mathbf{A})$ means that ϵ is applied to each entry of $\mathbf{A}$. Similar notation will be used in the remainder of the paper. Considering Theorem 3.6 we make the following definition.

Definition 3.7. The *rank* of an augmentation $\epsilon\colon \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ with $\epsilon(\mu) \neq 1$ is the rank of $\epsilon(\mathbf{A})$. Given a knot K, the *augmentation rank of* K, denoted $\mathrm{ar}(K)$, is the maximum rank among augmentations of K.

Remark 3.8. By Theorem 3.6 the set of ranks of augmentations of a given K does not depend on choice of braid representative.

It is the case that $\mathrm{ar}(K)$ is well-defined. That is, given K there is a bound on the maximal rank of an augmentation of K.

Theorem 3.9 ([Cor14b]). *Given a knot $K \subset S^3$, if $g_1, \dots, g_d$ are meridians that generate π_K and $\rho\colon \pi_K \to GL_r\mathbb{C}$ is a KCH irrep then $r \leq d$.*

As in the introduction, if we denote the meridional rank of π_K by $\mathrm{mr}(K)$, then Theorem 3.9 says that $\mathrm{ar}(K) \leq \mathrm{mr}(K)$. In addition, the geometric quantity $b(K)$ called the bridge index of K is never less than $\mathrm{mr}(K)$. Thus we have the following corollary.

Corollary 3.10 ([Cor14b]). *Given a knot $K \subset S^3$,*

$$ar(K) \le mr(K) \le b(K).$$

Hence to verify that $\mathrm{mr}(K) = b(K)$ it suffices to find a rank $b(K)$ augmentation of K. Herein we concern ourselves with a setting where $\mathrm{ar}(K) = n$ and there is a braid $\beta \in B_n$ which closes to K. This is a special situation, since $b(K)$ is strictly less than the braid index for many knots.

3.4. Finding augmentations

The following theorem concerns the behavior of the matrices Φ^L_β and Φ^R_β under the product in B_n. It is an essential tool for studying $HC_0(K)$ and is central to our arguments.

Theorem 3.11 ([Ng05a], Chain Rule). *Let β_1, β_2 be braids in B_n. Then $\Phi^L_{\beta_1\beta_2} = \phi_{\beta_1}(\Phi^L_{\beta_2}) \cdot \Phi^L_{\beta_1}$ and $\Phi^R_{\beta_1\beta_2} = \Phi^R_{\beta_1} \cdot \phi_{\beta_1}(\Phi^R_{\beta_2})$.*

Another property of Φ^L_β and Φ^R_β that is important to us is the following symmetry. Define an involution $x \mapsto \overline{x}$ on $\mathcal{A}_n$ (termed *conjugation*) as follows: first set $\overline{a_{ij}} = a_{ji}$; for x, y each some product of generators define $\overline{xy} = \overline{y}\,\overline{x}$; then extend the operation linearly to $\mathcal{A}_n$.

Theorem 3.12 ([Ng05a], Prop. 6.2). *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $\beta \in B_n$, Φ^R_β is the transpose of $\overline{\Phi^L_\beta}$.*

The main result of the paper concerns augmentations with rank equal to the braid index of K. Define the diagonal matrix $\Delta(\beta) = \mathrm{diag}[(-1)^{w(\beta)}, 1, \ldots, 1]$. From Section 5 of [Cor14a] we have the following.

Theorem 3.13 ([Cor14a]). *If K is the closure of $\beta \in B_n$ and has a rank n augmentation $\epsilon\colon \mathcal{A}_n \otimes R_0 \to \mathbb{C}$, then*

$$\epsilon(\Phi^L_\beta) = \Delta(\beta) = \epsilon(\Phi^R_\beta). \tag{4}$$

Furthermore, any homomorphism $\epsilon\colon \mathcal{A}_n \to \mathbb{C}$ which satisfies (4) can be extended to $\mathcal{A}_n \otimes R_0$ to produce a rank n augmentation of K.

4. Main result

The proof of Theorem 1.2 relies on Theorem 3.13. Given braids $\alpha \in B_k$ and $\gamma \in B_p$, recall the definition of $\imath_p(\alpha)$, Definition 1.1. We define a homomorphism $\psi\colon \mathcal{A}_{kp} \to \mathcal{A}_k \otimes \mathcal{A}_p$ which suitably simplifies $\Phi^L_{\imath_p(\alpha)}$ and $\Phi^R_{\imath_p(\alpha)}$ when applied to the entries. Theorem 3.11 then allows us to construct a map that satisfies (4) for $\beta = \gamma(\alpha)$. The map in question is "close to" the tensor product of an augmentation of $\hat{\alpha}$ and an augmentation of $\hat{\gamma}$, composed with ψ.

Section 4.1 begins with an intermediate result, Proposition 4.1, followed by the proofs of Theorem 1.2 and Corollary 1.3. In Section 4.2 we prove Lemma 4.2, which is needed to prove Proposition 4.1.

4.1. Proof of main result

We recall the statement of Theorem 1.2.

Theorem 1.2. If $\alpha \in B_k$ and $\gamma \in B_p$ are such that $\mathrm{ar}(\hat{\alpha}) = k$ and $\mathrm{ar}(\hat{\gamma}) = p$, then $\mathrm{ar}(K(\alpha,\gamma)) = kp$.

For $1 \le i \le kp$, write $i = (q_i - 1)p + r_i$, where $1 \le r_i \le p$ and $1 \le q_i \le k$. For each generator $a_{ij} \in \mathcal{A}_{kp}$, $1 \le i \ne j \le kp$, define

$$\psi(a_{ij}) = \begin{cases} 1 \otimes a_{r_i r_j} & : q_i = q_j \\ a_{q_i q_j} \otimes 1 & : r_i = r_j \\ 0 & : (q_i - q_j)(r_i - r_j) < 0 \\ a_{q_i q_j} \otimes a_{r_i r_j} & : (q_i - q_j)(r_i - r_j) > 0 \end{cases}, \tag{5}$$

which determines an algebra map $\psi\colon \mathcal{A}_{kp} \to \mathcal{A}_k \otimes \mathcal{A}_p$. Extend ψ to a map $\psi^*\colon \mathcal{A}^L_{kp} \to \mathcal{A}^L_k \otimes \mathcal{A}^L_p$ that carries one basis to another: $\psi^*(a_{i*}) = a_{q_i,*} \otimes a_{r_i,*}$ for any $1 \le i \le kp$. Note, if we extend conjugation to $\mathcal{A}_k \otimes \mathcal{A}_p$ by applying it to each factor, then $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$.

Proposition 4.1. *$\psi\left(\Phi^L_{\imath_p(\alpha)}\right) = \Phi^L_\alpha \otimes I_p$ and $\psi\left(\Phi^R_{\imath_p(\alpha)}\right) = \Phi^R_\alpha \otimes I_p$ for any braid α.*

A comment on notation is in order. The tensor product (over $\mathbb{Z}$) of $\mathcal{A}^L_k$ and $\mathcal{A}^L_p$ is a left $(\mathcal{A}_k \otimes \mathcal{A}_p)$-module with basis $\{a_{i*} \otimes a_{j*}\}$. By $\Phi^L_\alpha \otimes I_p$ we mean the matrix in this basis for the $(\mathcal{A}_k \otimes \mathcal{A}_p)$-linear map equal to the tensor product of the map corresponding to Φ^L_α with the identity on $\mathcal{A}^L_p$. Similarly for $\mathcal{A}^R_k$ and $\mathcal{A}^R_p$.

Proposition 4.1 hinges on the following lemma, proved in Section 4.2.

Lemma 4.2. *For $\alpha \in B_k$ the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{A}^L_{kp} \arrow[r, "\phi^*_{\imath_p(\alpha)}"] \arrow[d, "\psi^*"'] & \mathcal{A}^L_{kp} \arrow[d, "\psi^*"'] \\
\mathcal{A}^L_k \otimes \mathcal{A}^L_p \arrow[r, "\phi^*_\alpha \otimes \mathrm{id}"'] & \mathcal{A}^L_k \otimes \mathcal{A}^L_p
\end{tikzcd}$$

In particular, $\psi^*(\phi^*_{\imath_p(\alpha)}(a_{i,*})) = (\phi^*_\alpha \otimes \mathrm{id})(\psi^*(a_{i,*}))$ *for any* $1 \leq i \leq kp$.

Proof of Proposition 4.1. The proposition readily follows from Lemma 4.2. Fixing $\alpha \in B_k$ and $1 \leq i \leq kp$, we have

$$\begin{aligned}
\left(\sum_{l=1}^{k} \left(\Phi^L_\alpha\right)_{q_i l} a_{l*}\right) \otimes a_{r_i *} &= \left(\phi^*_\alpha \otimes \mathrm{id}\right) \psi^* \left(a_{i*}\right) \\
&= \psi^* \left(\phi^*_{\imath_p(\alpha)} \left(a_{i*}\right)\right) \\
&= \sum_{j=1}^{kp} \psi \left(\left(\Phi^L_{\imath_p(\alpha)}\right)_{ij}\right) \left(a_{q_j *} \otimes a_{r_j *}\right).
\end{aligned}$$

Hence $\psi((\Phi^L_{\imath_p(\alpha)})_{ij}) = 0$ if $r_i \neq r_j$ and $\psi((\Phi^L_{\imath_p(\alpha)})_{ij}) = (\Phi^L_\alpha)_{q_i q_j} \otimes 1$ if $r_i = r_j$, since for each $1 \leq l \leq k$ exactly one j satisfies both $r_j = r_i$ and $q_j = l$. We conclude $\psi(\Phi^L_{\imath_p(\alpha)}) = \Phi^L_\alpha \otimes I_p$. That $\psi(\Phi^R_{\imath_p(\alpha)}) = \Phi^R_\alpha \otimes I_p$ follows from $\Phi^R_\alpha = \overline{\Phi^L_\alpha}^t$ and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. □

Proof of Theorem 1.2. By Theorem 3.13 there exist augmentations $\epsilon_k \colon \mathcal{A}_k \otimes R_0 \to \mathbb{C}$ and $\epsilon_p \colon \mathcal{A}_p \otimes R_0 \to \mathbb{C}$, for the closures of α, γ respectively, such that $\epsilon_k\left(\Phi^L_\alpha\right) = \epsilon_k\left(\Phi^R_\alpha\right) = \Delta(\alpha)$ and $\epsilon_p\left(\Phi^L_\gamma\right) = \epsilon_p\left(\Phi^R_\gamma\right) = \Delta(\gamma)$. Theorem 3.13 also implies that it suffices to prove that there exists a map $\epsilon \colon \mathcal{A}_{kp} \to \mathbb{C}$ such that $\epsilon\left(\Phi^L_{\gamma(\alpha)}\right) = \epsilon\left(\Phi^R_{\gamma(\alpha)}\right) = \Delta(\gamma(\alpha))$.

Below we will define a homomorphism $\delta \colon \mathcal{A}_p \to \mathbb{C}$ such that for each generator a_{ij} we have $\delta(a_{ij}) = \pm\epsilon_p(a_{ij})$, the sign depending on the parity of $w(\alpha)$ and p. Let $\pi \colon \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ be the multiplication $a \otimes b \mapsto ab$. Our desired map is defined by $\epsilon = \pi \circ (\epsilon_k \otimes \delta) \circ \psi$.

Theorem 3.11 gives that

$$\pi \circ (\epsilon_k \otimes \delta) \circ \psi\left(\Phi^L_{\gamma(\alpha)}\right) = \pi \circ (\epsilon_k \otimes \delta)\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar\gamma}\right)\right)\psi\left(\Phi^L_{\imath_p(\alpha)}\right). \tag{6}$$

Consider the action of $\imath_p(\alpha)$ on spanning arcs c_{ij}, for $1 \le i \ne j \le p$. Since the points $\{1, \dots, p\} \in D_{kp}$ are moved as one block by $\imath_p(\alpha)$, there exists $0 \le m < k$ with $\phi_{\imath_p(\alpha)}(a_{ij}) = \chi(\imath_p(\alpha) \cdot c_{ij}) = a_{i+mp,j+mp}$. But then $\psi(a_{i+mp,j+mp}) = 1 \otimes a_{ij}$ and so

$$\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar\gamma}\right)\right) = \left[1 \otimes \left(\Phi^L_{\bar\gamma}\right)_{ij}\right], \tag{7}$$

the right hand side denoting the $kp \times kp$ matrix with (i, j) entry $1 \otimes \left(\Phi^L_{\bar\gamma}\right)_{ij} \in \mathcal{A}_k \otimes \mathcal{A}_{kp}$. While $\Phi^L_{\bar\gamma}$ is a matrix over $\mathcal{A}_{kp}$, its entries lie in the image of the natural inclusion of $\mathcal{A}_p$ into $\mathcal{A}_{kp}$. Thus we regard the entries of the matrix in (7) as elements of $\mathcal{A}_k \otimes \mathcal{A}_p$. Returning to the right hand side of (6), by Proposition 4.1 we have

$$\begin{aligned}
&\pi \circ (\epsilon_k \otimes \delta)\left(\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar\gamma}\right)\right)\psi\left(\Phi^L_{\imath_p(\alpha)}\right)\right)\\
&= \pi \circ (\epsilon_k \otimes \delta)\left(\left[1 \otimes \left(\Phi^L_{\bar\gamma}\right)_{ij}\right]\left(\Phi^L_\alpha \otimes I_p\right)\right)\\
&= \delta\left(\Phi^L_{\bar\gamma}\right)\pi\left(\Delta(\alpha) \otimes I_p\right).
\end{aligned}$$

We are done if we define δ so that $\delta\left(\Phi^L_{\bar\gamma}\right)\pi\left(\Delta(\alpha) \otimes I_p\right) = \Delta(\gamma(\alpha))$. When $w(\alpha)$ is even $w(\imath_p(\alpha))$ is also, and further $\Delta(\alpha) = I_k$. Letting $\delta = \epsilon_p$ makes

$$\delta\left(\Phi^L_{\bar\gamma}\right)\pi\left(\Delta(\alpha) \otimes I_p\right) = \epsilon_p\left(\Phi^L_{\bar\gamma}\right) = \Delta(\bar\gamma) = \Delta(\gamma(\alpha)).$$

Suppose $w(\alpha)$ is odd. Define $g\colon \{1, \dots, p\} \to \{\pm 1\}$ as follows. Let $x_1 = 1$, and $x_l = \mathrm{perm}(\gamma)(x_{l-1})$ for $1 < l \le p$. Since the closure of γ is a knot, $\mathrm{perm}(\gamma)$ is given by the p-cycle $(x_1x_2 \cdots x_p)$. If p is even, we let $g(x_1) = 1$, and $g(x_l) = -g(x_{l-1})$ for $1 < l \le p$. If p is odd, let $g(x_1) = g(x_2) = 1$ and $g(x_l) = -g(x_{l-1})$ for $2 < l \le p$.

Define $\delta\colon \mathcal{A}_p \to \mathbb{C}$ by setting $\delta(a_{ij}) = g(i)g(j)\epsilon_p(a_{ij})$ for $1 \le i \ne j \le p$. Fix i, j and consider a monomial M of $\left(\Phi^L_{\bar\gamma}\right)_{ij}$, which is constant if $i > p$ or $j > p$. For $i, j \le p$, write $i_0 = \mathrm{perm}(\gamma)(i)$. By Proposition 3.3 there is a constant $u_M \in \mathbb{Z}$ so that $M = u_M a_{i_0,j_1}a_{j_1,j_2} \cdots a_{j_m,j}$ for some $j_1, \dots j_m \in \{1, \dots, p\}$, (where possibly $M = u_M$ if $i_0 = j$), implying that

$$\delta(M) = g(i_0)g(j)\left(\prod_{k=1}^m g(j_k)^2\right)\epsilon_p(M) = g(i_0)g(j)\epsilon_p(M).$$

For M a constant, $\delta(M) = M = g(i_0)g(j)\epsilon_p(M)$ since $i_0 = j$. This holds for each monomial, thus

$$\delta\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ij}\right) = g(i_0)g(j)\epsilon_p\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ij}\right).$$

Recall that $w(\alpha)$ is odd. When p is even, $w(\imath_p(\alpha))$ is also even and so the opposite parity of $w(\alpha)$. Our definition of g gives $\delta\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ii}\right)$ for $i \leq p$. Thus

$$\delta\left(\Phi^L_{\bar{\gamma}}\right) = \begin{pmatrix} (-1)^{w(\bar{\gamma})+1} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{\bar{\gamma}}\right)(\Delta(\alpha)\otimes I_p) = \operatorname{diag}[(-1)^{w(\alpha)+w(\bar{\gamma})+1}, 1, \ldots, 1] = \Delta(\gamma(\alpha))$$

as desired.

When p is odd, $w(\imath_p(\alpha))$ is odd and therefore the same parity of $w(\alpha)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{\bar{\gamma}}\right)_{11}\right) = \epsilon\left(\left(\Phi^L_{\bar{\gamma}}\right)_{11}\right)$ and $\delta\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ii}\right)$ for $1 < i \leq p$, so

$$\delta\left(\Phi^L_{\bar{\gamma}}\right) = \begin{pmatrix} (-1)^{w(\bar{\gamma})} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{\bar{\gamma}}\right)(\Delta(\alpha)\otimes I_p) = \operatorname{diag}[(-1)^{w(\alpha)+w(\bar{\gamma})}, 1, \ldots, 1] = \Delta(\gamma(\alpha))$$

as desired.

There is little difference in the proof that $\epsilon(\Phi^R_{\gamma(\alpha)}) = \Delta(\gamma(\alpha))$, except that monomials in $(\Phi^R_{\bar{\gamma}})_{ij}$ are of the form $u_M a_{i,j_1}a_{j_1,j_2}\cdots a_{j_k,j_0}$ where $j_0 = \operatorname{perm}(\gamma)(j)$. Applying Theorem 3.13 now completes the proof. $\square$

Proof of Corollary 1.3. We prove the corollary by induction on the dimensions of the vectors $\mathbf{p}$ and $\mathbf{q}$. If $\mathbf{p}$ and $\mathbf{q}$ have one entry, then $T(\mathbf{p},\mathbf{q})$ is simply the (p_1,q_1)-torus knot, and by Theorem 1.3 from [Cor14b] we have $\operatorname{ar}(T(\mathbf{p},\mathbf{q})) = p_1$.

Suppose that $\mathbf{p}$ and $\mathbf{q}$ have n entries and $\operatorname{ar}(T(\hat{\mathbf{p}},\hat{\mathbf{q}})) = p_1p_2\cdots p_{n-1}$. Choose a braid $\alpha \in B_{p_1p_2\cdots p_{n-1}}$ such that $\hat{\alpha} = T(\hat{\mathbf{p}},\hat{\mathbf{q}})$ (this braid is automatically minimal), and let $\gamma = (\sigma_1\cdots\sigma_{p_n-1})^{q_n}$. Theorem 1.3 from [Cor14b] implies that $\mathrm{ar}(\hat{\gamma}) = p_n$, and since $T(\mathbf{p}, \mathbf{q}) = K(\alpha, \gamma)$, Theorem 1.2 gives the desired result. $\square$

4.2. Supporting lemmas

In this section we prove Lemma 4.2, for which we make some definitions. Set $X_{m,l} = \{m, m+1, \ldots, m+l-1\}$ for any $m, l > 0$. Given $Y \subseteq X_{m,l}$, write the cardinality as $\upsilon = |Y|$ and denote elements of Y by $\{y_1, \ldots, y_\upsilon\}$, so that $y_1 < \cdots < y_\upsilon$. Suppose $1 \leq i \neq j \leq kp+1$. If $i, j \notin X_{m,l}$ we define

$$A(i,j,X_{m,l}) = \sum_{Y \subseteq X_{m,l}} (-1)^\upsilon a_{iy_1} a_{y_1 y_2} \cdots a_{y_\upsilon j};$$
$$A'(i,j,X_{m,l}) = \sum_{Y \subseteq X_{m,l}} (-1)^\upsilon a_{iy_\upsilon} a_{y_\upsilon y_{\upsilon-1}} \cdots a_{y_1 j}.$$

If $j \in X_{m,l}$ and $i \notin X_{m,l}$ define

$$B'(i,j,X_{m,l}) = \sum_{Y \subseteq X_{m,l},\ y_1 \neq j} c_Y a_{iy_\upsilon} a_{y_\upsilon y_{\upsilon-1}} \cdots a_{y_1 j}$$

where $c_Y = (-1)^{\upsilon+1}$ if $y_1 > j$, and $c_Y = (-1)^\upsilon$ if $y_1 < j$. To prove Lemma 4.2 we need two lemmas. Furthermore, in the proof of Lemma 4.2 we will focus on generators a_{ij}, $i < j$. We write $*$ for $j = kp+1$. Recall the definition of the spanning arcs, in particular c_{ij}, and the map χ from Section 3.2.

Lemma 4.3. *Given $1 \leq n \leq k-1$ let $X_n^{(p)} = X_{(n-1)p+1,p}$. For $1 \leq i < j \leq kp+1$ we have*

$$\phi_{\iota_p(\sigma_n)}(a_{ij}) = \begin{cases} a_{i+p,j+p} & : i,j \in X_n^{(p)} \\ a_{i-p,j-p} & : i,j \in X_{n+1}^{(p)} \\ a_{i-p,j} & : i \in X_{n+1}^{(p)},\ j > (n+1)p \\ a_{i,j-p} & : i \leq (n-1)p,\ j \in X_{n+1}^{(p)} \\ A(i,j+p,X_n^{(p)}) & : i \leq (n-1)p,\ j \in X_n^{(p)} \\ A'(i+p,j,X_n^{(p)}) & : i \in X_n^{(p)},\ j > (n+1)p \\ B'(i+p,j-p,X_n^{(p)}) & : i \in X_n^{(p)},\ j \in X_{n+1}^{(p)} \\ a_{ij} & : \text{otherwise} \end{cases}.$$

Proof. Define $\tau_{m,p} = \sigma_m \sigma_{m+1} \cdots \sigma_{m+p-1}$ and let $\kappa_{m,l} = \tau_{m+l-1,p} \tau_{m+l-2,p} \cdots \tau_{m,p}$ (depicted in Figure 7). Note that for any $1 \leq n \leq k-1$ and $m = (n-$

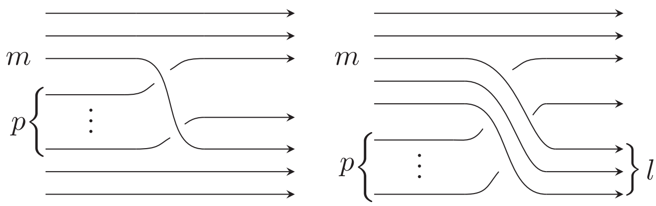


Figure 7: Left: the braid $\tau_{m,p}$. Right: the braid $\kappa_{m,l}$.

1)$p+1$, by thinking of $\kappa_{m,p}$ as an element of B_{kp} we have $\kappa_{m,p} = \imath_p(\sigma_n)$. We may prove the result, therefore, by showing that for $i<j$ if $l \le p$ then

$$\phi_{\kappa_{m,l}}(a_{ij}) = \begin{cases} a_{i+l,j+l} & : i,j \in X_{m,p} \\ a_{i-p,j-p} & : i,j \in X_{m+p,l} \\ a_{i-p,j} & : i \in X_{m+p,l},\ j \ge m+l+p \\ a_{i,j-p} & : i<m,\ j \in X_{m+p,l} \\ A(i,j+l,X_{m,l}) & : i<m,\ j \in X_{m,p} \\ A'(i+l,j,X_{m,l}) & : i \in X_{m,p},\ j \ge m+p+l \\ B'(i+l,j-p,X_{m,l}) & : i \in X_{m,p},\ j \in X_{m+p,l} \\ a_{ij} & : \text{otherwise} \end{cases} \tag{8}$$

The proof of (8) is by induction on l. For the case $l=1$, note that $\kappa_{m,1} = \tau_{m,p}$. It is relatively straightforward to calculate, for $1 \le m \le (k-1)p$ and $i<j$, that

$$\phi_{\tau_{m,p}}(a_{ij}) = \begin{cases} a_{i+1,j+1} & : m \le i < j < m+p \\ a_{mj} & : m+p = i < j \\ a_{im} & : i < m < m+p = j \\ a_{i,j+1} - a_{i,m}a_{m,j+1} & : i < m \le j < m+p \\ a_{i+1,j} - a_{i+1,m}a_{m,j} & : m \le i < m+p < j \\ -a_{i+1,m} & : m \le i < j = m+p \\ a_{ij} & : \text{otherwise} \end{cases} \tag{9}$$

Indeed, the effect of $\tau_{m,p}$ is to move points $\{m,\dots,m+p-1\}$ in (D,P) one to the right and the point at $m+p$ is carried through the upper half-disk to m. Figure 8 shows $\tau_{m,p} \cdot c_{ij}$ for two interesting cases in (9). Using the relation in Figure 4 at the point m, we get $\tau_{m,p} \cdot c_{ij} = c_{i,j+1} - c_{im}c_{m,j+1}$ if

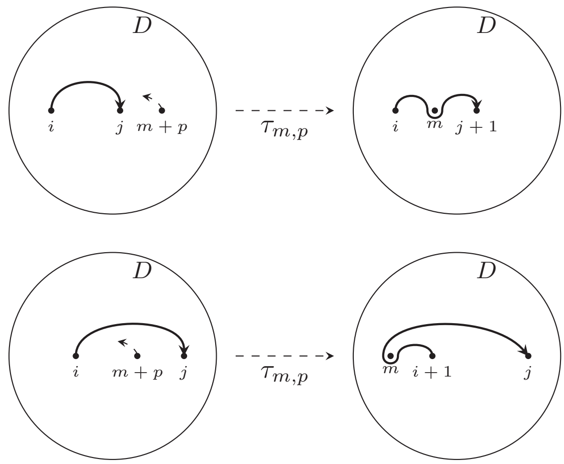


Figure 8: $\tau_{m,p} \cdot c_{ij}$, two possible cases.

$i < m \le j < m+p$, and $\tau_{m,p} \cdot c_{ij} = c_{i+1,j} + c_{i+1,m}c_{mj}$ if $m \le i < m+p < j$. Applying the map χ gives the calculation in (9) for these cases. Verification of the other cases are left to the reader.

Since $X_{m,1} = \{m\}$, we have $A(i, j+1, X_{m,1}) = a_{i,j+1} - a_{im}a_{m,j+1}$ and $A'(i+1, j, X_{m,1}) = a_{i+1,j} - a_{i+1,m}a_{mj}$. Also, when $j = m+p$ the subsets considered for $B'(i+1, j-p, X_{m,1})$ must be empty, so it is $-a_{i+1,m}$. The other cases clearly agree with (8) for $l = 1$, proving the base case (note $i, j \in X_{m+p,1} = \{m+p\}$ is impossible).

The argument for $l > 1$ should be handled in each case appearing in (8). The first four cases, and the last as well, are fairly straightforward, and we leave them to the reader. We present the argument in the cases $i < m, j \in X_{m,p}$ and $i \in X_{m,p}, j \ge m+p+l$ and when $i \in X_{m,p}, j \in X_{m+p,l}$.

If $i < m, j \in X_{m,p}$ then

$$
\begin{aligned}
&\phi_{\kappa_{m,l}}(a_{ij}) \\
&= \phi_{\tau_{m+l-1,p}}(\phi_{\kappa_{m,l-1}}(a_{ij})) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-2\}} (-1)^{|Y|} \phi_{\tau_{m+l-1,p}}(a_{i,y_1}a_{y_1y_2}\cdots a_{y_v,j+l-1}) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-2\}} (-1)^{|Y|} a_{iy_1}a_{y_1y_2}\cdots a_{y_{v-1}y_v}(a_{y_v,j+l} - a_{y_v,m+l-1}a_{m+l-1,j+l}) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-1\}} (-1)^{|Y|} a_{i,y_1}a_{y_1y_2}\cdots a_{y_v,j+l} \\
&= A(i, j+l, X_{m,l}).
\end{aligned}
$$

The third equality uses (9) and holds because $l \leq p$.

The case $i \in X_{m,p}, j \geq m+p+l$ is very similar, except that the indices of generators appearing in the sum are descending, so we also use that $\phi_{\tau_{m+l-1,p}}$ commutes with conjugation.

Finally, suppose $i \in X_{m,p}, j \in X_{m+p,l}$. Note $j-(m+p) \leq l-1$. If $j-m-p = l-1$, then by the preceding case

$$\phi_{\kappa_{m,l-1}}(a_{ij}) = A'(i+j-m-p, j, X_{m,j-m-p}).$$

We then have

$$\begin{aligned}
&\phi_{\tau_{m+(j-m-p),p}}(A'(i+j-m-p, j, X_{m,j-m-p}))\\
&= \sum_{Y \subseteq \{m,\ldots,j-p-1\}} (-1)^{|Y|} \phi_{\tau_{j-p,p}}(a_{i+j-m-p,y_v} a_{y_v y_{v-1}} \cdots a_{y_1,j})\\
&= -a_{i+j-m-p+1,j-p}\\
&\quad + \sum_{\substack{Y \subseteq \{m,\ldots,j-p-1\}\\ Y \neq \emptyset}} (-1)^{|Y|}(a_{i+j-m-p+1,y_v} - a_{i+j+m-p+1,j-p} a_{j-p,y_v})\\
&\qquad\qquad \times a_{y_v y_{v-1}} \cdots a_{y_2 y_1} a_{y_1,j-p}\\
&= B'(i+l, j-p, X_{m,l}).
\end{aligned}$$

If instead $j-m-p < l-1$, and $l \leq p$, we conclude the proof by checking

$$\begin{aligned}
&\phi_{\tau_{m+l-1,p}}(B'(i+l-1, j-p, X_{m,l-1}))\\
&= \sum_{\substack{Y \subseteq \{m,\ldots,m+l-2\}\\ Y \cap X_{m,j-m-p} \neq \emptyset}} (-1)^{|Y|} \phi_{\tau_{m+l-1,p}}(a_{i+l-1,y_v} a_{y_v y_{v-1}} \cdots a_{y_1,j-p})\\
&\quad - \sum_{\substack{Y \subseteq \{m,\ldots,m+l-2\}\\ Y \cap X_{m,j-m-p+1} = \emptyset}} (-1)^{|Y|} \phi_{\tau_{m+l-1,p}}(a_{i+l-1,y_v} a_{y_v y_{v-1}} \cdots a_{y_1,j-p})\\
&= \sum_{\substack{Y \subseteq \{m,\ldots,m+l-2\}\\ Y \cap X_{m,j-m-p} \neq \emptyset}} (-1)^{|Y|}(a_{i+l,y_v} - a_{i+l,m+l-1} a_{m+l-1,y_v}) a_{y_v y_{v-1}} \cdots a_{y_1,j-p}\\
&\quad - \sum_{\substack{Y \subseteq \{m,\ldots,m+l-2\}\\ Y \cap X_{m,j-m-p+1} = \emptyset}} (-1)^{|Y|}(a_{i+l,y_v} - a_{i+l,m+l-1} a_{m+l-1,y_v}) a_{y_v y_{v-1}} \cdots a_{y_1,j-p}\\
&= B'(i+l, j-p, X_{m,l}).
\end{aligned}$$

□

The last step needed to prove Lemma 4.2 is the following. For $1 \leq i < kp+1$, recall the notation $i = (q_i - 1)p + r_i$, $1 \leq r_i \leq p$ defined at the start of Section 4.1. It is worth noting here, and in the proof of Lemma 4.2, that $i \in X_n^{(p)}$ if and only if $q_i = n$.

Lemma 4.4. *Fix $1 \leq i < j \leq kp+1$ and define $s_i = (n-1)p + r_i \in X_n^{(p)}$. We have the following equalities.*

$$
\begin{array}{ll}
\psi(A(i, j+p, X_n^{(p)})) = \psi(a_{i,j+p} - a_{is_i}a_{s_i,j+p}) & : i \leq (n-1)p, j \in X_n^{(p)} \\
\psi(A'(i+p, j, X_n^{(p)})) = \psi(a_{i+p,j} - a_{i+p,i}a_{i,j}) & : i \in X_n^{(p)}, j > (n+1)p \\
\psi(B'(i+p, j-p, X_n^{(p)})) = \psi(-a_{i+p,j-p} + \delta a_{i+p,i}a_{i,j-p}) & : i \in X_n^{(p)}, j \in X_{n+1}^{(p)},
\end{array}
$$

where $\delta \in \{-1, 0, 1\}$ is 0 if $i = j - p$, and is the sign of $i - (j-p)$ otherwise.

Remark 4.5. It is possible to have $j = *$ in the case that $j > (n+1)p$. In this instance ψ should be replaced by ψ^*.

Proof of Lemma 4.4. Each of the three cases involves a sum over subsets $Y \subseteq X_n^{(p)}$.

In the case $i \leq (n-1)p$, any $y_1 < s_i$ satisfies $r_{y_1} < r_i$ and $q_i < q_{y_1}$. Hence $\psi(a_{iy_1}) = 0$. Thus we restrict to subsets $Y \subseteq \{s_i, \ldots, np\}$, i.e.

$$\psi(A(i, j+p, X_n^{(p)})) = \sum_{Y \subseteq \{s_i, \ldots, np\}} (-1)^{|Y|} \psi(a_{iy_1}a_{y_1y_2} \cdots a_{y_\upsilon, j+p}).$$

For any $y_1 \in \{s_i + 1, \ldots, np\}$ we get

$$\psi(a_{iy_1} - a_{is_i}a_{s_iy_1}) = a_{q_i,q_{y_1}} \otimes a_{r_i,r_{y_1}} - (a_{q_i,n} \otimes 1)(1 \otimes a_{r_i,r_{y_1}}) = 0,$$

and so

$$
\begin{aligned}
&\psi(A(i, j+p, X_n^{(p)}) \\
&= \psi(a_{i,j+p} - a_{is_i}a_{s_i,j+p}) \\
&\quad + \sum_{\substack{Y \subseteq \{s_i+1, \ldots, np\} \\ Y \neq \emptyset}} (-1)^{|Y|} \psi(a_{iy_1} - a_{is_i}a_{s_iy_1}) \psi(a_{y_1y_2} \cdots a_{y_\upsilon, j+p}) \\
&= \psi(a_{i,j+p} - a_{is_i}a_{s_i,j+p}).
\end{aligned}
$$

In the remaining cases $i \in X_n^{(p)}$, and so $s_i = i$. If $y_\upsilon > i$ then $r_{y_\upsilon} > r_{i+p}$ and $q_{i+p} > q_{y_\upsilon}$ so that $\psi(a_{i+p,y_\upsilon}) = 0$. Thus in these cases we restrict to $Y \subseteq \{(n-1)p+1, \ldots, i\}$. The argument for the second case then proceeds analogously to the first.

In the third case, with $j \in X_{n+1}^{(p)}$, we must account for the condition $y_1 \neq j - p$ in each summand. This causes the non-vanishing part of the sum to vary, depending on whether i is larger than, equal to, or smaller than $j-p$. The δ in the statement of the lemma incorporates the three situations. Noting that if $\emptyset \neq Y \subseteq \{(n-1)p+1,\ldots,i-1\}$ then $c_Y = -c_{Y\cup\{s_i\}}$ (recall $s_i = i$), the argument then proceeds analogously to the first. □

Proof of Lemma 4.2. The statement holds when α is the identity braid. We prove for $1 \leq n < k$ that

$$\psi^* \circ \phi^*_{\imath_p(\sigma_n)} = \left(\phi^*_{\sigma_n} \otimes \mathrm{id}\right) \circ \psi^*.$$

As the maps $B_k \to \mathrm{Aut}(\mathcal{A}^L_k \otimes \mathcal{A}^L_p)$, given by $\alpha \mapsto \phi^*_\alpha \otimes \mathrm{id}$, and $B_k \to \mathrm{Aut}(\mathcal{A}^L_{kp})$, given by $\alpha \mapsto \phi^*_{\imath_p(\alpha)}$, are homomorphisms, this suffices to prove the lemma.

Furthermore, for β any braid, ϕ_β and ψ each commute with conjugation, so we only need prove that

$$\psi^*(\phi^*_{\imath_p(\sigma_n)}(a_{ij})) = (\phi^*_{\sigma_n} \otimes \mathrm{id})\psi^*(a_{ij}) \tag{10}$$

for $i < j$, possibly $j = *$. We check (10) for each case in the statment of Lemma 4.3.

In the first two cases both sides of (10) equal $1 \otimes a_{r_i r_j}$ since $q_i = q_j$.

When $j > (n+1)p, i \in X^{(p)}_{n+1}$ there are two possibilities, either $j = *$ or $j \leq kp$. In either case $\phi^*_{\sigma_n}(a_{ij}) = a_{i-p,j}$ by Lemma 4.3. If $j = *$ then

$$\psi^*(a_{i-p,*}) = a_{q_i-1,*} \otimes a_{r_i,*} = (\phi^*_{\sigma_n} \otimes \mathrm{id})\psi^*(a_{i,*}),$$

the first equality from the definition of ψ^* and the second equality since $q_i = n+1$. If $j \leq kp$ then $\psi(a_{i-p,j}) = a_{q_i-1,q_j} \otimes x$ where $x = a_{r_i r_j}, 1$ or 0 depending on the relation of r_i to r_j. As $q_i = n+1$ and $q_j > n+1$ we get $a_{q_i-1,q_j} = \phi_{\sigma_n}(a_{q_i q_j})$, proving the statement. The case $i \leq (n-1)p, j \in X^{(p)}_{n+1}$ is similar.

Recall that $s_i = (n-1)p + r_i$ for all $1 \leq i \leq kp$. In the case that $\phi_{\imath_p(\sigma_n)}(a_{ij}) = A(i, j+p, X^{(p)}_n)$ we have by Lemma 4.4 that

$$\psi(\phi_{\imath_p(\sigma_n)}(a_{ij})) = \psi(a_{i,j+p} - a_{is_i}a_{s_i,j+p}).$$

We recall that by the definition of ϕ_{σ_n}, if $q_j = n$ and $q_i \neq n, n+1$ then $\phi_{\sigma_n}(a_{q_i q_j}) = a_{q_i,q_j+1} - a_{q_i q_j}a_{q_j,q_j+1}$ and $\phi_{\sigma_n}(a_{q_j q_i}) = a_{q_j+1,q_i} - a_{q_j+1,q_j}a_{q_j q_i}$.

In the current case since $q_i < q_j = n$ and $q_{s_i} = n$, we see that

$$\begin{aligned}\psi(a_{i,j+p} - a_{is_i}a_{s_i,j+p}) &= (a_{q_i,n+1} - a_{q_i,n}a_{n,n+1}) \otimes x \\ &= (\phi_{\sigma_n} \otimes \mathrm{id})(a_{q_iq_j} \otimes x) \\ &= (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij})),\end{aligned}$$

where $x = a_{r_ir_j}$ if $r_i < r_j$, $x = 1$ if $r_i = r_j$ and $x = 0$ if $r_i > r_j$.

In the case that $\phi^*_{\iota_p(\sigma_n)}(a_{ij}) = A'(i+p, j, X_n^{(p)})$ (here $j = *$ is possible),

$$\psi^*(\phi_{\iota_p(\sigma_n)}(a_{ij})) = \psi^*(a_{i+p,j} - a_{i+p,i}a_{ij}).$$

Then, as $q_{i+p} = n+1 < q_j$ we get (replace q_j with $*$ if $j = *$)

$$\psi^*(a_{i+p,j} - a_{i+p,i}a_{i,j}) = (a_{n+1,q_j} - a_{n+1,n}a_{n,q_j}) \otimes x = (\phi^*_{\sigma_n} \otimes \mathrm{id})(\psi^*(a_{ij})),$$

with x either as before, or $x = a_{r_i,*}$ if $j = *$.

Finally, suppose $i \in X_n^{(p)}, j \in X_{n+1}^{(p)}$. Then $q_{j-p} = q_i = n$ and $s_i = i$. Again there are three cases according to whether $r_i > r_j$, $r_i < r_j$, or $r_i = r_j$; however, each case alters the answer in Lemma 4.4 (by changing δ) and we will handle each separately.

If $i > j - p$ then $r_i > r_j$. By Lemmas 4.3 and 4.4

$$\begin{aligned}\psi(\phi_{\iota_p(\sigma_n)}(a_{ij})) = \psi(-a_{i+p,j-p} + \delta a_{i+p,i}a_{i,j-p}) &= \psi(-a_{i+p,j-p} + a_{i+p,i}a_{i,j-p}) \\ &= (-a_{q_i+1,q_i} + a_{q_i+1,q_i}) \otimes a_{r_i,r_j} \\ &= 0,\end{aligned}$$

which equals $(\phi_{\sigma_n} \otimes \mathrm{id})\psi(a_{ij})$ since $(q_i - q_j)(r_i - r_j) < 0$ makes $\psi(a_{ij}) = 0$. If $j - p > i$ then, as $r_i < r_j$, we have $\psi(a_{i+p,j-p}) = 0$ by definition of ψ. Moreover, $\psi(a_{ij}) = a_{q_i,q_i+1} \otimes a_{r_ir_j}$. Thus

$$\psi(-a_{i+p,j-p} - a_{i+p,i}a_{i,j-p}) = -(a_{q_i+1,q_i} \otimes 1)(1 \otimes a_{r_i,r_j}) = (\phi_{\sigma_n} \otimes \mathrm{id})\psi(a_{ij}).$$

Finally $j - p = i$ implies $r_i = r_j$ and then since $q_i = n$ and $q_j = n+1$ we have $\psi(-a_{i+p,j-p}) = -a_{n+1,n} \otimes 1 = (\phi_{\sigma_n} \otimes \mathrm{id})\psi(a_{ij})$. This finishes the proof. □

5. Comments on augmentation rank and multiplicativity

The section is in two parts. First we prove Theorem 1.4, showing some cables of torus knots have augmentation rank less than bridge number. In the second part we discuss how this result, and some computational evidence, might fit into Conjecture 1.5.

5.1. Cables of $(n, n+1)$ torus knots

Theorem 1.4. Given $p > 1$ and $n > 1$, we have

$$\mathrm{ar}(T((n,p),(n+1,1))) < np.$$

Remark 5.1. The remarks at the end of Section 2 imply that the bridge number of $T((n,p),(n+1,1))$ is np.

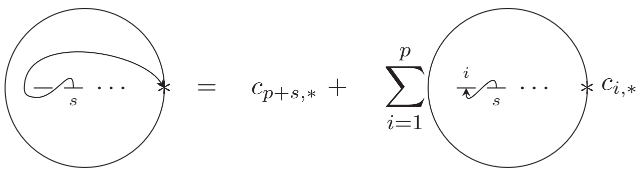


Figure 9: $\imath_p(\tau)\cdot c_{s*}$, $1 \le s \le p$ as an element of $\mathscr{S}_{np+1}$.

Proof. Let $\tau = \sigma_1 \cdots \sigma_{n-1} \in B_n$ and set $\alpha = \tau^{n+1}$, which has the $(n, n+1)$ torus knot as its braid closure. We have $T((n,p),(n+1,1)) = K(\alpha,\gamma)$ for $\gamma = \sigma_1 \cdots \sigma_{p-1} \in B_p$. Write $(\Phi^L_{\gamma(\alpha)})_i$ for the i^{th} row of $\Phi^L_{\gamma(\alpha)}$.

The structure of the proof is the following. By Theorem 3.13 we must prove there is no homomorphism $\epsilon\colon \mathcal{A}_{np} \to \mathbb{C}$ such that $\epsilon(\Phi^L_{\gamma(\alpha)}) = \Delta(\gamma(\alpha))$. Note that, since $\overline{\gamma}$ is in the image of the inclusion $B_p \hookrightarrow B_{np}$ given by $\sigma_i \mapsto \sigma_i$, Theorem 3.11 implies that $(\Phi^L_{\gamma(\alpha)})_i = (\Phi^L_{\imath_p(\alpha)})_i$ for $p < i \le np$. Hence, were such an ϵ to exist then $\epsilon((\Phi^L_{\imath_p(\alpha)})_i) = \mathbf{e}_i$ for $p < i \le np$ (here $\mathbf{e}_i$ denotes the i^{th} standard basis vector).

We will see that $(\Phi^L_{\overline{\gamma}})_p = \mathbf{e}_1$, implying that the entry $(\Phi^L_{\imath_p(\alpha)})_{1p}$ agrees with a diagonal entry of $\Phi^L_{\gamma(\alpha)}$. We then calculate that, for any ϵ satisfying $\epsilon((\Phi^L_{\imath_p(\alpha)})_i) = \mathbf{e}_i$ for $p < i \le np$, we must have $\epsilon((\Phi^L_{\imath_p(\alpha)})_{1p}) = 0$. This contradicts $\epsilon(\Phi^L_{\gamma(\alpha)}) = \Delta(\gamma(\alpha))$ and proves the result.

In consideration of the above, for the remainder of the proof $\epsilon\colon \mathcal{A}_{np} \to \mathbb{C}$ denotes a homomorphism with the property $\epsilon((\Phi^L_{\imath_p(\alpha)})_i) = \mathbf{e}_i$ for $p < i \le np$.

To prove that $\epsilon((\Phi^L_{\imath_p(\alpha)})_{1p}) = 0$ we first demonstrate, in **I** below, that $\epsilon((\Phi^L_{\imath_p(\alpha)})_{1p}) = -\epsilon(a_{p+1,p})$. This is followed in **II** by a proof that $\epsilon(a_{p+1,p}) = 0$, which completes the proof of the theorem (the equality $(\Phi^L_{\overline{\gamma}})_p = \mathbf{e}_1$ is derived in the process).

I. For $z \in \mathbb{Z}$, consider matrices $\Phi^L_{\imath_p(\mathcal{T})^z}$ and partition them into an $n \times n$ array of $p \times p$ submatrices. In notation, define for $1 \le i, j \le n$ the $p \times p$ matrix Ψ^z_{ij} so that

$$\Phi^L_{\imath_p(\mathcal{T})^z} = \begin{pmatrix} \Psi^z_{11} & \cdots & \Psi^z_{1n} \\ & \ddots & \\ \Psi^z_{n1} & \cdots & \Psi^z_{nn} \end{pmatrix}.$$

We claim that

(a) the $(n-1)p \times (n-1)p$ submatrix $(\Psi^1_{ij})_{i<n,j>1}$ is the identity matrix;

(b) Ψ^1_{n1} is the $p \times p$ identity matrix;

(c) Ψ^1_{nj} is the zero matrix for $j > 1$;

(d) finally, $(\Phi^L_{\overline{\gamma}})_p = (1, 0, \ldots, 0)$.

Verification of the claim is left to the reader. As an example, (a) requires identities in $\mathscr{S}_{np+1}$ (which are passed through to $\mathcal{A}_{np+1}$ by χ) similar to the identity in Figure 9, which can be used to calculate Ψ^1_{1j} for $1 \le j \le n$. By considering $\imath_p(\mathcal{T})$ when $p = 1$ (and disassociating the index of γ from the p in $\imath_p(\mathcal{T})$), we note that Φ^L_γ can be understood using (a)-(c). Since $\overline{\gamma} \in B_{np}$ only acts non-trivally on the first p strands, we obtain (d) from (b) and (c).

By Theorem 3.11 we have $\Phi^L_{\imath_p(\mathcal{T})^{z+1}} = \phi_{\imath_p(\mathcal{T})}(\Phi^L_{\imath_p(\mathcal{T})^z})\Phi^L_{\imath_p(\mathcal{T})}$. Thus by (a) and (c) above, for $1 \le j < n$,

$$\Psi^{z+1}_{i,j+1} = \phi_{\imath_p(\mathcal{T})}(\Psi^z_{ij}). \tag{11}$$

Theorem 3.11 also shows $\Phi^L_{\imath_p(\mathcal{T})^{z+1}} = \phi_{\imath_p(\mathcal{T})^z}(\Phi^L_{\imath_p(\mathcal{T})})\Phi^L_{\imath_p(\mathcal{T})^z}$. Hence by (b), (c)

$$\Psi^{z+1}_{nj} = \Psi^z_{1j} \tag{12}$$

for all $1 \le j \le n$, and, for $1 \le i < n$, we have by (a) that

$$\Psi^{z+1}_{ij} = \Psi^z_{i+1,j} + \phi_{\imath_p(\mathcal{T})^z}(\Psi^1_{i1})\Psi^z_{1j}.$$

Taking $z = n$ above, equations (11) and (12) thus imply

$$\Psi^{n+1}_{ij} = \phi^{-1}_{\imath_p(\mathcal{T})}(\Psi^{n+1}_{i+1,j+1}) + \phi_{\imath_p(\mathcal{T})^n}(\Psi^1_{i1})\Psi^{n+1}_{nj}.$$

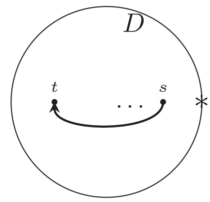


Figure 10: The spanning arc for w_{st}.

Hence $\epsilon(\Psi_{ij}^{n+1}) = \epsilon(\phi_{\imath_p(\tau)}^{-1}(\Psi_{i+1,j+1}^{n+1}))$ for $1 \leq j < n$, since $\epsilon(\Psi_{nj}^{n+1}) = \mathbf{0}$ by assumption. Utilizing (11) and (12) again we find that, for $i \geq j$,

$$\epsilon(\Psi_{ij}^{n+1}) = \epsilon(\phi_{\imath_p(\tau)}^{-n+i}(\Psi_{n,j+(n-i)}^{n+1})) = \epsilon(\Psi_{nj}^{i+1}) = \epsilon(\Psi_{1j}^{i}).$$

Taking $s = 1$ in Figure 9, we see that the $(1,p)$-entry of Ψ_{11}^1 is $\chi(c_{p+1,p}) = -a_{p+1,p}$. And so $\epsilon((\Phi_{\imath_p(\alpha)}^L)_{1p}) = \epsilon((\Psi_{11}^{n+1})_{1p}) = -\epsilon(a_{p+1,p})$, which we were to show in **I**.

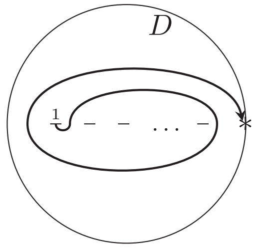
II. We must show that $\epsilon(a_{p+1,p}) = 0$. To do so we consider $\phi_{\imath_p(\alpha)}^*(a_{(n-1)p+1,*})$ in $\mathcal{A}_{np}^L \subset \mathcal{A}_{np+1}$ which, similar to above, we understand through the corresponding spanning arc (see Figure 11). Our assumption that $\epsilon((\Phi_{\imath_p(\alpha)}^L)_{(n-1)p+1}) = \mathbf{e}_{(n-1)p+1}$ means that if we write $\phi_{\imath_p(\alpha)}^*(a_{(n-1)p+1,*})$ in the basis $\{a_{1,*}, \ldots, a_{np,*}\}$ of $\mathcal{A}_{np}^L$ then ϵ sends all but the $(n-1)p+1$ coefficient to zero.

For $p < r \leq np$, define $v_r \in \mathcal{A}_{np}$ such that $\chi^{-1}(v_r)$ is the spanning arc shown on the right in Figure 11, which ends at r. Define w_{st} so that (as shown in Figure 10) $\chi^{-1}(w_{st})$ is contained in the lower half of D_{np}, and begins at s and ends at t. If $s = t$ then we define $w_{st} = 1$.

In **I** we showed $\epsilon(\Phi_{ij}^{n+1}) = \epsilon(\Psi_{1j}^{i})$ for any $i \geq j$. This has an important consequence for elements of the form $w_{ip+1,j}$. The entries of Ψ_{1j}^i are computed from $\imath_p(\tau)^i \cdot c_{s,*}$ where $1 \leq s \leq p$ (Figure 9 shows the case $i = 1$). Take $s = 1$. Let $1 \leq i \leq n-1$ and $1 \leq j = (q-1)p + r \leq ip$ (for some $1 \leq r \leq p$). Note this makes $i \geq q$. Then the $(1,r)$-entry of Ψ_{1q}^i is $w_{ip+1,j}$. Our assumption on ϵ implies, only for $1 < i \leq n-1$, that

$$\epsilon(w_{ip+1,j}) = \epsilon((\Psi_{iq}^{n+1})_{1r}) = \delta_{iq}\delta_{1r} = \delta_{(i-1)p+1,j}, \tag{13}$$

where δ is the Kronecker-delta.



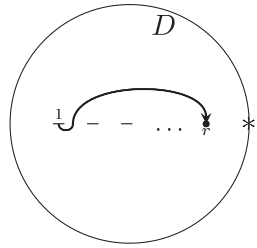


Figure 11: Row $(n-1)p+1$ of $\Phi^L_{\imath_p(\alpha)}$.

For $p < j \leq np$, the coefficient of $a_{j,*}$ in $\phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*})$ can be written as

$$x_j \coloneqq \langle \phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*}), a_{j,*}\rangle = \sum_{r=j}^{np} v_r w_{rj}. \tag{14}$$

Claim. For $p<j\leq np$, if $j=(n-i)p+1$ then $\epsilon(v_j)=(-1)^{i+1}$ and $\epsilon(v_j)=0$ otherwise.

Proof of Claim. The proof uses induction on i, proving the statement for each $(n-i)p+1 \leq j \leq (n-i+1)p$.

For $i=1$, by assumption $\epsilon(x_j)=0$ for $(n-1)p+1 < j \leq np$. Note that $x_{np} = v_{np}$. Thus $\epsilon(v_{np}) = \epsilon(x_{np}) = 0$. This, applied to $j = np-1$, then $j = np-2$, and so on, implies that $\epsilon(v_j) = \epsilon(x_j) = 0$ for $(n-1)p+1 < j \leq np$. Furthermore, we get that $\epsilon(v_{(n-1)p+1}) = \epsilon(x_{(n-1)p+1}) = 1$.

Now suppose for some $1 < i \leq n-1$ that $(n-i)p+1 \leq j \leq (n-i+1)p$. Assuming the claim holds for $v_{j'}$ with $j < j'$ we have

$$0 = \epsilon(x_j) = \sum_{r=j}^{np} \epsilon(v_r)\epsilon(w_{rj}) = \epsilon(v_j) + \sum_{k=1}^{i-1}(-1)^{k+1}\epsilon(w_{(n-k)p+1,j}).$$

Recalling (13), $\epsilon(w_{(n-k)p+1,j}) = \delta_{(n-k-1)p+1,j}$ (provided $n-k>1$), and thus $\epsilon(v_j)=0$ provided $j \neq (n-i)p+1$. When $j=(n-i)p+1$ we get that

$$0 = \epsilon(v_j) + (-1)^i \epsilon(w_{(n-i+1)p+1,(n-i)p+1}) = \epsilon(v_j) + (-1)^i$$

as claimed. $\square$

We finish the proof by considering $\langle \phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*}), a_{p,*}\rangle$; the spanning arc corresponding to $\phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*})$ indicates a small difference to the previous coefficients. We have

$$x_p := \langle \phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*}), a_{p,*}\rangle = \sum_{r=p+1}^{np} v_r w_{rp}.$$

Applying our previous claim, (13), and $w_{p+1,p} = -a_{p+1,p}$ we have

$$\begin{aligned} 0 = \epsilon(x_p) = \sum_{i=1}^{n-1}(-1)^{n-i+1}\epsilon(w_{ip+1,p}) &= (-1)^n \epsilon(w_{p+1,p}) \\ &= (-1)^{n-1}\epsilon(a_{p+1,p}). \end{aligned}$$

This implies the desired result and finishes the proof of the theorem. □

5.2. Augmentation rank does not multiply

In this section $\alpha \in B_n$ is a braid with smallest index such that $\hat{\alpha} = K$. As discussed in Section 2 the braid satellite $K(\alpha, \gamma)$ depends only on γ and K. Let ω denote the writhe of α and write P for the closure $\widehat{\Delta^{2\omega}\gamma}$, as in Section 2. Recall, $K(\alpha, \gamma)$ is the satellite $P(K)$.

The augmentation rank may be defined in a way that is wholly independent of braid representatives (see Remark 3.8). One might thus expect any relationship between the ranks $\mathrm{ar}(K(\alpha, \gamma))$ and $\mathrm{ar}(K)$ to involve $\mathrm{ar}(P)$, but this is not the case: from Theorem 1.4 we find examples where $\mathrm{ar}(K(\alpha, \gamma)) < \mathrm{ar}(K)\,\mathrm{ar}(P)$ and from Theorem 1.2 there are examples with $\mathrm{ar}(K(\alpha, \gamma)) > \mathrm{ar}(K)\,\mathrm{ar}(P)$ (take $\alpha = \sigma_1^3$ and $\gamma = \sigma_1^{-5}$, for example). We can show the following.

Proposition 5.2. *For α, K, and P as above,* $\mathrm{ar}(K(\alpha, \gamma)) \geq \mathrm{ar}(K)$ *and* $\mathrm{ar}(K(\alpha, \gamma)) \geq \mathrm{ar}(P)$.

Proof. Consider a regular neighborhood $n(K)$ of K that contains $K(\alpha, \gamma)$. Write $T = \partial(n(K))$. Choose the basepoint for $\pi_{K(\alpha,\gamma)}$ on T. Then inclusion makes $\pi_1(T)$ a subgroup and $\pi_{K(\alpha,\gamma)}$ is isomorphic to the product of π_K and π_P amalgamated along $\pi_1(T)$.

Let m_1 be the meridian of K determined by a based loop contained in T that is contractible in $n(K)$. Suppose that $\rho\colon\ \pi_K \to \mathrm{GL}_r\mathbb{C}$ is an irreducible KCH representation with $\widetilde{M} = \rho(m_1) = \mathrm{diag}[\widetilde{\mu}_0, 1, \ldots, 1]$ for some $\widetilde{\mu}_0 \in \mathbb{C} \setminus \{0\}$. Choose any p^{th} root μ_0 of $\widetilde{\mu}_0$.

Consider a collection of meridians $m_1, \ldots, m_s$ of K that generate π_K. For each $1 \leq i \leq s$ there are p meridians $m_{i1}, \ldots, m_{ip}$ of $K(\alpha, \gamma)$ such that $m_{i1}m_{i2}\cdots m_{ip} = m_i$. Set $\sigma(m_{1j}) = \mathrm{diag}[\mu_0, 1, \ldots, 1] = M$ for $1 \leq j \leq p$. Then, for each $1 < i \leq s$ find $w_i \in \pi_K$ so that $m_i = w_i m_1 w_i^{-1}$ and set $\sigma(m_{ij}) = \rho(w_i)M\rho(w_i)^{-1}$ for $1 \leq j \leq p$.

Due to the braid pattern of $K(\alpha, \gamma)$, $\pi_{K(\alpha,\gamma)}$ has a presentation so that each relation has the form $xm_{i,j}x^{-1} = w_i m_{1,k} w_i^{-1}$, where x is a word in $\{m_{i1}^{\pm}, \ldots, m_{ip}^{\pm}\}$ and $1 \leq j, k \leq p$, $1 \leq i \leq s$. Thus $\sigma\colon \pi_{K(\alpha,\gamma)} \to \mathrm{GL}_r\mathbb{C}$ is a well-defined KCH representation. Moreover, the image of σ contains that of ρ, implying it is irreducible, so $\mathrm{ar}(K(\alpha, \gamma)) \geq \mathrm{ar}(K)$.

That $\mathrm{ar}(K(\alpha, \gamma)) \geq \mathrm{ar}(P)$, follows from Proposition 2.1 and the existence of a surjection $\pi_{K(\alpha,\gamma)} \to \pi_P$, preserving peripheral structures (see Proposition 3.4 in [SW06], for example). □

It is possible that the statement of Conjecture 1.5 holds. We note an example where $\mathrm{ar}(K(\alpha, \gamma))$ is strictly larger than $\mathrm{ar}(K)\,\mathrm{ar}(\hat{\gamma})$, found in the $(2, 11)$-cable (Seifert framing) of the $(2, 5)$ torus knot. From computer-aided computations, we have a solution to (4) for $\alpha = \sigma_1^5 \in B_2$ and $\gamma = \sigma_1 \in B_2$, showing that $\mathrm{ar}(K(\sigma_1^5, \sigma_1)) = 4$, even though $\mathrm{ar}(\widehat{\sigma_1^5}) = 2$ and $\mathrm{ar}(\hat{\sigma_1}) = 1$. Unfortunately, other examples of cables of torus knots (not covered by Theorems 1.2 and 1.4) seem outside our computational abilities.

We end with computational observations and a question. By the inequalities in (3.10) if a knot has bridge number less than its minimal braid index n, it cannot have augmentation rank equal to n. Take a minimal index braid representative of such a knot, and multiply that braid by successively higher powers of $\Delta^2 \in B_n$, testing in each instance if the closure has augmentation rank equal to n. In examples, the power of Δ^2 need not be very high, compared to n, before a braid with augmentation rank n appears. Also, once such an augmentation appears, it seems to persist.

Dehornoy introduced a total, left-invariant order on B_n. By Theorem 1.4 the closure of $\sigma_1(\sigma_1^3)$ has augmentation rank less than 4. In comparison, $\sigma_1(\sigma_1^5)$ is larger in Dehornoy's order on B_4 and, as mentioned, has augmentation rank 4.

The relation in the order of a braid to powers of a full twist has been shown to carry significance for the braid closure. In fact, it was shown in [MN04] that there is a constant m_n such that if $\alpha > \Delta^{2m_n}$ (or $\alpha^{-1} > \Delta^{2m_n}$) then α does not admit one of the Birman-Menasco templates, and thus is a minimal index representative of $K = \hat{\alpha}$ by the MTWS [BM06]. Perhaps there is a similar result for augmentation rank.

Question. For a given braid index n, is there a number m_n so that $\mathrm{ar}(\hat{\alpha}) = n$ for any $\alpha \in B_n$ (with connected closure) greater than Δ^{2m_n} in Dehornoy's order?

If for each n such a number m_n exists, then every knot which is the closure of a sufficiently large braid has augmentation rank, meridional rank, and bridge number all equal.

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Received April 10, 2015