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On p-Bergman kernel for bounded domains in $\mathbb{C}^n$

Jiafu Ning, Huiping Zhang, and Xiangyu Zhou

In this paper, we obtain some properties of the p-Bergman kernels by applying L^p extension theorem. We prove that for any bounded domain in $\mathbb{C}^n$, it is pseudoconvex if and only if its p-Bergman kernel is an exhaustion function, for any $p \in (0, 2)$. As an application, we give a negative answer to a conjecture of Tsuji.

1. Introduction

T. Ohsawa and K. Takegoshi [16] proved the Ohsawa-Takegoshi L^2 extension theorem, which turns out to be useful in several complex variables and complex geometry. B. Berndtsson and M. Păun [2] proved the $L^{2/m}$ version of Ohsawa-Takegoshi theorem for $m \in \mathbb{N}$. Recently, Qi'an Guan and Xiangyu Zhou [9] obtained optimal estimate for L^p $(0 < p \leq 2)$ extension as an application of their solution of a sharp L^2 extension problem.

In the present paper, we study the p-Bergman kernels for bounded domains in $\mathbb{C}^n$, and apply L^p extension theorem to give some properties of p-Bergman kernels.

The definition of p-Bergman kernel is as follows:

Definition 1.1. For a domain $\Omega \subseteq \mathbb{C}^n$ and $p \in (0, 2]$, the p-Bergmann kernel $K_{\Omega,p}$ is denoted by

$$K_{\Omega,p}(z) = \sup_{f \in A^p(\Omega)} \frac{|f(z)|^p}{\int_\Omega |f|^p},$$

where

$$A^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) : \int_\Omega |f|^p < +\infty \right\}$$

(the integral is w.r.t. Lebesgue measure).

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According to the extreme property, the usual Bergman kernel is just 2-Bergman kernel for the case $p = 2$ in the above definition, which has been studied for years.

Let S be a closed complex subvariety of a domain $U \subset \mathbb{C}^n$. It's known that one has the same Bergman kernels on U and $U \setminus S$, since for any $f \in A^2(U \setminus S)$, one can holomorphically extend f to U. That is to say, one can not distinguish U and $U \setminus S$ by the Bergman kernel.

However, the p-Bergman kernel may give some distinction. We will prove that for a bounded domain, it is pseudoconvex if and only if its p-Bergman kernel is an exhaustion function for any $p \in (0, 2)$. Besides, the p-Bergman kernel is interesting per se. We'll also give estimate about the boundary behavior of the p-Bergman kernel for a bounded pseudoconvex domain. In the last section, we'll answer negatively a conjecture of H. Tsuji in [20].

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2. The *p*-Bergman kernel

Note that when $p = 2$, the p-Bergmann kernel is just the usual Bergman kennel. For simplicity, we write K_Ω for $K_{\Omega,2}$. The p-Bergmann kernel has some properties similar to the usual Bergman kernel, for example, it is easy to see that $K_{\Omega_1,p}(z) \geq K_{\Omega_2,p}(z)$ for $z \in \Omega_1$ and two domains $\Omega_1 \subseteq \Omega_2$, and the p-Bergmann kernels are plurisubharmonic.

We will study some more properties of $K_{\Omega,p}$.

Proposition 2.1. *Let $\Omega_1 \subset \mathbb{C}^n$ be simply connected domain and $\Omega_2 \subset \mathbb{C}^n$ be a domain. Then for any $\phi : \Omega_1 \to \Omega_2$ biholomorphism, we have $K_{\Omega_1,p}(z) = K_{\Omega_2,p}(\phi(z))|J\phi(z)|^2$, where $J\phi$ is the determinant of Jacobian of ϕ. In particular, if $p = \frac{2}{m}$, where $m \in \mathbb{N}$, there is no need for the condition that Ω_1 is simply connected.*

Proof. As Ω_1 is simply connected and $J\phi$ is nonvanishing, we can choose a single valued holomorphic function of $log\, J\phi$.

Then

$$\begin{aligned} \Phi : A^p(\Omega_2) &\to A^p(\Omega_1) \\ f &\mapsto f \circ \phi\, e^{\frac{2}{p} log(J\phi)} \end{aligned}$$

is isometric, since

$$\int_{\Omega_2} |f|^p = \int_{\Omega_1} |f \circ \phi|^p |J\phi|^2 = \int_{\Omega_1} |f \circ \phi\, e^{\frac{2}{p} log(J\phi)}|^p.$$

When $p = \frac{2}{m}$, $m \in \mathbb{N}$, we take

$$\begin{aligned} \Phi : A^p(\Omega_2) &\to A^p(\Omega_1) \\ f &\mapsto f \circ \phi\,(J\phi)^m, \end{aligned}$$

in this case, the simply connected condition is not needed any more.

By definition,

$$\begin{aligned} K_{\Omega_2,p}(\phi(z)) &= \sup_{f \in A^p(\Omega_2)} \frac{|f(\phi(z))|^p}{\int_{\Omega_2} |f|^p} \\ &= \sup_{f \in A^p(\Omega_2)} \frac{|f(\phi(z))|^p}{\int_{\Omega_1} |f \circ \phi|^p |J\phi|^2} \\ &= \frac{1}{|J\phi(z)|^2} \sup_{f \in A^p(\Omega_2)} \frac{|f \circ \phi(z) e^{\frac{2}{p} log(J\phi(z))}|^p}{\int_{\Omega_1} |f \circ \phi\, e^{\frac{2}{p} log(J\phi)}|^p} \\ &= \frac{K_{\Omega_1,p}(z)}{|J\phi(z)|^2}. \end{aligned}$$

□

It's easy to see that, if $J\phi$ is constant, then the above proposition is still true without the assumption that Ω_1 is simply connected. For example, if the domain Ω is a $G-$invariant domain w.r.t. a linear action of a semisimple Lie group G, then the p-Bergmann kernel is $G-$invariant.

The condition that Ω_1 is simply connected is necessary for some $p \in (0, 2)$ (see Remark 2.3).

Similar to the usual Bergman kernel, the following proposition holds for the p-Bergman kernel.

Proposition 2.2. *Suppose that $\Omega_j \subset \mathbb{C}^n$ are bounded domains and $\Omega_j \subset \Omega_{j+1}$ for $j \geq 1$, $\cup_{j=1}^{\infty} \Omega_j = \Omega$, where Ω is a bounded domain in $\mathbb{C}^n$. Then for $0 < p \leq 2$,*

$$\lim_{j \to \infty} K_{\Omega_j,p}(z) = K_{\Omega,p}(z),$$

and the convergence is uniform on compact subsets of Ω.

Proof. As $K_{\Omega_j,p}(z)$ is decreasing,

$$\lim_{j \to \infty} K_{\Omega_j,p}(z)$$

exists and $\geq K_{\Omega,p}(z)$.

For fixed $z \in \Omega$, we may assume $z \in \Omega_{j_0}$. There is $f_j \in \mathcal{O}(\Omega_j)$ such that

$$\int_{\Omega_j} |f_j|^p = 1$$

and

$$|f_j(z)|^p = K_{\Omega_j,p}(z)$$

for each $j \geq j_0$.

By the Montel theorem, there is a subsequence of j_k such that

$$\lim_{k\to\infty} f_{j_k}$$

is uniformly convergent to $f \in \mathcal{O}(\Omega)$.

It is easy to check that

$$\int_{\Omega} |f|^p \leq 1.$$

By the definition, we have

$$K_{\Omega,p}(z) \geq |f(z)|^p = \lim_{j\to\infty} K_{\Omega_j,p}(z).$$

As $K_{\Omega,p}(z)$ is continuous and $K_{\Omega_j,p}(z)$ is decreasing, it follows that $K_{\Omega_j,p}(z)$ converges uniformly to $K_{\Omega,p}(z)$ on compact subsets of Ω. □

Theorem 2.3. *Let Ω be one of the classical domains (see [11], [12], [13]):*

$$\begin{aligned}
\mathfrak{R}_1 &:= \{Z \in M(m,n) : I^{(m)} - Z\bar{Z}' > 0\},\\
\mathfrak{R}_2 &:= \{Z \in M(n,n) : I^{(n)} - Z\bar{Z}' > 0, Z = Z'\},\\
\mathfrak{R}_3 &:= \{Z \in M(n,n) : I^{(n)} - Z\bar{Z}' > 0, Z = -Z'\},\\
\mathfrak{R}_4 &:= \{Z \in M(1,n) : |ZZ'| + 1 - 2\bar{Z}Z' > 0, |ZZ'| < 1\}.
\end{aligned}$$

Then

$$K_{\Omega,p}(Z) = K_{\Omega,2}(Z)$$

for $Z \in \Omega$ for $p > 0$.

Proof. For $Z \in \Omega$ and $|t| \leq 1$, we have $tZ \in \Omega$.

For any $f \in \mathcal{O}(\Omega)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta}Z)|^p d\theta \geq |f(0)|^p.$$

Then by the Fubini Theorem,

$$\int_\Omega |f|^p = \frac{1}{2\pi}\int_0^{2\pi}\int_\Omega |f(e^{i\theta}Z)|^p dV_Z d\theta$$
$$= \int_\Omega dV_Z \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta}Z)|^p d\theta \geq |f(0)|^p Vol(\Omega),$$

we have

$$K_{\Omega,p}(0) = \frac{1}{Vol(\Omega)}.$$

As Ω is homogenous, it is well known that Ω is also simply connected, combining with the above proposition, we have $K_{\Omega,p}(Z) = K_{\Omega,2}(Z)$ for $Z \in \Omega$. □

Remark 2.1. The above result is true for any complete circular and bounded homogeneous domain. It's known that any bounded symmetric domain is such a domain.

For a general bounded homogenous domain Ω, we have $K_{\Omega,p}(z) \geq K_{\Omega,2}(z)$. It is well known that $K_\Omega(z,w)$ is zero free and Ω is simply connected, we can define a holomorphic function $logK_\Omega(z,w)$ for $z \in \Omega$ and fixed $w \in \Omega$. Then $e^{2/p\ logK_\Omega(z,w)} \in A^p(\Omega)$, and it is easy to get $K_{\Omega,p}(z) \geq K_{\Omega,2}(z)$.

It seems to be strange that the p-Bergmann kernel may be independent of p for some domains. From the following theorem, we can deduce that, in general, $K_{\Omega,p}$ is dependent on p.

Lemma 2.4. *For $\Omega \subset \mathbb{C}^n$, we have*

$$K_{\Omega,\frac{p}{m}}(z) \geq K_{\Omega,p}(z)$$

for any $p \in (0,2)$ and $m \in \mathbb{N}$.

Proof. If $f \in A^p(\Omega)$, then

$$f^m \in A^{\frac{p}{m}}(\Omega),$$

and

$$\int_\Omega |f|^p = \int_\Omega |f^m|^{\frac{p}{m}}.$$

By the definition of p-Bergman kernel, we have

$$K_{\Omega,\frac{p}{m}}(z) \geq K_{\Omega,p}(z).$$

□

The next theorem needs the L^p extension theorem. We state it in the following. For the proof, one can refer to [2] or [9].

Theorem 2.5. (*see [2] or [9]*) *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, L be a complex affine line in $\mathbb{C}^n$, and $\Omega \cap L \neq \emptyset$. For $0 < p \leq 2$, then for any $f \in A^p(\Omega \cap L)$, there is $F \in A^p(\Omega)$, such that $F|_{\Omega\cap L} = f$ and*

$$\int_{\Omega} |F|^p \leq C \int_{\Omega\cap L} |f|^p,$$

where C is a constant depending only on $diam\,\Omega$ and n.

Theorem 2.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, $p \in (0, 2)$ and $l = max\{s \in \mathbb{N}_+ : s < \frac{2}{p}\}$. Then we have*

$$K_{\Omega,p}(z) \geq \frac{c}{\delta(z)^{pl}},$$

where $\delta(z) = \inf\limits_{w\in\partial\Omega} d(z, w)$ and c is a constant positive number.

Proof. For any complex line L, after a unitary transform, we may assume $L = \{z_2 = \cdots = z_n = 0\}$.

Let $z^0 = (z_1^0, 0, \ldots, 0) \in \partial\Omega \cap L$, take

$$f = \frac{1}{(z_1 - z_1^0)^l} \in A^p(\Omega \cap L).$$

From the L^p extension theorem 2.5, we get $F \in A^p(\Omega)$ such that $F|_{\Omega\cap L} = f$, and

$$\int_{\Omega} |F|^p \leq C \int_{\Omega\cap L} |f|^p \leq 1/c$$

for some constant $c > 0$, c depends only on $diam\Omega$ and n.

Then

$$K_{\Omega,p}(z)|_{\Omega\cap L} \geq \frac{c}{|z_1 - z_1^0|^{pl}}.$$

As we can choose arbitrary complex line and boundary points, we get

$$K_{\Omega,p}(z) \geq \frac{c}{\delta(z)^{pl}}.$$

□

According to the above theorem and the fact that the p-Bergman kernel is plurisubharmonic, we can easily get the following interesting theorem.

Theorem 2.7. *For any bounded domain Ω in $\mathbb{C}^n$, Ω is pseudoconvex if and only if $K_{\Omega,p}(z)$ is an exhaustion function for $p \in (0,2)$.*

Remark 2.2. The condition that Ω is bounded is necessary. If we consider $\Omega = \mathbb{C} \setminus \Delta$, then $K_{\Omega,p}(z)$ is bounded near ∞ for $0 < p < 2$.

Theorem 2.8. *Let $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $1 \le p < 2$, then we have $K_{\Delta^*,p}(z) = O\big(\frac{1}{|z|^p}\big)$.*

Proof. For any $f \in A^p(\Delta^*)$, we have $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, then $g(z) := \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

In the present proof, we denote by $\|f\|_p = (\int_{\Delta^*} |f|^p)^{\frac{1}{p}}$ for $f \in A^p(\Delta^*)$.

Obviously, $\int_{\Delta^*_\tau} |g(z)|^p < \infty$, where $\Delta^*_\tau = \{z \in \mathbb{C} : 0 < |z| < \tau\}$ and $0 < \tau < 1$.

It's easy to see that

$$\int_{\Delta^*} \left|\frac{1}{z}\right|^p dxdy = \int_0^1 \int_0^{2\pi} r^{1-p} d\theta dr = \frac{2\pi}{2-p}.$$

From

$$\|g + h\|_p \le \|g\|_p + \|h\|_p,$$

we get

$$h(z) := \sum_{n=-\infty}^{-2} a_n z^n \in A^p(\Delta^*_\tau).$$

We want to prove $h = 0$.

$$\int_{\Delta^*_\tau} |h(z)|^p dxdy = \int_{\mathbb{C} \setminus \Delta_{\frac{1}{\tau}}} \left|h\left(\frac{1}{z}\right)\right|^p \frac{dxdy}{|z|^4} = \int_{\frac{1}{\tau}}^{\infty} \frac{1}{r^3} dr \int_0^{2\pi} \left|h\left(\frac{e^{i\theta}}{r}\right)\right|^p d\theta.$$

Let $\widetilde{h}(z) = h(\frac{1}{z})$, then $\widetilde{h}$ is holomorphic on $\mathbb{C} \setminus \Delta_{\frac{1}{\tau}}$ and

$$\widetilde{h}(z) = \sum_{n=2}^{\infty} a_{-n} z^n.$$

If $\widetilde{h}$ is not 0, then there is $n_0 > 1$ such that $a_{-n_0} \ne 0$ and $a_{-n} = 0$ for $1 < n < n_0$. Write $\widetilde{h}(z) = z^{n_0} f_1(z)$, where $f_1(z) = \sum_{n=n_0}^{\infty} a_{-n} z^{n-n_0}$.

By the submean property

$$\int_0^{2\pi} \left| f_1\left(\frac{e^{i\theta}}{r}\right)\right|^p d\theta \geq 2\pi |a_{-n_0}|^p,$$

and $n_0 p - 3 > -1$, it follows that

$$\int_{\Delta_\tau^*} |h(z)|^p dxdy \geq 2\pi |a_{-n_0}|^p \int_{\frac{1}{\tau}}^{\infty} r^{n_0 p - 3} dr = \infty.$$

Therefore, $h = 0$. That is to say, for any $f \in A^p(\Delta^*)$, we have $f(z) = \sum_{n=-1}^{\infty} a_n z^n$.

Note that

$$K_{\Delta^*,p}(z) \geq \frac{\frac{1}{|z|^p}}{\int_{\Delta^*} |\frac{1}{z}|^p} \geq \frac{2-p}{2\pi |z|^p}. \tag{1}$$

Since

$$\begin{aligned} |z|^p K_{\Delta^*,p}(z) &= |z|^p \sup_{f\in A^p(\Delta)} \frac{|\frac{a}{z} + f(z)|^p}{\int_{\Delta^*} |\frac{a}{z} + f(z)|^p dxdy} \\ &= \sup_{f\in A^p(\Delta)} \frac{|a + zf(z)|^p}{\int_{\Delta^*} |\frac{a}{z} + f(z)|^p dxdy}. \end{aligned} \tag{2}$$

From (1), for z near 0, we may take $a = 1$. For $f \in A^p(\Delta)$

(a) If $\|f\|_p^p > 2^p \frac{2\pi}{2-p}$, then $\|f(z) + \frac{1}{z}\|_p \geq \|f(z)\|_p - \|\frac{1}{z}\|_p > \frac{1}{2}\|f(z)\|_p$, so

$$\frac{|1 + zf(z)|^p}{\int_{\Delta^*} |\frac{1}{z} + f(z)|^p dxdy} < \frac{2^p(1 + |zf(z)|^p)}{(1/2^p)\int_{\Delta^*} |f|^p} < 2^{2p}\left(\frac{2-p}{2^{p+1}\pi} + |z|^p K_{\Delta,p}(z)\right).$$

(b) If $\|f\|_p^p \leq 2^p \frac{2\pi}{2-p}$, then $|f(z)| \leq C$ for all z near 0, where C is a positive constant independent on f.

Since

$$\begin{aligned} \int_{\Delta^*} \left|\frac{1}{z} + f(z)\right|^p dxdy &= \int_0^1 r^{1-p} dr \int_0^{2\pi} |1 + re^{i\theta} f(re^{i\theta})|^p d\theta \\ &\geq 2\pi \int_0^1 r^{1-p} dr = \frac{2\pi}{2-p}, \end{aligned}$$

then

$$\frac{|1 + zf(z)|^p}{\int_{\Delta^*} |\frac{1}{z} + f(z)|^p dxdy} < \frac{(2-p)(1 + |z|C)^p}{2\pi}.$$

According to (a) and (b), we get that $|z|^p K_{\Delta^*,p}(z)$ is bounded near 0.

□

From the above theorem, we know the lower bounds of Theorem 2.6 is optimal.

Remark 2.3. Let $D = \{z \in \mathbb{C} : |z| > 1\}$, for $p \in (1, 2)$, there is $c = c(p) > 0$ such that

$$K_{D,p}(z) \le \frac{c}{|z|^{2p}}$$

for $|z| \gg 1$.

Let $\varphi : \Delta^* \to D$, $z \mapsto 1/z$. For $p \in (4/3, 2)$,

$$K_{\Delta^*,p}(z) \ne K_{D,p}(1/z)\frac{1}{|z|^4}.$$

Proof of the Remark:

For any $f \in A^p(D)$, we have

$$f(z) = \sum_{n=-1}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^{-n}.$$

Let $f_1(z) = \sum_{n=-1}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=2}^{\infty} b_n z^{-n}$.

It is easy to check that there is $r \gg 1$ such that $\int_{\{|z|>r\}} |f_2|^p < \infty$ holds.

Hence $\int_{\{|z|>r\}} |f_1|^p < \infty$.

If f_1 is not 0, we may choose k to be the integer such that $a_n = 0$ for $n < k$, $a_k \ne 0$, then

$$\begin{aligned}\int_{\{|z|>r\}} |f_1|^p &= \int_{\{|z|>r\}} \left|\sum_{n=k}^{\infty} a_n z^n\right|^p \\ &= \int_r^{\infty} \rho d\rho \int_0^{2\pi} (\rho)^{kp} \left|\sum_{n=k}^{\infty} a_n z^{n-k}|^p \ge 2\pi|a_k\right|^p \int_r^{\infty} (\rho)^{1+kp} = \infty.\end{aligned}$$

Therefore, $f_1 = 0$.

We get $K_{D,p}(z) \le \frac{c}{|z|^{2p}}$ for $|z| \gg 1$.

By the above theorem, $K_{\Delta^*,p}(z) = O\big(\frac{1}{|z|^p}\big)$.

As

$$K_{D,p}(1/z)\frac{1}{|z|^4} \le \frac{c}{|z|^{4-2p}}$$

for $|z| << 1$,

if $p > 4/3$, then

$$K_{\Delta^*,p}(z) \neq K_{D,p}(z)\frac{1}{|z|^4}.$$

We have finished the proof of the remark.

3. A conjecture of H. Tsuji

We first recall a definition for complex manifolds, see H. Tsuji [20].

Definition 3.1. Let M be a complex manifold with the canonical line bundle K_M, for every positive integer m, we set

$$Z_m := \left\{\sigma \in \Gamma(M, \mathcal{O}_M(mK_M)) \,\middle|\, \left|\int_M (\sigma \wedge \overline{\sigma})^{\frac{1}{m}}\right| < +\infty\right\}$$

and

$$K_{M,m} := \sup\left\{|\sigma|^{\frac{2}{m}}; \sigma \in \Gamma(M, \mathcal{O}_M(mK_M)) \,\middle|\, \left|\int_M (\sigma \wedge \overline{\sigma})^{\frac{1}{m}}\right| \leq 1\right\},$$

where the sup denotes the pointwise supremum.

Then let

$$K_{M,\infty} := \limsup_{m\to\infty} K_{M,m}$$

and $h_{can,M}$:=the lower envelope of $\frac{1}{K_{M,\infty}}$.

Lemma 3.1. *For $\Omega \subset \mathbb{C}^n$, we have*

$$\sup_{m\in\mathbb{N}} K_{\Omega,\frac{2}{m}}(z) = \sup_{p\in(0,2]} K_{\Omega,p}(z).$$

Proof. By Lemma 2.4, we have

$$\sup_{m\in\mathbb{N}} K_{\Omega,\frac{2}{m}}(z) = \sup_{p\in(0,2]\cap\mathbb{Q}} K_{\Omega,p}(z).$$

If $f \in \mathcal{O}(\Omega)$ and $\int_\Omega |f|^p < \infty$, then

$$\lim_{q\to p, q<p} \int_\Omega |f|^q = \int_\Omega |f|^p.$$

So

$$\sup_{p\in(0,2]\cap\mathbb{Q}} K_{\Omega,p}(z) = \sup_{p\in(0,2]} K_{\Omega,p}(z)$$

and the lemma follows. □

For $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$, since the canonical bundle K_{Δ^*} is trivial, so when we consider $K_{\Delta^*,\infty}$ and h^{-1}_{can,Δ^*}, we can omit the form dt.

H. Tsuji [20] proposed the following conjecture (see Conjecture 2.16 in [20]):

$$h^{-1}_{can,\Delta^*} = O\left(\frac{1}{|z|^2(log|z|)^2}\right)$$

holds.

However, we get the following theorem:

Theorem 3.2. *One has*

$$h^{-1}_{can,\Delta^*}(z) \geq K_{\Delta^*,\infty}(z) \geq \frac{1}{2\pi e}\frac{1}{|z|^2|log|z||}$$

for $0 < |z| < e^{-1}$.

Proof. Since

$$\int_{\Delta^*}\left|\frac{1}{z}\right|^p = \frac{2\pi}{2-p},$$

by Lemma 2.4 and Lemma 3.1, we get

$$\begin{aligned} K_{\Delta^*,\infty}(z) &= \limsup_{m\to\infty} K_{\Delta^*,m}(z) = \sup_{m\geq 1} K_{\Delta^*,m}(z) \\ &= \sup_{p\in(0,2]} K_{\Delta^*,p}(z) \geq \sup_{p\in(0,2]}\frac{2-p}{2\pi}\frac{1}{|z|^p}. \end{aligned}$$

For $0 < |z| < e^{-1}$, let

$$p = 2 + \frac{1}{log|z|} \in [1, 2],$$

therefore

$$\frac{2-p}{2\pi}\frac{1}{|z|^p} = \frac{1}{2\pi e}\frac{1}{|z|^2|log|z||},$$

so

$$K_{\Delta^*,\infty}(z) \geq \frac{1}{2\pi e}\frac{1}{|z|^2|log|z||}.$$

Hence

$$h^{-1}_{can,\Delta^*}(z) \geq K_{\Delta^*,\infty}(z) \geq \frac{1}{2\pi e}\frac{1}{|z|^2|log|z||}.$$

□

From the above theorem, we know that h^{-1}_{can,Δ^*} is not integrable near 0.

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College of Mathematics and Statistics, Chongqing University
Chongqing 401331, China
E-mail address: ningjiafu@amss.ac.cn

Department of Mathematics, Information School
Renmin University of China
Beijing 100872, China
E-mail address: huipingzhang@ruc.edu.cn

Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100190, China
E-mail address: xyzhou@math.ac.cn

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