Regularity of shadows and the geometry of the singular set associated to a Monge-Ampère equation

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Illuminating the surface of a convex body with parallel beams of light in a given direction generates a shadow region. We prove sharp regularity results for the boundary of this shadow in every direction of illumination. Moreover, techniques are developed for investigating the regularity of the region generated by orthogonally projecting a convex domain onto another. As an application we study the geometry and Hausdorff dimension of the singular set corresponding to a Monge-Ampère equation.

1. Introduction

Shadows play an important role in many different branches of mathematics such as differential geometry, convex geometry, geometric combinatorics, and functional analysis [1–3, 8, 11, 12, 16, 18]. Our aim in this paper is to show that they also naturally appear in a free boundary problem associated to a Monge-Ampère equation. Indeed, it turns out that the regularity of certain shadow regions yields information on the Hausdorff dimension of the singular set appearing in the optimal partial transport problem [6, 7, 9, 10, 15].

1.1. Illumination shadows

Illumination shadows form powerful tools in the classification of surfaces. For instance, it is a well-known fact that if every shadow boundary generated by parallel illumination on a Blaschke surface embedded in $\mathbb{R}^3$ is a plane curve, then the surface is quadric [19, Theorem 5.8]. Moreover, in 2001 Ghomi [11] solved the shadow problem formulated in 1978 by Wente: if M is a closed

oriented 2-dimensional manifold and $f : M \to \mathbb{R}^3$ is a smooth immersion, then f is a convex embedding if and only if the shadow region generated by parallel illumination is simply connected in every direction.

Regularity properties of shadow boundaries have been investigated in [11, 12, 14, 21, 24, 25]. For example, given a smooth manifold it is well known that if the Gaussian curvature does not vanish at a given point, then the shadow boundary is locally smooth around that point (via the inverse function theorem). Moreover, using Sard's theorem, it is not difficult to prove that in almost every direction (in the sense of Lebesgue), the shadow boundary of a surface is continuous. Steenarts [24] showed that for a smooth convex body, the shadow boundary has finite $(n-2)$-dimensional Hausdorff measure in almost all directions (in the sense of Lebesgue). On the other hand, Gruber and Sorger [12] showed that when one considers the product space of convex bodies and directions $\mathcal{G} \times \mathbb{S}^{n-1}$, most pairs $(\Omega, u) \in \mathcal{G} \times \mathbb{S}^{n-1}$ (i.e. up to a meagre set in the sense of Baire) generate boundaries with infinite $(n-2)$-dimensional Hausdorff measure while having Hausdorff dimension $(n-2)$, see [12, Theorems 1 & 2]. These results suggest that in the general class of convex bodies, most shadow boundaries are highly irregular.

Moreover, in applications one may need regularity results for the shadow boundary of a generic convex body in every direction (an example is given in §4). In this weak setting, the tools of differential geometry do not apply due to lack of regularity and "almost everywhere" results in the sense of Lebesgue or Baire do not suffice since they may exclude a dense set of directions. In §2, we address this problem with tools from convex geometry and obtain the following results.

- [Theorem 2.5] For a strictly convex domain in $\mathbb{R}^n$, the boundary of the shadow generated by parallel illumination is locally a continuous graph in every direction.

- [§2.1] There exists a convex set and a direction so that the shadow boundary generated by parallel illumination is not locally a graph.

- [Theorem 2.6] For a p-uniformly convex $C^{1,\alpha}$ domain in $\mathbb{R}^n$, $\alpha \in (0, 1]$, $p \geq 2$, the boundary of the shadow generated by parallel illumination is locally a $C^{0, \frac{\alpha}{p-1}}$ graph in every direction.

- [§2.2] For every $\alpha \in (0, 1]$ and $p \geq 2$, there exists a C^∞ smooth convex set and a direction so that the shadow boundary generated by parallel illumination in that direction is in $C^{0,\beta} \setminus C^{0, \frac{\alpha}{p-1}}$ for some $\beta < \frac{\alpha}{p-1}$.

- [Remark 2.8] For a 2-uniformly convex C^{k+1} domain in $\mathbb{R}^n$, $k \geq 1$, the boundary of the shadow generated by parallel illumination is locally a C^k graph.

We note that shadows generated by another type of illumination process also appear in a well-known covering problem of Levi [17] and Hadwiger [13]: let $\Omega \subset \mathbb{R}^n$ be a convex body and $h(\Omega)$ the smallest number so that Ω can be covered by $h(\Omega)$ smaller homothetical copies of itself; the conjecture states that $h(\Omega) \leq 2^n$, with equality if and only if Ω is an n-parallelotope. Indeed, Boltyanskii [5] connected this conjecture with an illumination problem by showing that $h(\Omega) = l(\Omega)$ where $l(\Omega)$ is the smallest number of light sources outside of Ω required to illuminate $\partial\Omega$; a boundary point y of Ω is said to be illuminated from $x \notin \Omega$ if the half-line through y from x intersects the interior of Ω (at a point not between x and y). For further reading, we refer the reader to two survey articles [2, 18] and the references therein.

1.2. Projection shadows

In 1986 Kiselman [16] addressed the following question: what degree of smoothness does a two-dimensional projection of a three-dimensional smooth convex set possess? He proved that if the convex set is C^1, then its projection is also C^1; if it is $C^{2,1}$, then the boundary of the projection is twice differentiable; and, if it is real-analytic, then the boundary of the projection is $C^{2,\alpha}$ for some $\alpha > 0$. Moreover, he provided examples to show that these results are essentially sharp: in the real-analytic case, the boundary of the projection may be exactly $C^{2,\frac{2}{q}}$ for any odd integer $q \geq 3$ [16, Example 3.2], and the boundary of the shadow of a C^∞ set may not be C^2 [16, Example 3.3]. V. Sedykh [23] studied this question in higher dimensions and proved that the projection of a smooth closed convex surface in $\mathbb{R}^n$ onto a hyperplane is $C^{1,1}$ and showed that this result is sharp in the sense that there exists a hypersurface whose shadow is not twice differentiable; this contrasts with Kiselman's result in $\mathbb{R}^3$. Moreover, the analytic case also displays a loss of regularity in higher dimensions: Bogaevsky [4] showed the existence of a real-analytic closed convex hypersurface, whose shadow does not belong to the class C^2. These results are all compiled and discussed in the book "Arnold's problems" by V.I. Arnold [1] (Arnold calls these types of shadows "apparent contours").

In applications, however, one may require regularity results of this shadow when projecting onto a strictly convex domain (as opposed to a

hyperplane as in the results above), see e.g. §1.3. Indeed, this situation is quite different in the sense that the projection no longer occurs in just one direction, but in many different directions determined by the normal of the set onto which the projection takes place. Therefore, the variation of this normal dictates the regularity and geometry of the boundary of the projection and this requires a new approach in contrast with the affine case. Here is the precise statement of the problem: given two convex domains $\Omega \subset \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^n$, if $P_\Lambda(\Omega)$ denotes the orthogonal projection of Ω onto Λ, then how smooth is $\partial(P_\Lambda(\Omega) \cap \partial\Lambda)$? The following results are established in §3.

- [Theorem 3.1] Let $\Omega \subset \mathbb{R}^n$ be a bounded strictly convex domain and $\Lambda \subset \mathbb{R}^n$ a convex domain whose boundary is $C^{1,1}$. If $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, then $\partial P_\Lambda(\Omega)$ is finitely $(n-2)$-rectifiable.

- [Remark 3.2] The disjointness assumption in Theorem 3.1 is necessary: there exist two bounded convex domains Ω and Λ in $\mathbb{R}^2$ for which $\mathcal{H}^0(\partial(P_\Lambda(\Omega) \cap \partial\Lambda)) = \infty$.

- [Theorem 3.3] If Ω and Λ are C^{k+1} convex domains in $\mathbb{R}^n$ with disjoint closures, $k \geq 1$, and Ω is bounded and 2-uniformly convex, then $\partial P_\Lambda(\Omega)$ is an $(n-2)$-dimensional C^k_{loc} graph.

We point out that when one takes Λ to be a hyperplane, Theorem 3.1 is immediate: the projection of a convex set onto a hyperplane is convex, so $\partial P_\Lambda(\Omega)$ is locally Lipschitz. However, the situation is different if Λ is curved. Here is the idea of our method: we take a point $y \in \partial P_\Lambda(\Omega)$ and represent Λ locally by a bi-Lipschitz graph with respect to the tangent space at y, $\mathbb{T}_y \partial\Lambda = \mathbb{R}^{n-1}$. Then we consider $P_{\mathbb{R}^{n-1}} \partial P_\Lambda(\Omega)$ and cook up an auxiliary uniformly convex $C^{1,1}$ function that touches this set at $P_{\mathbb{R}^{n-1}} \partial P_\Lambda(y)$. By applying our results from Theorem 2.6 (or rather, the idea in the proof), we show that there exists a Lipschitz function which touches $P_{\mathbb{R}^{n-1}} \partial P_\Lambda(\Omega)$ at $P_{\mathbb{R}^{n-1}} \partial P_\Lambda(y)$ and bounds $P_{\mathbb{R}^{n-1}} \partial P_\Lambda(\Omega)$ from one side (in a suitable coordinate system). This yields the existence of a cone whose opening can be shown to depend only on the initial data (i.e. Ω and Λ) and that touches $P_{\mathbb{R}^{n-1}} \partial P_\Lambda(\Omega)$ only at $P_{\mathbb{R}^{n-1}} \partial P_\Lambda(y)$; the rest follows by iterating the argument above and locally transporting cones at all the other points in $P_{\mathbb{R}^{n-1}} \partial P_\Lambda(\Omega)$ from the surrounding tangent spaces via the $C^{1,1}$ charts representing Λ and applying a standard covering argument from geometric measure theory.

The idea of this argument in terms of finding a cone was employed by Indrei [15], although he assumed Λ to be uniformly convex. The novelty in this paper is that we construct our barrier-type function without requiring

uniform convexity of Λ. Indeed, this support function is constructed by using the boundary of the shadow generated by illuminating Λ in the direction of some normal of Ω at the point $y + t(y)N_\Lambda(y)$, where $t(y)$ is the first hitting time of Ω. However, in contrast with [15, Proposition 4.1], we require a strict convexity assumption on Ω. Nevertheless, this tradeoff turns out to be more useful when applying our theory to a free boundary problem that has a strict convexity assumption on Ω naturally built into it, see §1.3.

On the other hand, the method we employ to prove Theorem 3.3 is completely different. The starting point is that we may represent the sets $\partial\Omega$ and $\partial\Lambda$ locally as level sets of two convex functions $G : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}$. By exploiting the geometry of the problem, we construct a function $\phi : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+3}$ so that $\partial P_\Lambda(\Omega)$ is locally a coordinate projection of a level set of ϕ (herein lies the novelty of our approach since we are connecting the two sets and the unknown shadow boundary by a single function); next, we compute the differential of this map and show that it has full rank and conclude via the implicit function theorem.

1.3. Shadows and a Monge-Ampère equation

The optimal partial transport problem is a generalization of the classical Monge-Kantorovich problem: given two non-negative functions $f = f\chi_\Omega$, $g = g\chi_\Lambda \in L^1(\mathbb{R}^n)$ and a number $0 < m \leq \min\{\|f\|_{L^1}, \|g\|_{L^1}\}$, the objective is to find an optimal transference plan between f and g with mass m. A transference plan is a non-negative, finite Borel measure γ on $\mathbb{R}^n \times \mathbb{R}^n$, whose first and second marginals are controlled by f and g respectively: for any Borel set $A \subset \mathbb{R}^n$,

$$\gamma(A \times \mathbb{R}^n) \leq \int_A f(x)dx, \quad \gamma(\mathbb{R}^n \times A) \leq \int_A g(x)dx.$$

An optimal transference plan is a minimizer of the functional

$$(1.1) \qquad \gamma \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)d\gamma(x,y),$$

where c is a non-negative cost function. Issues of existence, uniqueness, and regularity of optimal transference plans have been addressed by Caffarelli & McCann [6], Figalli [9, 10], Indrei [15], and Chen & Indrei [7].

If

$$\|f \wedge g\|_{L^1(\mathbb{R}^n)} \leq m \leq \min\{\|f\|_{L^1(\mathbb{R}^n)}, \|g\|_{L^1(\mathbb{R}^n)}\},$$

then by the results in [10, Section 2], there exists a convex function Ψ_m and non-negative functions f_m, g_m for which

$$\gamma_m := (Id \times \nabla\Psi_m)_{\#} f_m = (\nabla\Psi_m^* \times Id)_{\#} g_m,$$

is the unique solution of (1.1) and $\nabla\Psi_{m\#} f_m = g_m$ (see [10, Theorem 2.3]). Ψ_m is known as the *Brenier solution* of the Monge-Ampère equation

$$\det(D^2\Psi_m)(x) = \frac{f_m(x)}{g_m(\nabla\Psi_m(x))},$$

with $x \in F_m :=$ set of density points of $\{f_m > 0\}$, and $\nabla\Psi_m(F_m) \subset G_m :=$ set of density points of $\{g_m > 0\}$. Moreover, as in [10, Remark 3.2], we set

$$U_m := (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x},\bar{y})\in\Gamma_m} B_{|\bar{x}-\bar{y}|}(\bar{y}),$$

$$V_m := (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x},\bar{y})\in\Gamma_m} B_{|\bar{x}-\bar{y}|}(\bar{x}),$$

where Γ_m is the set

$$(Id \times \nabla\Psi_m)(F_m \cap D_{\nabla\Psi_m}) \cap (\nabla\Psi_m^* \times Id)(G_m \cap D_{\nabla\Psi_m^*}),$$

with $D_{\nabla\Psi_m}$ and $D_{\nabla\Psi_m^*}$ denoting the set of continuity points for $\nabla\Psi_m$ and $\nabla\Psi_m^*$, respectively, where Ψ_m^* is the Legendre transform of Ψ_m.

The free boundary associated to f_m is denoted by $\overline{\partial U_m \cap \Omega}$ and the free boundary associated to g_m by $\overline{\partial V_m \cap \Lambda}$. They correspond to $\overline{\partial F_m \cap \Omega}$ and $\overline{\partial G_m \cap \Lambda}$, respectively [10, Remark 3.3]. One method of obtaining free boundary regularity is to first prove regularity results on Ψ_m and then utilize that $\nabla\Psi_m$ gives the direction of the normal to the free boundary $\overline{\partial U_m \cap \Omega}$ (by symmetry and duality, this also implies a similar result for $\overline{\partial V_m \cap \Lambda}$).

Indeed, this method was employed by Caffarelli & McCann [6] to deduce $C_{loc}^{1,\alpha}$ free boundary regularity away from a singular set $\tilde{S}$ in the case when Ω and Λ are strictly convex and separated by a hyperplane. Indrei [15] generalized an improvement of this result in the overlapping case: he obtains $C_{loc}^{1,\alpha}$ free boundary regularity away from $\overline{\Omega \cap \Lambda}$ and a singular set S which in the disjoint case is a subset of $\tilde{S}$. Moreover, he developed a method to study the Hausdorff dimension of S and utilized it to prove that if the domains are $C^{1,1}$, then S has dimension at most $(n-2)$.

In §4, we connect the shadow boundaries with this singular set and show that one may replace the uniform convexity assumption with a strict convexity assumption to obtain that the singular set has Hausdorff dimension

at most $(n-2)$, see Theorem 4.3. The precise connection is this: the singular set breaks up into two parts; one of these can be handled using notions from transport theory and non-smooth analysis; the other can be shown to be trapped on the boundary of $P_\Lambda(\Omega)$. Thus, understanding the Hausdorff dimension of the boundary of this shadow is a way to obtain bounds on the Hausdorff dimension of the singular set. This is where the rectifiability result of Theorem 3.1 comes into play. Since Theorem 2.6 was used in the proof of Theorem 3.1, this highlights the interplay between the shadow generated by parallel illumination, the shadow generated by orthogonal projections, and the Monge-Ampère free boundary problem arising in optimal transport theory.

2. Regularity of shadows generated by parallel illumination

In this section we investigate the regularity of the shadow region of a convex domain (i.e. an open, convex set) $\Lambda \subset \mathbb{R}^n$ under parallel illumination. For $u \in \mathbb{S}^{n-1}$, the *shadow* of Λ generated by u is the set S_u of points $x \in \partial \Lambda$ such that there exists an outward-pointing normal vector $\nu(x)$ at x (i.e. $\langle \nu(x), y - x \rangle \le 0$ for all $y \in \Lambda$) for which $\langle \nu(x), u \rangle > 0$. In other words,

Definition 2.1. $S_u := \{ x \in \partial \Lambda : \langle \nu(x), u \rangle > 0$ for some outward-pointing normal $\nu(x)$ at $x \}$.

Our aim is to prove that for a strictly convex domain Λ, the boundary ∂S_u (in the topology of $\partial \Lambda$) is locally a continuous graph and that this regularity is optimal in the sense that if Λ is not strictly convex, then ∂S_u might fail to locally be a graph.

Given $k \in \mathbb{N}$ and $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, we denote an arbitrary vector in $\mathbb{R}^{k-1}$ by $x' := (x_1, x_2, \ldots, x_{k-1})$. Furthermore, $x'' := (x')' = (x_1, x_2, \ldots, x_{k-2}) \in \mathbb{R}^{k-2}$, and given a set $A \subset \mathbb{R}^k$, $A' := \{x' : x \in A\}$ and $A'' := (A')'$.

Let $x_0 \in \partial \Lambda$ be a boundary point of S_u. Without loss, we assume that in a neighborhood of x_0, say U, Λ is parametrized as $x_n \le \phi(x')$, for some strictly concave function ϕ. Consequently, $\partial \Lambda$ is locally given by $x_n = \phi(x')$ where the domain of ϕ is a bounded, open set $U' \subset \mathbb{R}^{n-1}$. Note that $x' \mapsto (x', \phi(x'))$ is a homeomorphism between the spaces U' and $\partial \Lambda \cap U$.

We denote the superdifferential of the concave function ϕ at a point $x' \in U'$ by

$$\partial^+ \phi(x') := \{ z' \in \mathbb{R}^{n-1} : \forall y' \in U', \phi(y') \le \phi(x') + \langle z', y' - x' \rangle \}.$$

For every $y' \in U'$ there is a one-to-one correspondence between superdifferentials $w \in \partial^+\phi(y')$ and outward-pointing normals ν at $(y', \phi(y'))$ given by $\nu = \frac{(-w,1)}{(|w|^2+1)^{1/2}}$. Therefore $(y', \phi(y')) \in S_u$ if and only if $\langle w, u' \rangle < u_n$, for some $w \in \partial^+\phi(y')$.

In this section we prove that $\partial S_u' \cap U'$ (in the usual $\mathbb{R}^{n-1}$ topology) is locally a continuous graph. By rotating the coordinate system, if necessary, we may assume $x_0 = 0$ and $u' = (0, 0, \ldots, u_{n-1}) \in \mathbb{R}^{n-1}$ with $u_{n-1} > 0$; moreover, $\mathbb{R}^{n-2}$ is identified with $(u')^\perp$. Under these assumptions, the condition $\langle w, u' \rangle < u_n$ takes the form $w_{n-1} < \frac{u_n}{u_{n-1}}$. We begin our analysis with the following lemma.

Lemma 2.1. *Let $\Lambda \subset \mathbb{R}^n$ be a strictly convex domain and $y' \in S_u' \cap U'$. Then $(y'', t) \in S_u'$, for every $t > y_{n-1}$ such that $(y'', t) \in U'$.*

Proof. Since $y' \in S_u'$, there exists $w^1 \in \partial^+\phi(y')$ such that $w^1_{n-1} < \frac{u_n}{u_{n-1}}$. Let $t > y_{n-1}$ be such that $(y'', t) \in U'$ and consider $w^2 \in \partial^+\phi(y'', t)$. By the monotonicity formula,

$$\langle w^2 - w^1, (y'', t) - y' \rangle < 0,$$

or equivalently $(w^2_{n-1} - w^1_{n-1})(t - y_{n-1}) < 0$. Therefore, $w^2_{n-1} < w^1_{n-1} < \frac{u_n}{u_{n-1}}$ and this implies $(y'', t) \in S_u'$. □

Lemma 2.2. *Let $\Lambda \subset \mathbb{R}^n$ be a strictly convex domain. Then there exists $t_1 > 0 > t_2$ and a ball $V'' \subset \mathbb{R}^{n-2}$ centered at $0''$ with the following properties: for every $y'' \in V''$, $(y'', t_1), (y'', t_2) \in U'$, and for any $t \leq t_2$ such that $(y'', t) \in U'$, any $\zeta^t \in \partial^+\phi(y'', t)$, and $\eta \in \partial^+\phi(y'', t_1)$, one has $\eta_{n-1} < \frac{u_n}{u_{n-1}} < \zeta^t_{n-1}$.*

Proof. Since the set $\partial^+\phi(0')$ is convex, one of the following is true:

(i) $w_{n-1} < \frac{u_n}{u_{n-1}}$ for every $w \in \partial^+\phi(0')$;

(ii) $w_{n-1} > \frac{u_n}{u_{n-1}}$ for every $w \in \partial^+\phi(0')$;

(iii) $w_{n-1} = \frac{u_n}{u_{n-1}}$ for some $w \in \partial^+\phi(0')$.

However, by continuity properties of the superdifferential of a concave function (see e.g. [20, Corollary 24.5.1]), if (i) or (ii) holds, then the same strict inequality will be satisfied for all $w \in \partial^+\phi(y')$, for all y' in some neighborhood of $0'$. This contradicts $0' \in \partial S_u'$. Hence, $w_{n-1} = \frac{u_n}{u_{n-1}}$ for some $w \in \partial^+\phi(0')$. Pick $t_1 > 0 > t_2$ such that $(0'', t_1), (0'', t_2) \in U'$. The monotonicity formula

combined with the strict convexity of Λ implies that every $\eta \in \partial^+\phi(0'', t_1)$ satisfies $\eta_{n-1} < w_{n-1} = \frac{u_n}{u_{n-1}}$; similarly, for $t \leq t_2$ such that $(0'', t) \in U'$, every $\zeta^t \in \partial^+\phi(0'', t)$ satisfies $\zeta^t_{n-1} > w_{n-1} = \frac{u_n}{u_{n-1}}$, and by utilizing the continuity of the superdifferential, the same inequalities hold in some ball V'' centered at $0''$. $\qquad\square$

Remark 2.3. Rephrased in terms of the shadow set S_u, Lemma 2.2 states that $((y'', t_1), \phi(y'', t_1)) \in S_u$ while $((y'', t), \phi(y'', t)) \notin S_u$ for any $y'' \in V''$ and $t \leq t_2$.

Let V'' be the ball from Lemma 2.2. For every $y'' \in V''$, define

(2.1)
$$\gamma(y'') := \inf\{t \ : \ (y'', t) \in S'_u \cap U'\}.$$

By Lemma 2.2, γ is well-defined with $(y'', \gamma(y'')) \in U'$. Moreover,

$$t_1 \leq \gamma(y'') \leq t_2$$

for all $y'' \in V''$.

Lemma 2.4. *(Properties of γ) Let $\Lambda \subset \mathbb{R}^n$ be a strictly convex domain and V'' the ball from Lemma 2.2. For every $y'' \in V''$ there exists $w \in \partial^+\phi(y'', \gamma(y''))$ such that $w_{n-1} = \frac{u_n}{u_{n-1}}$. Moreover, if $t > \gamma(y'')$ and $(y'', t) \in U'$, then $\zeta_{n-1} < \frac{u_n}{u_{n-1}}$ for every $\zeta \in \partial^+\phi(y'', t)$. Similarly, if $t < \gamma(y'')$ and $(y'', t) \in U'$, then $\zeta_{n-1} > \frac{u_n}{u_{n-1}}$ for every $\zeta \in \partial^+\phi(y'', t)$. Furthermore, γ is continuous.*

Proof. Suppose $w_{n-1} < \frac{u_n}{u_{n-1}}$ for all $w \in \partial^+\phi(y'', \gamma(y''))$. By continuity of the superdifferential, $\zeta_{n-1} < \frac{u_n}{u_{n-1}}$ for all $\zeta \in \partial^+\phi(y'', t)$ if t is sufficiently close to $\gamma(y'')$; therefore, $(y'', t) \in S'_u \cap U'$ for some $t < \gamma(y'')$, and this contradicts the definition of γ. On the other hand, if $w_{n-1} > \frac{u_n}{u_{n-1}}$ for all $w \in \partial^+\phi(y'', \gamma(y''))$, then again by continuity of the superdifferential, $\zeta_{n-1} > \frac{u_n}{u_{n-1}}$ for all $\zeta \in \partial^+\phi(y'', t)$ if t is sufficiently close to $\gamma(y'')$. This implies $(y'', t) \in (S'_u)^c \cap U'$ for $\gamma(y'') < t \leq t_0$, with t_0 sufficiently close to $\gamma(y'')$; again, this produces a contradiction. Therefore, there exist $w^0, w^1 \in \partial^+\phi(y'', \gamma(y''))$ such that $w^1_{n-1} \leq \frac{u_n}{u_{n-1}}$ and $w^0_{n-1} \geq \frac{u_n}{u_{n-1}}$. Hence, for some $s \in [0, 1]$, $w_{n-1} = \frac{u_n}{u_{n-1}}$ where $w = (1 - s)w^0 + sw^1$. Since $\partial^+\phi(y'', \gamma(y''))$ is a convex set, it follows that $w \in \partial^+\phi(y'', \gamma(y''))$. Now pick any $t > \gamma(y'')$. By the monotonicity formula, for every $\zeta \in \partial^+\phi(y'', t)$, $\zeta_{n-1} < w_{n-1} = \frac{u_n}{u_{n-1}}$. Similarly, if $t < \gamma(y'')$, then $\eta_{n-1} > w_{n-1} = \frac{u_n}{u_{n-1}}$ for all $\eta \in \partial^+\phi(y'', t)$. Next let $y''_k, y'' \in V''$ and $y''_k \to y''$. Since $\{\gamma(y''_k)\}$ is a bounded sequence, every subsequence has a further subsequence that converges. Take such a subsequence

and suppose it converges to, say, $t \in \mathbb{R}$. If $t > \gamma(y'')$, then by what has already been proved, we have that $\zeta_{n-1} < \frac{u_n}{u_{n-1}}$ for all $\zeta \in \partial^+ \phi(y'', t)$. Therefore, by the continuity of the superdifferential, this condition is satisfied in some neighborhood of (y'', t), but this contradicts the fact that $(y_k'', \gamma(y_k'')) \to (y'', t)$ (along this subsequence) and that there exists $w^k \in \partial^+ \phi(y_k'', \gamma(y_k''))$ such that $w_{n-1}^k = \frac{u_n}{u_{n-1}}$. The case $t < \gamma(y'')$ may be excluded in the same manner. Hence, $t = \gamma(y'')$, and we proved that every subsequence of $\{\gamma(y_k'')\}$ admits a further subsequence converging to $\gamma(y'')$; this implies the continuity of γ. $\qquad\square$

Now we have all the ingredients to prove the following theorem which may be seen as the first step towards investigating the regularity of the boundary of the shadow region.

Theorem 2.5. *Let $\Lambda \subset \mathbb{R}^n$ be a strictly convex domain and $u \in \mathbb{S}^{n-1}$. Then the boundary of the shadow region generated by parallel illumination in the direction u is locally the graph of a continuous function. More precisely,*

$$(2.2) \qquad \partial S_u' \cap U' \cap (V'' \times \mathbb{R}) = \{(y'', \gamma(y'')) \; : \; y'' \in V''\}.$$

Proof. Lemma 2.4 implies the continuity of γ and that $(y'', t) \in S_u'$ for every $t > \gamma(y'')$ and $(y'', t) \in (S_u')^c$ for every $t < \gamma(y'')$, where $(y'', t) \in U' \cap (V'' \times \mathbb{R})$. $\qquad\square$

In some configurations, one may identify a modulus of continuity of the shadow boundary. To make this more precise, we utilize the notion of p-uniform convexity: let $p > 0$ and consider a domain $E \subset \mathbb{R}^k$. A locally Lipschitz function $\Psi : E \to \mathbb{R}$ is said to be *p-uniformly concave on E* if there exists $C > 0$ such that all points of differentiability $x, x' \in E \cap dom\nabla\Psi$ satisfy

$$\langle \nabla\Psi(x) - \nabla\Psi(x'), x - x' \rangle < -C|x - x'|^p,$$

where $dom\nabla\Psi$ is the domain of $\nabla\Psi$.

If the convex domain to be illuminated is p-uniformly convex with $p \geq 2$ (i.e. its boundary can be locally represented as the graph of a p-uniformly concave function), then the shadow boundary is locally Hölder continuous under mild regularity assumptions. The next theorem quantifies this statement.

Theorem 2.6. *If $\Lambda \subset \mathbb{R}^n$ is a p-uniformly convex $C^{1,\alpha}$ domain with $p \geq 2$ and $\alpha \in (0, 1]$, then $\partial S_u'$ is locally a $C^{0, \frac{\alpha}{p-1}}$ graph.*

Proof. From Theorem 2.5 it follows that $\partial S'_u$ is the graph of a continuous function γ defined on the ball V''. Therefore, it suffices to show that γ is Hölder continuous on V''. Lemma 2.4 implies that for every $y'' \in V''$, $t = \gamma(y'')$ is the only solution of the equation $\frac{\partial \phi}{\partial y_{n-1}}(y'', t) = \frac{u_n}{u_{n-1}}$, hence

$$(2.3) \qquad \frac{\partial \phi}{\partial y_{n-1}}(y'', \gamma(y'')) = \frac{u_n}{u_{n-1}},$$

(recall that ϕ generates the local chart representing $\partial \Lambda$). Since Λ is $C^{1,\alpha}$ and p-uniformly convex, ϕ is $C^{1,\alpha}$ and p-uniformly concave, i.e.

$$(2.4) \qquad |\nabla \phi(y') - \nabla \phi(z')| \leq L|y' - z'|^\alpha$$

and

$$(2.5) \qquad \langle \nabla \phi(y') - \nabla \phi(z'), y' - z' \rangle \leq -\theta |y' - z'|^p,$$

for some $L, \theta > 0$ and all $y', z' \in V'$. To prove that γ is Hölder, it suffices to show that at every point on the graph of γ, we can place a cusp with uniform opening that stays above the graph (by symmetry this implies an analogous result below the graph). We show this at just one point since the proof is identical for any other point. Without loss, we assume $0' \in \partial S'_u$ and show that a cusp can be placed at $0'$ that stays above the graph: fix a point (y'', y_{n-1}) such that $y_{n-1} > (\frac{L}{\theta})^{\frac{1}{p-1}} |y''|^{\frac{\alpha}{p-1}}$. By (2.4) we have

$$\frac{\partial \phi}{\partial y_{n-1}}(y'', y_{n-1}) \leq \frac{\partial \phi}{\partial y_{n-1}}(0, y_{n-1}) + L|y''|^\alpha.$$

On the other hand, the monotonicity formula (2.5) and the assumption $0' \in \partial S'_u$ imply

$$\frac{\partial \phi}{\partial y_{n-1}}(0, y_{n-1}) \leq \frac{\partial \phi}{\partial y_{n-1}}(0, 0) - \theta y_{n-1}^{p-1} = \frac{u_n}{u_{n-1}} - \theta y_{n-1}^{p-1}.$$

By combining the previous two inequalities, it follows that

$$\frac{\partial \phi}{\partial y_{n-1}}(y'', y_{n-1}) \leq \frac{u_n}{u_{n-1}} - \theta y_{n-1}^{p-1} + L|y''|^\alpha < \frac{u_n}{u_{n-1}},$$

which means $(y'', y_{n-1}) \in S'_u$ or equivalently $y_{n-1} > \gamma(y'')$. Hence, the epigraph of the cusp $y_{n-1} = (\frac{L}{\theta})^{\frac{1}{p-1}} |y''|^{\frac{\alpha}{p-1}}$ touches the graph of γ from above. $\qquad \square$

Remark 2.7. Note that the opening of the cusp in the proof of Theorem 2.3 is determined by $(\frac{L}{\theta})^{\frac{1}{p-1}}$.

Remark 2.8. If Λ is a uniformly (i.e. 2-uniformly) convex domain with a C^{k+1}, $k \geq 1$, smooth boundary then it is not difficult to see that $\partial S_u'$ is a C^k graph. Indeed, it suffices to show that γ is a C^k function. Since Λ is 2-uniformly convex, ϕ is uniformly concave or $D^2\phi \leq -\theta Id$, for some $\theta > 0$. In particular, $\frac{\partial^2\phi}{\partial y_{n-1}^2} \leq -\theta < 0$. Since $\gamma(y'')$ is the only solution of the equation $\frac{\partial\phi}{\partial y_{n-1}}(y'', y_{n-1}) = 0$, by the implicit function theorem, γ is as regular as $\frac{\partial\phi}{\partial y_{n-1}}$, i.e. C^k (note that in the smooth case, we may assume without loss of generality that $u_n = 0$ in (2.3)).

2.1. Theorem 2.5 is sharp

In Theorem 2.5, it was shown that for a strictly convex set, the boundary of the shadow is locally a continuous graph in any given direction. It is natural to wonder if this result extends to merely convex sets. Indeed, the following counterexample shows that this is not so: in $\mathbb{R}^3$, consider the circle

$$\{(x, y, z) : (x - 1)^2 + z^2 = 1, y = 0\}$$

and construct a cone-like set by connecting this circle to the point $(0, 1, 0)$ with line segments. It is not difficult to see that this process generates a convex body so that when it is illuminated in the direction $(0, 1, 0)$, the boundary of the resulting shadow is

$$\{(x, y, z) : (x - 1)^2 + z^2 = 1\} \cup \{(0, t, 0) : 0 \leq t \leq 1\}.$$

In particular, the boundary of the shadow is not a graph near the origin in any coordinate system, see Figure 2.1.

2.2. Theorem 2.6 is sharp

Here, we show that Theorem 2.6 is optimal in the following sense: given a direction $u \in \mathbb{S}^{n-1}$, $p \geq 2$, and $\alpha \in (0, 1]$, there exists a smooth (i.e. C^∞) convex body $\Lambda \subset \mathbb{R}^n$ for which the boundary of the shadow is not locally $C^{0, \frac{\alpha}{p-1}}$. Indeed, the key observation in the construction of the counterexample is that for a smooth, strictly convex set, $\gamma(y'')$ is the unique solution of the equation $\frac{\partial\phi}{\partial y_{n-1}}(y'', y_{n-1}) = 0$ (see (2.3)), and since we are working locally, it suffices to find a smooth, strictly convex function $\phi : \mathbb{R}^2 \to \mathbb{R}$ whose
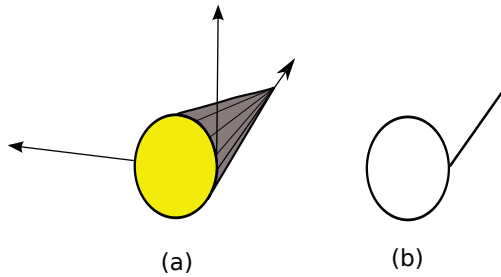
(a)   (b)

Figure 2.1: Figure (a) depicts the cone. Figure (b) depicts the shadow boundary.

level set

$$\{(x,y) : \partial_y \phi(x,y) = 0\}$$

is far from smooth. In fact, an example like this already appeared in work of Kiselman [16] in which the regularity of the projection of a three dimensional convex set onto a 2-dimensional plane is analyzed (see [16, Example 3.2]): let q be an odd natural number and set

$$\phi(x,y) = x^2 \left(4 - y + \frac{1}{2}y^2 \right) + \frac{1}{q+1}y^{q+1} - \frac{1}{q+2}y^{q+2};$$

note that ϕ is convex in the strip $|y| < \frac{1}{2}$ and

$$\partial_y \phi(x,y) = (y^q - x^2)(1 - y).$$

Thus, one may construct a smooth convex set in $\Lambda \subset \mathbb{R}^3$ whose boundary is locally given by ϕ in a neighborhood of the origin. In particular, at the local level

$$\{(x,y) : \partial_y \phi(x,y) = 0\}$$

is represented by

$$\left\{ (x,y) : y = |x|^{\frac{2}{q}} \right\},$$

and by selecting $u = (0,1,0)$ it becomes evident that illuminating Λ in the direction u generates a shadow boundary which is of class $C^{0,\frac{2}{q}}$; hence, we may pick q large enough to generate a convex body whose shadow is not $C^{0,\frac{\alpha}{p-1}}$. Note that this level set method also suggests a way of constructing shadows on the surface of convex bodies with a specified degree of regularity.

3. Regularity of shadows generated by convex projections

Let $\Omega \subset \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^n$ be two convex domains and suppose that we wish to project Ω onto Λ via the metric projection: for a given $x \in \Omega$,

$$P_\Lambda(x) := \operatorname{argmin}\{|x - z| : z \in \Lambda\} = \{z \in \overline{\Lambda} : |x - z| \le |x - w| \ \ \forall w \in \Lambda\}.$$

This operation generates a shadow region $P_\Lambda(\Omega) \cap \partial\Lambda$ on the boundary of Λ. The purpose of this section is to study the regularity of this shadow. In other words, given $z_0 \in \partial(P_\Lambda(\Omega) \cap \partial\Lambda)$, we wish to understand how smooth $\partial(P_\Lambda(\Omega) \cap \partial\Lambda)$ is in a neighborhood of z_0. For the convenience of the reader, we first collect several properties of the metric projection.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^n$ be two convex domains such that $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$ and Ω is bounded. Then the following holds.*

(a) If $x \in \Omega$, then $P_\Lambda(x) \in \partial\Lambda$ is single-valued and for any $z \in P_\Lambda(\Omega)$ there exists $x \in \partial\Omega \cap P_\Lambda^{-1}(z)$.

(b) $P_\Lambda(\Omega)$ is bounded.

(c) For $x \ne z$, $z \in P_\Lambda(x)$ if and only if $x - z$ is an outward normal vector for Λ at z.

(d) If $x \in \partial\Omega$, ν is an outward unit normal to Ω at x, and $x - P_\Lambda(x) \perp \nu$, then $P_\Lambda(x) \in \partial P_\Lambda(\Omega)$.

(e) If $\partial\Lambda$ is C^1 and $z \in \partial P_\Lambda(\Omega)$, then there exists $x \in \partial\Omega$ and an outward normal ν to Ω at x such that $z = P_\Lambda(x)$ and $x - z \perp \nu$.

Proof. (a) Firstly, by considering a minimizing sequence $|x - z_k| \to |x - P_\Lambda(x)|$, it follows that $\{z_k\}$ is bounded and so there exists $z \in \overline{\Lambda}$, such that $z \in P_\Lambda(x)$. If $z \in \Lambda$, then $z_t = z + t(x - z) \in \Lambda$ for $t > 0$ small and since $|x - z_t| = (1 - t)|x - z|$, this produces a contradiction. Thus, $z \in \partial\Lambda$. Next if there exist $x_1 \ne x_2 \in P_\Lambda(x)$, then the convexity of Λ, the disjointness assumption, and the fact that $|x - x_1| = |x - x_2|$, imply that x is not on the line generated by the direction $x_1 - x_2$. With this in mind, let $w := \frac{1}{2}(x_1 + x_2) \in \Lambda$ and note that $|w - x| < |x - x_1|$, a contradiction. Now let $z = P_\Lambda(x)$ for some $x \in \Omega$ and set $x_t := tx + (1 - t)z$, $t \in [0, 1]$. Since $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, it follows that $x_t \in \partial\Omega$ for some $t \in (0, 1)$. Suppose that there exists $w \in \Lambda$ such

that $|x_t - w| < |x_t - z|$. Note

$$|x_t - z| = t|x - z| \leq t|x - w| \leq t|x - x_t| + t|x_t - w|$$
$$< t(1 - t)|x - z| + t|x_t - z| = |x_t - z|,$$

a contradiction. Thus $z = P_\Lambda(x_t)$.

(b) Let $z_0 \in \Lambda$ and pick $R > 0$ such that $\Omega \subset B_R(0)$. Then for all $x \in \Omega$,

$$|P_\Lambda(x)| \leq 2|x| + |z_0| \leq 2R + |z_0|.$$

(c) Without loss, assume $z = 0$. If there exists $w \in \Lambda$ such that $\langle w, x \rangle > 0$, set $w_t := tw \in \Lambda$. Pick $t > 0$ small such that $t|w|^2 < 2\langle x, w \rangle$. This implies $|x - w_t|^2 < |x|^2$, a contradiction. Thus, for all $w \in \Lambda$, $\langle w, x \rangle \leq 0$ and hence x is an outward normal for Λ at 0. Conversely, if $\langle w, x \rangle \leq 0$ for all $w \in \Lambda$, $|x|^2 = x \cdot (x - w) + x \cdot w \leq |x||x - w|$ and thus $|x| \leq |x - w|$ for all $w \in \Lambda$, i.e. $P_\Lambda(x) = 0$.

(d) Without loss, assume $P_\Lambda(x) = 0$. There exists a concave function ϕ whose graph locally represents $\partial \Lambda$. Assume the positive e_n direction is given by $x - P_\Lambda(x)$ so that in particular $x = |x|e_n$ and $0' \in \partial^+\phi(0')$. Note that $\nu = (\nu', 0)$ in these coordinates. Fix $t > 0$ small, $w'_t \in \partial^+\phi(t\nu')$, and note that the ray

$$\{(t\nu', \phi(t\nu')) + \lambda(-w'_t, 1) : \lambda > 0\}$$

is outward-pointing in the direction of the normal to Λ at the origin. Monotonicity implies $\langle \nu', w'_t \rangle \leq 0$, so for any $\lambda > 0$,

$$\langle (t\nu', \phi(t\nu')) + \lambda(-w'_t, 1) - x, \nu \rangle = \langle t\nu' - \lambda w'_t, \nu' \rangle \geq t > 0.$$

In particular, the above ray does not intersect Ω and since w'_t is an arbitrary supergradient, (c) implies $(t\nu', \phi(t\nu')) \notin P_\Lambda(\Omega)$ for any $t > 0$ sufficiently small. Thus $P_\Lambda(x)$, being a limit of points in $\partial \Lambda \setminus P_\Lambda(\Omega)$, is a boundary point.

(e) Consider a sequence $z_k \in P_\Lambda(\Omega)$ such that $z_k \to z$. The disjointness assumption and (c) yield $z_k = P_\Lambda(x_k)$, $x_k \neq z_k$, $x_k - z_k$ is an outward normal to Λ, and thanks to (a) we may assume without loss that $x_k \in \partial \Omega$. Thus up to a subsequence, $x_k \to x_0 \in \partial \Omega$ and $x_0 - z$ is parallel to the outward normal to Λ at z so (c) implies $z = P_\Lambda(x_0)$. Next consider $\tilde{z}_k \notin P_\Lambda(\Omega)$ such that $\tilde{z}_k \to z$. The C^1 regularity assumption implies that the outward unit normal vectors to Λ at the points $\tilde{z}_k$ converge to the outward unit normal

at z. Thus the ray in the outward normal direction at z intersects Ω only along its boundary (otherwise some $\tilde{z}_k \in P_\Lambda(\Omega)$) and x can be chosen to be any of these intersection points (e.g. x_0 above). The existence of the claimed outward normal follows by applying [20, Theorem 11.2] with $C = \Omega$. $\square$

3.1. Weak case

A set $E \subset \mathbb{R}^m$ is said to be *finitely $(m-1)$-rectifiable* if there are finitely many Lipschitz maps $f_i : A_i \subset \mathbb{R}^{m-1} \to \mathbb{R}^m$ such that $E \subset \bigcup_{i=1}^k f_i(A_i)$ for some $k \in \mathbb{N}$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded strictly convex domain and $\Lambda \subset \mathbb{R}^n$ a convex domain whose boundary is $C^{1,1}$. If $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, then $\partial P_\Lambda(\Omega)$ is finitely $(n-2)$-rectifiable.*

Proof. Consider an arbitrary $y \in \partial P_\Lambda(\Omega)$, and let $\phi : \mathbb{T}_y(\partial\Lambda) \to \mathbb{R}$ be a $C^{1,1}$ concave function whose graph locally represents $\partial\Lambda$ in a neighborhood B_{r_y} around y so that $\nabla\phi(y) = 0$. In this way, $\mathbb{T}_y(\partial\Lambda)$ is identified with $\mathbb{R}^{n-1}$. By translating the coordinate system, if necessary, we may also assume $y = 0$. Lemma 3.1 (c) & (e) imply that the half-line L at the origin in the direction of the unit outward normal to Λ at 0, $N_\Lambda(0)$, touches Ω tangentially at some point x and $\langle \nu, N_\Lambda(0) \rangle = 0$ for some unit outward normal ν to Ω at x. It follows that ν lives on the tangent space of Λ at 0. Let $e_{n-1} := \nu$ and $\{e_1, \dots, e_{n-1}\}$ be a basis for $\mathbb{R}^{n-1}$; set

$$t^* := \max_{x \in \overline{\Omega}, z \in P_\Lambda(\Omega)} |x - z|$$

($t^* < \infty$ by Lemma 3.1 (b)) and

$$\Psi(z') := \Psi(z'', z_{n-1}) = \phi(z') - \frac{1}{2t^*}|z|^2.$$

Note that Ψ is $C^{1,1}$ and 2-uniformly concave, so by Theorem 2.6, it follows that locally around the origin, the level set

$$\{(z'', z_{n-1}) : 0 = \partial_{z_{n-1}} \Psi(z'', z_{n-1})\},$$

is a Lipschitz graph which will be denoted by $\tilde{\gamma}(z'') = z_{n-1}(z'')$ (see (2.3)). Now let $\gamma(z'') := \max\{\tilde{\gamma}(z''), 0\}$, and note that γ is Lipschitz. We claim that

locally around the origin,

$$(3.1) \qquad \Phi^{-1}(P_\Lambda(\Omega)) \subset \{(z'', z_{n-1}) : z_{n-1} \le \gamma(z'')\},$$

where $\Phi(w') := (w', \phi(w'))$. Indeed, let

$$z' := (z'', z_{n-1}) \in \Phi^{-1}(P_\Lambda(\Omega)) \setminus \{0\};$$

if $z_{n-1} \le 0$, then since $\gamma \ge 0$, the result follows. So without loss assume $z_{n-1} > 0$. Since $z' \in \Phi^{-1}(P_\Lambda(\Omega))$ and $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, Lemma 3.1 (c) implies the existence of $t(z') > 0$ such that

$$\Phi(z') + t(z')N_\Lambda(\Phi(z')) \in \partial\Omega$$

and since the ray at $\Phi(z')$ in the direction of $N_\Lambda(\Phi(z'))$ collapses to the same point via the metric projection (see e.g. [22, Lemma 1.2.2]), it follows that

$$\Phi(z') + t(z')N_\Lambda(\Phi(z')) \ne x.$$

Next, consider

$$P_{\mathbb{R}^{n-1}}(\Phi(z') + t(z')N_\Lambda(\Phi(z'))) \in \mathbb{R}^{n-1}$$

and note that the e_{n-1} component of this point is negative (since Ω is strictly convex and e_{n-1} is one of its outer normal vectors). In other words, $z_{n-1} - t(z')\frac{\partial_{z_{n-1}}\phi(z')}{\sqrt{1+|\nabla\phi(z')|^2}} < 0$. Thus, $\partial_{z_{n-1}}\phi(z') > 0$ (recall $z_{n-1} > 0$) and since $t(z) \le t^*$, it follows that

$$(3.2) \quad \partial_{z_{n-1}}\Psi(z'', \tilde\gamma(z'')) = 0 < \frac{\partial_{z_{n-1}}\phi(z')}{\sqrt{1+|\nabla\phi(z')|^2}} - \frac{1}{t^*}z_{n-1}$$
$$< \partial_{z_{n-1}}\phi(z') - \frac{1}{t^*}z_{n-1} = \partial_{z_{n-1}}\Psi(z'', z_{n-1});$$

now assume by contradiction that $z_{n-1} > \gamma(z'')$. In particular, $z_{n-1} > \tilde\gamma(z'')$ so by monotonicity,

$$\langle \nabla\Psi(z'', z_{n-1}) - \nabla\Psi(z'', \tilde\gamma(z'')), (0, z_{n-1} - \tilde\gamma(z'')) \rangle \le 0.$$

Thus,

$$\partial_{z_{n-1}}\Psi(z'', z_{n-1}) \le \partial_{z_{n-1}}\Psi(z'', \tilde\gamma(z'')) = 0,$$

and this contradicts (3.2) and proves the claim (i.e. (3.1)). Next, note that

$$0 \in \Phi^{-1}(P_\Lambda(\Omega)) \cap \{(z'', z_{n-1}) : z_{n-1} \leq \gamma(z'')\},$$

and since γ is Lipschitz, (3.1) implies that we can place a cone with vertex at $0'$ and oriented in the positive e_{n-1} direction so that it lies in

$$\mathbb{R}^{n-1} \setminus \Phi^{-1}(P_\Lambda(\Omega)).$$

The opening of the cone depends on the Lipschitz constant of $\nabla\Psi$ and the uniform convexity constant of $-\Psi$; in particular, it depends on t_y^*. However, since $P_\Lambda(\Omega)$ is bounded (recall that Ω is bounded) and the domains have disjoint closures, it follows that t_y^* has a uniform positive lower bound. The existence of this cone implies the claim within the proof of [15, Proposition 4.1]. Indeed, this is the only part where Indrei used the uniform convexity of Λ, which we were able to replace with strict convexity of Ω in our proof above; thus, the rest of the proof follows exactly as [15, Proposition 4.1] (the idea is that once we have a cone at a point, we can use the $C^{1,1}$ regularity to transition between charts to get a cone at every point of $\partial P_\Lambda(\Omega)$; nevertheless, the cones may be oriented in different directions, but this readily implies rectifiability via a covering argument). □

Remark 3.2. The disjointness assumption in Theorem 3.1 is necessary: indeed, consider a Cantor set $\mathcal{C}$ on $[1, 2]$ and let g be a smooth function whose zero level set is $\mathcal{C}$. For $\epsilon > 0$ small, it follows that $f(x) := x^2 + \epsilon g(x)$ is convex, so its epigraph is a convex set in $\mathbb{R}^2$. Moreover, consider the epigraph of the function $h(x) := x^2$; of course, it is likewise convex. Now it is not difficult to see that using these epigraphs, one may obtain two bounded convex sets, say Ω and Λ, with the property that their boundaries intersect on the image of $\mathcal{C}$ under h. In this case, $\partial(P_\Lambda(\Omega) \cap \partial\Lambda)$ does not have finite $\mathcal{H}^0$ measure. Nevertheless, in the general case one may still prove a local version of Theorem 3.1 away from $\partial(\partial(\Omega \cap \Lambda) \cap \partial\Lambda)$.

3.2. Smooth case

In Theorem 3.1, we utilized a geometric method for investigating the regularity of shadow boundaries generated by orthogonal projections. In what follows, we develop an analytical approach to attack this problem. The idea is to represent the unknown boundary as the level set of a function defined in terms of local charts. However, since the differential of this function contains the information regarding the regularity of the level set, we need to

ensure that this function is smooth enough; this leads us to impose higher regularity on the domains.

Theorem 3.3. *Suppose $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are C^{k+1}, $k \geq 1$, convex domains separated by a hyperplane with Ω bounded and 2-uniformly convex. If $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, then $\partial P_\Lambda(\Omega)$ is locally an $(n-2)$-dimensional C^k graph.*

Proof. Given a point $y \in \partial \Lambda$ let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be the C^{k+1} concave function which represents $\partial \Lambda$ locally around y. Likewise, for $x \in \partial \Omega$ let g denote the C^{k+1} uniformly concave function locally representing Ω around x. Set

$$F(y_1, \ldots, y_n) = y_n - f(y_1, \ldots, y_{n-1}),$$
$$G(x_1, \ldots, x_n) = x_n - g(x_1, \ldots, x_{n-1})$$

and consider the function

$$\phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+3}$$

given by

$$\phi(x, y, t) := (G(x), F(y), \nabla G(x) \cdot \nabla F(y), y + t\nabla F(y) - x).$$

Lemma 3.1 implies that $\partial P_\Lambda(\Omega)$ is locally the y-projection of $\phi^{-1}(0, 0, 0, 0)$: indeed, let $(x, y, t) \in \phi^{-1}(0, 0, 0, 0)$; then, $y \in \{F(y) = 0\}$ and since this set locally describes the boundary of Λ, it follows that $y \in \partial \Lambda$; likewise $x \in \{G(x) = 0\}$ implies $x \in \partial \Omega$. Then, since $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$ and $x - y = t\nabla F(y)$, it follows that $t \neq 0$, and as $\nabla F(y)$ is an outward normal for Λ at y, $t > 0$ and Lemma 3.1 (c) implies $y = P_\Lambda(x)$; next, $\nabla G(x) \cdot \nabla F(y) = 0$ and part (d) of the same lemma yield $y \in \partial P_\Lambda(\Omega)$ (since $\nabla G(x)$ is an outward normal to Ω at x). Conversely, based on (c) and (e) of Lemma 3.1, the set $\partial P_\Lambda(\Omega)$ is locally a subset of the y-projection of $\phi^{-1}(0, 0, 0, 0)$. With this in mind, our strategy is to prove that $\phi^{-1}(0, 0, 0, 0)$ is locally an $(n-2)$-dimensional C^k graph where the $(n-2)$ parameters are taken from the y variables. The first step is to compute the differential of ϕ and show that it has full rank $n+3$: let

$$\phi_1(x, y, t) := G(x)$$
$$\phi_2(x, y, t) := F(y)$$
$$\phi_3(x, y, t) := \nabla G(x) \cdot \nabla F(y)$$
$$\Phi(x, y, t) := [\phi_4(x, y, t), \ldots, \phi_{n+3}(x, y, t)]^T := y + t\nabla F(y) - x.$$

Thus,

$$\nabla_x \phi_1 = \nabla G(x)$$
$$\nabla_y \phi_1 = 0$$
$$\partial_t \phi_1 = 0$$
$$\nabla_x \phi_2 = 0$$
$$\nabla_y \phi_2 = \nabla F(y)$$
$$\partial_t \phi_2 = 0$$
$$\nabla_x \phi_3 = D^2 G(x) \nabla F(y)$$
$$\nabla_y \phi_3 = D^2 F(y) \nabla G(x)$$
$$\partial_t \phi_3 = 0$$
$$D_x \Phi = -Id \in \mathbb{R}^{n \times n}$$
$$D_y \Phi = Id + t D^2 F(y) \in \mathbb{R}^{n \times n}$$
$$D_t \Phi = \nabla F(y).$$

Therefore,

$$D\phi(x, y, t) = \begin{bmatrix} \nabla G(x)^T & 0 & 0 \\ 0 & \nabla F(y)^T & 0 \\ (D^2 G(x) \nabla F(y))^T & (D^2 F(y) \nabla G(x))^T & 0 \\ -Id & Id + t D^2 F(y) & \nabla F(y) \end{bmatrix}$$

(note that this is an $(n+3) \times (2n+1)$ matrix). Next we show $\ker(D\phi)^T = \{0\}$ at points $(x, y, t) \in \phi^{-1}(0, 0, 0, 0)$. Indeed, let

$$(\alpha_1, \alpha_2, \alpha_3, v) \in \ker(D\Phi)^T,$$

and note that since

$$D\phi(x, y, t)^T = \begin{bmatrix} \nabla G(x) & 0 & D^2 G(x) \nabla F(y) & -Id \\ 0 & \nabla F(y) & D^2 F(y) \nabla G(x) & Id + t D^2 F(y) \\ 0 & 0 & 0 & \nabla F^T(y) \end{bmatrix},$$

we have

(3.3) $0 = \alpha_1 \nabla G(x) + \alpha_3 D^2 G(x) \nabla F(y) - v;$

(3.4) $0 = \alpha_2 \nabla F(y) + \alpha_3 D^2 F(y) \nabla G(x) + v + t D^2 F(y) v;$

(3.5) $0 = \nabla F(y) \cdot v.$

In particular,

$$0 = \nabla F(y) \cdot v$$
$$= \nabla F(y) \cdot (\alpha_1 \nabla G(x) + \alpha_3 D^2 G(x) \nabla F(y))$$
$$= \alpha_1 \nabla F(y) \cdot \nabla G(x) + \alpha_3 \nabla F^T(y) D^2 G(x) \nabla F(y)$$
$$= \alpha_3 \nabla F^T(y) D^2 G(x) \nabla F(y),$$

(note $\nabla F(y) \cdot \nabla G(x) = 0$ since $(x, y, t) \in \phi^{-1}(0, 0, 0, 0)$). Since G is uniformly convex, it follows that $\alpha_3 = 0$ and so (3.3) implies

$$v = \alpha_1 \nabla G(x);$$

plugging this information into (3.4) and taking a dot product with $\nabla G(x)$ yields

$$0 = \alpha_1 \left(t \nabla G(x)^T D^2 F(y) \nabla G(x) + |\nabla G(x)|^2 \right).$$

Since $|\nabla G(x)| > 0$, and F is convex, it follows that $\alpha_1 = 0$ which readily implies $v = 0$ and so $\alpha_2 = 0$. Thus, we proved $\ker(D\phi^T) = \{0\}$; in particular, $\text{rank}(D\phi) = n + 3$ for each point of interest (x, y, t). Now for $(x, y, t) \in \phi^{-1}(0, 0, 0, 0)$ consider the $(n + 3) \times (n + 1)$ matrix

$$D_{xt}\phi := \begin{bmatrix} \nabla G(x)^T & 0 \\ 0 & 0 \\ (D^2 G(x) \nabla F(y))^T & 0 \\ -Id & \nabla F(y) \end{bmatrix}.$$

Let $(w, \beta) \in \ker(D_{xt}\phi)$. Then,

$$0 = D^2 G(x) \nabla F(y) \cdot w;$$
$$0 = -w + \beta \nabla F(y),$$

so uniform convexity of G readily yields $\beta = |w| = 0$. Thus $D_{xt}\phi$ has rank $n + 1$. Let

$$\{v_1, \ldots, v_{2n+1}\}$$

denote the columns of $D\phi$, i.e., $v_i = D_{x_i}\phi$ for $1 \le i \le n$; $v_i = D_{y_{i-n}}\phi$ for $n + 1 \le i \le 2n$ and $v_{2n+1} = D_t\phi$. Since $D_{xt}\phi$ has rank $n + 1$, it follows that the set $\{v_1, \ldots, v_n, v_{2n+1}\}$ consists of linearly independent vectors. If for all $j \in \{n + 1, \ldots, 2n\}$, v_j is in the span of $\{v_1, \ldots, v_n, v_{2n+1}\}$, then the rank of $D\phi$ would be $n + 1$, a contradiction (we know $\text{rank}(D\phi) = n + 3$). Thus, there exists some $j \in \{n + 1, \ldots, 2n\}$ so that $\{v_1, \ldots, v_n, v_{2n+1}, v_j\}$ consists

of linearly independent vectors and repeating this rationale it follows that there exists $j \neq j' \in \{n+1, \ldots, 2n\}$ so that $\{v_1, \ldots, v_n, v_{2n+1}, v_j, v_{j'}\}$ consists of $n+3$ linearly independent vectors. [1] Thus $\det D_{xy_{j-n}y_{j'-n}t}\phi \neq 0$, and by the implicit function theorem it follows that $\phi^{-1}(0,0,0,0)$ is locally an $(n-2)$-dimensional C^k graph where the $(n-2)$ parameters are $\{y_1, y_2, \ldots, y_n\} \setminus \{y_{j-n}, y_{j'-n}\}$; hence, the y-projection of $\phi^{-1}(0,0,0,0)$ is locally an $(n-2)$-dimensional C^k graph. $\square$

Remark 3.4. The disjointness assumption in Theorem 3.3 is necessary, cf. Remark 3.2.

4. The singular set associated to a Monge-Ampère equation

In this section, a connection is established between the illumination shadow, the projection shadow, and the singular set associated to a Monge-Ampère equation arising in mass transfer theory. More precisely, we apply the results of the previous sections to improve a result of Indrei [15] (see §1.3 for a description of the optimal partial transport problem and relevant notation).

4.1. The structure of the singular set

In order to analyze the singular set for the free boundaries, we recall two sets which play a crucial role in the subsequent analysis; cf. [15, Equations (2.2) and (2.3)]. The nonconvex part of the free boundary $\overline{\partial U_m \cap \Omega}$ is the closed set

$$(4.1) \quad \partial_{nc}U_m := \{x \in \overline{\Omega \cap U_m} : \Omega \cap U_m \text{ fails to be locally convex at } x\},$$

(a set E fails to be locally convex at x if $E \cap B_r(x)$ is not convex for all $r > 0$). Moreover, the nontransverse intersection points are defined by

$$(4.2) \quad \partial_{nt}\Omega := \{x \in \partial\Omega \cap \overline{\Omega \cap \partial U_m} : \langle \nabla\tilde{\Psi}_m(x) - x, z - x \rangle \leq 0 \ \ \forall z \in \Omega\},$$

where $\tilde{\Psi}_m$ is the extension of Ψ_m given by [10, Theorem 4.10]. By duality, $\partial_{nc}V_m$ and $\partial_{nt}\Lambda$ are similarly defined. Now, for $x \in \partial(\Omega \cap U_m)$, if

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[1]In fact, by explicit computation it follows that $v_{2n} = (0, 1, 0, e_n)$ where $e_n = (0, \ldots, 1)$ so that either v_j or $v_{j'}$ must be v_{2n} since it is linearly independent from the columns of $D_{xt}\phi$.

$x \neq \nabla\tilde{\Psi}_m(x)$ (e.g. if Ω and Λ have disjoint closurers), let

$$L(x) := \left\{ \nabla\tilde{\Psi}_m(x) + \frac{x - \nabla\tilde{\Psi}_m(x)}{|x - \nabla\tilde{\Psi}_m(x)|} t : t \geq 0 \right\};$$

$$K := \left\{ x \in \partial(\Omega \cap U_m) : L(x) \cap \overline{\Omega \cap U_m} \subset \partial(\overline{\Omega \cap U_m}) \right\};$$

$$S_1 := \nabla\tilde{\Psi}_m^{-1}(\partial_{nt}\Lambda) \cap K;$$

$$A_1 := S_1 \cap \partial U_m;$$

$$A_2 := S_1 \setminus \partial U_m.$$

If Ω and Λ are strictly convex and have disjoint closures, the singular points of the free boundary $\overline{\partial V_m \cap \Lambda}$ are contained in the set

$$S = (\nabla\tilde{\Psi}_m(\partial_{nc}U_m) \cup \nabla\tilde{\Psi}_m(S_1)) \cap \partial V_m \cap \partial\Lambda$$
$$= (\nabla\tilde{\Psi}_m(\partial_{nc}U_m) \cap \partial V_m \cap \partial\Lambda) \cup (\nabla\tilde{\Psi}_m(S_1) \cap \partial V_m \cap \partial\Lambda)$$
$$= (\nabla\tilde{\Psi}_m(\partial_{nc}U_m) \cap \partial V_m \cap \partial\Lambda) \cup \nabla\tilde{\Psi}_m(A_1) \cup \nabla\tilde{\Psi}_m(A_2),$$

see [15, Corollary 3.13 and Remark 3.11; cf. Theorem 4.9]. The next lemma describes the first two sets appearing in S.

Lemma 4.1. *Assume $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are strictly convex bounded domains with disjoint closures. Then*

$$(\nabla\tilde{\Psi}_m(\partial_{nc}U_m) \cap \partial V_m \cap \partial\Lambda) \cup \nabla\tilde{\Psi}_m(A_1)$$

is $\mathcal{H}^{n-2}$ σ-finite. Moreover, if Ω is C^1, then

$$\mathcal{H}^{n-2}((\nabla\tilde{\Psi}_m(\partial_{nc}U_m) \cap \partial V_m \cap \partial\Lambda) \cup \nabla\tilde{\Psi}_m(A_1)) < \infty.$$

Proof. Firstly, $\nabla\tilde{\Psi}_m$ is a homeomorphism between $\overline{U_m \cap \Omega}$ and $\overline{V_m \cap \Lambda}$ by [10, Theorem 4.10]. Thus, for $y \in (\nabla\tilde{\Psi}_m(\partial_{nc}U_m) \cap \partial V_m \cap \partial\Lambda)$ set $x := \nabla\tilde{\Psi}_m^*(y)$; since Ω is convex and $x \in \partial_{nc}U_m$, it follows that $x \notin \Omega \setminus \partial U_m$. Moreover, since free boundary never maps to free boundary (see e.g. [15, Proposition 2.15]), we also have $x \notin \partial U_m \cap \Omega$, which implies $x \in \partial U_m \cap \partial\Omega$. Therefore,

$$(\nabla\tilde{\Psi}_m(\partial_{nc}U_m) \cap \partial V_m \cap \partial\Lambda) \subset \nabla\tilde{\Psi}_m(\partial U_m \cap \partial\Omega) \cap \partial V_m \cap \partial\Lambda.$$

An application of [15, Proposition 4.8] yields that

$$\nabla\tilde{\Psi}_m(\partial_{nc}U_m) \cap \partial V_m \cap \partial\Lambda$$

is $\mathcal{H}^{n-2}$ - finite; the fact that $\nabla\tilde{\Psi}_m(A_1)$ is $\mathcal{H}^{n-2}$ σ-finite ($\mathcal{H}^{n-2}$ finite if Ω is C^1) follows from [15, Corollary 4.6]. $\square$

In the following lemma, we establish a connection between the singular set S and the boundary of the projection of Ω onto Λ studied in §3.

Lemma 4.2. *Assume $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are strictly convex bounded domains with disjoint closures. Then*

$$(4.3) \qquad\qquad\qquad \nabla\tilde{\Psi}_m(A_2) \subset \partial P_\Lambda(\Omega).$$

Proof. Let $y := \nabla\tilde{\Psi}_m(x) \in \nabla\tilde{\Psi}_m(A_2)$, $L_t := \nabla\tilde{\Psi}_m(x) + \frac{x-\nabla\tilde{\Psi}_m(x)}{|x-\nabla\tilde{\Psi}_m(x)|}t$ and note that the half-line $\{L_t\}_{t\geq 0}$ is tangent to the active region $U_m \cap \Omega$ due to the fact that $x \in A_2 \subset K$. Since $x \in \partial\Omega \setminus \partial U_m$, it follows that L_t is tangent to Ω at x; hence, [20, Theorem 11.2] gives the existence of an outward normal to Ω at x, perpendicular to L_t so that L_t is a subset of a tangent space to Ω at x, $\mathbb{T}_x\partial\Omega$. Let $z = P_\Lambda(x) \in \partial\Lambda$ (recall that P_Λ is the orthogonal projection operator). Then by Lemma 3.1 (c), $x - z$ is parallel to some unit outward normal $N_\Lambda(z)$ to Λ at z. Since $x \in S_1$, it follows that $\nabla\tilde{\Psi}_m(x) \in \partial_{nt}\Lambda$; in particular, $x - \nabla\tilde{\Psi}_m(x)$ is parallel to some outward normal to Λ at $\nabla\tilde{\Psi}_m(x)$, say $N_\Lambda(\nabla\tilde{\Psi}_m(x))$. Thus, invoking Lemma 3.1 (a) & (c), it follows that $z = y$. Combining $\{L_t\}_{t\geq 0} \subset \mathbb{T}_x\partial\Omega$ and $y = P_\Lambda(x)$ yields $y \in \partial P_\Lambda(\Omega)$ by Lemma 3.1 (d). $\square$

Lemmas 4.1 & 4.2 imply that the singular set S is contained in the union of an $\mathcal{H}^{n-2}$ σ-finite set and $\partial P_\Lambda(\Omega)$ under a strict convexity and disjointness assumption on the domains. Thus, a way to obtain bounds on the Hausdorff dimension of the singular set is by studying the Hausdorff dimension of $\partial P_\Lambda(\Omega)$. In [15, Proposition 4.1], Indrei shows that if Ω is a bounded convex domain and Λ is uniformly convex, bounded, and $C^{1,1}$ smooth, then $P_\Lambda(\Omega) \cap \partial\Lambda$ is $(n-2)$-rectifiable away from $\partial(\partial(\Omega \cap \Lambda) \cap \partial\Lambda)$; in particular, if the domains have disjoint closures, then

$$\mathcal{H}^{n-2}(\partial P_\Lambda(\Omega)) < \infty.$$

The proof of [15, Proposition 4.1] is technical but relies on a simple idea which we describe in the language developed in this paper in order to further highlight the connection with shadows: let $y \in \partial P_\Lambda(\Omega)$ and $x \in \partial\Omega$ be such that $y = P_\Lambda(x)$. Then $\partial P_\Lambda(\Omega) \subset \partial\Lambda \setminus S_{N_\Omega(x)}$, where $N_\Omega(x)$ is some outward normal of Λ at the point x and $S_{N_\Omega(x)}$ is the shadow from §2

(see also Lemma 3.1 (e)). In other words, $P_\Lambda(\Omega)$ is trapped in the illuminated portion of $\partial\Lambda$ under parallel illumination in the direction $N_\Omega(x)$. Since $y \in \partial P_\Lambda(\Omega) \cap \partial S_{N_\Omega(x)}$, it follows that $\partial S_{N_\Omega(x)}$ acts as a one-sided support for $\partial P_\Lambda(\Omega)$ locally around y. Therefore, if one can place a cone in the shadow portion $S_{N_\Omega(x)}$, a compactness argument would yield the desired rectifiability result. Indeed, this is where the uniform convexity and $C^{1,1}$ assumptions come into play in the proof of [15, Proposition 4.1]. However, the results of §2 shed new light on the regularity and uniform convexity assumptions; more specifically, they imply that by this method, the uniform convexity assumption is necessary to obtain the desired cone and the $C^{1,1}$ regularity assumption is irrelevant: indeed, §2.2 shows that there exists a C^∞ strictly convex set whose shadow is Hölder, but not Lipschitz (with arbitrarily small Hölder exponent). In particular, this discussion shows that one may not hope to remove the uniform convexity assumption by the same method (i.e. by using $\partial S_{N_\Omega(x)}$ as a support function).

However, Theorem 3.1 implies that one may obtain the cone without a uniform convexity assumption; this is achieved by cooking up a new type of support function related to the distance between the two sets. Moreover, Theorem 3.3 yields a higher regularity result. Thus, Theorem 3.1, Theorem 3.3, Lemma 4.1, Lemma 4.2, and the definition of S imply the following theorem.

Theorem 4.3. *Assume $\Omega \subset \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^n$ are bounded strictly convex domains and that Λ has a $C^{1,1}$ boundary. If $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, then the free boundary $\overline{\partial V_m \cap \Lambda}$ is a $C^{1,\alpha}_{loc}$ hypersurface away from the compact, $\mathcal{H}^{n-2}$ σ-finite set:*

$$S := (\nabla\tilde{\Psi}_m(\partial_{nc}U_m) \cap \partial V_m \cap \partial\Lambda) \cup \nabla\tilde{\Psi}_m(A_1) \cup \nabla\tilde{\Psi}_m(A_2).$$

If Ω has a C^1 boundary, then S is $\mathcal{H}^{n-2}$ finite. Moreover, if Ω and Λ are C^{k+1}, $k \geq 1$, and Ω is 2-uniformly convex, then $\nabla\tilde{\Psi}_m(A_2)$ is contained on an $(n-2)$-dimensional C^k_{loc} graph.

Remark 4.4. By duality and symmetry, an analogous statement holds for $\overline{\partial U_m \cap \Omega}$.

Remark 4.5. In the case when Ω and Λ overlap, a careful analysis may be conducted to obtain regularity of the free boundary away from $\overline{\Omega \cap \Lambda}$. The idea is as follows: a result of Figalli [10] asserts that the free boundary does not intersect $\Omega \cap \Lambda$ and he shows that away from $\overline{\Omega \cap \Lambda}$, the free boundary is locally C^1. Starting from this result, Indrei localized the method utilized by Caffarelli and McCann [6] in the disjoint case to deduce local $C^{1,\alpha}$ regularity

away from $\overline{\Omega \cap \Lambda}$. Even in the disjoint case, the method of Caffarelli and McCann may fail at points where the free boundary of say Λ intersects the fixed boundary $\partial \Lambda$. A method was developed in [15, §4] to estimate the Hausdorff dimension of such points. By utilizing the results developed in this paper, one may employ the strategy in [15, §4] to obtain a result analogous to Theorem 4.3 in the case when there is overlap (cf. [15, Proposition 4.1, Theorem 4.9]).

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