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Invariant Einstein metrics on three-locally-symmetric spaces

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In this paper, we classify three-locally-symmetric spaces for a connected, compact and simple Lie group. Furthermore, we study invariant Einstein metrics on these spaces.

1. Introduction

A Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is called Einstein if the Ricci tensor Ric of the metric $\langle\cdot,\cdot\rangle$ satisfies $\mathrm{Ric} = c\langle\cdot,\cdot\rangle$ for some constant c. The above Einstein equation reduces to a system of nonlinear second-order partial differential equations. But it is difficult to get general existence results. Under the assumption that M is a homogeneous Riemannian manifold, the Einstein equation reduces to a more manageable system of (nonlinear) polynomial equations, which in some cases can be solved explicitly. There is a lot of progress in the study on invariant Einstein metrics of homogeneous manifolds, such as the articles [4–6, 10–12, 17–19, 21, 22, 24–28, 31–33, 35, 36], and the survey article [30] and so on.

Consider a homogeneous compact space G/H with a semisimple connected Lie group G and a connected Lie subgroup H. Denote by $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G, H respectively. Assume that $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to B, where B is the Killing from of $\mathfrak{g}$. Every G-invariant metric on G/H generates an $\mathrm{ad}\mathfrak{h}$-invariant inner product on $\mathfrak{p}$ and vice versa [9]. This makes it possible to identify invariant Riemannian metrics on G/H with $\mathrm{ad}\mathfrak{h}$-invariant inner products on $\mathfrak{p}$. Note that the metric generated by the inner product $-B|_{\mathfrak{p}}$ is called standard. Furthermore, if G acts almost effectively on the homogeneous space G/H, and $\mathfrak{p}$ is the

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direct sum of three ad$\mathfrak{h}$-invariant irreducible modules pairwise orthogonal with respect to B, i.e.

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3,$$

with $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for any $i \in \{1, 2, 3\}$, then G/H is called a three-locally-symmetric space.

The notation of a three-locally-symmetric space is introduced by Nikonorov in [28]. There have been a lot of studies on invariant Einstein metrics for certain three-locally-symmetric spaces. For example, invariant Einstein metrics on the flag manifold $SU(3)/T_{max}$ are given in [16], on

$$Sp(3)/Sp(1) \times Sp(1) \times Sp(1) \text{ and } F_4/Spin(8)$$

are obtained in [32], on the Kähler C-spaces

$$SU(n_1 + n_2 + n_3)/S(U(n_1) \times U(n_2) \times U(n_3)),$$
$$SO(2n)/U(1) \times U(n-1),$$
$$E_6/U(1) \times U(1) \times Spin(8)$$

are classified in [22], another approach to

$$SU(n_1 + n_2 + n_3)/S(U(n_1) \times U(n_2) \times U(n_3))$$

is given in [5]. The existence is proved in [28] of at least one invariant Einstein metric for every three-locally-symmetric space. Furthermore in [25], invariant Einstein metrics on

$$Sp(l + m + n)/Sp(l) \times Sp(m) \times Sp(m),$$
$$SO(l + m + n)/SO(l) \times SO(m) \times SO(m)$$

are studied. Recently in [1, 2], invariant Einstein metrics on three-locally-symmetric spaces are considered from the point of view of the normalized Ricci flows.

But the classification of three-locally-symmetric spaces is still incomplete, which leads to the incomplete classification of invariant Einstein metrics. In this paper, we complete the classification of three-locally-symmetric spaces for G simple, and then classify invariant Einstein metrics on all previously unexplored three-locally-symmetric spaces for G simple.

The paper is organized as follows. In Section 2, we give the correspondence between the classification of three-locally-symmetric spaces G/H and that of certain involution pairs of G. In Section 3, the classification of three-locally-symmetric spaces G/H is given for G simple based on the theory on involutions of compact Lie groups. We list them in Table 1 in Theorem 3.15. Furthermore we prove the isotropy summands are pairwise nonisomorphic for the new three-locally-symmetric spaces. It makes the method of [25] valid to give the classification of invariant Einstein metrics on these spaces in Section 4.

Remark 1.1. The minute that we uploaded this paper on www.arXiv.org, we received an Email from Prof. Nikonorov with the paper [29] on three-locally-symmetric spaces which were called generalized Wallach spaces. The classification of three-locally-symmetric spaces for G simple was obtained in [29] based on the classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces [7, 20, 23]. We note that the classification by Nikonorov is more general since it includes the case of non-simple G.

2. Three-locally-symmetric spaces G/H and involution pairs of G

Assume that G/H is a three-locally-symmetric space. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$, and it is easy to see that the Lie brackets satisfy

$$[\mathfrak{h}, \mathfrak{p}_i] \subset \mathfrak{p}_i, \quad [\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}, \quad [\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_k \tag{2.1}$$

for any $i \in \{1, 2, 3\}$ and $\{i, j, k\} = \{1, 2, 3\}$. Define a linear map θ_1 on $\mathfrak{g}$ by

$$\theta_1|_{\mathfrak{h}\oplus\mathfrak{p}_1} = id, \quad \theta_1|_{\mathfrak{p}_2\oplus\mathfrak{p}_3} = -id.$$

By the equation (2.1), $\theta_1[X, Y] = [\theta_1(X), \theta_1(Y)]$ for any $X, Y \in \mathfrak{g}$. It follows that θ_1 is an automorphism, and then an involution of $\mathfrak{g}$. Similarly, define another linear map θ_2 on $\mathfrak{g}$ by

$$\theta_2|_{\mathfrak{h}\oplus\mathfrak{p}_2} = id, \quad \theta_2|_{\mathfrak{p}_1\oplus\mathfrak{p}_3} = -id,$$

which is also an involution of $\mathfrak{g}$. Moreover we have $\theta_1\theta_2 = \theta_2\theta_1$, $\mathfrak{h} = \{X \in \mathfrak{g} | \theta_1(X) = X, \theta_2(X) = X\}$, $\mathfrak{p}_1 = \{X \in \mathfrak{g} | \theta_1(X) = X, \theta_2(X) = -X\}$, $\mathfrak{p}_2 = \{X \in \mathfrak{g} | \theta_1(X) = -X, \theta_2(X) = X\}$, and $\mathfrak{p}_3 = \{X \in \mathfrak{g} | \theta_1(X) = -X, \theta_2(X) = -X\}$.

On the other hand, let G be a compact semisimple connected Lie group with the Lie algebra $\mathfrak{g}$ and ρ, φ be involutions of $\mathfrak{g}$ satisfying $\rho\varphi = \varphi\rho$. Then we have a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3,$$

corresponding to ρ, φ, where $\mathfrak{h} = \{X \in \mathfrak{g} | \rho(X) = X, \varphi(X) = X\}$, $\mathfrak{p}_1 = \{X \in \mathfrak{g} | \rho(X) = X, \varphi(X) = -X\}$, $\mathfrak{p}_2 = \{X \in \mathfrak{g} | \rho(X) = -X, \varphi(X) = X\}$, and $\mathfrak{p}_3 = \{X \in \mathfrak{g} | \rho(X) = -X, \varphi(X) = -X\}$. It is easy to check that

$$[\mathfrak{h}, \mathfrak{p}_i] \subset \mathfrak{p}_i, \quad [\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}, \quad [\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_k$$

for any $i \in \{1, 2, 3\}$ and $\{i, j, k\} = \{1, 2, 3\}$. Let H denote the connected Lie subgroup of G with the Lie algebra $\mathfrak{h}$. If every $\mathfrak{p}_i$ for $i \in \{1, 2, 3\}$ is an irreducible ad$\mathfrak{h}$-module, then G/H is a three-locally-symmetric space.

In summary, there is a one-to-one correspondence between the set of three-locally-symmetric spaces and the set of commutative involution pairs of $\mathfrak{g}$ such that every $\mathfrak{p}_i$ for $i \in \{1, 2, 3\}$ is an irreducible ad$\mathfrak{h}$-module.

3. The classification of three-locally-symmetric spaces

The section is to give the classification of three-locally-symmetric spaces for a compact simple Lie group. By the discussion in Section 2, it turns to the classification of certain commutative involution pairs.

Let G be a compact simple connected Lie group with the Lie algebra $\mathfrak{g}$ and (θ, τ) be an involution pair of G with $\theta\tau = \tau\theta$. Then for θ, we have a decomposition,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m},$$

where $\mathfrak{k} = \{X \in \mathfrak{g} | \theta(X) = X\}$ and $\mathfrak{m} = \{X \in \mathfrak{g} | \theta(X) = -X\}$. Since $\theta\tau = \tau\theta$, we know $\tau(X) \in \mathfrak{k}$ for any $X \in \mathfrak{k}$, which implies that $\tau|_{\mathfrak{k}}$ is an involution of $\mathfrak{k}$. Roughly to say, we can classify commutative involution pairs of $\mathfrak{g}$ by studying the extension of an involution of $\mathfrak{k}$ to $\mathfrak{g}$. But an important problem is when an involution of $\mathfrak{k}$ can be extended to an involution of $\mathfrak{g}$.

Cartan and Gantmacher made great attributions on the classification of involutions on compact Lie groups. The theory on the extension of involutions of $\mathfrak{k}$ to $\mathfrak{g}$ can be found in [8], which is different in the method from that in [37]. There are also some related discussion in [13–15]. The following are the theories without proof.

Let $\mathfrak{t}_1$ be a Cartan subalgebra of $\mathfrak{k}$ and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}_1$.

Theorem 3.1 (Gantmacher Theorem). *With the above notations, θ is conjugate with $\theta_0 e^{\mathrm{ad}H}$ under* $\mathrm{Aut}\mathfrak{g}$*, where $H \in \mathfrak{t}_1$ and θ_0 is an involution which keeps the Dynkin diagram invariant.*

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a fundamental system of $\mathfrak{t}$ and $\phi = \sum_{i=1}^{n} m_i \alpha_i$ be the maximal root respectively. Let $\alpha'_i = \frac{1}{2}(\alpha_i + \theta_0(\alpha_i))$. Then $\Pi' = \{\alpha'_1, \ldots, \alpha'_l\}$ consisting different elements in $\{\alpha'_1, \ldots, \alpha'_n\}$ is a fundamental system of $\mathfrak{g}_0$, where $\mathfrak{g}_0 = \{X \in \mathfrak{g} | \theta_0(X) = X\}$. Denote by $\phi' = \sum_{i=1}^{l} m'_i \alpha'_i$ the maximal root of $\mathfrak{g}_0$ respectively. Furthermore we have

Theorem 3.2 ([37]). *If $H \neq 0$, then for some i, we can take H satisfying*

$$\alpha'_i = \alpha_i; \quad \langle H, \alpha'_i \rangle = \pi\sqrt{-1}; \quad \langle H, \alpha'_j \rangle = 0, \forall j \neq i. \tag{3.1}$$

Here $m'_i = 1$ or $m'_i = 2$.

Moreover, $\mathfrak{k}$ is described as follows.

Theorem 3.3 ([37]). *Let the notations be as above. Assume that α_i satisfies the identity (3.1).*

1) *If $\theta_0 = Id$ and $m_i = 1$, then $\Pi - \{\alpha_i\}$ is a fundamental system of $\mathfrak{k}$, and ϕ and $-\alpha_i$ are the highest weights of $\mathrm{ad}_{\mathfrak{m}^{\mathbb{C}}}\mathfrak{k}$ corresponding to the fundamental system.*
2) *If $\theta_0 = Id$ and $m_i = 2$, then $\Pi - \{\alpha_i\} \cup \{-\phi\}$ is a fundamental system of $\mathfrak{k}$, and $-\alpha_i$ is the highest weight of $\mathrm{ad}_{\mathfrak{m}^{\mathbb{C}}}\mathfrak{k}$ corresponding to the fundamental system.*
3) *If $\theta_0 \neq Id$, then $\Pi' - \{\alpha'_i\} \cup \{\beta_0\}$ is a fundamental system of $\mathfrak{k}$, and $-\alpha_i$ is the highest weight of $\mathrm{ad}_{\mathfrak{m}^{\mathbb{C}}}\mathfrak{k}$ corresponding to the fundamental system.*

Remark 3.4. In Theorem 3.3, the dimension of $C(\mathfrak{k})$, i.e. the center of $\mathfrak{k}$, is 1 for case (1); 0 for cases (2) and (3), β_0 in case (3) is the highest weight of $\mathrm{ad}_{\mathfrak{m}^{\mathbb{C}}}\mathfrak{k}$ for $\theta = \theta_0$ corresponding to Π'.

Now for any involution $\tau^{\mathfrak{k}}$ of $\mathfrak{k}$, we can write $\tau^{\mathfrak{k}} = \tau_0^{\mathfrak{k}} e^{\mathrm{ad}H^{\mathfrak{k}}}$, where $\tau_0^{\mathfrak{k}}$ is an involution on $\mathfrak{k}$ which keeps the Dynkin diagram of $\mathfrak{k}$ invariant, $H^{\mathfrak{k}} \in \mathfrak{t}_1$ and $\tau_0^{\mathfrak{k}}(H^{\mathfrak{k}}) = H^{\mathfrak{k}}$. Since $e^{\mathrm{ad}H^{\mathfrak{k}}}$ is an inner-automorphism, naturally we can extend $e^{\mathrm{ad}H^{\mathfrak{k}}}$ to an automorphism of $\mathfrak{g}$. Moreover,

Theorem 3.5 ([37]). *The involution $\tau_0^{\mathfrak{k}}$ can be extended to an automorphism of $\mathfrak{g}$ if and only if $\tau_0^{\mathfrak{k}}$ keeps the weight system of $\mathrm{ad}_{\mathfrak{m}^{\mathbb{C}}}\mathfrak{k}$ invariant.*

If $C(\mathfrak{k}) \neq 0$, then $\dim C(\mathfrak{k}) = 1$. Thus $\tau_0^{\mathfrak{k}}(Z) = Z$ or $\tau_0^{\mathfrak{k}}(Z) = -Z$ for any $Z \in C(\mathfrak{k})$.

Theorem 3.6 ([37]). *Assume that $C(\mathfrak{k}) \neq 0$ and $\tau_0^{\mathfrak{k}}(Z) = Z$ for any $Z \in C(\mathfrak{k})$. If $\tau^{\mathfrak{k}}$ can be extended to an automorphism of $\mathfrak{g}$, then $\tau^{\mathfrak{k}}$ can be extended to an involution of $\mathfrak{g}$.*

For the other cases, we have the following theorems.

Theorem 3.7 ([37]). *Assume that $C(\mathfrak{k}) = 0$, or $C(\mathfrak{k}) \neq 0$ but $\tau_0^{\mathfrak{k}}(Z) = -Z$ for any $Z \in C(\mathfrak{k})$. If τ is an automorphism of $\mathfrak{g}$ extending an involution $\tau^{\mathfrak{k}}$ of $\mathfrak{k}$, then $\tau^2 = Id$ or $\tau^2 = \theta$. Furthermore, the following conditions are equivalent:*

1) *There exists an automorphism τ of $\mathfrak{g}$ extending $\tau^{\mathfrak{k}}$ which is an involution.*
2) *Every automorphism τ of $\mathfrak{g}$ extending $\tau^{\mathfrak{k}}$ is an involution.*

Then it is enough to determine when the automorphism extending $\tau^{\mathfrak{k}}$ is an involution.

Theorem 3.8 ([37]). *Let τ_0 be the automorphism of $\mathfrak{g}$ extending the involution $\tau_0^{\mathfrak{k}}$ on $\mathfrak{k}$. Then $\tau_0^2 = Id$ except $\mathfrak{g} = A_n^i$ and n is even. For $e^{\mathrm{ad} H^{\mathfrak{k}}}$, we have:*

1) *If $\theta_0 \neq Id$, then the natural extension of $e^{\mathrm{ad} H^{\mathfrak{k}}}$ is an involution.*
2) *Assume that $\theta_0 = Id$. Let $\alpha'_{i_1}, \ldots, \alpha'_{i_k}$ be the roots satisfying $\langle \alpha'_j, H \rangle \neq 0$. Then the natural extension of $e^{\mathrm{ad} H^{\mathfrak{k}}}$ is an involution if and only if $\sum_{j=i_1}^{i_k} m'_j$ is even.*

In particular, for every case in Theorem 3.3,

Theorem 3.9 ([8, 37]). *If τ is an involution of $\mathfrak{g}$ extending an involution $\tau^{\mathfrak{k}}$ on $\mathfrak{k}$, then every extension of $\tau^{\mathfrak{k}}$ is an involution of $\mathfrak{k}$, which is equivalent with τ or $\tau\theta$.*

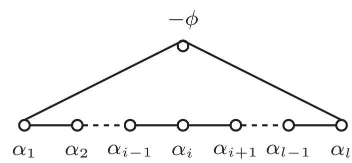
Up to now, we can obtain the classification of commutative involution pairs by that of $(\theta, \tau^{\mathfrak{k}})$ based on the above theory. By Theorem 3.9, for an involution $\tau^{\mathfrak{k}}$ on $\mathfrak{k}$ which can be extended to an involution τ of $\mathfrak{g}$, we have two involution pairs (θ, τ) and $(\theta, \theta\tau)$ which determine the same three-locally-symmetric space. So, without loss of generality, we denote by τ the natural extension of $\tau^{\mathfrak{k}}$. Let $\mathfrak{h} = \{X \in \mathfrak{k} | \tau^{\mathfrak{k}}(X) = X\}$, $\mathfrak{p}_1 = \{X \in \mathfrak{k} | \tau(X) = -X\}$, $\mathfrak{p}_2 = \{X \in \mathfrak{m} | \tau(X) = X\}$, and $\mathfrak{p}_3 = \{X \in \mathfrak{m} | \tau(X) = -X\}$. We shall pick up certain pairs $(\theta, \tau^{\mathfrak{k}})$ by the following steps:

Step 1: we obtain the classification of the pairs $(\theta, \tau^{\mathfrak{k}})$ satisfying that the extension τ of $\tau^{\mathfrak{k}}$ is an involution and $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are irreducible as ad$\mathfrak{h}$-modules.

Step 2: among the pairs given by Step 1, we remain only one if several pairs determine the same three-locally-symmetric space.

Here we don't list all the pairs satisfying Steps 1 and 2 but give some examples and remarks.

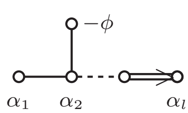
Example 3.10. Let $G = A_l (l \geq 1)$ and the Dynkin diagram with the maximal root is



We have the following cases:

1) $l = 1$, $\theta = e^{\mathrm{ad}H}$, where $\langle H, \alpha_1 \rangle = \pi\sqrt{-1}$; $\tau^{\mathfrak{k}}|_{C(\mathfrak{k})} = -Id$.
2) $l \geq 3$ is odd, $\theta = e^{\mathrm{ad}H}$, where $\langle H, \alpha_{\frac{l+1}{2}} \rangle = \pi\sqrt{-1}$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i\}$; $\tau^{\mathfrak{k}}(\alpha_k) = \alpha_{l+1-k}$ for $k \neq \frac{l+1}{2}$ and $\tau^{\mathfrak{k}}|_{C(\mathfrak{k})} = Id$.
3) $l \geq 2$, $\theta = e^{\mathrm{ad}H}$, where $\langle H, \alpha_i \rangle = \pi\sqrt{-1}$ for some $1 \leq i \leq [\frac{l+1}{2}]$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i\}$; $\tau^{\mathfrak{k}} = e^{\mathrm{ad}H_1}$, where $\langle H_1, \alpha_j \rangle = \pi\sqrt{-1}$ for some $\alpha_j \in \Pi - \{\alpha_i\}$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i, \alpha_j\}$. Furthermore, we may require that $1 \leq i \leq [\frac{l+1}{3}]$ and $2i \leq j \leq [\frac{l+i+1}{2}]$ for Step 2.

Example 3.11. Let $G = B_l (l \geq 2)$ and the Dynkin diagram with the maximal root is



We have the following cases:

1) $\theta = e^{\mathrm{ad}H}$, where $\langle H, \alpha_i \rangle = \pi\sqrt{-1}$ for some $2 \leq i \leq l$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i\}$; $\tau^{\mathfrak{k}} = e^{\mathrm{ad}H_1}$, where $\langle H_1, \alpha_j \rangle = \pi\sqrt{-1}$ for some

$2 \le j < i$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i, \alpha_j\} \cup \{-\phi\}$. Moreover, we may assume that $2 < i \le l$ and $\frac{i}{2} \le j \le i-1$.

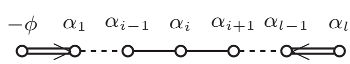
2) $\theta = e^{\mathrm{ad}H}$, where $\langle H, \alpha_i \rangle = \pi\sqrt{-1}$ for some $2 \le i \le l$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i\}$; $\tau^{\mathfrak{k}}(\alpha_1) = -\phi$, $\tau^{\mathfrak{k}}(-\phi) = \alpha_1$, and $\tau^{\mathfrak{k}}(\alpha_k) = \alpha_k$ for any $\alpha_k \in \Pi - \{\alpha_1, \alpha_i\}$. Moreover, we may assume that $\frac{l+1}{2} \le i \le l$.

3) $\theta = e^{\mathrm{ad}H}$, where $\langle H, \alpha_i \rangle = \pi\sqrt{-1}$ for some $2 \le i \le l$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i\}$; $\tau^{\mathfrak{k}} = \tau_0^{\mathfrak{k}} e^{\mathrm{ad}H_1}$, where $\tau_0^{\mathfrak{k}}(\alpha_1) = -\phi$, $\tau_0^{\mathfrak{k}}(-\phi) = \alpha_1$, and $\tau_0^{\mathfrak{k}}(\alpha_k) = \alpha_k$ for any $\alpha_k \in \Pi - \{\alpha_1, \alpha_i\}$, $\langle H_1, \alpha_j \rangle = \pi\sqrt{-1}$ for some $2 \le j < i$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i, \alpha_j\} \cup \{-\phi\}$. Furthermore, we may require that $[\frac{2l+3}{3}] \le i \le l$ and $[\frac{i+2}{2}] \le j \le 2i - l$.

Remark 3.12. Let $G = A_l$ where $l \ge 1$ is odd. Take the involution $\theta = e^{\mathrm{ad}H}$, where $\langle H, \alpha_{\frac{l+1}{2}} \rangle = \pi\sqrt{-1}$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i\}$. Then $\tau^{\mathfrak{k}}$ defined by $\tau^{\mathfrak{k}}(\alpha_k) = \alpha_{\frac{l+1}{2}+k}$ and $\tau^{\mathfrak{k}}(\alpha_{\frac{l+1}{2}+k}) = \alpha_k$ for any $1 \le k \le \frac{l-1}{2}$ is an involution on $\mathfrak{k}$. By the above theory, if $\tau^{\mathfrak{k}}$ can be extended to an involution of $\mathfrak{g}$, we obtain $\tau^{\mathfrak{k}}(-\alpha_{\frac{l+1}{2}}) = \phi$. It is equivalent to $\tau^{\mathfrak{k}}|_{C(\mathfrak{k})} = -Id$. If $l \ge 3$, $\mathfrak{p}_1$ is reducible. If $l = 1$, we get the case (1) of Example 3.10.

Remark 3.13. The case (1) of Example 3.11 is valid for $i = 1$ and j, which determines the same three-locally-symmetric space with $i' = j$ and $j' = j - 1$. If we take θ as that of the case (1) of Example 3.11 for $i = 1$, then $\tau^{\mathfrak{k}}|_{C(\mathfrak{k})} = -Id$ can be extended to an involution of $\mathfrak{g}$. This pair determines the same three-locally-symmetric space as the case (1) of Example 3.11 for $i = l$. Every involution $\tau^{\mathfrak{k}}$ in cases (2) and (3) of Example 3.11 is an outer-automorphism on $\mathfrak{k}$, which can be extended to an inner-automorphism of $\mathfrak{g}$.

Example 3.14. Let the Dynkin diagram with the maximal root of $G = C_l (l \ge 3)$ be

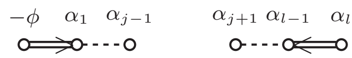


Consider the following involution pair (θ, τ) of $G = C_l$. Here $\theta = e^{\mathrm{ad}H}$ is an involution of $\mathfrak{g}$, where $\langle H, \alpha_l \rangle = \pi\sqrt{-1}$ and others zero. Then $\mathfrak{k}$ is the direct sum of A_{l-1} and the center of one-dimension. Given an involution $\tau^{\mathfrak{k}} = e^{\mathrm{ad}H_1}$ of $\mathfrak{k}$, where $\langle H_1, \alpha_j \rangle = \pi\sqrt{-1}$ for some $j \in \{1, 2, \ldots, l-1\}$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_j, \alpha_l\}$. Then the extension of $\tau_{\mathfrak{k}}$ is an involution of $\mathfrak{g}$ and

1) $\mathfrak{p}_1$ is an irreducible ad$\mathfrak{h}$-module.

2) By a result in [8], the extension of $\tau^{\mathfrak{k}}$ is τ or $\tau\theta$. Here τ is also denoted by $e^{\mathrm{ad}H_1}$, where $\langle H_1, \alpha_j \rangle = \pi\sqrt{-1}$ and others zero.

We can prove that $\mathfrak{p}_2$ is a reducible ad$\mathfrak{h}$-module. In fact, the Dynkin diagram of $\mathfrak{h} + \mathfrak{p}_2$, i.e. the set of fixed points of τ, is



Then by the definition of θ, the Dynkin diagram of $\mathfrak{h}$ is

and on $\mathfrak{h} + \mathfrak{p}_2$, $\langle H, \alpha_l \rangle = \pi\sqrt{-1}$, $\langle H, -\phi \rangle = \pi\sqrt{-1}$, and others zero. That is, $\mathfrak{p}_2$ is reducible.

By the similar discussions case by case, we classify three-locally-symmetric spaces as follows.

Theorem 3.15. *The classification of three-locally-symmetric spaces G/H for a connected, compact and simple Lie group G is given in Table 1. In Table 1, $A_1 = B_1 = C_1$, $B_2 = C_2$, $A_3 = D_3$, $D_1 = T$ and $A_0 = B_0 = C_0 = D_0 = \{e\}$.*

Remark 3.16. The well-known examples of three-locally-symmetric spaces are the following:

1) $SU(2) = SU(2)/\{e\}$,
2) $SU(n_1 + n_2 + n_3)/S(U(n_1) \times U(n_2) \times U(n_3))$,
3) $SO(l + m + n)/SO(l) \times SO(m) \times SO(n)$,
4) $SO(2n)/U(1) \times U(n - 1)$,
5) $Sp(l + m + n)/Sp(l) \times Sp(m) \times Sp(m)$,
6) $E_6/U(1) \times U(1) \times Spin(8)$,
7) $F_4/Spin(8)$.

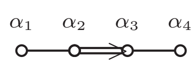
The first one is the three-locally-symmetric space of type A-I in Table 1 of Theorem 3.15, the second one is of type A-III, the third one corresponds to types B-I, B-II, B-III, D-I, D-II, D-III, and D-IV, the fourth one is of type D-V, the fifth one is of type C-I, the sixth one is of type E_6-I, and the seventh one is of type F_4-I. For the above cases, $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ have been

Type	G	H	*Type*	G	H
A-I	A_1	$\{e\}$	*A-II*	A_l	$T \times A_{\frac{l-1}{2}}$ $l \geq 3$ *is odd*
A-III	A_l	$T^2 \times A_{i-1} \times A_{j-i-1} \times A_{l-j}$ $1 \leq i \leq [\frac{l+1}{3}]$ $2i \leq j \leq [\frac{l+i+1}{2}]$	*B-I*	B_l	$B_{l-i} \times D_j \times D_{i-j}$ $2 < i \leq l$ $\frac{i}{2} \leq j \leq i-1$
B-II	B_l	$B_{i-1} \times B_{l-i}$ $\frac{l+1}{2} \leq i \leq l$	*B-III*	B_l	$B_{i-1} \times B_{i-j} \times B_{l-i}$ $[\frac{2l+3}{3}] \leq i \leq l$ $[\frac{i+2}{2}] \leq j \leq 2i-l$
C-I	C_l	$C_i \times C_{j-i} \times C_{l-j}$ $1 \leq i \leq [\frac{l}{3}]$ $2i \leq j \leq [\frac{l+i}{2}]$	*D-I*	D_l	$D_i \times D_{j-i} \times D_{l-j}$ $1 \leq i \leq [\frac{l}{3}]$ $2i \leq j \leq [\frac{l+i}{2}]$
D-II	D_l	$B_{i-1} \times D_{l-i}$ $1 \leq i \leq l-2$	*D-III*	D_l	$D_i \times B_{j-i} \times B_{l-j-1}$ $1 \leq i \leq l-2$ $i \leq j \leq [\frac{l+i-1}{2}]$
D-IV	D_l	D_{l-1}	*D-V*	D_l	$T^2 \times A_{l-2}$
E_6*-I*	E_6	$T^2 \times D_4$	E_6*-II*	E_6	$T \times A_1 \times A_1 \times A_3$
E_6*-III*	E_6	$A_1 \times C_3$	E_7*-I*	E_7	$A_1 \times A_1 \times A_1 \times D_4$
E_7*-II*	E_7	$T \times A_1 \times A_5$	E_7*-III*	E_7	D_4
E_8*-I*	E_8	$A_1 \times A_1 \times D_6$	E_8*-II*	E_8	$D_4 \times D_4$
F_4*-I*	F_4	D_4	F_4*-II*	F_4	$A_1 \times A_1 \times C_2$

Table 1: Classification of three-locally-symmetric spaces.

proved to be irreducible, and pairwise nonisomorphic with respect to the adjoint action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{p}$ except $SO(n+2)/SO(n)$, which is of type B-II for $i = l$ and D-IV.

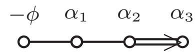
Example 3.17. ([14]) Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a fundamental system of F_4 such that the Dynkin diagram of F_4 is



Consider the involution $\theta = e^{\mathrm{ad}H}$ defined by

$$\langle H, \alpha_4 \rangle = \pi\sqrt{-1}; \quad \langle H, \alpha_j \rangle = 0, \forall j \neq 4.$$

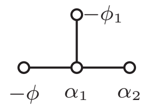
Let $\phi = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Then the Dynkin diagram of $\mathfrak{k}$ is



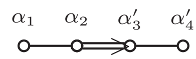
The involution τ is the extension of $\tau^{\mathfrak{k}}$, where $\tau^{\mathfrak{k}} = e^{\mathrm{ad}H_1}$ satisfies

$$\langle H_1, \alpha_3 \rangle = \pi\sqrt{-1}; \quad \langle H, \alpha_1 \rangle = \langle H_1, \alpha_2 \rangle = \langle H_1, -\phi \rangle = 0.$$

Then $\phi_1 = -(\alpha_2 + 2\alpha_3 + 2\alpha_4)$ be the maximal root of $\mathfrak{k}$. Then the Dynkin diagram of $\mathfrak{h}$ is



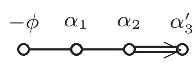
Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ be the decomposition of $\mathfrak{g}$ corresponding to (θ, τ). Here $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_1$, which is the decomposition of $\mathfrak{k}$ corresponding to the involution $\tau^{\mathfrak{k}}$. By Theorem 3.3, $\mathfrak{p}_1$ is the irreducible representation of $\mathfrak{h}$ with the highest weight $-\alpha_3$, which is a fundamental dominant weight corresponding to α_2. Let $\mathfrak{k}^1 = \{x \in \mathfrak{g} | \theta\tau(x) = x\} = \mathfrak{h} \oplus \mathfrak{p}_3$, which is the decomposition of $\mathfrak{k}^1$ corresponding to the involution $\tau|_{\mathfrak{k}^1}$. The Dynkin diagram of F_4 corresponding to the fundamental system $\{\alpha_1, \alpha_2, \alpha_3' = \alpha_3 + \alpha_4, \alpha_4' = -\alpha_4\}$ is



The involution $\theta\tau = e^{\mathrm{ad}(H+H_1)}$ satisfies

$$\langle H + H_1, \alpha_4' \rangle = \pi\sqrt{-1}; \quad \langle H + H_1, \alpha_1 \rangle = \langle H + H_1, \alpha_2 \rangle = \langle H + H_1, \alpha_3' \rangle = 0.$$

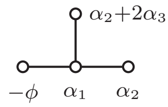
Here the maximal root $2\alpha_1 + 3\alpha_2 + 4\alpha_3' + 2\alpha_4' = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \phi$. It follows that the Dynkin diagram of $\mathfrak{k}^1$ is



The involution $\tau = e^{\mathrm{ad}H_1}$ restricted on $\mathfrak{k}^1$ satisfies

$$\langle H_1, \alpha_3' \rangle = \pi\sqrt{-1}; \quad \langle H, \alpha_1 \rangle = \langle H_1, \alpha_2 \rangle = \langle H_1, -\phi \rangle = 0.$$

Then $-(\alpha_2 + 2\alpha_3)$ is the maximal root of $\mathfrak{k}$, and the Dynkin diagram of $\mathfrak{h}$ is



By Theorem 3.3, $\mathfrak{p}_3$ is the irreducible representation of $\mathfrak{h}$ with the highest weight $-\alpha_3'$. For the fundamental system $\{-\phi, \alpha_1, \alpha_2, -\phi_1\}$, $\mathfrak{p}_3$ is the irreducible representation of $\mathfrak{h}_1$ with the highest weight $-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)$, which is a fundamental dominant weight corresponding to $-\phi$. The discussion for $\mathfrak{p}_2$ is similar.

Hence, for the fundamental system $\{-\phi, \alpha_1, \alpha_2, -\phi_1\}$ of $\mathfrak{h}$, we conclude that the highest weights of $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ as $\mathrm{ad}\mathfrak{h}$ modules are fundamental dominant weights of $\mathfrak{h}$ corresponding to α_2, $-\phi_1$ and $-\phi$ respectively, which are pairwise nonisomorphic.

By Theorem 3.15, we obtain in Table 2 the dimensions of $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ for the following cases. Clearly $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are pairwise nonisomorphic for

Type	$\dim \mathfrak{p}_1$	$\dim \mathfrak{p}_2$	$\dim \mathfrak{p}_3$	Type	$\dim \mathfrak{p}_1$	$\dim \mathfrak{p}_2$	$\dim \mathfrak{p}_3$
A-II	$\frac{(l-1)(l+3)}{4}$	$\frac{(l+1)(l+3)}{4}$	$\frac{(l-1)(l+1)}{4}$	E_6-II	16	16	24
E_6-III	14	28	12	E_7-I	32	32	32
E_7-II	24	30	40	E_7-III	35	35	35
E_8-I	48	64	64	E_8-II	64	64	64
F_4-II	20	8	8				

Table 2: The dimensions of $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$.

types A-II, E_6-III and E_7-II. We can prove the same result for the other cases similar to Example 3.17.

In summary, we have the following theorem.

Theorem 3.18. *Let G/H be a three-locally-symmetric space in Theorem 3.15 with the decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$. Then $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are pairwise nonisomorphic with respect to the adjoint action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{p}$ except types B-II for $i = l$ and D-IV.*

4. Einstein metrics on three-locally-symmetric spaces

There are some studies on the geometry of three-locally-symmetric spaces. In particular, a lot of studies on invariant Einstein metrics have been done for some three-locally-symmetric spaces independently. For example,

(1) The flag manifolds $SU(3)/T_{max}, Sp(3)/Sp(1)\times Sp(1)\times Sp(1)$, $F_4/Spin(8)$ known as Wallach spaces admit invariant Riemannian metrics of positive section curvature ([35]). The invariant Einstein metrics on the first space are classified in [16], on the other two spaces in [32]. In any case, there are exactly four invariant Einstein metrics up to proportionality.

(2) The invariant Einstein metrics on the Kähler C-spaces $SU(n_1+n_2+n_3)/S(U(n_1)\times U(n_2)\times U(n_3))$, $SO(2n)/U(1)\times U(n-1)$, $E_6/U(1)\times U(1)\times Spin(8)$ are classified in [22]. Every space admits four invariant Einstein metrics up to proportionality. Another approach to $SU(n_1+n_2+n_3)/S(U(n_1)\times U(n_2)\times U(n_3))$ is given in [5].

(3) The Lie group $SU(2)$ considered as $SU(2)/\{e\}$ admits only one left-invariant Einstein metric which is a metric of constant curvature [9].

(4) It is proved in [28] that every three-locally-symmetric space admits at least on invariant Einstein metric. Furthermore, it is proved in [25] that $Sp(l+m+n)/Sp(l)\times Sp(m)\times Sp(n)$ admits exactly four invariant Einstein metrics up to proportionality and that $SO(l+m+n)/SO(l)\times SO(m)\times SO(n)$ admits one, two, three or four invariant Einstein metrics up to proportionality. In particular, it is demonstrated in [21] that $SO(n+2)/SO(n)$ admits just one Einstein metric up to isometry and homothety for $n\geq 3$, the space $SO(4)/SO(2)$ has two such metrics from the classification theorem for five dimensional homogeneous compact Einstein manifolds [3].

In summary, invariant Einstein metrics on three-locally-symmetric spaces are studied for types A-I, A-III, B-I, B-II, B-III, C-I, D-I, D-II, D-III, D-IV, D-V, E_6-I and F_4-I in Theorem 3.15. The following is to classify invariant Einstein metrics on every three-locally-symmetric space in Theorem 3.15 except the above cases.

By Theorem 3.18, in the decomposition $\mathfrak{p}=\mathfrak{p}_1\oplus\mathfrak{p}_2\oplus\mathfrak{p}_3$, $\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3$ are pairwise nonisomorphic with respect to the adjoint action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{p}$. Then we can give the classification of invariant Einstein metrics on these spaces following the theory in [25, 28, 36].

Let d_i denote the dimension of $\mathfrak{p}_i$, and let $\{e_i^j\}$ be an orthonormal basis in $\mathfrak{p}_i$ with respect to $\langle\cdot,\cdot\rangle=-B(\cdot,\cdot)$, where $i=1,2,3$ and $1\leq j\leq d_i=\dim\mathfrak{p}_i$. Let $\left[{k \atop ij}\right]=\sum_{\alpha,\beta,\gamma}\langle[e_i^\alpha,e_j^\beta],e_k^\gamma\rangle^2$, where α,β,γ range from 1 to d_i,d_j,d_k respectively. Then $\left[{k \atop ij}\right]$ are symmetric in all three indices and $\left[{k \atop ij}\right]=0$ if two indices coincide. Let c_i be the Casimir constant of the adjoint representation of $\mathfrak{h}$ on $\mathfrak{p}_i$. If $\{e_0^j\}_{1\leq j\leq\dim\mathfrak{h}}$ is an orthonormal basis in $\mathfrak{h}$ with respect to $\langle\cdot,\cdot\rangle$ and e is an arbitrary unit vector in $\mathfrak{p}_i$, then $c_i=\sum_j\langle[e_0^j,e],[e_0^j,e]\rangle$. For three-locally-symmetric spaces, by [25, 36],

$$2A = \begin{bmatrix} k \\ ij \end{bmatrix} + \begin{bmatrix} j \\ ik \end{bmatrix} = d_i(1-2c_i). \tag{4.1}$$

Let ρ be an invariant metric on G/H. We identity it with the corresponding $\mathrm{ad}\mathfrak{h}$-invariant $(\cdot,\cdot)$ on $\mathfrak{p}$. Since $\mathfrak{p}_i$ are irreducible and pairwise nonisomorphic, we have

$$(\cdot,\cdot) = x_1\langle\cdot,\cdot\rangle|_{\mathfrak{p}_1} \oplus x_2\langle\cdot,\cdot\rangle|_{\mathfrak{p}_2} \oplus x_3\langle\cdot,\cdot\rangle|_{\mathfrak{p}_3}$$

for some positive real numbers x_i. The Ricci curvature $\mathrm{Ric}(\cdot,\cdot)$ of the metric $(\cdot,\cdot)$ is also $\mathrm{ad}\mathfrak{h}$-invariant. It is easy to see $\mathrm{Ric}(\cdot,\cdot)|_{\mathfrak{p}_i} = r_i(\cdot,\cdot)|_{\mathfrak{p}_i}$ for some real numbers r_i. As that given in [28], we have the following formula

$$r_i = \frac{1}{2x_i} + \frac{A}{2d_i}\left(\frac{x_i}{x_jx_k} - \frac{x_k}{x_ix_j} - \frac{x_j}{x_ix_k}\right).$$

Here $\{i,j,k\} = \{1,2,3\}$. Put $a_i = \frac{A}{d_i}$. Then we have

$$\begin{cases} r_1 = \dfrac{1}{2x_1} + \dfrac{a_1}{2}\left(\dfrac{x_1}{x_2x_3} - \dfrac{x_2}{x_1x_3} - \dfrac{x_3}{x_1x_2}\right) \\ r_2 = \dfrac{1}{2x_2} + \dfrac{a_2}{2}\left(\dfrac{x_2}{x_1x_3} - \dfrac{x_1}{x_2x_3} - \dfrac{x_3}{x_1x_2}\right) \\ r_3 = \dfrac{1}{2x_3} + \dfrac{a_3}{2}\left(\dfrac{x_3}{x_1x_2} - \dfrac{x_1}{x_2x_3} - \dfrac{x_2}{x_1x_3}\right) \end{cases}$$

Now the invariant metric $(\cdot,\cdot)$ is Einstein if and only if $r_1 = r_2 = r_3$. If $a_i = a_j$ for $i \neq j$, then the equations $r_i = r_j$ and $r_i = r_k$ for $k \neq i,j$ become

$$\begin{cases} (x_j - x_i)(x_k - 2a_i(x_i + x_j)) = 0, \\ x_j(x_k - x_i) + (a_i + a_k)(x_i^2 - x_k^2) + (a_k - a_i)x_j^2 = 0 \end{cases}$$

If $x_j = x_i$, then the second equation is

$$(1-2a_k)x_i^2 - x_ix_k + (a_i + a_k)x_k^2 = 0. \tag{4.2}$$

If $a_k = 1/2$, we have only one family of proportional Einstein metrics. Otherwise, $1 - 2a_k > 0$, hence, all real solutions of the equation (4.2) are positive. Then there exist one family of proportional Einstein metrics for $\Delta_1 = 0$, two families for $\Delta_1 > 0$, and none for $\Delta_1 < 0$. Here Δ_1 is the discriminant of (4.2). If $x_k = 2a_i(x_i + x_j)$, then the second equation is

$$(4.3) \qquad (a_i + a_k)(1 - 4a_i^2)x_i^2 - (1 - 2a_i + 8a_i^2(a_i + a_k))x_i x_j + (a_i + a_k)(1 - 4a_i^2)x_j^2 = 0.$$

If $a_i = 1/2$ then the equation (4.3) has no solution. Otherwise, $1 - 4a_i^2 > 0$, hence, all real roots of the equation (4.3) are positive. Then there exist one family of proportional Einstein metrics for the discriminant $\Delta_2 = 0$, two families for $\Delta_2 > 0$, and none for $\Delta_2 < 0$. Here Δ_2 is the discriminant of (4.3).

In particular, if $a_1 = a_2 = a_3$, then we have the following theorem.

Theorem 4.1 ([25] Theorem 3). *If G/H is a three-locally-symmetric space in Theorem 3.15 satisfying $a_1 = a_2 = a_3$, then, for $a_1 \notin \{\frac{1}{2}, \frac{1}{4}\}$, G/H admits exactly four nonproportional invariant Einstein metrics. The parameters $\{x_1, x_2, x_3\}$ has the form (t, t, t), $((1 - 2a_1)t, 2a_1 t, 2a_1 t)$, $(2a_1 t, (1 - 2a_1)t, 2a_1 t)$, or $(2a_1 t, 2a_1 t, (1 - 2a_1)t)$. For $a_1 = \frac{1}{2}$ and $a_1 = \frac{1}{4}$, every invariant Einstein metric is proportional to the standard metric.*

The following is the method for calculating c_i given in [25]. In detail, $\mathfrak{k}_i = \mathfrak{h} \oplus \mathfrak{p}_i$ is a subalgebra of $\mathfrak{g}$. Let K_i be the connect Lie subgroup in G with the Lie algebra $\mathfrak{k}_i$. In this case, the homogeneous spaces K_i/H and G/K_i are locally symmetric [9]. If K_i does not act almost effectively on $M = K/H$, consider its subgroup acting on $M = K_i/H = \widetilde{K_i}/\widetilde{H}$ almost effectively, here $\widetilde{H}$ denotes the corresponding isotropy group. The pair of algebras $(\widetilde{\mathfrak{k}}_i, \widetilde{\mathfrak{h}})$ is irreducible symmetric [9]. If $\widetilde{\mathfrak{k}}_i$ is simple, then its Killing form $B_{\widetilde{k_i}}$ is proportional to the restriction of the Killing form of $\mathfrak{g}$ to $\widetilde{\mathfrak{k}}_i$, i.e. $B_{\widetilde{k_i}} = \gamma_i B|_{\widetilde{k_i}}$. By Lemma 1 in [25], $c_i = \gamma_i/2$. It follows that

$$a_i = \frac{A}{d_i} = \frac{1 - \gamma_i}{2}.$$

From the above formulae and results in [25, 28, 36], we can classify invariant Einstein metrics on three-locally-symmetric spaces case by case.

4.1. Invariant Einstein metrics on the three-locally-symmetric space of type A-II

For this case, $\mathfrak{h} \oplus \mathfrak{p}_2 = C_k$, where $l = 2k - 1$ for $k \geq 2$. By the table for $\gamma_i \geq \frac{1}{2}$ given in [17],

$$\gamma_2 = \frac{k+1}{2k}.$$

In fact, there is a method to compute every γ_i in [17]. Here for three-locally-symmetric spaces, from (4.1) and the dimensions in Table 2, we calculate directly

$$\gamma_1 = \frac{1}{2}, \quad \gamma_3 = \frac{k-1}{2k}.$$

It follows that

$$a_1 = \frac{1}{4}, \quad a_2 = \frac{k-1}{4k}, \quad a_3 = \frac{k+1}{4k}.$$

Let $x_1 = 1$. The equations $r_1 = r_2 = r_3$ are equivalent to

$$\begin{cases} x_2^2 - (2k+1)x_3^2 + 4kx_2x_3 - 4kx_2 + 2k + 1 = 0, \\ x_2^2 - x_3^2 + 2x_3 - 2x_2 + \dfrac{1}{k} = 0. \end{cases}$$

Excluding the summand containing x_2^2 form the first equation, we obtain

$$x_2(4kx_3 - 4k + 2) = 2kx_3^2 + 2x_3 + \frac{1}{k} - 2k - 1. \tag{4.4}$$

For this case, $4kx_3 - 4k + 2 \neq 0$. Expressing x_2 by x_3 from (4.4) and inserting it into second one, we obtain

$$\begin{aligned} &12k^4x_3^4 - (48k^4 - 8k^3)x_3^3 + (72k^4 - 36k^3 - 4k^2)x_3^2 \\ &- (48k^4 - 48k^3 + 4k^2 + 4k)x_3 + 12k^4 - 20k^3 + 7k^2 + 2k - 1 = 0. \end{aligned}$$

Denote by $U_0(x_3)$ the left side of the above equation. Similar to [25], define Sturm's series by

$$\begin{aligned} U_1(x_3) &= 48k^4x_3^3 - (144k^4 - 24k^3)x_3^2 + (144k^4 - 72k^3 - 8k^2)x_3 \\ &\quad - 48k^4 + 48k^3 - 4k^2 - 4k, \\ U_2(x_3) &= (6k^3 + 3k^2)x_3^2 - \left(12k^3 - 2k^2 - \frac{8}{3}k\right)x_3 + 6k^3 - 4k^2 - \frac{7}{6}k + \frac{5}{6}, \\ U_3(x_3)7 &= \frac{32k}{27(2k+1)^2}[(108k^4 - 18k^3 - 12k^2 + 4k)x_3 - 108k^4 \\ &\qquad + 36k^3 + 27k^2 - 7k - 1], \\ U_4(x_3) &= -\frac{27(4k^2-1)^2(27k^4 - 9k^2 + 1)}{4(-54k^3 + 9k^2 + 6k - 2)^2}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &U_0(0) > 0, U_1(0) < 0, U_2(0) > 0, U_3(0) < 0, U_4(0) < 0; \\ &U_0(\infty) > 0, U_1(\infty) > 0, U_2(\infty) > 0, U_3(\infty) > 0, U_4(\infty) < 0. \end{aligned}$$

Denote by $Z(0)$ the number of the sign changes in the series $U_0(0)$, $U_1(0)$, $U_2(0)$, $U_3(0)$, $U_4(0)$ (neglecting zeros) and $Z(\infty)$ the number of sign changes in $U_0(\infty)$, $U_1(\infty)$, $U_2(\infty)$, $U_3(\infty)$, $U_4(\infty)$, where $U_i(\infty)$ denotes the leading coefficient of $U_i(x_3)$ which defines the sign of $U_i(x_3)$ as $x_i \to \infty$. By Sturm's theorem [34], the number of real roots of (4.1) on $(0, \infty)$ is equal to $Z(0) - Z(\infty) = 2$ since $U_0(0) \neq 0$. By the discussion in [25], the homogeneous manifold of type A-II in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality.

4.2. Invariant Einstein metrics on the three-locally-symmetric space of type E_6-II

For this case, $\mathfrak{h} \oplus \mathfrak{p}_1 = A_1 \oplus A_5$ and $\mathfrak{p}_1 \subset A_5$. By the tables in [17], $\gamma_1 = \frac{1}{2}$. From (4.1) and the dimensions in Table 2, $\gamma_2 = \frac{1}{2}$ and $\gamma_3 = \frac{2}{3}$. It follows that

$$a_1 = a_2 = \frac{1}{4}, \quad a_3 = \frac{1}{6}.$$

If $x_1 = x_2$, then the equation (4.2) is

$$\frac{2}{3}x_1^2 - x_1x_3 + \frac{5}{12}x_3^2 = 0.$$

The discriminant $1 - \frac{40}{36} < 0$, which implies that the equation has no solution. If $x_3 = \frac{1}{2}(x_1 + x_2)$, then the equation (4.2) equals with

$$15x_1^2 - 34x_1x_2 + 15x_2^2 = 0.$$

The discriminant $34^2 - 30^2 = 16^2 > 0$, which implies that the equation has two solutions.

That is, the homogeneous manifold of type E_6-II in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters (x_1, x_2, x_3) have the form $(\frac{5}{3}t, t, \frac{4}{3}t)$, or $(\frac{3}{5}t, t, \frac{4}{5}t)$, where $t > 0$.

4.3. Invariant Einstein metrics on the three-locally-symmetric space of type E_6-III

For this case, $\mathfrak{h} \oplus \mathfrak{p}_1 = A_1 \oplus A_5$ and $\mathfrak{p}_1 \subset A_5$. By the tables in [17], $\gamma_1 = \frac{1}{2}$. From (4.1) and the dimensions in Table 2, $\gamma_2 = \frac{3}{4}$ and $\gamma_3 = \frac{5}{12}$. It follows that

$$a_1 = \frac{1}{4}, \quad a_2 = \frac{1}{8}, \quad a_3 = \frac{7}{24}.$$

Let $x_1 = 1$. The equations $r_1 = r_2 = r_3$ are equivalent to

$$\begin{cases} x_2^2 - 13x_3^2 + 24x_2x_3 - 24x_2 + 13 = 0, \\ 5x_2^2 - 5x_3^2 + 12x_3 - 12x_2 + 2 = 0. \end{cases}$$

Excluding the summand containing x_2^2 form the first equation, we obtain

$$x_2(40x_3 - 36) = 20x_3^2 + 4x_3 - 21. \tag{4.5}$$

For this case, $40x_3 - 36 \neq 0$. Expressing x_2 by x_3 from (4.5) and inserting it into second one, we obtain

$$1200x_3^4 - 4960x_3^3 + 7048x_3^2 - 4152x_3 + 855 = 0,$$

which has two real solutions $x_3 \approx 1.8845$ or $x_3 \approx 0.4838$. From (4.5), $x_2 \approx 1.4618$ or $x_2 \approx 0.8640$.

That is, the homogeneous manifold of type E_6-III in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters $(x_1, x_2, x_3) \approx (t, 1.4618t, 1.8845t)$ or $(x_1, x_2, x_3) \approx (t, 0.8640t, 0.4838t)$, where $t > 0$.

4.4. Invariant Einstein metrics on the three-locally-symmetric space of type E_7-I

For this case, $\mathfrak{h} \oplus \mathfrak{p}_1 = A_1 \oplus D_6$ and $\mathfrak{p}_1 \subset D_6$. By the tables in [17], $\gamma_1 = \frac{5}{9}$. From (4.1) and the dimensions in Table 2, $\gamma_2 = \gamma_3 = \frac{5}{9}$. It follows that

$$a_1 = a_2 = a_3 = \frac{2}{9}.$$

By Theorem 4.1, i.e. Theorem 3 in [25], the homogeneous manifold of type E_7-I in Theorem 3.15 admits exactly four invariant Einstein metrics up to proportionality. The parameters (x_1, x_2, x_3) have the form (t, t, t), $\left(\frac{5}{9}t, \frac{4}{9}t, \frac{4}{9}t\right)$, $\left(\frac{4}{9}t, \frac{5}{9}t, \frac{4}{9}t\right)$, or $\left(\frac{4}{9}t, \frac{4}{9}, \frac{5}{9}t\right)$, where $t > 0$.

4.5. Invariant Einstein metrics on the three-locally-symmetric space of type E_7-II

For this case, $\mathfrak{h} \oplus \mathfrak{p}_2 = A_1 \oplus D_6$ and $\mathfrak{p}_2 \subset D_6$. By the tables in [17], $\gamma_2 = \frac{5}{9}$. From (4.1) and the dimensions in Table 2, $\gamma_1 = \frac{4}{9}$ and $\gamma_3 = \frac{2}{3}$. It follows that

$$a_1 = \frac{5}{18}, \quad a_2 = \frac{2}{9}, \quad a_3 = \frac{1}{6}.$$

Let $x_1 = 1$. The equations $r_1 = r_2 = r_3$ are equivalent to

$$\begin{cases} x_2^2 + 4x_3^2 - 9x_2x_3 + 9x_2 - 4 = 0, \\ 7x_2^2 - 7x_3^2 + 18x_3 - 18x_2 - 1 = 0. \end{cases}$$

Excluding the summand containing x_2^2 form the first equation, we obtain

$$x_2(63x_3 - 81) = 35x_3^2 - 18x_3 - 27. \tag{4.6}$$

For this case, $63x_3 - 81 \neq 0$. Expressing x_2 by x_3 from (4.6) and inserting it into second one, we obtain

$$2744x_3^4 - 13482x_3^3 + 24732x_3^2 - 19926x_3 + 5832 = 0,$$

which has two real solutions $x_3 \approx 1.5535$ or $x_3 \approx 0.7302$. From (4.6), $x_2 \approx 1.7489$ or $x_2 \approx 0.6139$.

That is, the homogeneous manifold of type E_7-II in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters $(x_1, x_2, x_3) \approx (t, 1.7489t, 1.5535t)$ or $(x_1, x_2, x_3) \approx (t, 0.6139t, 0.7302t)$, where $t > 0$.

4.6. Invariant Einstein metrics on the three-locally-symmetric space of type E_7-III

For this case, $\mathfrak{h} \oplus \mathfrak{p}_1 = A_7$. It is the same as that of type E_7-II. That is, $\gamma_1 = \frac{4}{9}$. From (4.1) and the dimensions in Table 2, $\gamma_2 = \gamma_3 = \frac{4}{9}$. It follows that

$$a_1 = a_2 = a_3 = \frac{5}{18}.$$

By Theorem 4.1, i.e. Theorem 3 in [25], the homogeneous manifold of type E_7-III in Theorem 3.15 admits exactly four invariant Einstein metrics up to proportionality. The parameters (x_1, x_2, x_3) have the form (t, t, t), $\left(\frac{4}{9}t, \frac{5}{9}t, \frac{5}{9}t\right)$, $\left(\frac{5}{9}t, \frac{4}{9}t, \frac{5}{9}t\right)$, or $\left(\frac{5}{9}t, \frac{5}{9}t, \frac{4}{9}t\right)$, where $t > 0$.

4.7. Invariant Einstein metrics on the three-locally-symmetric space of type E_8-I

For this case, $\mathfrak{h} \oplus \mathfrak{p}_2 = A_1 \oplus E_7$ and $\mathfrak{p}_2 \subset E_7$. By the tables in [17], $\gamma_2 = \frac{3}{5}$. From (4.1) and the dimensions in Table 2, $\gamma_1 = \frac{7}{15}$ and $\gamma_3 = \frac{3}{5}$. It follows that

$$a_1 = \frac{4}{15}, \quad a_2 = a_3 = \frac{1}{5}.$$

If $x_2 = x_3$, then the equation (4.2) is

$$\frac{7}{15}x_2^2 - x_1x_2 + \frac{7}{15}x_1^2 = 0.$$

The discriminant $1 - (\frac{14}{15})^2 > 0$, which implies that the equation has two solutions. If $x_1 = \frac{2}{5}(x_2 + x_3)$, then the equation (4.2) equals with

$$147x_2^2 - 281x_2x_3 + 147x_3^2 = 0.$$

The discriminant $281^2 - 294^2 < 0$, which implies that the equation has no solution.

That is, the homogeneous manifold of type E_8-I in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters (x_1, x_2, x_3) have the form (qt, t, t), where $t > 0$ and q is the root of the equation $7x^2 - 15x + 7 = 0$.

4.8. Invariant Einstein metrics on the three-locally-symmetric space of type E_8-II

For this case, $\mathfrak{h} \oplus \mathfrak{p}_1 = D_8$. It is the same as that of type E_8-I. That is, $\gamma_1 = \frac{7}{15}$. From (4.1) and the dimensions in Table 2, $\gamma_2 = \gamma_3 = \frac{7}{15}$. It follows that

$$a_1 = a_2 = a_3 = \frac{4}{15}.$$

By Theorem 4.1, i.e. Theorem 3 in [25], the homogeneous manifold of type E_8-II in Theorem 3.15 admits exactly four invariant Einstein metrics up to proportionality. The parameters (x_1, x_2, x_3) have the form (t, t, t), $\left(\frac{7}{15}t, \frac{8}{15}t, \frac{8}{15}t\right)$, $\left(\frac{8}{15}t, \frac{7}{15}t, \frac{8}{15}t\right)$, or $\left(\frac{8}{15}t, \frac{8}{15}t, \frac{7}{15}t\right)$, where $t > 0$.

4.9. Invariant Einstein metrics on the three-locally-symmetric space of type F_4-II

For this case, $\mathfrak{h} \oplus \mathfrak{p}_1 = D_4$. By the tables in [17], $\gamma_1 = \frac{7}{9}$. From (4.1) and the dimensions in Table 2, $\gamma_2 = \gamma_3 = \frac{4}{9}$. It follows that

$$a_1 = \frac{1}{9}, \quad a_2 = a_3 = \frac{5}{18}.$$

If $x_2 = x_3$, then the equation (4.2) is

$$\frac{7}{9}x_2^2 - x_1 x_2 + \frac{7}{18}x_1^2 = 0.$$

The discriminant $1 - \frac{98}{81} < 0$, which implies that the equation has no solution. There exists none Einstein metrics. If $x_1 = \frac{5}{9}(x_2 + x_3)$, then the equation (4.2) equals with

$$196x_2^2 - 499x_2x_3 + 196x_3^2 = 0.$$

The discriminant $499^2 - 392^2 > 0$, which implies that the equation has two solutions.

That is, the homogeneous manifold of type F_4-II in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters (x_1, x_2, x_3) have the form $(\frac{5}{9}(q+1)t, qt, t)$, where $t > 0$ and q is the root of the equation $196x^2 - 499x + 196 = 0$.

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References

[1] N. A. Abiev, A. Arvanitoyeorgos, Yu. G. Nikonorov, and P. Siasos, *The dynamics of the Ricci flow on generalized Wallach spaces*, Diffrential Geom. Appl. **35** (2014), Supplement, 26–43.

[2] N. A. Abiev, A. Arvanitoyeorgos, Yu. G. Nikonorov, and P. Siasos, *The Ricci flow on some generalized Wallach spaces*, in: V. Rovenski,

P. Walczak (eds.), Geometry and its Applications, Springer Proceedings in Mathematics and Statistics, V. 72, Switzerland: Springer, 2014, VIII+243 p. 3–37.

[3] D. Alekseevsky, I. Dotti, and C. Ferraris, *Homogeneous Ricci positive 5-manifolds*, Pacific J. Math. **175** (1996), 1–12.

[4] D. Alekseevsky and B. Kimel'fel'd, *Structure of homogeneous Riemannian spaces with zero Ricci curvature*, Functional Anal. Appl. **9** (1975), 97–102.

[5] A. Arvanitoyeorgos, *New invariant Einstein metrics on generalized flag manifolds*, Trans. Amer. Math. Soc. **337** (1993), 981–995.

[6] A. Arvanitoyeorgos and I. Chrysilos, *Invariant Einstein metrics on genelalized flag manifolds with two isotropy summands*, J. Aust. Math. Soc. **90** (2011), 237–251.

[7] Y. Bahturin and M. Goze, $\mathbb{Z}_2 \times \mathbb{Z}_2$*-symmetric spaces*, Pacific J. Math. **236** (2008), 1–21.

[8] M. Berger, *Les espaces symétriques noncompacts*, Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177.

[9] A. Besse, *Einstein manifolds*, Ergeb. Math. **10** (1987), Springer-Verlag, Berlin-Heidelberg.

[10] C. Böhm, *Homogeneous Einstein metrics and simplicial complexes*, J. Differetial Geom. **67** (2004), 79–165.

[11] C. Böhm and M. Kerr, *Low-dimensional homogeneous Einstein manifolds*, Trans. Amer. Math. Soc. **358** (2006), no. 4, 1455–1468.

[12] C. Böhm, M. Wang, and W. Ziller, *A variational approach for homogeneous Einstein metrics*, Geom. Funct. Anal. **14** (2004), 681–733.

[13] Z. Chen and K. Liang, *Classification of analytic involution pairs of Lie groups (in Chinese)*, Chinese Ann. Math. Ser. A **26** (2005), no. 5, 695–708; translation in Chinese J. Contemp. Math. **26** (2006), no. 4, 411–424.

[14] Z. Chen and K. Liang, *Non-naturally reductive Einstein metrics on the compact simple Lie group* F_4, Ann. Glob. Anal. Geom. **46** (2014), 103–115.

[15] M. K. Chuan and J. S. Huang, *Double Vogan diagrams and semisimple symmetric spaces*, Trans. Amer. Math. Soc. **362** (2010), 1721–1750.

[16] J. E. D'Atri and H. K. Nickerson, *Geodesic symmetries in spaces with special curvature tensors*, J. Differetial Geom. **9** (1974), 252–262.

[17] J. E. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Memoirs of Amer. Math. Soc. **215** (1979).

[18] G. W. Gibbons, H. Lü, and C. N. Pope, *Einstein metrics on group manifolds and cosets*, J. Geom. Phys. **61** (2011), no. 5, 947–960.

[19] J. Heber, *Noncompact homogeneous Einstein spaces*, Invent. math. **133** (1998), 279–352.

[20] J. S. Huang and J. Yu, *Klein four-subgroups of Lie algebra automorphisms*, Pacific J. Math. **262** (2013), 397–420.

[21] M. Kerr, *New examples of homogeneous Einstein metrics*, Michigan J. Math. **45** (1998), 115–134.

[22] M. Kimura, *Homogeneous Einstein metrics on certain Kahler C-spaces*, Adv. Stud. Pure Math., **18** (1990), 303–320.

[23] A. Kollross, *Exceptional* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-symmetric spaces*, Pacific J. Math. **242** (2009), 113–130.

[24] J. Lauret, *Einstein solvmanifolds are standard*, Ann. Math. (2) **172** (2010), no. 3, 1859–1877.

[25] A. M. Lomshakov, Yu. G. Nikonorov, and E. V. Firsov, *Invariant Einstein metrics on three-locally-symmetric spaces*, Mat. Tr. **6** (2003), no. 2, 80–101; translation in Siberian Adv. Math. **14** (2004), no. 3, 43–62.

[26] K. Mori, *Left invariant Einstein metrics on* $SU(n)$ *that are not naturally reductive*, Master Thesis (in Japanese) Osaka University 1994, English translation Osaka University RPM 96-10 (preprint series) 1996.

[27] A. H. Mujtaba, *Homogeneous Einstein metrics on* $SU(n)$, J. Geom. Phys. **62** (2012), no. 5, 976–980.

[28] Yu. G. Nikonorov, *On a class of homogeneous compact Einstein manifolds*, Sibirsk. Mat. Zh., **41** (2000), 200–205; translation in Siberian Math. J. **41** (2000), 168–172.

[29] Yu. G. Nikonorov, *Classification of generalized Wallach spaces*, Geom. Dedicata **181** (2016), 193–212.

[30] Yu. G. Nikonorov, E. D. Rodionov, and V. V. Slavskii, *Geometry of homogeneous Riemannian manifolds*, J. Math. Sci. **146** (2007), no. 6, 6313–6390.

[31] C. N. Pope, *Homogeneous Einstein metrics on* $SO(n)$, arXiv:1001.2776, 2010.

[32] E. D. Rodionov, *Einstein metrics on even-dimensional homogeneous spaces admitting a homogeneous Riemannian metric of positive sectional curvature*, Siberian Math. J. **41** (1991), 168–172.

[33] A. Sagle, *Some homogeneous Einstein manifolds*, Nagoya Math. J. **39** (1970), 81–106.

[34] B. L. van der Waerden, *Algebra*, Springer-Verlag, Berlin, 1964.

[35] N. R. Wallach, *Compact homognenous Riemannian manifolds with strictly positive curvature*, Ann. Math. (2) **96** (1972), 277–295.

[36] M. Wang and W. Ziller, *Existence and non-existence of homogeneous Einstein metrics*, Invent. Math. **84** (1986), 177–194.

[37] Z. D. Yan, *Real semisimple Lie algebras (in chinese)*, Nankai University Press, Tianjin, 1998.

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