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Boundedness of Laplacian eigenfunctions on manifolds of infinite volume

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In a Hadamard manifold M, it is proved that if u is a λ-eigenfunction of the Laplacian that belongs to $L^p(M)$ for some $p \geq 2$, then u is bounded and $\|u\|_{L^\infty} \leq C\|u\|_{L^p}$, where C depends only on p, λ and the dimension of M. This result is obtained in the more general context of a Riemannian manifold endowed with an isoperimetric function H satisfying some integrability condition. In this case, the constant C depends on p, λ and H.

1. Introduction

Let M be a Riemannian manifold with Laplace-Beltrami operator Δ. Given an open smooth subset $U \subset M$ and $\lambda \in \mathbb{R}$, we call λ-eigenfunction of U any nontrivial $u \in C^2(U)$ that satisfies

$$\Delta u + \lambda u = 0 \text{ in } U. \tag{1}$$

If in addition U has nonempty boundary, we require that u vanishes continuously on ∂U.

When U is compact, according to the spectral theory for elliptic operators, the numbers λ for which (1) has a nontrivial classical solution are terms of an increasing unbounded sequence. They are called the eigenvalues of $-\Delta$ and are the only elements of its spectrum. If U is noncompact, the situation is more delicate since the spectrum of $-\Delta$ can contain elements that are not eigenvalues. Furthermore, a λ-eigenfunction may not belong to $L^2(U)$ or even to $L^\infty(U)$. This raises the questions whether a λ-eigenfunction is in some $L^p(U)$ and whether this implies its boundedness.

In this setting, A. Cianchi and V.G. Maz'ya [5] investigate bounds for L^2 eigenfunctions in noncompact manifolds of finite measure. They considered a slightly different eigenvalue problem, which coincides with (1) in the case

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of empty boundary. Under assumptions on the isoperimetric profile of M (see Definition 2.1), the authors obtained the estimate

$$\|u\|_{L^\infty} \leq C\|u\|_{L^2} \tag{2}$$

for any u λ-eigenfunction of M, where $C > 0$ is a constant that depends only on the isoperimetric profile and on λ. Furthermore, their result is sharp in the sense that if H is a suitable isoperimetric profile that does not satisfy the assumptions above mentioned, then there is a manifold M with isoperimetric profile close to H that admits an eigenfunction $u \in L^2(M)\backslash L^\infty(M)$.

In the present work we prove that, under the same assumptions on the isoperimetric profile considered by A. Cianchi and V.G. Maz'ya in [5], the estimate (2) holds with no restriction on the volume of M. Moreover, we also obtain the more general estimate $\|u\|_{L^\infty} \leq C\|u\|_{L^p}$ for λ-eigenfunctions in L^p, $p \geq 2$.

Another tool to obtain this kind of estimate is to apply the Moser iteration technique ([11], [9]). In the classical case, this technique uses Sobolev inequalities for solutions that belong to H^1. However there are Riemannian manifolds that do not admit standard Sobolev inequalities or λ-eigenfunctions that are not L^2. In the final section, we present a manifold where we can apply our results, but for which the inequalities of the form

$$\|u\|_{L^s} \leq C\|\nabla u\|_{L^t}$$

do not hold for any $s, t \geq 1$.

These new results establishing a bound for the L^∞ norm are useful since for some interesting manifolds with infinite volume there exist λ-eigenfunctions in L^2. This is the case of Hadamard manifolds with curvature going to minus infinity, for which the existence of eigenfunctions in L^2 is proved in [8]. Another application is for manifolds that do not admit eigenfunctions in L^2, but in L^p for $p > 2$. The hyperbolic spaces $\mathbb{H}^n$ are examples of such manifolds, where for $\lambda \leq \lambda_1(\mathbb{H}^n)$ the λ-eigenfunctions belong to L^p for $p > 2/\sqrt{1 - \lambda/\lambda_1}$, see [1].

2. Main result

The statement of our main result requires the definition of isoperimetric functions in manifolds. This concept is a generalisation of isoperimetric profile, which is the largest isoperimetric function of a manifold.

Definition 2.1. Consider M a Riemannian manifold. An isoperimetric function on M is a function $H : [0, |M|] \to \mathbb{R}$ that satisfies

$$H(|\Omega|) \leq |\partial\Omega| \; \forall\, \Omega \subset\subset M, \tag{3}$$

where $|\cdot|$ stands for the Hausdorff measure. If M has infinite measure, H is defined in $[0, \infty)$.

The number $H(s)$ gives a lower bound for the measure of the boundary of any set of volume s. Since any Riemannian manifold is locally Euclidean, it is expectable that there exists H such that

$$H(s) \approx Cs^{1-1/n},$$

for s close to zero. This is true in compact manifolds, but might fail for noncompact ones. Nevertheless, even a noncompact manifold may admit an isoperimetric function good enough for our result. This is the case when there exists H for which the associated isoperimetric function (a.i.f.), given by

$$H_a(t) := \int_0^t \frac{s}{H(s)^2} ds, \quad t \in [0, |M|),$$

is well-defined.

Theorem 2.2. *Let M be a Riemannian manifold and let H be an isoperimetric function on M with a well-defined a.i.f. H_a. Consider a smooth domain $U \subset M$, possibly unbounded, $\lambda > 0$ and $p \geq 2$. There exists a constant $C = C(\lambda, p, H)$ such that for any nontrivial solution $w \in C^2(U) \cap C^0\left(\overline{U}\right)$ of*

$$\begin{cases} -\Delta w = \lambda w & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases} \tag{4}$$

that belongs to $L^p(U)$, it holds

$$\|w\|_\infty \leq C\|w\|_p.$$

Moreover, this constant is given by $C(\lambda, p, H) = 2\left(H_a^{-1}(\frac{1}{2\lambda})\right)^{-1/p}$.

2.1. Lemmata

In this section we prove some lemmas required in the proof of Theorem 2.2. Henceforth M will be a Riemannian manifold endowed with an isoperimetric function H that has a well defined a.i.f. H_a.

The first lemma is a kind of Alexandrov's Maximum Principle that bounds the supremum of a function using the a.i.f. H_a. The idea of the proof is the same as that presented by Talenti [13], among others, and we write it here for completeness.

Lemma 2.3. *Let $\Omega \subset M$ be a domain with finite measure and non empty boundary. If u is a solution of $-\Delta u = 1$ in Ω, $u = 0$ on $\partial\Omega$, then*

$$\sup u \leq H_a\left(|\Omega|\right).$$

Proof. We first consider the case that Ω is bounded and hence the solution u must be nonnegative.

Since $-\Delta u = 1$ in Ω, it holds that $u \in C^\infty(\Omega)$. Considering μ the distribution function of u, defined as $\mu(t) = |\{x \in \Omega \mid u(x) > t\}|$, we have

$$\mu'(t) = -\int_{\{u=t\}} \frac{1}{|\nabla u|} da_t, \tag{5}$$

for almost all t, where da_t stands for the Hausdorff measure of $\{u = t\}$.

For all $t > 0$, the set $\{u(x) > t\}$ is at positive distance from the boundary $\partial\Omega$. It is compactly contained in Ω, has boundary $\{u = t\}$ with inner normal vector $\frac{\nabla u}{|\nabla u|}$ well defined for almost all $t > 0$ by Sard's Theorem. If the inner normal vector is well defined for t, we apply the Divergence Theorem on the differential equation $-\Delta u = 1$, obtaining

$$\mu(t) = \int_{\{u>t\}} 1\, dV = \int_{\{u=t\}} |\nabla u| da_t. \tag{6}$$

Applying the isoperimetric function H in $\mu(t)$ and using Cauchy-Schwarz inequality, we obtain

$$H(\mu(t)) \leq |\{u = t\}| \leq \left(\int_{\{u=t\}} \frac{1}{|\nabla u|} da_t\right)^{1/2} \left(\int_{\{u=t\}} |\nabla u| da_t\right)^{1/2}.$$

Hence, for almost all $t > 0$, expressions (5) and (6) imply

$$H^2(\mu(t)) \leq -\mu'(t)\mu(t) \text{ and } 1 \leq \frac{\mu(t)(-\mu'(t))}{H^2(\mu(t))} = -\frac{d}{dt} H_a(\mu(t)).$$

Integrating the above inequality in $[0, \sup u]$,

$$\sup u \leq -\left(H_a(\mu(\sup u)) - H_a(|\Omega|)\right) = H_a(|\Omega|),$$

the proof is complete for bounded Ω.

To the general case, let u_n be the solution of $-\Delta u_n = 1$ in Ω_n, $u_n = 0$ on $\partial\Omega_n$, where (Ω_n) is an increasing sequence of bounded sets such that $\Omega = \cup\Omega_n$. From the first case, (u_n) is uniformly bounded by $H_a(|\Omega|)$ and, from the Maximum Principle, is an increasing sequence. Hence u_n converges to a solution u that is bounded by $H_a(|\Omega|)$, proving the result. □

We remind that the first (Dirichlet) eigenvalue of the Laplacian operator of a domain U is the infimum of the Rayleigh quotient,

$$\lambda_1(U) = \inf\left\{ \frac{\|\nabla v\|_2^2}{\|v\|_2^2} \;\middle|\; v \in H_0^1(U) \right\}.$$

If U is a bounded smooth domain, $\lambda_1(U)$ is the smallest positive real number λ for which U admits a nontrivial $\lambda-$eigenfunction. If $U = M$, a compact Riemannian manifold without boundary, $\lambda_1(M) = 0$, since the constant functions are $0-$eigenfunctions in M. More generally, for any domain U, possibly unbounded, its first eigenvalue is

$$\lambda_1(U) = \inf\{\lambda_1(\Omega) \mid \Omega \subset\subset U,\ \Omega \text{ smooth}\}.$$

It is well known that if $\Omega \subset M$ is a bounded domain with $\partial\Omega \neq \emptyset$, $\lambda \leq \lambda_1(\Omega)$ and u satisfies $\Delta u + \lambda u \leq 0$ in Ω, $u \geq 0$ on $\partial\Omega$ and $u \not\equiv 0$ on $\partial\Omega$, then $u \geq 0$ in Ω. Although this property may be false for unbounded domains, its converse is true.

Lemma 2.4. *Let $\Omega \subset M$ be a domain, possibly unbounded. Given $\lambda > 0$, if there is a function $u > 0$, $u \in C^2(\Omega)$ satisfying $\Delta u + \lambda u \leq 0$, then $\lambda \leq \lambda_1(\Omega)$.*

Proof. Suppose that $\lambda > \lambda_1(\Omega)$. There exists a bounded smooth domain $\Omega_0 \subset\subset \Omega$ with

$$\lambda > \lambda_1(\Omega_0) > \lambda_1(\Omega).$$

Let u_0 be a positive eigenfunction associated to $\lambda_1(\Omega_0)$ in Ω_0. Since Ω_0 is smooth, $u_0 \in C^0(\overline{\Omega}_0) \cap C^2(\Omega_0)$ and $u_0 = 0$ on $\partial\Omega_0$. Since $u \in C^2(\Omega)$, $u \in C^2(\overline{\Omega}_0)$ and the positivity of u in Ω implies that

$$\alpha := \sup_{\Omega_0} \frac{u_0}{u}$$

is finite. Therefore the function $\alpha u - u_0$ defined in $\overline{\Omega}_0$ contradicts the Maximum Principle because its minimum value is 0, attained in an interior point of Ω_0, but $-\Delta(\alpha u - u_0) \geq 0$ in Ω_0. □

The next proposition is a preliminary version of the main result for bounded domains and it is crucial in the proof. Not only it establishes a bound to the maximum of the solution of some Dirichlet problem by its L^p norm, but gives an estimate that does not depend on the diameter, measure or boundary of its domain Ω. The idea in the proof of Theorem 2.2 is to build a sequence of functions defined in bounded domains that converges to the solution. The uniform boundedness of this sequence is guaranteed by the proposition below. We remark that the positive boundary data is necessary, because there is no λ-eigenfunction in a bounded domain Ω for $\lambda < \lambda_1(\Omega)$.

The proof follows ideas from an estimate for quotient relating different norms of eigenfunctions of some elliptic operators in domains of $\mathbb{R}^n$ presented in [2] (Section 7). The associated isomperimetric function (a.i.f.) arises to adapt these ideas for manifolds. This was used in Lemma 2.3 to estimate the supremum of a function with constant Laplacian by the measure of its domain.

Proposition 2.5. *Let $\Omega \subset M$ be a bounded domain with non empty boundary and $w \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a nontrivial solution of*

$$\begin{cases} -\Delta w = \lambda w & \text{in } \Omega \\ w = \gamma & \text{on } \partial\Omega \end{cases}$$

for some $0 < \lambda \leq \lambda_1(\Omega)$ and $\gamma \geq 0$. Then, for any $p > 1$,

$$\|w\|_p^p \geq \left(\frac{\|w\|_\infty + \gamma}{2}\right)^p {H_a}^{-1}\left(\frac{\|w\|_\infty - \gamma}{2\lambda\|w\|_\infty}\right). \tag{7}$$

Proof. Denote by $K = \|w\|_\infty$. Notice that w cannot change sign, because $\lambda \leq \lambda_1(\Omega)$. We therefore assume without loss of generality that $w \geq 0$.

Let

$$\widetilde{\Omega} = \left\{ x \in \Omega \;\middle|\; w(x) > \frac{K+\gamma}{2} \right\}.$$

Then,

$$\|w\|_p^p = \int_\Omega |w|^p dV \geq \int_{\widetilde{\Omega}} |w|^p dV \geq \left(\frac{K+\gamma}{2}\right)^p |\widetilde{\Omega}| \tag{8}$$

On the other hand,

$$-\Delta w = \lambda w \leq \lambda K.$$

By the Comparison Principle, $w \leq u$ in $\widetilde{\Omega}$ where u satisfies

$$\begin{cases} -\Delta u = \lambda K \text{ in } \widetilde{\Omega} \\ u = \frac{K+\gamma}{2} \text{ on } \partial\widetilde{\Omega}. \end{cases}$$

Lemma 2.3 gives an upper bound for u

$$K \leq \sup u \leq \frac{K+\gamma}{2} + \lambda K H_a\left(|\widetilde{\Omega}|\right).$$

Hence,

$$|\widetilde{\Omega}| \geq H_a{}^{-1}\left(\frac{K-\gamma}{2\lambda K}\right).$$

Therefore, from inequality (8), we obtain

$$\|w\|_p^p \geq \left(\frac{K+\gamma}{2}\right)^p H_a{}^{-1}\left(\frac{K-\gamma}{2\lambda K}\right).$$

□

Remark. For the case $\gamma = 0$, the result is true for any $\lambda > 0$, because we may assume $\max w = \max |w|$.

Lemma 2.6. *Let $\Omega \subset M$ be a domain with non empty boundary and finite volume, possibly unbounded, and $u \in C^2(\overline{\Omega})$ be a classical solution of*

$$\begin{cases} -\Delta u = \lambda u \ \textit{in } \Omega \\ u > \gamma \ \textit{in } \Omega \\ u = \gamma \ \textit{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ and $\gamma > 0$. If $u \in L^p(\Omega)$ for $p \geq 2$, then $u \in H^1(\Omega)$ and

$$\int_\Omega |\nabla u|^2 dV \leq \lambda \int_\Omega u^2 dV.$$

Proof. For $\varepsilon > 0$, let $\psi = \psi_\varepsilon$ be defined in Ω by $\psi(x) = u(x) - \gamma - \varepsilon$. Let $A = A_\varepsilon = \{\psi > 0\} = \{u > \gamma + \varepsilon\}$. We claim that $u \in H^1(A)$ and

$$\int_A |\nabla u|^2 \, dV \leq \lambda \int_A u^2 \, dV.$$

Hence, letting ε go to zero, the result holds.

In order to prove the claim, fix $o \in M$ and, for each $R > 1$, let $\eta = \eta_R : M \to \mathbb{R}$ be a smooth radial function satisfying: $\eta_R(x) \in [0,1]$, $\eta_R(x) = 1$ if $x \in B_{R-1}(o)$, $\eta_R(x) = 0$ if $x \notin B_R(o)$ and $|\nabla \eta_R(x)| < 2$.

Let $w = w_R$ be defined in Ω by $w(x) = \eta^2(x)\psi(x) = \eta^2(x)(u(x) - \gamma - \varepsilon)$. Since $u, \eta \in C^2(\overline{\Omega})$, we have $w \in H^1(\Omega \cap B_{R+1})$. Hence, the positive part of w, $w^+ = \max\{w, 0\}$, is in $H^1(\Omega \cap B_{R+1})$. Besides, $w^+ = 0$ in some neighborhood of $\partial(\Omega \cap B_{R+1})$ because $\psi \leq -\varepsilon$ on $\partial\Omega$ and $\eta = 0$ on $B_{R+1} \setminus B_R$. This implies $w^+ \in H_0^1(\Omega \cap B_{R+1})$ and

$$\int_\Omega \nabla w^+ \nabla u \, dV = \lambda \int_\Omega w^+ u \, dV$$

since u is a weak solution.

Observe that

$$w^+ = [\eta^2 \psi]^+ = \eta^2 \psi^+ \quad \text{and} \quad \nabla w^+ = [\eta^2 \nabla u + (u - \gamma - \varepsilon) 2\eta \nabla \eta]\chi_A$$

almost everywhere in Ω. Therefore

$$\int_A \eta^2 |\nabla u|^2 + 2\psi\eta\nabla\eta\nabla u \, dV = \lambda \int_A \eta^2 \psi u \, dV$$

and, from Cauchy-Schwarz inequality,

$$\begin{aligned} \int_A \eta^2 |\nabla u|^2 \, dV &\leq \lambda \int_A \eta^2 \psi u \, dV \\ &\quad + 2\left(\int_A \eta^2 |\nabla u|^2 dV\right)^{1/2} \left(\int_A \psi^2 |\nabla \eta|^2 dV\right)^{1/2}. \end{aligned} \tag{9}$$

Therefore, using that $|\eta| \leq 1$, $|\nabla \eta| \leq 2$, and $|\psi| \leq u$ in A, we get

$$\int_A \eta^2 |\nabla u|^2 \, dV \leq \lambda \int_A u^2 \, dV + 4\left(\int_A \eta^2 |\nabla u|^2 dV\right)^{1/2} \left(\int_A u^2 dV\right)^{1/2}.$$

The integral $\int_A u^2 dV$ is finite since $u \in L^p(\Omega)$ for $p \geq 2$, $A \subset \Omega$ and $|\Omega| < \infty$. Hence, for any $R > 0$, the corresponding $\eta = \eta_R$ satisfies

$$\int_A \eta_R^2 |\nabla u|^2 \, dV \leq \left(2 + \sqrt{4 + \lambda}\right)^2 \int_A u^2 dV < \infty.$$

Since $\eta_R \to 1$ as $R \to \infty$, it follows that $\int_A |\nabla u|^2\, dV < \infty$. Moreover, $u \in L^2(A)$ implies that

$$\int_A u^2 |\nabla \eta|^2 dV \leq 4 \int_{A \setminus B_R} u^2 dV \to 0 \quad \text{as} \quad R \to \infty,$$

then making $R \to \infty$ in (9), we get

$$\int_A |\nabla u|^2\, dV \leq \lambda \int_A \psi u\, dV \leq \lambda \int_A u^2\, dV,$$

concluding the proof. □

The next lemma is some uniqueness result based on one established by Brézis and Oswald [3].

Lemma 2.7. *Let $\Omega \subset M$ be a domain with non empty boundary and finite measure, possibly unbounded. If $u_1, u_2 \in H^1(\Omega) \cap C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $0 < u_1 \leq u_2$ are classical solutions of*

$$\begin{cases} -\Delta u = au + b & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

for a, b positive constants, then $u_1 = u_2$.

Proof. Since $u_1, u_2 \in H^1(\Omega) \cap C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u_1 = u_2 = 0$ on $\partial\Omega$, we can prove that $u_1, u_2 \in H^1_0(\Omega)$. Hence, from the definition of weak solution,

$$\int_\Omega \nabla u_1 \nabla u_2\, dV = \int_\Omega (au_1 + b) u_2\, dV$$
$$\text{and} \quad \int_\Omega \nabla u_2 \nabla u_1\, dV = \int_\Omega (au_2 + b) u_1\, dV.$$

Thus

$$\int_\Omega \left(\frac{au_1 + b}{u_1} - \frac{au_2 + b}{u_2} \right) u_1 u_2\, dV = 0.$$

It is then clear that the integrand is nonnegative, which implies that it is equal to zero. Hence

$$\frac{au_1 + b}{u_1} = \frac{au_2 + b}{u_2},$$

and, therefore, $u_1 = u_2$ completing the proof. □

Remark. Lemma 2.7 holds in a more general setting: if $\beta \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that $f(t+\beta)/t$ is decreasing, then the same conclusion holds considering the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = \beta & \text{on } \partial\Omega. \end{cases}$$

2.2. Proof of Theorem 2.2

We assume that $w > 0$ in U, otherwise we split U into two subsets. Fix a point $o \in U$, let $\gamma_0 = w(o)/2$ and, for $0 < \gamma \leq \gamma_0$, consider the set $\Omega = \{x \in U \mid w(x) > \gamma\}$. Then Ω is not empty, $\overline{\Omega} \subset U$ and it has finite measure since

$$\gamma^p |\Omega| \leq \int_\Omega |w|^p \, dV < \infty.$$

Moreover, since $\Delta w + \lambda w = 0$ and w is positive in Ω, it follows from Lemma 2.4 that $\lambda \leq \lambda_1(\Omega)$. Now define $\Omega_k = \Omega \cap B_k(o)$ for $k \in \mathbb{N}$ and let $z_k \in H_0^1(\Omega_k)$ be a weak solution of

$$\begin{cases} -\Delta v = \lambda v + \lambda\gamma & \text{in } \Omega_k \\ v = 0 & \text{on } \partial\Omega_k \end{cases} \tag{10}$$

If $\partial\Omega_k \cap \Omega \neq \emptyset$, the existence and uniqueness of solution to this problem is a consequence of $\lambda \leq \lambda_1(\Omega) < \lambda_1(\Omega_k)$ and the classical theory for eigenvalue problems in PDE. Observe that $w_k := z_k + \gamma$ is a weak solution of

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega_k \\ v = \gamma & \text{on } \partial\Omega_k \end{cases}$$

and that $w_k = \gamma \leq w$ on $\partial\Omega_k$. Then $w_k \leq w$ in Ω_k because $\lambda \leq \lambda_1(\Omega) \leq \lambda_1(\Omega_k)$ and $w_k < w$ at some point of $\partial\Omega$. If $\partial\Omega_k \subset \partial\Omega$, we take $z_k = w - \gamma < w$. By the same argument, $w_k \geq w_m > \gamma$ for any $k > m$. Hence from Proposition 2.5, for each $k \in \mathbb{N}$,

$$\left(\frac{\|w_k\|_\infty + \gamma}{2}\right)^p {H_a}^{-1}\left(\frac{\|w_k\|_\infty - \gamma}{2\lambda\|w_k\|_\infty}\right) \leq \|w_k\|_p^p \leq \|w\|_p^p. \tag{11}$$

Since ${H_a}^{-1}$ is an increasing positive function, the left-hand side would diverge to infinity if $\|w_k\|_\infty$ went to infinity. Hence $(\|w_k\|_\infty)$ is a bounded sequence and, since it is also increasing, it converges pointwise to some bounded function $\bar{w}$ defined on Ω. Moreover $w_k \leq \bar{w} \leq w$. To prove that w is bounded, it is sufficient to show that $w = \bar{w}$ in $H^1(\Omega)$. We observe that since Ω has finite measure and $w \in L^p(\Omega)$, $p \geq 2$, Lemma 2.6 implies that $w \in H^1(\Omega)$.

Let $z = w - \gamma$ and $\bar{z} = \bar{w} - \gamma$. Then $z_k \leq \bar{z} \leq z$ and z_k converges pointwise to $\bar{z}$. We have to show that $\bar{z} = z$. The idea is to verify that $z, \bar{z} \in H^1(\Omega) \cap C^2(\Omega) \cap C^0(\overline{\Omega})$ and satisfy the same Dirichlet problem.

First, notice that $z \in C^2(\overline{\Omega})$ since $z = w - \gamma$, $w \in C^2(U)$ and $\overline{\Omega} \subset U$. Besides $z \in H^1(\Omega)$ because $0 \leq z = w - \gamma \leq w$ and $w \in H^1(U)$. Moreover z is a solution of

$$\begin{cases} -\Delta v = \lambda v + \lambda\gamma & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

We now prove similar results for $\bar{z}$. Since z_k is solution of (10), $0 \leq w_k - \gamma = z_k \leq w_k \leq w$ and $\Omega_k \subset \Omega$, it follows that

$$\begin{aligned} \int_{\Omega_k} |\nabla z_k|^2 dV &= \lambda \int_{\Omega_k} z_k^2 \, dV + \lambda\gamma \int_{\Omega_k} z_k \, dV \\ &\leq \lambda \int_{\Omega} w^2 \, dV + \lambda\gamma \int_{\Omega} w \, dV < \infty. \end{aligned}$$

Hence the sequence $(\|\nabla z_k\|_2)$ is bounded. Moreover, $z_k \leq w$ implies that $(\|z_k\|_2)$ is bounded. Therefore, up to some subsequence, z_k converges weakly to some function in $H_0^1(\Omega)$. This limit is $\bar{z}$, since z_k converges pointwise to $\bar{z}$. Thus $\bar{z}$ is a weak solution of (12). Then $\bar{z}$ is a classical solution and it is of class $C^2(\Omega)$. The continuity of $\bar{z}$ on $\partial\Omega$ is a consequence of $0 \leq \bar{z} \leq z$ and of the fact that z vanishes continuously on $\partial\Omega$.

Therefore, z and $\bar{z}$ satisfy the hypotheses of Lemma 2.7, which implies uniqueness of solution of problem (12). Hence $z = \bar{z}$ and $w = \bar{w}$ in Ω, proving that w is bounded and $\|w\|_{L^\infty(\Omega)} = \|\bar{w}\|_{L^\infty(\Omega)}$.

Furthermore, since $w_k \to \bar{w}$ and $w_k \leq \bar{w}$, we have

$$\|w_k\|_{L^\infty(\Omega)} \to \|\bar{w}\|_{L^\infty(\Omega)} = \|w\|_{L^\infty(\Omega)}.$$

This implies that (11) also holds replacing w_k by w. Observe also that $\gamma > 0$ can be chosen as small as we want and then it can be omitted in (11), obtaining

$$\left(\frac{\|w\|_\infty}{2}\right)^p {H_a}^{-1}\left(\frac{1}{2\lambda}\right) \leq \|w\|_{L^p(\Omega)}^p,$$

which completes the proof. □

3. Applications

3.1. Application to Hadamard manifolds

A Hadamard manifold is a complete simply connected Riemannian manifold with nonpositive sectional curvature. Hadamard manifolds admit some isoperimetric function with well-defined a.i.f. and, therefore, Theorem 2.2 applies to them. Indeed it is a consequence of Theorem 3.1 proved by Christopher B. Croke [7] for $n \geq 3$. For $n = 2$ the theorem also holds even for more general manifolds according to E. F. Beckenbach and T. Radó [4].

Theorem 3.1 ([4], [7]). *Let N be a compact $n-$dimensional, $n \geq 2$, Riemannian manifold (with boundary) of nonpositive sectional curvature. Suppose that any geodesic ray in N minimizes length up to the point it hits the boundary. Then there exists a positive constant $D(n)$, that depends only on n, such that*

$$\text{Vol}(\partial N) \geq D(n)(\text{Vol}(N))^{1-1/n}. \tag{13}$$

If M is a Hadamard manifold, any smooth compact subdomain satisfies the hypotheses of the theorem. Hence $H(s) = D(n)s^{1-1/n}$ is an isoperimetric function on M and

$$H_a(t) = \int_0^t \frac{s}{H(s)^2} ds = \frac{n}{2(D(n))^2} t^{2/n}.$$

As a consequence of this and Theorem 2.2, we have

Theorem 3.2. *Suppose that M is a Hadamard manifold, U is an unbounded domain of M and w is a $\lambda-$eigenfunction in U. If $w \in L^p(U)$ for some $p \geq 2$, then w is bounded and*

$$\|w\|_{L^\infty(U)} \leq \frac{2\,(n\lambda)^{n/2p}}{(D(n))^{n/p}} \|w\|_{L^p(U)}.$$

For instance, this result holds for manifolds like $\mathbb{R}^n$ and hyperbolic spaces $\mathbb{H}^n$. For $M = \mathbb{R}^n$, this estimate was obtained by Chiti [6] with the best constant.

In [8], Donnelly and Li proved the following theorem about the existence of L^2 eigenfunctions on some manifolds with infinite volume. According to Theorem 3.2, these eigenfunctions are bounded.

Theorem 3.3 (Theorem 1.1 of [8]). *Let M be a complete simply connected negatively curved Riemannian manifold. Fix $p \in M$ and write $K(r) = \sup\{K(x,\pi) \,|\, d(x,p) \geq r\}$, where K is the sectional curvature of M and π denotes a 2−plane in T_xM. If $\lim K(r) = -\infty$ as $r \to \infty$, then Δ has pure point spectrum.*

3.2. Manifolds without classical Sobolev inequality

The existence of an isoperimetric inequality with the same expoent as (13) is equivalent to the existence of the Sobolev inequality $\|u\|_{L^q(M)} \leq C\|\nabla u\|_{L^p(M)}$ for $u \in H_0^1(M)$, where $p \geq 1$ and $q = np/(n-p)$ (see [12]). However, differently from the Hadamard case, some manifolds do not admit such isoperimetric inequality. In turn we cannot use Moser iterations with Sobolev inequalities in a usual way to control the supremum of a λ-eigenfunction by its L^p-norm. Nevertheless, Theorem 2.2 estimates these functions provided M has a suitable isoperimetric profile. In the sequel, we present an example of this situation.

Consider the rotationally symmetric surface $M = \mathbb{R} \times S^1$ with metric given by $ds^2 = dr^2 + f(r)^2 d\theta^2$, where S^1 is the unit circle and $d\theta^2$ is its standard metric, for

$$f(r) = \begin{cases} \sqrt{\alpha}\arctan(r/\sqrt{\alpha}), & \text{if } r \in [0,1] \\ B\sqrt{r+a}, & \text{if } r \geq 1, \end{cases}$$

where α, B and a are appropriated constants that turn M into a C^2 manifold. Precisely, $0 < \alpha < \pi/2$ is given by $\tan(\sqrt{\alpha}/2) = 1/\sqrt{\alpha}$, $B = \alpha(\alpha+1)^{-1/2}$ and $a = (\alpha - 3)/4$.

First we show that M does not admit any inequality of the type $\|u\|_{L^s} \leq C\|\nabla u\|_{L^t}$. Let $r = r(x) = dist(x, 0)$ and define for $R > 1$ the function $v_R(x) = v_R(r(x))$ given by $v_R(x) = 1$ for $r \in [0,1]$, $v_R(r) = 1 - \frac{r-1}{R}$ for $r \in (1, R+1)$ and $v_R(r) = 0$ for $r \geq R+1$. Since for $r \geq 1$ the volume element is given by $dS = B(r+a)^{1/2} dr d\theta$, we have for $R > 1$

$$\frac{\|\nabla v_R\|_{L^t}}{\|v_R\|_{L^s}} \leq CR^{\frac{3}{2t} - \frac{3}{2s} - 1}.$$

Hence s and t must satisfy $3/2t - 3/2s - 1 \geq 0$, otherwise the manifold does not have a Sobolev inequality. On the other hand, for $R < 1$, consider $w_R(r) = 1$ if $0 \leq r \leq R/2$, $w_R(r) = 2 - \frac{2r}{R}$ if $R/2 < r \leq R$ and $w_R(r) = 0$ if

$r > R$. Using that $dS \approx r dr d\theta$ for sufficiently small $r < 1$, we get

$$\frac{\|\nabla w_R\|_{L^t}}{\|w_R\|_{L^s}} \leq C R^{\frac{2}{t} - \frac{2}{s} - 1}.$$

Making $R \to 0$, it is necessary that $2/t - 2/s - 1 \leq 0$. Both conditions on s and t lead to $3(2/t - 2/s) - 3 \leq 0 \leq 4(3/2t - 3/2s) - 4$, which is an absurdity.

However, M has an isoperimetric function with well-defined a.i.f.. Indeed, due to Theorem 1.2 of [10], if N is a 2-dimensional rotationally symmetric manifold with strictly decreasing radial sectional curvature, then all the disks centered at the origin satisfy equality in the isoperimetric inequality. Therefore H depends only on f and hence $H(s) \approx C s^{1/2}$ near the origin, since $f(r) \approx r$ for $r \approx 0$. This implies that M has a well-defined a.i.f..

Finally we observe that M admits $\lambda-$eigenfunctions in L^p for $p > 6$, and therefore Theorem 2.2 may be applied. Indeed, a straightforward computation gives that any radially symmetric λ-eigenfunction $u(r)$ must satisfy

$$u'' + \frac{f'}{f} u' + \lambda u = 0.$$

Since $f(r) = \sqrt{r + a}$ for $r > 1$, this ODE has the form $u'' + u'/2(r + a) + \lambda u = 0$ for $r > 1$. Defining $v(r) = u(r)/(r + a)^{1/4}$ and making a change of variable, we conclude that

$$u(r) = c_1 \, (r + a)^{\frac{1}{4}} J_{\frac{1}{4}}(\sqrt{\lambda}(r + a)) + c_2 \, (r + a)^{\frac{1}{4}} J_{-\frac{1}{4}}(\sqrt{\lambda}(r + a)), \; r > 1,$$

where $J_{1/4}$ and $J_{-1/4}$ are Bessel functions of the first and second kind respectively, c_1 and c_2 are suitable constants. Since $|J_{1/4}(s)| < C s^{-1/2}$ and $|J_{-1/4}(s)| < C s^{-1/2}$ for $s > 1 + a > 0$ and some $C > 0$ [14], then we can prove that

$$\int_0^{2\pi} \int_1^{\infty} |u(r)|^p \sqrt{r + a} dr d\theta$$

is finite for any $p > 6$.

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