

On non-topological solutions of the \mathbf{G}_2 Chern-Simons system

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For any rank 2 of simple Lie algebra, the relativistic Chern-Simons system has the following form:

$$(0.1) \quad \begin{cases} \Delta u_1 + \left(\sum_{i=1}^2 K_{1i} e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{1i} e^{u_j} K_{ij} \right) = 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + \left(\sum_{i=1}^2 K_{2i} e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{2i} e^{u_j} K_{ij} \right) = 4\pi \sum_{j=1}^{N_2} \delta_{q_j} \end{cases} \quad \text{in } \mathbb{R}^2,$$

where K is the Cartan matrix of rank 2. There are three Cartan matrix of rank 2: \mathbf{A}_2 , \mathbf{B}_2 and \mathbf{G}_2 . A long-standing open problem for (0.1) is the question of the existence of non-topological solutions. In a previous paper [1], we have proved the existence of non-topological solutions for the \mathbf{A}_2 and \mathbf{B}_2 Chern-Simons system. In this paper, we continue to consider the \mathbf{G}_2 case. We prove the existence of non-topological solutions under the condition that either $N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j$ or $N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j$ and $N_1, N_2 > 1$, $|N_1 - N_2| \neq 1$. We solve this problem by a perturbation from the corresponding \mathbf{G}_2 Toda system with one singular source. Combining with [1], we have proved the existence of non-topological solutions to the Chern-Simons system with Cartan matrix of rank 2.

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This is the shortened version of the paper. The full and detailed version with full expressions of all computations is posted on arXiv.org([3])

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1. Introduction

1.1. Background

There are four types of simple non-exceptional Lie Algebra: \mathbf{A}_m , \mathbf{B}_m , \mathbf{C}_m and \mathbf{D}_m which Cartan subalgebra are $sl(m+1)$, $so(2m+1)$, $sp(m)$ and $so(2m)$ respectively. To each of them, a Toda system is associated. In geometry, solutions of Toda system are closely related to holomorphic curves in projective spaces. For example, the Toda system of type \mathbf{A}_m can be derived from the classical Plücker formulas, and any holomorphic curve gives rise to a solution u of the Toda system, whose branch points correspond to the singularities of u . Conversely, we could integrate the Toda system, and any solution u gives rise to a holomorphic curve in $\mathbb{C}\mathbb{P}^n$ at least locally. See [11], [23] and references therein. It is very interesting to note that the reverse process holds globally if the domain for the equation is \mathbb{S}^2 or \mathbb{C} . Any solution u of type \mathbf{A}_m Toda system on \mathbb{S}^2 or \mathbb{C} could produce a global holomorphic curve into $\mathbb{C}\mathbb{P}^n$. This holds even when the solution u has singularities. We refer the readers to [23] for more precise statements of these results.

In physics, the Toda system also plays an important role in non-Abelian gauge field theory. One example is the relativistic Chern-Simons model proposed by Dunne [12–14] in order to explain the physics of high critical temperature superconductivity. See also [20], [21] and [22].

The model is defined in the $(2+1)$ Minkowski space $\mathbb{R}^{1,2}$, the gauge group is a compact Lie group with a semi-simple Lie algebra \mathcal{G} . The Chern-Simons Lagrangian density \mathcal{L} is defined by:

$$\mathcal{L} = -k\epsilon^{\mu\nu\rho}tr\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) - tr((D_\mu\phi)^\dagger D^\mu\phi) - V(\phi, \phi^\dagger)$$

for a Higgs field ϕ in the adjoint representation of the compact gauge group G , where the associated semi-simple Lie algebra is denoted by \mathcal{G} and the \mathcal{G} -valued gauge fields A_α are defined on $2+1$ dimensional Minkowski space $\mathbb{R}^{1,2}$ with metric $\text{diag}\{-1,1,1\}$. Here $k > 0$ is the Chern-Simons coupling parameter, tr is the trace in the matrix representation of \mathcal{G} and V is the

potential energy density of the Higgs field given by

$$V(\phi, \phi^\dagger) = \frac{1}{4k^2} \text{tr}([\phi, \phi^\dagger], \phi - v^2\phi)^\dagger([\phi, \phi^\dagger], \phi - v^2\phi),$$

where $v > 0$ is a constant which measures either the scale of the broken symmetry or the subcritical temperature of the system.

In general, the Euler-Lagrangian equation corresponding \mathcal{L} is very difficult to study. So we restrict to consider solutions to be energy minimizers of the Lagrangian functional, and then a self-dual system of first order derivatives could be derived from minimizing the energy functional:

$$(1.1) \quad \begin{aligned} D_- \phi &= 0, \\ F_{+-} &= \frac{1}{k^2} \left[v^2 \phi - [[\phi, \phi^\dagger], \phi], \phi^\dagger \right], \end{aligned}$$

where $D_- = D_1 - iD_2$, and $F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-]$ with $A_\pm = A_1 \pm iA_2$, $\partial_\pm = \partial_1 \pm i\partial_2$. Here ∂_i and D_i are respectively the partial derivative and the gauge-covariant derivative w.r.t z_i , $i = 1, 2$.

In order to find non-trivial solutions which are not algebraic solutions of $[[\phi, \phi^\dagger], \phi] = v^2\phi$, Dunne [13] has considered a simplified form of the self-dual system (1.1) in which both the gauge potential A and the Higgs field ϕ are algebraically restricted, for example, ϕ has the following form:

$$\phi = \sum_{a=1}^r \phi^a E_{\pm a},$$

where r is the rank of the Lie algebra \mathcal{G} , $\{E_{\pm a}\}$ is the family of the simple root step operators (with $E_{-a} = E_a^+$), and ϕ^a are complex-valued functions.

In this paper, we consider the case of rank 2. Let

$$u_a = \ln |\phi^a|^2.$$

By using this ansatz, then Equation (1.1) can be reduced to

$$(1.2) \quad \begin{cases} \Delta u_1 + \frac{v^4}{k^2} \left(\sum_{i=1}^2 K_{1i} e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{1i} e^{u_j} K_{ij} \right) = 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + \frac{v^4}{k^2} \left(\sum_{i=1}^2 K_{2i} e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{2i} e^{u_j} K_{ij} \right) = 4\pi \sum_{j=1}^{N_2} \delta_{q_j} \end{cases} \quad \text{in } \mathbb{R}^2,$$

where $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$ is the Cartan matrix of rank 2 of the Lie algebra \mathcal{G} , $\{p_1, \dots, p_{N_1}\}$ and $\{q_1, \dots, q_{N_2}\}$ are given vortex points. For the details of

the process to derive (1.2) from (1.1), we refer to [13],[29],[38] and [39]. In this paper, without loss of generality, we assume $\frac{v_1}{k^2} = 1$.

It is known that there are only three types of Cartan matrix of rank 2, given by

$$(1.3) \quad \mathbf{A}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{B}_2(= \mathbf{C}_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad \mathbf{G}_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

In the previous paper [1], we have constructed non-topological solutions in the case of \mathbf{A}_2 and \mathbf{B}_2 . In this paper, we will construct non-topological solutions for the \mathbf{G}_2 case, i.e. the following equation:

$$(1.4) \quad \begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} = 4e^{2u_1} - 2e^{2u_2} + e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + 2e^{u_2} - 3e^{u_1} = 4e^{2u_2} - 6e^{2u_1} - 3e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_2} \delta_{q_j}. \end{cases}$$

1.2. Previous results

In the literature, a solution $\mathbf{u} = (u_1, u_2)$ to system (1.2) is called a *topological* solution if \mathbf{u} satisfies

$$u_a(z) \rightarrow \ln \sum_{j=1}^2 (K^{-1})_{aj} \quad \text{as } |z| \rightarrow +\infty, \quad a = 1, 2,$$

and is called a *non-topological* solution if \mathbf{u} satisfies

$$(1.5) \quad u_a(z) \rightarrow -\infty \quad \text{as } |z| \rightarrow +\infty, \quad a = 1, 2.$$

The existence of topological solutions with arbitrary multiple vortex points was proved by Yang [39] more than fifteen years ago, not only for Cartan matrix of rank 2, but also for general Cartan matrix including $SU(N+1)$ case, $N \geq 1$. However, the existence of non-topological solutions is more difficult to prove. The first result was due to Chae and Imanuvilov [5] for the $SU(2)$ Abelian Chern-Simons equation which is obtained by letting $u_1(z) = u_2(z) = u(z)$ in the \mathbf{A}_2 system where u satisfies

$$(1.6) \quad \Delta u + e^u(1 - e^u) = 4\pi \sum_{j=1}^N \delta_{p_j} \quad \text{in } \mathbb{R}^2.$$

Equation (1.6) is the $SU(2)$ Chern-Simons equation for the Abelian case. This relativistic Chern-Simons model was proposed by Jackiw-Weinberg [19] and Hong-Kim-Pac [18]. For the past more than twenty years, the existence and multiplicity of solutions to (1.6) with different nature (e.g. topological, non-topological, periodically constrained etc.) have been studied, see [4], [5], [6], [7], [8], [10], [18], [24], [25], [26], [27], [31], [32], [33], [34], [35] and references therein.

In [5], Chae and Imanuvilov proved the existence of non-topological solutions for (1.6) for any vortex points (p_1, \dots, p_N) . For the question of existence of non-topological solutions for the \mathbf{A}_2 system, an “answer” was given by Wang and Zhang [37] but their proof contains serious gaps. In fact they used a special solution of the Toda system as the approximate solution, but they did not have the full non-degeneracy of the linearized equation of the Toda system and their analysis for the linearized equation is incorrect. Thus, the existence of non-topological solutions has remained a long-standing open problem. Even for radially symmetric solutions (the case when all the vortices coincide), the ODE system is much more subtle than Equation (1.6). The classification of radial solution is an important issue for future study as long as bubbling solutions are concerned, see [5], [6], [7], [8], [15], [24], [25], [28], [30], [33], [34], [36] in this direction.

For the rank 2 Chern-Simons system of Lie type, Huang and the second author [16, 17] studied the structure of radial solutions. Among other things, they proved the following result:

Theorem A *If (u_1, u_2) is a radially symmetric non-topological solution to the rank 2 Chern-Simons system of Lie type with all vortices at the origin, then*

$$(1.7) \quad u_1(r) = -2\alpha_1 \log r + O(1), \quad u_2(r) = -2\alpha_2 \log r + O(1)$$

at infinity for some $\alpha_1, \alpha_2 > 1$. Furthermore,

$$J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1),$$

where $J(x, y)$ is the quadratic form associated to K^{-1} .

For the existence of radially symmetric non-topological solutions, Choe, Kim and the second author [9] recently proved the following result:

Theorem B *If (α_1, α_2) defined in (1.7) satisfies*

$$\begin{aligned} -2N_1 - N_2 - 3 < \alpha_2 - \alpha_1 < 2N_2 + N_1 + 3, \\ 2\alpha_1 + \alpha_2 > N_1 + 2N_2 + 6 \quad \text{and} \quad \alpha_1 + 2\alpha_2 > 2N_1 + N_2 + 6, \end{aligned}$$

then the \mathbf{A}_2 Chern-Simons system has a radially symmetric solution (u_1, u_2) subject to the boundary condition (1.7).

For general configuration vortices in \mathbb{R}^2 , we first got the existence of non-topological solutions for the \mathbf{A}_2 and \mathbf{B}_2 Chern-Simons system by perturbation from the \mathbf{A}_2 and \mathbf{B}_2 Toda system with a singular source. In [1], we proved the following:

Theorem C Let $\{p_j\}_{j=1}^{N_1}, \{q_j\}_{j=1}^{N_2} \subset \mathbb{R}^2$. If either

$$(a) \quad N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j ;$$

or

$$(b) \quad N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j \text{ and } N_1, N_2 > 1, |N_1 - N_2| \neq 1, \text{ then there ex-}$$

ists a non-topological solution (u_1, u_2) of the \mathbf{A}_2 and \mathbf{B}_2 Chern-Simons system respectively.

1.3. Main results

In this paper, we continue our work on the rank 2 Chern-Simons system. We consider the remaining \mathbf{G}_2 Chern-Simons system. We give an affirmative answer to the existence of non-topological solutions for the system with Cartan matrix \mathbf{G}_2 . Our main theorem can be stated as follows.

Theorem 1.1. Let $\{p_j\}_{j=1}^{N_1}, \{q_j\}_{j=1}^{N_2} \subset \mathbb{R}^2$. If either

$$(a) \quad N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j ;$$

or

$$(b) \quad N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j \text{ and } N_1, N_2 > 1, |N_1 - N_2| \neq 1, \text{ then there ex-}$$

ists a non-topological solution (u_1, u_2) of problem (1.4).

Remark 1.1. Note that if $N_1 = 0$ or $N_2 = 0$, by translation, assumption (a) in Theorem 1.1 is always satisfied.

Non-topological solutions play very important role in the bubbling analysis of solutions to (1.2). Therefore, our result is only the first step towards understanding the solution structure of non-topological solution of (1.2). For further study on non-topological solutions for the Abelian case, we refer to [6] and [8].

We will prove Theorem 1.1 in three cases which we describe below:

Assumption (i):

$$(1.8) \quad \sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j, \quad N_1 = N_2;$$

Assumption (ii):

$$(1.9) \quad \begin{aligned} & N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j, \quad N_1 \neq N_2, \quad N_1, N_2 \neq 1 \\ \text{or } & N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j, \quad |N_1 - N_2| \neq 1, \quad N_1, N_2 > 1; \end{aligned}$$

Assumption (iii):

$$(1.10) \quad N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j, \quad N_1 \neq N_2, \quad N_1 = 1 \text{ or } N_2 = 1.$$

If we can prove the existence of non-topological solutions under the above three assumptions separately, then it is easy to see that Theorem 1.1 is proved. So in the following, we will prove the theorem under the three assumptions respectively.

1.4. Sketch of the Proof

In the following, we will outline the sketch of our proof. We follow exactly the same idea as in the proof for the \mathbf{A}_2 and \mathbf{B}_2 case.

As in [1], we will view Equation (1.4) as a small perturbation of the \mathbf{G}_2 Toda system with a singular source at the origin.

After a suitable scaling transformation and some manipulations (see Section 2.1), the system (1.4) is transformed to

$$(1.11) \quad \begin{cases} \Delta \tilde{U}_1 + \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 e^{2\tilde{U}_1 - \tilde{U}_2} = 2\varepsilon^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^4 e^{4\tilde{U}_1 - 2\tilde{U}_2} \\ \quad - \varepsilon^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 e^{\tilde{U}_2 - \tilde{U}_1}, \\ \Delta \tilde{U}_2 + \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 e^{2\tilde{U}_2 - 3\tilde{U}_1} = 2\varepsilon^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^4 e^{4\tilde{U}_2 - 6\tilde{U}_1} \\ \quad - 3\varepsilon^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 e^{\tilde{U}_2 - \tilde{U}_1}. \end{cases}$$

When $\varepsilon = 0$, we obtain the following limiting system

$$(1.12) \quad \begin{cases} \Delta \tilde{U}_1 + |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} = 0 & \text{in } \mathbb{R}^2 \\ \Delta \tilde{U}_2 + |z|^{2N_2} e^{2\tilde{U}_2 - 3\tilde{U}_1} = 0 & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} < +\infty, \int_{\mathbb{R}^2} |z|^{2N_2} e^{2\tilde{U}_2 - 3\tilde{U}_1} < +\infty \end{cases}$$

which is the \mathbf{G}_2 Toda system with a single source at the origin.

An immediate problem is the classification and non-degeneracy of the above system. In [23], Lin, Wei and Ye obtained the classification and non-degeneracy results of the $SU(N + 1)$ Toda system with a singular source. In [2], we used the results of [23] to obtain a complete classification and non-degeneracy of the \mathbf{G}_2 Toda system (1.12). In fact, the \mathbf{G}_2 Toda system can be embedded into the \mathbf{A}_6 Toda system under suitable group action. The solutions to (1.12) depend on fourteen parameters

$$(\mathbf{a}, \lambda) = (c_{43,1}, c_{43,2}, c_{52,1}, c_{52,2}, c_{53,1}, c_{53,2}, c_{54,1}, c_{54,2}, c_{61,1}, c_{61,2}, c_{62,1}, c_{62,2}, \lambda_4, \lambda_5) \in \mathbb{R}^{14}.$$

The dimension of the linearized operator is **fourteen**.

The main difficulty of dealing with this system is the large dimension of kernels. There are no explicit formula for the coefficients, except in the case $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j$, $N_1 = N_2$ which can be considered as the reminiscent of the $SU(2)$ scalar equation. To get over this difficulty, we make use of the two scaling parameters λ_4, λ_5 for solutions of the Toda system and introduce two more free parameters. Instead of solving the coefficient matrices for fixed scaling parameters, we only need to compute the two matrices in front of the two free parameters we introduce.

Now let us be more specific. We will view (1.11) as a perturbation of (1.12) and expand it in powers of ε . The term of order $O(\varepsilon)$ will satisfy (2.23) and (2.24) in Section 2.4. The $O(\varepsilon^2)$ term will satisfy (2.25). In this $O(\varepsilon^2)$ term ψ , we introduce **two free parameters** ξ_1, ξ_2 which play an important role in our proof. See Section 5 and 6. At last the solution we find will be of the form

$$(1.13) \quad \tilde{U} = \tilde{U}_{\mathbf{b}} + \varepsilon \Psi + \varepsilon^2 \psi + \varepsilon^2 v,$$

where $\tilde{U}_{\mathbf{b}} = (\tilde{U}_{1,\mathbf{b}}, \tilde{U}_{2,\mathbf{b}})$ is the solution of (1.12), and \mathbf{b} denote the parameters (λ, \mathbf{a}) for simplicity of notations. In order to solve in v , we need to solve

a linearized problem:

$$(1.14) \quad \begin{cases} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\phi_1 - \phi_2) = f_1 \\ \Delta\phi_2 + |z|^{2N_1} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\phi_2 - 3\phi_1) = f_2 \end{cases}$$

where f_1 and f_2 are explicitly given. Due to the existence of the kernels of the linearized equation, (f_1, f_2) must satisfy some extra conditions in order to have a solution. See Lemma 2.2 for the necessary and sufficient conditions. After that, we use the Liapunov-Schmidt reduction method to solve the nonlinear equation. It turns out that we can choose the perturbation \mathbf{a} and λ such that we can get the solution.

Now we comment on the technical conditions. In the proof, we will choose (λ_4, λ_5) first, depending on the assumptions. As we mentioned above, we will use the Liapunov-Schmidt reduction method to solve it. In the reduced problem, one main problem is to calculate the projection of the error E (see (2.30) and (2.31)) to the conjugate kernels $(Z_j^*, j = 3, \dots, 14)$ of the linearized equation of (1.12), i.e. we have to calculate the following:

$$(1.15) \quad \langle E, Z_j^* \rangle_{j=3, \dots, 14}.$$

In general, by Taylor’s expansion of E , we have

$$(1.16) \quad \langle E, Z_j^* \rangle_{j=3, \dots, 14} = \frac{1}{\varepsilon} \mathbf{B}\mathbf{a} + \frac{1}{\varepsilon} \mathbf{A}\mathbf{a} \cdot \mathbf{a} + \mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) + \mathbf{a}_0,$$

where $\mathbf{A}, \mathbf{B}, \mathbf{Q}$ are matrices of size 12×12 , and $\mathbf{a}_0 \in \mathbb{R}^{12}$. Furthermore, the matrix \mathbf{Q} can be decomposed into

$$(1.17) \quad \mathbf{Q} = \xi_1 \mathbf{Q}_1 + \xi_2 \mathbf{Q}_2 + \mathcal{T}$$

where ξ_1 and ξ_2 are two free parameters introduced in the $O(\varepsilon^2)$ -term $\varepsilon^2\psi$. Thus the reduced problem becomes:

$$(1.18) \quad \frac{1}{\varepsilon} \mathbf{B}\mathbf{a} + \frac{1}{\varepsilon} \mathbf{A}\mathbf{a} \cdot \mathbf{a} + \mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) + \mathbf{a}_0 = O(\varepsilon),$$

As we said before, we shall not attempt to compute the matrices \mathbf{A}, \mathbf{B} and \mathcal{T} . Instead we focus on the two matrices \mathbf{Q}_1 and \mathbf{Q}_2 . All we need to show is that at least of one of these two matrices is non-degenerate. In the proof, the $O(\varepsilon)$ term Ψ brings about a lot of troubles. Both $\frac{1}{\varepsilon} \mathbf{B}\mathbf{a}$ and $\frac{1}{\varepsilon} \mathbf{A}\mathbf{a} \cdot \mathbf{a}$ terms are related to Ψ . Now let us explain why the assumptions (a) and (b) are needed. In fact, it depends on whether or not there is the $O(\varepsilon)$ term.

When $N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j$, i.e. case (a) of Theorem 1.1, by a shift of origin, we may assume that $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$. In this case, the ε -term $\varepsilon\Psi$ vanishes and both \mathbf{A} and \mathbf{B} vanish. In this case, the reduced problem becomes much simpler:

$$(1.19) \quad \mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) + \mathbf{a}_0 = O(\varepsilon).$$

If $N_1 \neq N_2, N_1, N_2 \neq 1$ or $N_1 = N_2$, then \mathbf{a}_0 vanishes and the reduced problems (in terms of \mathbf{a}) becomes:

$$(1.20) \quad \mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) = O(\varepsilon).$$

If $N_1 \neq N_2, N_1 = 1$, or $N_2 = 1$, we use a different $O(\varepsilon^2)$ approximation ψ in (1.13) and we obtain the reduced problem as follows:

$$(1.21) \quad \mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) + \mathbf{a}_0 = O(\varepsilon).$$

See Section 6. In both cases, we can show that the matrix \mathbf{Q} is non-degenerate and (1.18) can be solved by contraction mapping.

When $N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j$, this becomes much more difficult. Since $N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j$, the ε -term exists and presents great difficulty in solving the reduced problem (1.18) in \mathbf{a} . In order to solve it, we first need to make \mathbf{B} vanish, and this requires that $|N_1 - N_2| \neq 1$, see Section 2.4. Since in general the ε -term exists, \mathbf{A} does not vanish. We need to make the \mathbf{a}_0 term vanish in order to solve (1.18), and for this we need $N_1, N_2 > 1$. In this case the reduced problem now takes the form

$$(1.22) \quad \frac{1}{\varepsilon} \mathbf{A}\mathbf{a} \cdot \mathbf{a} + \mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) = O(\varepsilon)$$

where \mathbf{Q} has the form of (1.17). By choosing large ξ_1 and $\xi_2 = 0$, we can solve (1.22) such that $|\mathbf{a}| \leq O(\varepsilon)$.

In summary, the technical condition we have imposed is to make sure that $\mathbf{B} = 0$ and that the quadratic term $\frac{1}{\varepsilon} \mathbf{A}\mathbf{a} \cdot \mathbf{a}$ and the $O(1)$ term can not coexist such that we can solve the reduced problem.

The organization of the paper is the following. In Section 2, we present several important preliminaries of analysis. We first formulate our problem in terms of the functional equations (Section 2.1). Then we apply the classification and nondegeneracy result of \mathbf{G}_2 Toda system (Section 2.2). In Section 2.3, we establish the invertibility properties of the linearized operator. Finally we obtain the next two orders $O(\varepsilon)$ and $O(\varepsilon^2)$ in Section 2.4. In

Section 3, we solve a projected nonlinear problem based on the preliminary results in Section 2. In Section 4, we prove our main theorem under the **Assumption (i)**. In Section 5, we prove the theorem under the **Assumption (ii)**. In Section 6, we prove the theorem under the **Assumption (iii)**.

2. Preliminaries

In this section, we consider the following \mathbf{G}_2 system in \mathbb{R}^2 :

$$(2.1) \quad \begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} = 4e^{2u_1} - 2e^{2u_2} + e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + 2e^{u_2} - 3e^{u_1} = 4e^{2u_2} - 6e^{2u_1} - 3e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_2} \delta_{q_j}. \end{cases}$$

2.1. Functional formulation of the problem

Defining

$$u_1 = \sum_{j=1}^{N_1} \ln |z - p_j|^2 + \tilde{u}_1, \quad u_2 = \sum_{j=1}^{N_2} \ln |z - q_j|^2 + \tilde{u}_2,$$

and $z = \frac{\tilde{z}}{\varepsilon}$, and let U_i and \tilde{U}_i to be

$$\tilde{u}_1(z) = U_1(\tilde{z}) + (2N_1 + 2) \ln \varepsilon, \quad \tilde{u}_2(z) = U_2(\tilde{z}) + (2N_2 + 2) \ln \varepsilon,$$

and

$$\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{pmatrix} = \mathbf{G}_2^{-1} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

where \mathbf{G}_2 is the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. Then $(\tilde{U}_1, \tilde{U}_2)$ will satisfy

$$(2.2) \quad \begin{cases} \Delta \tilde{U}_1 + \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 e^{2\tilde{U}_1 - \tilde{U}_2} = 2\varepsilon^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^4 e^{4\tilde{U}_1 - 2\tilde{U}_2} \\ \quad - \varepsilon^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 e^{\tilde{U}_2 - \tilde{U}_1}, \\ \Delta \tilde{U}_2 + \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 e^{2\tilde{U}_2 - 3\tilde{U}_1} = 2\varepsilon^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^4 e^{4\tilde{U}_2 - 6\tilde{U}_1} \\ \quad - 3\varepsilon^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 e^{\tilde{U}_2 - \tilde{U}_1}. \end{cases}$$

From now on, we shall work with (2.2). For simplicity of notations, we still denote the variable by z instead of \tilde{z} .

2.2. First approximate solution

When $\varepsilon = 0$, (2.2) becomes

$$(2.3) \quad \begin{cases} \Delta \tilde{U}_1 + |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} = 0 \\ \Delta \tilde{U}_2 + |z|^{2N_2} e^{2\tilde{U}_2 - 3\tilde{U}_1} = 0 \\ \int_{\mathbb{R}^2} |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} < \infty, \quad \int_{\mathbb{R}^2} |z|^{2N_2} e^{2\tilde{U}_2 - 3\tilde{U}_1} < \infty. \end{cases}$$

For this system, we have gotten the classification and non-degeneracy results in [2]. From Theorem 2.1 in [2], we see that all the solutions of (2.3) depend on fourteen parameters $(c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62}, \lambda_4, \lambda_5) \in \mathbb{C}^6 \times (\mathbb{R}^+)^2$, and all the solutions of (2.3) are of the form

$$(2.4) \quad e^{-\tilde{U}_1} = 2 \left(\lambda_0 + \sum_{i=1}^6 \lambda_i |P_i(z)|^2 \right),$$

where

$$(2.5) \quad P_i(z) = z^{\mu_1 + \dots + \mu_i} + \sum_{j=0}^{i-1} c_{ij} z^{\mu_1 + \dots + \mu_j},$$

and

$$(2.6) \quad \mu_1 = \mu_3 = \mu_4 = \mu_6 = N_1 + 1, \quad \mu_2 = \mu_5 = N_2 + 1.$$

Here $c_{10}, c_{20}, c_{21}, c_{30}, c_{31}, c_{32}, c_{40}, c_{41}, c_{42}, c_{50}, c_{51}, c_{60}, c_{63}, c_{64}, c_{65}$ can be expressed by $c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62}$. $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_6$ can be written in terms of μ_1, μ_2, λ_4 and λ_5 . Thus the solutions depend on fourteen free parameters $(c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62}, \lambda_4, \lambda_5) \in \mathbb{C}^6 \times (\mathbb{R}^+)^2$. For details, see Theorem 2.1 in [2].

We denote by

$$(2.7) \quad (\lambda, \mathbf{a}) = (\lambda_4, \lambda_5, c_{43,1}, c_{43,2}, c_{52,1}, c_{52,2}, c_{53,1}, c_{53,2}, c_{54,1}, c_{54,2}, c_{61,1}, c_{61,2}, c_{62,1}, c_{62,2}).$$

When $\mathbf{a} = 0$, the radially symmetric solution of (2.3) can be expressed as follows:

$$(2.8) \quad e^{-\tilde{U}_{1,0}} := \rho_{1,G}^{-1} = 2\rho_1^{-1}(r),$$

$$(2.9) \quad e^{-\tilde{U}_{2,0}} := \rho_{2,G}^{-1} = 4\rho_2^{-1}(r)$$

where we use $\tilde{U}_{i,0}$ to denote $\tilde{U}_{i,\lambda_4,\lambda_5,\mathbf{a}}|_{\mathbf{a}=0}$ and for the explicit expression of $\rho_1(r)$ and $\rho_2(r)$, please refer to Theorem 2.1 in [2].

Observe that the radial solution $(\tilde{U}_{1,0}, \tilde{U}_{2,0})$ depends on two scaling parameters (λ_4, λ_5) . Later we shall choose (λ_4, λ_5) in different ways.

Next we have the following non-degeneracy result:

Lemma 2.1. *(Non-degeneracy) The previous solutions of (2.3) are non-degenerate, i.e., the set of solutions corresponding to the linearized operator at $(\tilde{U}_{1,0}, \tilde{U}_{2,0})$ is exactly fourteen dimensional. More precisely, if $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ satisfies $|\phi(z)| \leq C(1 + |z|)^\alpha$ for some $0 \leq \alpha < 1$, and*

$$(2.10) \quad \begin{cases} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\phi_1 - \phi_2) = 0 \\ \Delta\phi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\phi_2 - 3\phi_1) = 0, \end{cases}$$

then ϕ belongs to the following linear space \mathcal{K} : the span of

$$\{Z_{\lambda_4}, Z_{\lambda_5}, Z_{c_{43,1}}, Z_{c_{43,2}}, Z_{c_{52,1}}, Z_{c_{52,2}}, Z_{c_{53,1}}, \\ Z_{c_{53,2}}, Z_{c_{54,1}}, Z_{c_{54,2}}, Z_{c_{61,1}}, Z_{c_{61,2}}, Z_{c_{62,1}}, Z_{c_{62,2}}\},$$

where $Z_{\lambda_4} = \partial_{\lambda_4} \tilde{U}_0, \dots$ etc.

For a proof, we refer to Corollary 2.3 in [2]. (The full explicit expressions of all kernel functions can be found in [3].) We also denote by $(Z_1, Z_2, \dots, Z_{14})$ the kernels $(Z_{\lambda_4}, Z_{\lambda_5}, \dots, Z_{c_{62,2}})$. Because $\{Z_i\}$ are linearly independent, we have

$$(2.11) \quad \det \left[\left(\int_{\mathbb{R}^2} \Delta Z_i \cdot Z_j \right)_{i,j=1,\dots,14} \right] \neq 0.$$

We have the following corollary:

Corollary 2.1. *If $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ satisfies $|\phi(z)| \leq C(1 + |z|)^\alpha$ for some $0 \leq \alpha < 1$, and*

$$(2.12) \quad \begin{cases} \Delta\phi_1 + 2|z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} \phi_1 - 3|z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} \phi_2 = 0 \\ \Delta\phi_2 + 2|z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} \phi_2 - |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} \phi_1 = 0, \end{cases}$$

then ϕ belongs to the following linear space \mathcal{K}^* : the span of

$$\{Z_{\lambda_4}^*, Z_{\lambda_5}^*, Z_{c_{43,1}}^*, Z_{c_{43,2}}^*, Z_{c_{52,1}}^*, Z_{c_{52,2}}^*, Z_{c_{53,1}}^*, \\ Z_{c_{53,2}}^*, Z_{c_{54,1}}^*, Z_{c_{54,2}}^*, Z_{c_{61,1}}^*, Z_{c_{61,2}}^*, Z_{c_{62,1}}^*, Z_{c_{62,2}}^*\},$$

where

$$(2.13) \quad Z_i^* = \begin{pmatrix} Z_{i,1}^* \\ Z_{i,2}^* \end{pmatrix} = \begin{pmatrix} 2Z_{i,1} - Z_{i,2} \\ \frac{2}{3}Z_{i,2} - Z_{i,1} \end{pmatrix}.$$

We have

$$(2.14) \quad \det\left[\left(\int_{\mathbb{R}^2} Z_i^* \cdot Z_j^*\right)_{i,j=3,\dots,14}\right] \neq 0.$$

We will choose the first approximate solution to be

$$\begin{pmatrix} \tilde{U}_{1,(\lambda,c_{43},c_{52},c_{53},c_{54},c_{61},c_{62})} \\ \tilde{U}_{2,(\lambda,c_{43},c_{52},c_{53},c_{54},c_{61},c_{62})} \end{pmatrix},$$

where the parameters $\lambda, c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62}$ satisfy

$$(2.15) \quad |\mathbf{a}| := |c_{43}| + |c_{52}| + |c_{53}| + |c_{54}| + |c_{61}| + |c_{62}| \leq C_0\varepsilon, \quad |\lambda| = O(1)$$

for some fixed constant $C_0 > 0$.

For the simplicity of notations, we also denote $\mathbf{b} = (\lambda, \mathbf{a})$, and $\tilde{U}_{i,\mathbf{b}} = \tilde{U}_{i,(\lambda,c_{43},c_{52},c_{53},c_{54},c_{61},c_{62})}$. We want to look for solutions of the form

$$(2.16) \quad \tilde{U}_1 = \tilde{U}_{1,\mathbf{b}} + \varepsilon\Psi_1 + \varepsilon^2\phi_1, \quad \tilde{U}_2 = \tilde{U}_{2,\mathbf{b}} + \varepsilon\Psi_2 + \varepsilon^2\phi_2,$$

where \mathbf{b} is fixed.

To obtain the next order term, we need to study the linearized operator around the solution $\begin{pmatrix} \tilde{U}_{1,0} \\ \tilde{U}_{2,0} \end{pmatrix}$.

2.3. Invertibility of the linearized operator

Now we consider the invertibility of the linearized operator in some suitable Sobolev spaces. To this end, we use the technical framework introduced by Chae-Imanuvilov [5] and which has been used in [1]. Let $\alpha \in (0, 1)$ and

$$(2.17) \quad X_\alpha = \left\{ u \in L^2_{loc}(\mathbb{R}^2), \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})|u|^2 dx < +\infty \right\},$$

$$(2.18) \quad Y_\alpha = \left\{ u \in W^{2,2}_{loc}(\mathbb{R}^2), \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})|\Delta u|^2 + \frac{|u|^2}{1 + |x|^{2+\alpha}} < +\infty \right\}.$$

On X_α and Y_α , we equip with two norms respectively:

$$(2.19) \quad \|f\|_{**} = \sup_{y \in \mathbb{R}^2} (1 + |y|)^{2+\alpha} |f(y)|, \quad \|h\|_* = \sup_{y \in \mathbb{R}^2} (\log(2 + |y|))^{-1} |h(y)|.$$

Clearly, the linearized operator in (2.10) is bounded from Y_α to X_α .

For $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, we denote by $\langle f, g \rangle = \int_{\mathbb{R}^2} f \cdot g dx$.

Note that $Z_j^* \in X_\alpha$ for all j except $j = 1, 2$. Using the non-degeneracy result we get, i.e. Lemma 2.1 and Corollary 2.1 in Section 2.2 and noting that (2.12) is the adjoint operator of (2.10), we have the following:

Lemma 2.2. *Assume that $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X_\alpha$ be such that*

$$(2.20) \quad \langle Z_i^*, h \rangle = 0, \quad \text{for } i = 3, \dots, 14.$$

Then one can find a unique solution $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = T^{-1}(h) \in Y_\alpha$ satisfying

$$(2.21) \quad \begin{cases} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\phi_1 - \phi_2) = h_1 \\ \Delta\phi_2 + |z|^{2N_1} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\phi_2 - 3\phi_1) = h_2 \end{cases} \quad \|\phi\|_* \leq C \|h\|_{**},$$

such that $\langle \Delta Z_i, \phi \rangle = 0$ for $i = 1, \dots, 14$. Moreover, the map $h \xrightarrow{T} \phi$ can be made continuous and smooth .

We note that the uniqueness in Lemma 2.2 is due to (2.11). In the next subsections, we will use Lemma 2.2 to obtain our approximate solution up to $O(\varepsilon^2)$. However our approximation solution would be chosen according to our assumption of the vortex configuration.

2.4. Improvements of the approximate solution

Similar to the \mathbf{A}_2 and \mathbf{B}_2 case, we need to find the $O(\varepsilon)$ and $O(\varepsilon^2)$ improvement of our approximate solutions. So we need to find solutions of the following equations.

Denote by

$$f(\varepsilon, z) = \Pi_1^{N_1} |z - \varepsilon p_i|^2, \quad g(\varepsilon, z) = \Pi_1^{N_2} |z - \varepsilon q_i|^2.$$

Then by Taylor’s expansion, we have

$$f(\varepsilon, z) = f(0, z) + \varepsilon f_\varepsilon(0, z) + \frac{\varepsilon^2}{2} f_{\varepsilon\varepsilon}(0, z) + O(\varepsilon^3)$$

where

$$(2.22) \quad f(0, z) = |z|^{2N_1}, \quad f_\varepsilon(0, z) = -2|z|^{2N_1-2} \left\langle \sum_{j=1}^{N_1} p_j, z \right\rangle.$$

Similarly we can get the expansion for $g(\varepsilon, z)$.

Let $\begin{pmatrix} \Psi_{0,1} \\ \Psi_{0,2} \end{pmatrix}$ be the solution of

$$(2.23) \quad \begin{cases} \Delta \Psi_{0,1} + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2}) \\ = -f_\varepsilon(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}, \\ \Delta \Psi_{0,2} + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\Psi_{0,2} - 3\Psi_{0,1}) \\ = -g_\varepsilon(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}}. \end{cases}$$

Let $\begin{pmatrix} \Psi_{i,1} \\ \Psi_{i,2} \end{pmatrix}$ be the solution of

$$(2.24) \quad \begin{cases} \Delta \Psi_{i,1} + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{i,1} - \Psi_{i,2}) \\ = -|z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2}) (2Z_{i,1} - Z_{i,2}) \\ \quad - f_\varepsilon(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{i,1} - Z_{i,2}) \\ \Delta \Psi_{i,2} + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\Psi_{i,2} - 3\Psi_{i,1}) \\ = -|z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\Psi_{0,2} - 3\Psi_{0,1}) (2Z_{i,2} - 3Z_{i,1}) \\ \quad - g_\varepsilon(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{i,2} - 3Z_{i,1}) \end{cases}$$

for $i = 3, \dots, 14$.

Let $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ be the solution of

$$(2.25) \quad \begin{cases} \Delta \psi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1 - \psi_2) \\ = 2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} \\ \quad - \frac{1}{2}|z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2})^2 \\ \quad - f_\varepsilon(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2}) - \frac{f_{\varepsilon\varepsilon}(0, z)}{2} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} \\ \Delta \psi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2 - 3\psi_1) \\ = 2|z|^{4N_2} e^{4\tilde{U}_{2,0} - 6\tilde{U}_{1,0}} - 3|z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} \\ \quad - \frac{1}{2}|z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\Psi_{0,2} - 3\Psi_{0,1})^2 \\ \quad - g_\varepsilon(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\Psi_{0,2} - 3\Psi_{0,1}) - \frac{g_{\varepsilon\varepsilon}(0, z)}{2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}}. \end{cases}$$

Obviously, if $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$, then $f_\varepsilon(0, z) = g_\varepsilon(0, z) = 0$ by (2.22), in this case, $\Psi_0 = \Psi_i = 0$ for $i = 3, \dots, 14$. Note that if $N_2 \sum_{j=1}^{N_1} p_j =$

$N_1 \sum_{j=1}^{N_2} q_j$, we can always shift the origin such that $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$. Hence $f_\varepsilon(0, z) \neq 0$ occurs only when assumption (b) of Theorem 1.1 holds. In this case, using Lemma 2.2, we know that there exists a unique solution $\begin{pmatrix} \Psi_{0,1} \\ \Psi_{0,2} \end{pmatrix} \in Y_\alpha$ of (2.23) such that $\langle \Psi_0, \Delta Z_j \rangle = 0$ for $j = 1, \dots, 14$. Moreover, by the uniqueness of solution, since the right hand side of (2.23) is the linear combination of functions of the form $h(r) \cos \theta$ and $h(r) \sin \theta$, by considering the Fourier series, we know that the solution Ψ_0 must be the linear combination of functions of the form $h(r) \cos \theta$ and $h(r) \sin \theta$. If $N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j$, then $|N_1 - N_2| \neq 1$ implies that the right hand side of Equation (2.24) is orthogonal to Z_i^* for $i = 3, \dots, 14$. By Lemma 2.2, there exists a unique solution $\begin{pmatrix} \Psi_{i,1} \\ \Psi_{i,2} \end{pmatrix} \in Y_\alpha$ of (2.24) such that $\langle \Psi_i, \Delta Z_j \rangle = 0$ for $j = 1, \dots, 14$. And if $N_1, N_2 > 1$, the right hand side of (2.25) is orthogonal to Z_i^* for $i = 3, \dots, 14$. By Lemma 2.2, we can find a unique solution $\psi_0 = \begin{pmatrix} \psi_{0,1} \\ \psi_{0,2} \end{pmatrix} \in Y_\alpha$ such that $\langle \psi_0, \Delta Z_i \rangle = 0$ for $i = 1, \dots, 14$.

Similar to the \mathbf{A}_2 and \mathbf{B}_2 case, the solution we will use later is $\psi = \psi_0 + \xi_1 Z_{\lambda_4} + \xi_2 Z_{\lambda_5}$ where ξ_1, ξ_2 are two constants independent of \mathbf{a} and will be determined later.

Finally, the approximate solution with all the terms of $O(\varepsilon)$ and $O(\varepsilon^2)$ is

$$(2.26) \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \tilde{U}_{1,\mathbf{b}} + \varepsilon(\Psi_{0,1} + \sum_{i=3}^{14} \Psi_{i,1} \mathbf{a}_i) + \varepsilon^2 \psi_1 \\ \tilde{U}_{2,\mathbf{b}} + \varepsilon(\Psi_{0,2} + \sum_{i=3}^{14} \Psi_{i,2} \mathbf{a}_i) + \varepsilon^2 \psi_2 \end{pmatrix},$$

with the notation

$$\begin{aligned} \mathbf{b} &= (\lambda_4, \lambda_5, \mathbf{a}), \\ \mathbf{a} &= (\mathbf{a}_3, \mathbf{a}_4, \dots, \mathbf{a}_{14}) \\ &= (c_{43,1}, c_{43,2}, c_{52,1}, c_{52,2}, c_{53,1}, c_{53,2}, c_{54,1}, c_{54,2}, c_{61,1}, c_{61,2}, c_{62,1}, c_{62,2}). \end{aligned}$$

Parameters λ_4, λ_5 will be chosen later according to different assumptions in Theorem 1.1. After λ_4, λ_5 are fixed, \mathbf{a} will be chosen in order to find a solution of (2.2) in the form of (2.26). Then $\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{pmatrix} = \begin{pmatrix} V_1 + \varepsilon^2 v_1 \\ V_2 + \varepsilon^2 v_2 \end{pmatrix}$ is a solution of (2.2) if $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfies

$$(2.27) \quad \begin{cases} \Delta v_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2v_1 - v_2) = G_1 \\ \Delta v_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2v_2 - 3v_1) = G_2, \end{cases}$$

where

$$(2.28) \quad G_1 = E_1 + N_{11}(v) + N_{12}(v),$$

$$(2.29) \quad G_2 = E_2 + N_{21}(v) + N_{22}(v),$$

and E is the error of the approximate solution $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$, and N_{ij} below are the higher order terms of the perturbation $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$:

$$\begin{aligned} N_{11}(v) &= 2\Pi|z - \varepsilon p_j|^4(e^{4\tilde{U}_1-2\tilde{U}_2} - e^{4V_1-2V_2}) \\ &\quad - \Pi|z - \varepsilon p_j|^2\Pi|z - \varepsilon q_j|^2(e^{\tilde{U}_2-\tilde{U}_1} - e^{V_2-V_1}), \\ N_{12}(v) &= \frac{-f(\varepsilon, z)e^{2\tilde{U}_1-\tilde{U}_2} + f(\varepsilon, z)e^{2V_1-V_2}}{\varepsilon^2} \\ &\quad + f(0, z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2v_1 - v_2), \\ N_{21}(v) &= 2\Pi|z - \varepsilon q_j|^4(e^{4\tilde{U}_2-6\tilde{U}_1} - e^{4V_2-6V_1}) \\ &\quad - 3\Pi|z - \varepsilon p_j|^2\Pi|z - \varepsilon q_j|^2(e^{\tilde{U}_2-\tilde{U}_1} - e^{V_2-V_1}), \\ N_{22}(v) &= \frac{-g(\varepsilon, z)e^{2\tilde{U}_2-3\tilde{U}_1} + g(\varepsilon, z)e^{2V_2-3V_1}}{\varepsilon^2} \\ &\quad + g(0, z)e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2v_2 - 3v_1). \end{aligned}$$

From the expressions of N_{ij} , we can see that N_{ij} are higher order terms and satisfy the following estimate:

$$\|N_{ij}\|_{**} \leq C(\varepsilon\|v\|_* + \|v\|_*^2).$$

By Taylor's expansion, the error E will be of the following form:

$$(2.30) \quad E_1 = \frac{A_1\mathbf{a} \cdot \mathbf{a}}{\varepsilon} + \mathbf{Q}_1\mathbf{a} + O(\varepsilon) + O(|\mathbf{a}|^2),$$

$$(2.31) \quad E_2 = \frac{A_2\mathbf{a} \cdot \mathbf{a}}{\varepsilon} + \mathbf{Q}_2\mathbf{a} + O(\varepsilon) + O(|\mathbf{a}|^2),$$

for some 12×12 matrices A_i, \mathbf{Q}_i . We only give the formula of E_1 here, that of E_2 is similar:

$$\begin{aligned}
(2.32) \quad E_1 = & -f(0, z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \left\{ \frac{1}{\varepsilon} \sum_{i=3}^{14} (2\Psi_{i,1} - \Psi_{i,2}) \mathbf{a}_i \sum_{j=3}^{14} (2Z_{j,1} - Z_{j,2}) \mathbf{a}_j \right. \\
& + (2\psi_1 - \psi_2) \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\
& + \frac{1}{2} (2\Psi_{0,1} - \Psi_{0,2})^2 \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\
& \left. + (2\Psi_{0,1} - \Psi_{0,2}) \sum_{i=3}^{14} (2\Psi_{i,1} - \Psi_{i,2}) \mathbf{a}_i \right\} \\
& - \frac{f(0, z)}{2\varepsilon} \sum_{i,j=3}^{14} \partial_{\mathbf{a}_i \mathbf{a}_j}^2 (e^{(2\tilde{U}_{1,0}-\tilde{U}_{2,0})}) (2\Psi_{0,1} - \Psi_{0,2}) \mathbf{a}_i \mathbf{a}_j \\
& - f_\varepsilon(0, z) \left\{ e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2}) \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \right. \\
& + e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \sum_{i=3}^{14} (2\Psi_{i,1} - \Psi_{i,2}) \mathbf{a}_i \\
& \left. + \frac{1}{2\varepsilon} \sum_{i,j=3}^{14} \partial_{\mathbf{a}_i \mathbf{a}_j}^2 (e^{(2\tilde{U}_{1,0}-\tilde{U}_{2,0})}) \mathbf{a}_i \mathbf{a}_j \right\} \\
& - \frac{1}{2} f_{\varepsilon\varepsilon}(0, z) e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\
& + 4f(0, z)^2 e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}} \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\
& - f(0, z)g(0, z) \sum_{i=3}^{14} e^{\tilde{U}_{2,0}-\tilde{U}_{1,0}} (Z_{i,2} - Z_{i,1}) \mathbf{a}_i \\
& + O((1 + |\xi|)\varepsilon) + O(\varepsilon^2 + (1 + |\xi|)|\mathbf{a}|^2),
\end{aligned}$$

The explicit formulas for the Taylor expansion of the errors E_1 and E_2 are very important for us to deal with the reduced problem. In Section 4 and Section 5 we will make use of the above explicit formulas for E_1 and E_2 to calculate the error projections.

3. A nonlinear projected problem

Similar to Proposition 2.1 in [1], we have the following result:

Proposition 3.1. *For \mathbf{a} satisfying (2.15), there exists a solution $(v, \{m_i\})$ to the following system*

$$(3.1) \quad \begin{cases} \Delta v_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2v_1 - v_2) = G_1 + \sum_{i=3}^{14} m_i(v) Z_{i,1}^* \\ \Delta v_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2v_2 - 3v_1) = G_2 + \sum_{i=3}^{14} m_i(v) Z_{i,2}^* \\ \langle \Delta Z_i, v \rangle = 0, \text{ for } i = 1, \dots, 14, \end{cases}$$

where $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ and $m_i(v)$ can be determined by

$$(3.2) \quad \left\langle G + \sum_{i=3}^{14} m_i(v) Z_i^*, Z_j^* \right\rangle = 0, \text{ for } j = 3, \dots, 14.$$

Furthermore, v satisfies the following estimate

$$(3.3) \quad \|v\|_* \leq C\varepsilon,$$

for some constant C independent of ε .

By Proposition 3.1, the full solvability for (2.2) is reduced to $m_i = 0$ for $i = 3, \dots, 14$. Since by (2.14), $\det(\langle Z_i^*, Z_j^* \rangle_{i,j=3,\dots,14}) \neq 0$, and recall the definition of m_i in (3.2), $m_i = 0$ is equivalent to

$$(3.4) \quad \int_0^{+\infty} \int_0^{2\pi} G \cdot Z_i^* r d\theta dr = 0 \quad \text{for } i = 3, \dots, 14.$$

To solve (3.4), we observe all N_{ij} terms in (2.28) and (2.29) are small. In fact, we have the following lemma:

Lemma 3.1. *Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be a solution of (3.1). Then we have the following estimates:*

$$(3.5) \quad \int_{\mathbb{R}^2} (N_{11}(v) + N_{12}(v)) Z_{i,1}^* + (N_{21}(v) + N_{22}(v)) Z_{i,2}^* dx = O(\varepsilon^2),$$

for $i = 3, \dots, 14$, where $O(\varepsilon^2) \leq C_1 \varepsilon^2$ for some positive C_1 independent of \mathbf{a} provided $|\mathbf{a}| \leq 1$.

Proof. The proof is similar to that of Lemma 2.4 in [1]. □

From the Taylor’s expansions in (2.30) and (2.31), we obtain that the error projection can be expressed as

$$(3.6) \quad (\langle E, Z_i^* \rangle)_{i=3, \dots, 14} = \frac{1}{\varepsilon} \tilde{\mathbf{A}} \mathbf{a} \cdot \mathbf{a} + \tilde{\mathbf{Q}} \mathbf{a} + O((1 + |\xi|)|\mathbf{a}|^2) + O((1 + |\xi|)\varepsilon).$$

4. Proof of Theorem 1.1 under Assumption (i)

Suppose the **Assumption (i)** holds. By a translation, we might assume that $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j = 0$ and $N_1 = N_2$, and we choose $(\xi_1, \xi_2) = (0, 0)$ in this section. This case is the reminiscent of $SU(2)$ case, even though, the proof is considerably harder since there are fourteen dimensional kernels instead of a three-dimensional one for the $SU(2)$ case.

Lemma 4.1. *Let (v_2) be a solution of (3.1). The following estimates hold:*

$$(4.1) \quad \begin{aligned} & (\langle E, Z_3^* \rangle, \langle E, Z_4^* \rangle, \langle E, Z_5^* \rangle, \langle E, Z_6^* \rangle, \\ & \quad \langle E, Z_7^* \rangle, \langle E, Z_8^* \rangle, \langle E, Z_9^* \rangle, \langle E, Z_{10}^* \rangle, \\ & \quad \langle E, Z_{11}^* \rangle, \langle E, Z_{12}^* \rangle, \langle E, Z_{13}^* \rangle, \langle E, Z_{14}^* \rangle)^t \\ & = \tilde{\mathcal{T}}(\mathbf{a}) + O|\mathbf{a}|^2 + O(\varepsilon), \end{aligned}$$

where $\tilde{\mathcal{T}}$ is an 12×12 matrix. Moreover, $\tilde{\mathcal{T}}$ is non-degenerate where N is defined as $N = N_1 = N_2$.

Proof. Without loss of generality, we may assume that $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$ and $N_1 = N_2 = N$, and denote by $\mu = N + 1$. Now we choose the parameters $(\lambda_4, \lambda_5, \xi_1, \xi_2) = (\frac{1}{3 \times 2^{10} \mu^6}, \frac{1}{15 \times 2^9 \mu^6}, 0, 0)$, so that we have

$$(4.2) \quad \begin{aligned} e^{\tilde{U}_{1,0}} &= \rho_{1,G} = \frac{1}{2} \rho_1 = \frac{45 \times 2^9 \mu^6}{(1 + r^{2\mu})^6}, \\ e^{\tilde{U}_{2,0}} &= \rho_{2,G} = \frac{1}{4} \rho_2 = \frac{3^3 \times 5^2 \times 2^{15} \mu^{10}}{(1 + r^{2\mu})^{10}}. \end{aligned}$$

Since $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i = 0$, we have $f_\varepsilon(0, z) = g_\varepsilon(0, z) = 0$, $\Psi_{0,1} = \Psi_{0,2} = 0$. By (2.24), we have $\Psi_{i,1} = \Psi_{i,2} = 0$, $i = 3, \dots, 14$. Recall that

$$(4.3) \quad E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

where by (2.32) we have

$$\begin{aligned}
 (4.4) \quad E_1 &= -|z|^{2N} \frac{\rho_{1,G}^2}{\rho_{2,G}} (2\psi_1 - \psi_2) \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\
 &\quad - \frac{1}{2} f_{\varepsilon\varepsilon}(0, z) \frac{\rho_{1,G}^2}{\rho_{2,G}} \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\
 &\quad + 4|z|^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i - |z|^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \sum_{i=3}^{14} (Z_{i,2} - Z_{i,1}) \mathbf{a}_i \\
 &\quad + O(\varepsilon) + O(\varepsilon^2 + |\mathbf{a}|^2),
 \end{aligned}$$

where $f_{\varepsilon\varepsilon}(0, z)$ is

$$\begin{aligned}
 (4.5) \quad f_{\varepsilon\varepsilon}(0, z) &= 2|z|^{2(N-1)} \left(\sum_{i \neq j} (p_{i1}p_{j1} - p_{i2}p_{j2}) \cos 2\theta \right. \\
 &\quad \left. + (p_{i1}p_{j2} + p_{i2}p_{j1}) \sin 2\theta \right).
 \end{aligned}$$

Similarly we obtain the expression E_2 involving $g_{\varepsilon\varepsilon}(0, z)$. Since the kernels are the product of radial functions and trigonometric functions, in the calculations below, we will use the above expansion for E and the orthogonality of trigonometric functions to simplify our computation.

Since for $N > 1$ we have

$$\begin{aligned}
 &\int h(r) \cos 2\theta (2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta \\
 &= \int h(r) \sin 2\theta (2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta = 0,
 \end{aligned}$$

and

$$\begin{aligned}
 &\int h(r) \cos 2\theta (2Z_{i,2} - 3Z_{i,1})(2Z_{j,2} - 3Z_{j,1}) r dr d\theta \\
 &= \int h(r) \sin 2\theta (2Z_{i,2} - 3Z_{i,1})(2Z_{j,2} - 3Z_{j,1}) r dr d\theta = 0,
 \end{aligned}$$

for $i, j = 3, \dots, 14$, from (4.5), we have

$$(4.6) \quad \int_0^\infty \int_0^{2\pi} f_{\varepsilon\varepsilon}(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta = 0,$$

and

$$(4.7) \quad \int_0^\infty \int_0^{2\pi} g_{\varepsilon\varepsilon}(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{i,2} - 3Z_{i,1})(2Z_{j,2} - 3Z_{j,1}) r dr d\theta = 0,$$

for $i, j = 3, \dots, 14$. Note that $f_{\varepsilon\varepsilon}(0, z) = g_{\varepsilon\varepsilon}(0, z) = 0$ if $N = 0, 1$.

Another important observation is the following:

$$(4.8) \quad \begin{aligned} & \int_0^\infty \int_0^{2\pi} r^{2N} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1 - \psi_2)(2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta \\ &= \int_0^\infty \int_0^{2\pi} f(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1^0 - \psi_2^0)(2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta, \end{aligned}$$

$$(4.9) \quad \begin{aligned} & \int_0^\infty \int_0^{2\pi} r^{2N} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2 - 3\psi_1)(2Z_{i,2} - 3Z_{i,1})(2Z_{j,2} - 3Z_{j,1}) r dr d\theta \\ &= \int_0^\infty \int_0^{2\pi} g(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2^0 - 3\psi_1^0)(2Z_{i,2} - 3Z_{i,1})(2Z_{j,2} - 3Z_{j,1}) r dr d\theta, \end{aligned}$$

where (ψ_1^0, ψ_2^0) is the radial solution of the following system:

$$(4.10) \quad \begin{cases} \Delta\psi_1 + |z|^{2N} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1 - \psi_2) \\ = 2|z|^{4N} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} - |z|^{4N} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} \\ \Delta\psi_2 + |z|^{2N} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2 - 3\psi_1) \\ = 2|z|^{4N} e^{4\tilde{U}_{2,0} - 6\tilde{U}_{1,0}} - 3|z|^{4N} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}}. \end{cases}$$

Here we decompose $\psi = \psi^0 + \tilde{\psi}$ into radial part ψ^0 and nonradial part $\tilde{\psi}$. For the nonradial part, similar to (2.23), we see that $\tilde{\psi}$ is the linear combination of radial functions times $\cos 2\theta$ and $\sin 2\theta$. This part will contribute zero in the integrals in (4.8) and (4.9).

By the above argument, (ψ_1^0, ψ_2^0) is the radial part of $(\psi_{0,1}, \psi_{0,2})$. Because of this observation, when dealing with the $O(\varepsilon^2)$ approximation, we only need to consider the radial part of the solutions.

In fact we can choose $\psi_1^0 = \psi$, $\psi_2^0 = \frac{5}{3}\psi$ such that ψ is the solution of the following ODE:

$$(4.11) \quad \Delta\psi + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2} \psi = r^{4N} \frac{3 \times 2^6 (N+1)^4}{(1+r^{2N+2})^4}.$$

Combining (4.6), (4.7), (4.8) and (4.9), and using the expansion (4.4) of E , one can get the following:

$$\begin{aligned} & \int E \cdot Z_k^* r dr d\theta \\ = & \int_0^\infty \int_0^{2\pi} \sum_{i=3}^{14} \left(\left[2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} (4Z_{i,1} - 2Z_{i,2}) \mathbf{a}_i \right. \right. \\ & - |z|^{2N_1 + 2N_2} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} (Z_{i,2} - Z_{1,i}) \mathbf{a}_i \\ & - f(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1^0 - \psi_2^0) (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \left. \right] Z_{k,1}^* \\ & + \left[4|z|^{4N_2} e^{4\tilde{U}_{2,0} - 6\tilde{U}_{1,0}} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i \right. \\ & - 3|z|^{2N_1 + 2N_2} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} (Z_{i,2} - Z_{i,1}) \mathbf{a}_i \\ & \left. \left. - g(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2^0 - 3\psi_1^0) (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i \right] Z_{k,2}^* \right) r dr d\theta \\ & + O(|\mathbf{a}|^2 + |\varepsilon|^2) + O(\varepsilon), \end{aligned}$$

where $O(\varepsilon)$ is independent of \mathbf{a} .

From the above expressions, we know that $\langle E, Z_k^* \rangle_{k=3, \dots, 14}$ is a system of linear combinations of \mathbf{a}_i , and one can write it as

$$\langle E, Z_k^* \rangle_{k=3, \dots, 14} = \tilde{\mathcal{T}} \mathbf{a} + O(\varepsilon) + O(|\mathbf{a}|^2)$$

where $\tilde{\mathcal{T}}$ is a 12×12 matrix with constant entries, so one need to calculate the entries for the matrix $\tilde{\mathcal{T}}$. Our aim is to show that the matrix $\tilde{\mathcal{T}}$ is non-degenerate. Since we know the explicit expression for the conjugate kernels Z_k^* but we do not know the expression for ψ , in the expression of $\langle E, Z_k^* \rangle$, we can calculate every term except the terms containing ψ . Next we will first write out the expressions of the entries of $\tilde{\mathcal{T}}$, and explain how we deal with the terms containing ψ . As an example, we can compute

$$\begin{aligned} & \int E \cdot Z_{c_{43,i}}^* r dr d\theta \\ = & \int \left[\left(\left(4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} (2Z_{c_{43,i},1} - Z_{c_{43,i},2})^2 \right. \right. \right. \\ & - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{43,i},2} - Z_{c_{43,i},1}) (2Z_{c_{43,i},1} - Z_{c_{43,i},2}) \\ & + \frac{4}{3} r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} (2Z_{c_{43,i},2} - 3Z_{c_{43,i},1})^2 \\ & \left. \left. \left. - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{43,i},2} - Z_{c_{43,i},1}) (2Z_{c_{43,i},2} - 3Z_{c_{43,i},1}) \right) \right) \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{3}r^{2N}\left(\frac{\rho_{1,G}^2}{\rho_{2,G}}(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})^2\right. \\
& \quad \left. + \frac{1}{3}\frac{\rho_{2,G}^2}{\rho_{1,G}^3}(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})^2\right)\psi\Big]c_{43,i} \\
& + \left(\left(4r^{4N}\frac{\rho_{1,G}^4}{\rho_{2,G}^2}(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}})\right.\right. \\
& \quad - r^{4N}\frac{\rho_{2,G}}{\rho_{1,G}}(Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}}) \\
& \quad + \frac{4r^{4N}}{3}\frac{\rho_{2,G}^4}{\rho_{1,G}^6}(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) \\
& \quad \left. - \frac{\rho_{2,G}r^{4N}}{\rho_{1,G}}(Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})\right) \\
& \quad - \frac{r^{2N}}{3}\left(\frac{\rho_{1,G}^2}{\rho_{2,G}}(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}})\right. \\
& \quad \left. + \frac{1}{3}\frac{\rho_{2,G}^2}{\rho_{1,G}^3}(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}})\right)\psi\Big]c_{54,i} \Big] r dr d\theta \\
& + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2) \\
& = \pi[(J_1 + \int q_1\psi r dr)c_{43,i} + (J_2 + \int q_7\psi r dr)c_{54,i}] \\
& + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2).
\end{aligned}$$

In order to know the entries for $\tilde{\mathcal{T}}$, one need to calculate $(J_1 + \int q_1\psi r dr)$, $(J_2 + \int q_7\psi r dr)$ in the above expressions. Similarly, we can get that

$$\begin{aligned}
\int E \cdot Z_{c_{54,i}}^* r dr d\theta & = \pi \left[\left(J_3 + \int q_7\psi r dr \right) c_{43,i} + \left(J_4 + \int q_4\psi r dr \right) c_{54,i} \right] \\
& + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$

where J_3, J_4, q_4, q_7 are explicitly given in the expression of $\langle E, Z_{c_{54,i}}^* \rangle$, and

$$\int E \cdot Z_{c_{52,i}}^* r dr d\theta = \pi \left(J_5 + \int q_2\psi r dr \right) c_{52,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),$$

where J_5, q_2 are explicitly given in the expression of $\langle E, Z_{c_{52,i}}^* \rangle$, and

$$\int E \cdot Z_{c_{53,i}}^* r dr d\theta = \pi \left(J_6 + \int q_3 \psi r dr \right) c_{53,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),$$

where J_6, q_3 are explicitly given in the expression of $\langle E, Z_{c_{53,i}}^* \rangle$, and

$$\int E \cdot Z_{c_{61,i}}^* r dr d\theta = \pi \left(J_7 + \int q_5 \psi r dr \right) c_{61,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),$$

where J_7, q_5 are explicitly given in the expression of $\langle E, Z_{c_{61,i}}^* \rangle$, and

$$\int E \cdot Z_{c_{62,i}}^* r dr d\theta = \pi \left(J_8 + \int q_6 \psi r dr \right) c_{62,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),$$

where J_8, q_6 are explicitly given in the expression of $\langle E, Z_{c_{62,i}}^* \rangle$, and all the terms $O(\varepsilon) \leq C\varepsilon$, $O(\varepsilon^2) \leq C\varepsilon^2$, $O(|\mathbf{a}|^2) \leq C|\mathbf{a}|^2$ for some positive constant C independent of \mathbf{a} and ε provided that they are small enough. For the explicit expressions of $J_1 - J_8$, and $q_1 - q_7$, we refer the readers to [3]. Recall that once we can calculate the terms containing ψ , then we can calculate all the entries of $\tilde{\mathcal{T}}$.

So we get that

$$\begin{aligned} & (\langle E, Z_3^* \rangle, \langle E, Z_4^* \rangle, \langle E, Z_5^* \rangle, \langle E, Z_6^* \rangle, \langle E, Z_7^* \rangle, \langle E, Z_8^* \rangle, \langle E, Z_9^* \rangle, \langle E, Z_{10}^* \rangle, \\ & \langle E, Z_{11}^* \rangle, \langle E, Z_{12}^* \rangle, \langle E, Z_{13}^* \rangle, \langle E, Z_{14}^* \rangle)^t \\ & = \pi \tilde{\mathcal{T}}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon), \end{aligned}$$

where the entries of the matrix $\tilde{\mathcal{T}}$ are given by the coefficients containing $J_i, \int q_j \psi r dr$ in the above expressions. Moreover, the determinant of the matrix $\tilde{\mathcal{T}}$ is

$$\begin{aligned} (4.12) \quad & (T_1 T_6 - T_2 T_5)^2 T_3^3 T_4^2 T_7^2 T_8^2 \\ & = \left(J_5 + \int q_2 \psi r dr \right)^2 \left(J_6 + \int q_3 \psi r dr \right)^2 \left(J_7 + \int q_5 \psi r dr \right)^2 \\ & \quad \times \left(J_8 + \int q_6 \psi r dr \right)^2 \left[\left(J_1 + \int q_1 \psi r dr \right) \left(J_4 + \int q_4 \psi r dr \right) \right. \\ & \quad \left. - \left(J_2 + \int q_7 \psi r dr \right) \left(J_3 + \int q_7 \psi r dr \right) \right]^2 \end{aligned}$$

where q_1, \dots, q_7 are explicit radial functions.

Next we will calculate the entries of $\tilde{\mathcal{T}}$. As we mentioned before, we need to calculate the integrals J_5 to J_8 , and $\int q_1\psi r dr$ to $\int q_7\psi r dr$. J_5, \dots, J_8 are explicit integrals. For $\int q_i\psi r dr$ the difficulty is that we do not have explicit formula for ψ . In order to get rid of ψ , we use integration by parts. The key observation is that for any radial ϕ satisfying $\phi(\infty) = 0$, we have

$$(4.13) \quad \int_0^\infty \left[\left(\Delta + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2} \right) \psi \right] \phi r dr = \int_0^\infty \left[\left(\Delta + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2} \right) \phi \right] \psi r dr.$$

As a consequence we have

$$\int \psi q_i r dr = \int_0^\infty r^{4N} \frac{3 \times 2^6 (N+1)^4}{(1+r^{2N+2})^4} \phi_i r dr$$

where ψ_i solves

$$(4.14) \quad \Delta \phi_i + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2} \phi_i = q_i, \quad i = 1, \dots, 7$$

which can be solved as in the proof of Lemma 3.4 in [1]. For the details of the computations of q_i, ϕ_i we refer to [3].

Since all the terms in the integrals are explicit now, by direct calculation, we obtain all the explicit formulas for the integrals $J_5 + \int q_2\psi r dr, J_7 + \int q_3\psi r dr$ and $J_8 + \int q_6\psi r dr$. Substituting these formulas into (4.12) we conclude that the matrix $\tilde{\mathcal{T}}$ is non-degenerate. For full computations of these integrals we refer to [3]. □

As a consequence of Proposition 3.1 and Lemma 4.1, the coefficients m_i vanish if and only if the parameters \mathbf{a} satisfy

$$(4.15) \quad \tilde{\mathcal{T}}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon) = 0,$$

Obviously, (4.15) can be solved immediately with $|\mathbf{a}| \leq C\varepsilon$, for some C large but fixed.

5. Proof of Theorem 1.1 under Assumption (ii)

In this section, we are going to prove Theorem 1.1 under **Assumption (ii)**. This situation is more complicated than the previous one, since the $O(\varepsilon)$

approximation and $O(\varepsilon^2)$ approximation induce several difficulties. The problem is that we cannot obtain the explicit expressions for these terms. In this case, we will see that the two free parameters ξ_1, ξ_2 we introduced in Section 2.4 for the improvement of the $O(\varepsilon^2)$ approximate solution play an important role. A key observation is that we only need to consider the terms involving ξ_1 and ξ_2 . This is contained in the following lemma.

Lemma 5.1. *Let $(\frac{v_1}{v_2})$ be a solution of (3.1). The following estimates hold:*

$$(5.1) \quad \langle E, Z_{c_{43,i}}^* \rangle = \xi_1(\tilde{\mathcal{A}}_1 c_{43,i} + \tilde{\mathcal{B}}_1 c_{54,i}) + \xi_2(\tilde{\mathcal{A}}_2 c_{43,i} + \tilde{\mathcal{B}}_2 c_{54,i}) + \tilde{\mathcal{T}}_{1i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O((1 + |\xi|)\varepsilon),$$

$$(5.2) \quad \langle E, Z_{c_{52,i}}^* \rangle = \xi_1 \tilde{\mathcal{C}}_1 c_{52,i} + \xi_2 \tilde{\mathcal{C}}_2 c_{52,i} + \tilde{\mathcal{T}}_{3i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O((1 + |\xi|)\varepsilon),$$

$$(5.3) \quad \langle E, Z_{c_{53,i}}^* \rangle = \xi_1 \tilde{\mathcal{D}}_1 c_{53,i} + \xi_2 \tilde{\mathcal{D}}_2 c_{53,i} + \tilde{\mathcal{T}}_{4i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O((1 + |\xi|)\varepsilon),$$

$$(5.4) \quad \langle E, Z_{c_{54,i}}^* \rangle = \xi_1(\tilde{\mathcal{E}}_1 c_{54,i} + \tilde{\mathcal{F}}_1 c_{43,i}) + \xi_2(\tilde{\mathcal{E}}_2 c_{54,i} + \tilde{\mathcal{F}}_2 c_{43,i}) + \tilde{\mathcal{T}}_{2i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O((1 + |\xi|)\varepsilon),$$

$$(5.5) \quad \langle E, Z_{c_{61,i}}^* \rangle = \xi_1 \tilde{\mathcal{G}}_1 c_{61,i} + \xi_2 \tilde{\mathcal{G}}_2 c_{61,i} + \tilde{\mathcal{T}}_{4i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O((1 + |\xi|)\varepsilon),$$

$$(5.6) \quad \langle E, Z_{c_{62,i}}^* \rangle = \xi_1 \tilde{\mathcal{H}}_1 c_{62,i} + \xi_2 \tilde{\mathcal{H}}_2 c_{62,i} + \tilde{\mathcal{T}}_{4i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O((1 + |\xi|)\varepsilon),$$

for $i = 1, 2$, where

$$(5.7) \quad \tilde{\mathcal{A}}_j = \int_0^{+\infty} \int_0^{2\pi} \left[r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{c_{43,1,1}} - Z_{c_{43,1,2}})^2 + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) \times (2Z_{c_{43,1,2}} - 3Z_{c_{43,1,1}})^2 \right] r dr d\theta,$$

$$(5.8) \quad \tilde{\mathcal{B}}_j = \int_0^{+\infty} \int_0^{2\pi} \left[r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{c_{43,1,1}} - Z_{c_{43,1,2}}) \times (2Z_{c_{54,1,1}} - Z_{c_{54,1,2}}) + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) \times (2Z_{c_{54,1,2}} - 3Z_{c_{54,1,1}})(2Z_{c_{43,1,2}} - 3Z_{c_{43,1,1}}) \right] r dr d\theta,$$

$$(5.9) \quad \tilde{C}_j = \int_0^{+\infty} \int_0^{2\pi} \left[r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{c_{52,1,1}} - Z_{c_{52,1,2}})^2 \right. \\ \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1})(2Z_{c_{52,1,2}} - 3Z_{c_{52,1,1}})^2 \right] r dr d\theta,$$

$$(5.10) \quad \tilde{D}_j = \int_0^{+\infty} \int_0^{2\pi} \left[r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{c_{53,1,1}} - Z_{c_{53,1,2}})^2 \right. \\ \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1})(2Z_{c_{53,1,2}} - 3Z_{c_{53,1,1}})^2 \right] r dr d\theta,$$

$$(5.11) \quad \tilde{E}_j = \int_0^{+\infty} \int_0^{2\pi} \left[r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{c_{54,1,1}} - Z_{c_{54,1,2}})^2 \right. \\ \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1})(2Z_{c_{54,1,2}} - 3Z_{c_{54,1,1}})^2 \right] r dr d\theta,$$

$$\tilde{F}_j = \int_0^{+\infty} \int_0^{2\pi} \left[r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{c_{43,1,1}} - Z_{c_{43,1,2}}) \right. \\ \left. \times (2Z_{c_{54,1,1}} - Z_{c_{54,1,2}}) + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) \right. \\ \left. \times (2Z_{c_{54,1,2}} - 3Z_{c_{54,1,1}})(2Z_{c_{43,1,2}} - 3Z_{c_{43,1,1}}) \right] r dr d\theta,$$

$$(5.12) \quad \tilde{G}_j = \int_0^{+\infty} \int_0^{2\pi} \left[r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{c_{61,1,1}} - Z_{c_{61,1,2}})^2 \right. \\ \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1})(2Z_{c_{61,1,2}} - 3Z_{c_{61,1,1}})^2 \right] r dr d\theta,$$

$$(5.13) \quad \tilde{H}_j = \int_0^{+\infty} \int_0^{2\pi} \left[r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{c_{62,1,1}} - Z_{c_{62,1,2}})^2 \right. \\ \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1})(2Z_{c_{62,1,2}} - 3Z_{c_{62,1,1}})^2 \right] r dr d\theta,$$

for $j = 1, 2$, and \tilde{T}_{ij} are 12×1 vectors which are uniformly bounded as ε tends to 0, and are independent of ξ_1, ξ_2 .

Proof. By (2.30) and (2.31), E is of the form

$$(5.14) \quad \frac{1}{\varepsilon} (\dots) \mathbf{a} \cdot \mathbf{a} + ((\dots) \mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O((1 + |\xi|)\varepsilon).$$

Recall that $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_{0,1} \\ \psi_{0,2} \end{pmatrix} + \xi_1 Z_{\lambda_4} + \xi_2 Z_{\lambda_5}$. In the following computations, we only need to consider the terms involving ξ_1 and ξ_2 , since all other terms are independent of ξ_1 and ξ_2 . We use the Taylor expansion of E .

By (2.30) and (2.31), we know that E can be expanded in the form

$$(5.15) \quad E = \tilde{\mathbf{Q}}\mathbf{a} + \frac{\tilde{\mathbf{A}}\mathbf{a} \cdot \mathbf{a}}{\varepsilon} + O((1 + |\xi|)\varepsilon) + O(1 + |\xi|)|\mathbf{a}|^2 + O(|\varepsilon|^2),$$

where the $|\xi||\mathbf{a}|^2, |\xi|\varepsilon$ terms come from the Taylor expansion of E which contains the $O(\varepsilon^2)$ -term ψ , since by the definition of ψ , we have ξ_1, ξ_2 in it, and

$$\begin{aligned} \tilde{\mathbf{Q}} = & \left(\begin{array}{c} \xi_1(2Z_{1,1} - Z_{1,2}) \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2})\mathbf{a}_i + \xi_2(2Z_{2,1} - Z_{2,2}) \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2})\mathbf{a}_i \\ \xi_1(2Z_{1,2} - 3Z_{1,1}) \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1})\mathbf{a}_i + \xi_2(2Z_{2,2} - 3Z_{2,1}) \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1})\mathbf{a}_i \end{array} \right) \\ & + \text{the remaining linear combination of } \mathbf{a}_i. \end{aligned}$$

By the orthogonality of $\cos(k\theta)$ and $\cos(l\theta)$ for $k \neq l$, we obtain

$$(5.16) \quad \begin{aligned} & - \int_0^{+\infty} \int_0^{2\pi} E \cdot Z_{c_{43,1}}^* r d\theta dr \\ & = \xi_1(\tilde{\mathcal{A}}_1 c_{43,1} + \tilde{\mathcal{B}}_1 c_{54,1}) + \xi_2(\tilde{\mathcal{A}}_2 c_{43,1} + \tilde{\mathcal{B}}_2 c_{54,1}) \\ & \quad + \tilde{\mathcal{T}}_{11}(\mathbf{a}) + O\left(\frac{|\mathbf{a}|^2}{\varepsilon}\right) + O((1 + |\xi|)\varepsilon) + O((1 + |\xi|)|\mathbf{a}|^2 + \varepsilon^2), \end{aligned}$$

where $\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2, \tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2$ are in (5.7) and (5.8) which come from the inner product of $\tilde{\mathbf{Q}}$ with $Z_{c_{43,1}}^*$. In the above Equation (5.16), $\tilde{\mathcal{T}}_{11}(\mathbf{a})$ denotes the inner product of the remaining linear combination of \mathbf{a}_i in $\tilde{\mathbf{Q}}$ and $Z_{c_{43,1}}^*$, and the coefficients of the linear combinations are uniformly bounded and are independent of ξ_1, ξ_2, \mathbf{a} . The $O(\frac{|\mathbf{a}|^2}{\varepsilon})$ terms comes from the $O(\frac{|\mathbf{a}|^2}{\varepsilon})$ term of E which is independent of ξ . Similarly, we can get the other estimates. \square

From the above lemma, we have the following result:

Lemma 5.2. *Let $(\frac{v_1}{v_2})$ be a solution of (3.1). Then the coefficients $m_i = 0$ if and only if the parameters \mathbf{a} satisfy*

$$(5.17) \quad \mathbf{Q}(\mathbf{a}) = \tilde{\mathcal{T}}(\mathbf{a}) + O\left(\frac{|\mathbf{a}|^2}{\varepsilon}\right) + O((1 + |\xi|)|\mathbf{a}|^2) + O((1 + |\xi|)\varepsilon),$$

where $\mathbf{Q} = \xi_1 \mathbf{Q}_1 + \xi_2 \mathbf{Q}_2$, and $\tilde{\mathcal{T}}$ is a 12×12 matrix which is uniformly bounded and independent of ξ_1, ξ_2 and moreover

$$\begin{aligned} \det(\mathbf{Q}_1) &= \tilde{\mathcal{C}}_1 \tilde{\mathcal{D}}_1 \tilde{\mathcal{G}}_1 \tilde{\mathcal{H}}_1 (\tilde{\mathcal{A}}_1 \tilde{\mathcal{E}}_1 - \tilde{\mathcal{B}}_1 \tilde{\mathcal{F}}_1), \\ \text{and } \det(\mathbf{Q}_2) &= \tilde{\mathcal{C}}_2 \tilde{\mathcal{D}}_2 \tilde{\mathcal{G}}_2 \tilde{\mathcal{H}}_2 (\tilde{\mathcal{A}}_2 \tilde{\mathcal{E}}_2 - \tilde{\mathcal{B}}_2 \tilde{\mathcal{F}}_2). \end{aligned}$$

Proof of Theorem 1.1 under Assumption (ii). Under the **Assumptions (ii)**, we will choose $\lambda_4 = \frac{\tilde{\lambda}}{(2^{7/2}\mu_1\mu_2(\mu_1+\mu_2))^2}$ and $\lambda_5 = \frac{1}{\tilde{\lambda}}$ for $\tilde{\lambda}$ large enough. Similar to the computations in the appendix of [1], by direct but tedious computation, we can get that for $\tilde{\lambda}$ large,

$$\begin{aligned} \tilde{A}_1 &= \gamma_1 \tilde{\lambda}^{-1} + o(\tilde{\lambda}^{-1}), & \tilde{C}_1 &= \gamma_2 \tilde{\lambda}^{-1} + o(\tilde{\lambda}^{-1}), \\ \tilde{D}_1 &= \gamma_3 \tilde{\lambda}^{-2} + o(\tilde{\lambda}^{-2}), & \tilde{E}_1 &= \gamma_4 \tilde{\lambda}^{-4} + o(\tilde{\lambda}^{-4}), \\ \tilde{G}_1 i &= \gamma_5 \tilde{\lambda}^{-3} + o(\tilde{\lambda}^{-3}), & \tilde{H}_1 &= \gamma_6 + o(1), \end{aligned}$$

and

$$(5.18) \quad \tilde{B}_1 = \tilde{F}_1 = \begin{cases} O(\tilde{\lambda}^{-3}) & \text{if } N_1 = N_2, \\ 0 & \text{if } N_1 \neq N_2, \end{cases}$$

where γ_1 to γ_6 are nonzero constants whose explicit expressions can be found in [3]. So we have

$$\begin{aligned} \tilde{A}_1 \tilde{E}_1 - \tilde{B}_1 \tilde{F}_1 &= \gamma_1 \gamma_4 \tilde{\lambda}^{-5} + o(\tilde{\lambda}^{-5}) \\ &\neq 0, \end{aligned}$$

and $\tilde{C}_1, \tilde{D}_1, \tilde{E}_1, \tilde{G}_1, \tilde{H}_1$ are all non-zero if $\tilde{\lambda}$ is large enough. Therefore, we choose ξ_1 large and $\xi_2 = 0$ to conclude that $\mathbf{Q}(\xi_1, \xi_2) - \tilde{T}$ is non-degenerate. After fixing $(\lambda_4, \lambda_5), (\xi_1, \xi_2)$, it is easy to see that (5.17) can be solved with $\mathbf{a} = O(\varepsilon)$. □

6. Proof of Theorem 1.1 under Assumption (iii)

We are left to prove the theorem for $N_2 \sum_{i=1}^{N_1} p_i = N_1 \sum_{j=1}^{N_2} q_j, N_1 \neq N_2$ and one of N_i is 1. Without loss of generality, assume $N_1 = 1$ and $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j = 0$. In this case, for the improvement of approximate solution in the $O(\varepsilon^2)$ term, we can not solve Equation (2.25) in Section 2.4. Since in general, we use Lemma 2.2 to solve (2.25), i.e. we have to verify that the right hand side of Equation (2.25) is orthogonal to Z_i^* in L^2 -norm for $i = 3, \dots, 14$, and basically we use the orthogonality of trigonometric functions to verify it. But when $N_1 = 1, Z_3^* = h(r) \cos 2\theta$, we can not get the orthogonality of the right hand side with Z_3^*, Z_4^* directly. So we can not solve (2.25) by using Lemma 2.2. Instead of solving (2.25), we can find a unique solution of the

following equation which is guaranteed by Lemma 2.2:

$$(6.1) \quad \begin{cases} \Delta\psi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1 - \psi_2) \\ = 2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} \\ \Delta\psi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2 - 3\psi_1) \\ = 2|z|^{4N_2} e^{4\tilde{U}_{2,0} - 6\tilde{U}_{1,0}} - 3|z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}}. \end{cases}$$

We use this unique solution as the new ψ_0 , and proceed as before. Then by checking the previous proof, we can get that in this case, the error $\|E\|_* \leq C_0$ and we can get a solution v of (3.1) which satisfies

$$(6.2) \quad \|v\|_* \leq C_0,$$

for some positive constant C_0 , and the following estimates hold:

$$(6.3) \quad \int_{\mathbb{R}^2} (N_{11}(v) + N_{12}(v))Z_{i,1}^* + (N_{21}(v) + N_{22}(v))Z_{i,2}^* dx = O(\varepsilon),$$

for $i = 3, \dots, 14$.

Then the reduced problem we get is

$$(6.4) \quad \mathbf{Q}(\mathbf{a}) + \tilde{\mathcal{T}}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(1) + O((1 + |\xi|)\varepsilon) = 0,$$

where the $O(1)$ term comes from the $O(1)$ term of the error E since we use the solution of (6.1) instead of (2.25) as the $O(\varepsilon^2)$ improvement. Recalling that $\mathbf{Q} = \mathbf{Q}(\xi_1, \xi_2)$ depends on two free parameters ξ_1, ξ_2 and arguing as before, we can choose ξ_1 large enough. Then it is easy to get a solution of (6.4) with $\mathbf{a} = O(\xi_1^{-\alpha})$ for any $0 < \alpha < 1$. □

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