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Two Morse functions and singularities of the product map

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For two Morse functions on a manifold, some distances between them are naturally defined from a topological point of view. We give some upper bounds for the distances in terms of singularities of the product map of the two Morse functions.

1. Introduction

We consider distances between two Morse functions on a manifold. Let X be a closed smooth manifold, and $F, G : X \to \mathbb{R}$ be Morse functions on X. By the Cerf theory [2], the functions F, G can be connected by a generic homotopy, which has finitely many births and deaths of canceling pairs of critical points and passings of critical values. This allows us to define $D(F, G)$ as the minimal number of births, deaths and passings over all generic homotopies between F and G, and define $d(F, G)$ as the minimal number of births and deaths by ignoring passings. Indeed, D gives a metric for isotopy classes of Morse functions, and d gives a metric for quasi-isotopy classes of Morse functions. (See Section 2 for basic definitions.)

In this paper, we give some upper bounds for the distances $D(F, G)$ and $d(F, G)$ in terms of singularities of the product map of the Morse functions F and G. By the product map, we mean the map $\varphi : X \to \mathbb{R}^2$ defined by $\varphi(x) = (F(x), G(x))$ for $x \in X$. We suppose that the dimension of X is at least 2 and that φ is a stable map (cf. Remark 3). See Theorem 5 for our bounds, one of which implies the following:

$$(\sharp) \qquad d(F, G) \leq \frac{1}{2}\left(m(F) + m(G)\right) + c(\varphi),$$

where $m(*)$ denotes the number of critical points of a Morse function and $c(\varphi)$ denotes the number of cusp points of φ.

We remark that the above bound might be improved by a certain development in singularity theory. Significantly, it is known by Levine [11] that a stable map from X to $\mathbb{R}^2$ can be homotoped so that the number of cusp points is at most 1. The inequality ($\sharp$) and Levine's theorem suggest the following conjecture:

$$(\sharp') \qquad d(F,G) \leq \frac{1}{2}\,(m(F)+m(G))+1.$$

This cannot be proved immediately because Levine's homotopy may not preserve the product structure of φ. Still, some local moves of φ reducing the number of cusp points can be performed preserving the product structure, at least when X is 3-dimensional [23].

In the case where the source manifold is 2-dimensional, the distance d is well understood, though the distance D is not so. Let F, G be Morse functions on a compact connected surface. Maksymenko [12] showed that F, G are quasi-isotopic if and only if $m_\lambda(F) = m_\lambda(G)$ for each $\lambda \in \{0,1,2\}$, where $m_\lambda(*)$ denotes the number of critical points of Morse index λ of a Morse function. This implies that $d(F,G) = |m_0(F) - m_0(G)| + |m_2(F) - m_2(G)|$.

In the case where the source manifold is 3-dimensional, the distance d is well studied in the context of Heegaard theory, though the distance D is not so. There are many results about the Reidemeister–Singer distance between two Heegaard splittings for a closed connected orientable 3-dimensional manifold, and they can be interpreted as results about the distance d. For example, see [8–10, 19, 22] for bounds for the Reidemeister–Singer distance, and see [1, 6, 9, 20] for relevant examples of Heegaard splittings. The inequality ($\sharp$) can be regarded as a generalization of [22, Theorem 1]. The conjecture ($\sharp'$) corresponds to [22, Conjecture 5], which is best possible by the example given in [6].

In the case where the source manifold is 4-dimensional or higher, the distances d and D are of interest in the context of the Kirby calculus, but few things are known about them. We remark that the Kirby calculus is related to transformations of group presentations (see [5, Solution of Exercise 5.1.10 (d)]). We hope that we can also approach problems of group presentations from a singularity theoretic point of view.

2. Preliminaries and main result

In this section, we review basic definitions and facts concerning singularities of smooth maps, and we state the main result of this paper.

The notion of stable map is defined as follows. Let X, Y be smooth manifolds and $C^\infty(X,Y)$ denote the space of smooth maps from X to Y endowed with the Whitney C^∞ topology (see [4] or [7]). A smooth homotopy $\{\phi_t : X \to Y\}_{t\in[0,1]}$ is said to be an *isotopy* if there exist smooth ambient isotopies $\{H_t^X : X \to X\}_{t\in[0,1]}$ and $\{H_t^Y : Y \to Y\}_{t\in[0,1]}$ such that $\phi_t = H_t^Y \circ \phi_0 \circ H_t^X$ for $t \in [0,1]$. Two smooth maps are said to be *isotopic* if they are connected by an isotopy. A smooth map $\phi : X \to Y$ is said to be *stable* if there exists an open neighborhood U of ϕ in $C^\infty(X, Y)$ such that ϕ and every map in U are isotopic.

A *Morse function* on a closed smooth manifold is a stable map from the manifold to $\mathbb{R}$. The critical points of a Morse function are all non-degenerate and have pairwise distinct values. The *Morse index* is well-defined for each non-degenerate critical point (see [15]).

The notion of generic homotopy for smooth functions is defined as follows. Let X be a closed smooth manifold, n denote the dimension of X, and $\{F_t : X \to \mathbb{R}\}_{t\in[-1,1]}$ be a smooth homotopy. Note that $\{F_t\}_{t\in[-1,1]}$ is an isotopy if F_t is a Morse function for every $t \in [-1,1]$. The homotopy $\{F_t\}_{t\in[-1,1]}$ is said to be a *quasi-isotopy* if the critical points of F_t are all non-degenerate for every $t \in [-1,1]$. A *birth* (resp. *death*) *of a canceling pair of critical points*, or simply a *birth* (resp. *death*), of $\{F_t\}_{t\in[-1,1]}$ is the pair (o, p) of $o \in (-1,1)$ and $p \in X$ such that $F_s(x_1, x_2, \ldots, x_n) = sx_1 - x_1^3 - x_2^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_n^2$ for a local coordinate system $(x_1, x_2, \ldots, x_n)$ at p and a local coordinate s at o whose direction agrees (resp. disagrees) with that of t. Note that, by a birth (resp. death) with this local form, a pair of non-degenerate critical points of Morse indices λ and $\lambda+1$ appears (resp. disappears). A *passing of critical values*, or simply a *passing*, of $\{F_t\}_{t\in[-1,1]}$ is the pair $(o, \{p, q\})$ of $o \in (-1,1)$ and $\{p, q\} \subset X$ such that p, q are distinct non-degenerate critical points of F_o with the same value, and the Cerf graphic $\{(t, v) \in \mathbb{R}^2 \mid v \text{ is a critical value of } F_t|_{U\cup V}\}$ has a transverse crossing at $(o, F_o(p))$ for small neighborhoods U, V of p, q, respectively. The homotopy $\{F_t\}_{t\in[-1,1]}$ is said to be *generic* if it is an isotopy except that, at each of finitely many t in $(-1,1)$, it has either a single birth, a single death or a single passing.

Some distances between two Morse functions are defined as follows. Let F, G be Morse functions on a closed smooth manifold. We define $d(F, G)$ (resp. $D(F, G)$) as the minimal number of births and deaths (resp. births, deaths and passings) over all generic homotopies between F and G. Also we define $d_{\lambda,\lambda+1}(F, G)$ as the minimal number of births and deaths of canceling pairs of critical points of Morse indices λ and $\lambda+1$ over all generic homotopies between F and G.

From now on, we consider a smooth map ϕ from a closed smooth n-dimensional manifold X with $n \geq 2$ to a smooth surface Y without boundary.

A *singular point* of ϕ is a point in X at which the differential of ϕ has rank less than two. The *singular set* of ϕ is the set of singular points of ϕ, and is denoted by S_ϕ.

A *fold point* of ϕ is a singular point $p \in X$ such that $\phi(x_1, x_2, \ldots, x_n) = (x_1,\ -x_2^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_n^2)$ for local coordinate systems at $p = (0, 0, \ldots, 0)$ and $\phi(p) = (0, 0)$. The minimum of $\{\lambda,\ n - \lambda - 1\}$ does not depend on the choice of coordinate systems, and is called the *absolute index* of the fold point p. One can see that, in a small neighborhood U of p, the singular set $S_\phi \cap U$ is a smooth arc consisting of fold points of the same absolute index as p, and the restriction of ϕ to the arc $S_\phi \cap U$ is an embedding.

A *cusp point* of ϕ is a singular point $p \in X$ such that $\phi(x_1, x_2, \ldots, x_n) = (x_1,\ x_1 x_2 - x_2^3 - x_3^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_n^2)$ for local coordinate systems at $p = (0, 0, \ldots, 0)$ and $\phi(p) = (0, 0)$. The minimum of $\{\lambda - 1,\ n - \lambda - 1\}$ does not depend on the choice of coordinate systems, and is called the *absolute index* of the cusp point p. One can see that, in a small neighborhood U of p, the singular set $S_\phi \cap U$ is a smooth arc, and the restriction of ϕ to the arc $S_\phi \cap U$ has an ordinary cusp at p. One component of $(S_\phi \cap U) \setminus \{p\}$ consists of fold points of absolute index equal to that of p, and the other component consists of fold points of absolute index equal to that of p plus one, except when $\lambda = \frac{n}{2}$. In the exceptional case, both of the components consist of fold points of absolute index $\lambda - 1$.

If every singular point of ϕ is either a fold point or a cusp point, the situation is as follows. By the above local observations and the compactness of X, the singular set S_ϕ is a collection of smooth circles and includes finitely many cusp points. The absolute index is well-defined for each complementary subarc of the cusp points in S_ϕ. The restriction of ϕ to S_ϕ is an immersion except that it has an ordinary cusp at each cusp point of ϕ, and is called the *discriminant curve* of ϕ.

Stable maps from X to Y are characterized as in the next theorem. This follows from Mather's theorems [13, Theorem A, Proposition 1.8] and [14, Theorem 4.1].

Theorem 1 (Mather). *A smooth map from a closed smooth n-dimensional manifold with $n \geq 2$ to a smooth surface without boundary is stable if and only if it satisfies the following:*

- *every singular point is either a fold point or a cusp point,*
- *every multiple point of the discriminant curve is a transverse crossing of exactly two subarcs outside of the cusp points.*

From now on, we consider the product map of two functions.

Notation 2. Let X be a closed smooth n-dimensional manifold with $n \geq 2$, and $F, G : X \to \mathbb{R}$ be smooth functions. Fix a global coordinate system (f, g) of the plane $\mathbb{R}^2$, and let $\pi_f, \pi_g : \mathbb{R}^2 \to \mathbb{R}$ denote the projections $(f, g) \mapsto f$, $(f, g) \mapsto g$, respectively. Let $\varphi : X \to \mathbb{R}^2$ denote the product map of F, G with respect to the coordinate system (f, g), that is, $\pi_f \circ \varphi = F$ and $\pi_g \circ \varphi = G$. Let S_φ denote the singular set of φ.

Remark 3. *If F, G are Morse functions, then φ is stable after arbitrarily small isotopies of F and G.*

Proof. Let U_F (resp. U_G) be an arbitrarily small open neighborhood of F (resp. G) in $C^\infty(X, \mathbb{R})$. Since F (resp. G) is stable, there exists an open neighborhood U'_F of F in U_F (resp. U'_G of G in U_G) such that every function in U'_F (resp. U'_G) is isotopic to F (resp. G). Since π_f (resp. π_g) is a smooth map, the induced map $\pi_{f*} : C^\infty(X, \mathbb{R}^2) \to C^\infty(X, \mathbb{R})$, $\psi \mapsto \pi_f \circ \psi$ (resp. $\pi_{g*} : C^\infty(X, \mathbb{R}^2) \to C^\infty(X, \mathbb{R})$, $\psi \mapsto \pi_g \circ \psi$) is continuous by [4, Chapter II, Proposition 3.5]. Hence $\pi_{f*}^{-1}(U'_F) \cap \pi_{g*}^{-1}(U'_G)$ is an open neighborhood of φ in $C^\infty(X, \mathbb{R}^2)$. Since stable maps are dense in $C^\infty(X, \mathbb{R}^2)$ by [14], there exists a stable map $\tilde{\varphi}$ in $\pi_{f*}^{-1}(U'_F) \cap \pi_{g*}^{-1}(U'_G)$. The functions $\pi_{f*}(\tilde{\varphi})$, $\pi_{g*}(\tilde{\varphi})$ are Morse functions isotopic to F, G, respectively, whose product map $\tilde{\varphi}$ is stable. $\square$

We analyze the discriminant curve of the product map. With the above notation, we suppose that the product map φ is stable, and let $C_\varphi : S_\varphi \to \mathbb{R}^2$ denote the discriminant curve of φ. A cusp point of φ is also called a *cusp point* of C_φ. By a *double point* of C_φ, we mean a point in $\mathbb{R}^2$ at which C_φ has one of the transverse crossings. Note that each point in S_φ uniquely determines a tangent line in $\mathbb{R}^2$ of C_φ. This allows us to define the *slope* of C_φ at a point in S_φ as the slope of the tangent line with respect to the coordinate system (f, g). In particular, a point in S_φ with slope zero (resp. infinity) is called a *horizontal* (resp. *vertical*) *point* of C_φ. A tangent line of C_φ is said to be a *double tangent line* of C_φ if it has two or more tangent points. We may regard the curve C_φ as parametrized so that the first derivative is not zero at every fold point of φ. A fold point is said to be an *inflection point* of C_φ if the first and the second derivatives are linearly dependent. The following are generic conditions of the curve C_φ, that is to say, hold after an arbitrarily small isotopy of C_φ in $\mathbb{R}^2$:

(i) there are only finitely many inflection points and double tangent lines,

(ii) each double tangent line has exactly two tangent points.

Remark 4. *If F, G are Morse functions, then any sufficiently small isotopy of C_φ in $\mathbb{R}^2$ can be realized by isotopies of F and G.*

Proof. The maps $\pi_{f*}, \pi_{g*} : C^\infty(X, \mathbb{R}^2) \to C^\infty(X, \mathbb{R})$ are continuous as mentioned in the proof of Remark 3. The map $\varphi^* : C^\infty(\mathbb{R}^2, \mathbb{R}^2) \to C^\infty(X, \mathbb{R}^2)$, $H \mapsto H \circ \varphi$ is also continuous by [4, Chapter II, Proposition 3.9]. The maps $\pi_{f*} \circ \varphi^*$ and $\pi_{g*} \circ \varphi^*$ are therefore continuous. $\square$

We give some upper bounds for the distances between two Morse functions in terms of the discriminant curve of the product map as in the following theorem.

Theorem 5. *Let X be a closed smooth n-dimensional manifold with $n \geq 2$, let $F, G : X \to \mathbb{R}$ be Morse functions, and let $\varphi : X \to \mathbb{R}^2$ denote the product map of F, G. Suppose that φ is stable and its discriminant curve satisfies the conditions* (i) *and* (ii)*. Then, the following inequalities hold:*

$$d(F, G) \leq i^-(\varphi), \tag{1}$$

$$D(F, G) \leq i^-(\varphi) + t^-(\varphi), \tag{2}$$

$$d(F, G) \leq \frac{1}{2}\left(m(F) + m(G)\right) + c^-(\varphi), \tag{3}$$

$$\begin{cases} d_{0,1}(F, G) \leq m_0(F) + m_0(G) \\ \quad d_{\lambda-1,\lambda}(F, G) + d_{\lambda,\lambda+1}(F, G) \\ \leq m_\lambda(F) + m_\lambda(G) + 2c^-_{\lambda-1}(\varphi) \qquad (0 < \lambda \leq \frac{n}{2}) \\ \quad d_{\lambda-1,\lambda}(F, G) + d_{\lambda,\lambda+1}(F, G) \\ \leq m_\lambda(F) + m_\lambda(G) + 2c^-_{n-\lambda-1}(\varphi) \quad (\frac{n}{2} \leq \lambda < n) \\ d_{n-1,n}(F, G) \leq m_n(F) + m_n(G), \end{cases} \tag{4}$$

$$\begin{aligned} D(F, G) \leq \frac{1}{2}\big(m(F) + m(G) + 2c^-(\varphi)\big)\left(m(F) + m(G) - 1\right) \\ + \frac{1}{2} m(F) m(G) + d^-(\varphi), \end{aligned} \tag{5}$$

where $m(*)$, $m_\lambda(*)$ denote the numbers of critical points, critical points of Morse index λ, respectively, of a Morse function, and $i^-(\varphi)$, $t^-(\varphi)$, $c^-(\varphi)$, $c^-_\lambda(\varphi)$, $d^-(\varphi)$ denote the numbers of negative slope inflection points, negative slope double tangent lines, negative slope cusp points, negative slope cusp points of absolute index λ, double points of negative slope subarcs, respectively, of the discriminant curve of φ.

The inequality (1) can be regarded as a generalization of Johnson's result [8] about Heegaard splittings for orientable 3-dimensional manifolds. The inequality (3) implies the inequality (♯) in Section 1 by ignoring the slopes at the cusp points. Compare this theorem with Fabricius-Bjerre's relation [3] about plane curves, and with the Morse type inequality by Motta–Porto–Saeki [17] about stable maps.

3. Reading the discriminant curve

In this section, we describe how to read information about two functions from the discriminant curve of the product map, and consider homotopies of the functions and the curve.

We use Notation 2 and suppose that the product map φ is stable. Let $C_\varphi : S_\varphi \to \mathbb{R}^2$ denote the discriminant curve of φ. Note that we do not assume F, G to be Morse functions.

Throughout this section, the two functions F, G are symmetric, and hence all the assertions for G also hold for F by replacing "horizontal" with "vertical".

We can read information about the function G from the discriminant curve C_φ of φ as follows. One can prove the next three lemmas by straightforward generalization of the proofs of [22, Lemmas 11, 12, 14], or see [21].

Lemma 6. *A point in X is a critical point of G if and only if it is a singular point of φ and is a horizontal point of C_φ.*

Lemma 7. *A point in X is a degenerate critical point of G if and only if it is a fold point of φ and is a horizontal inflection point of C_φ.*

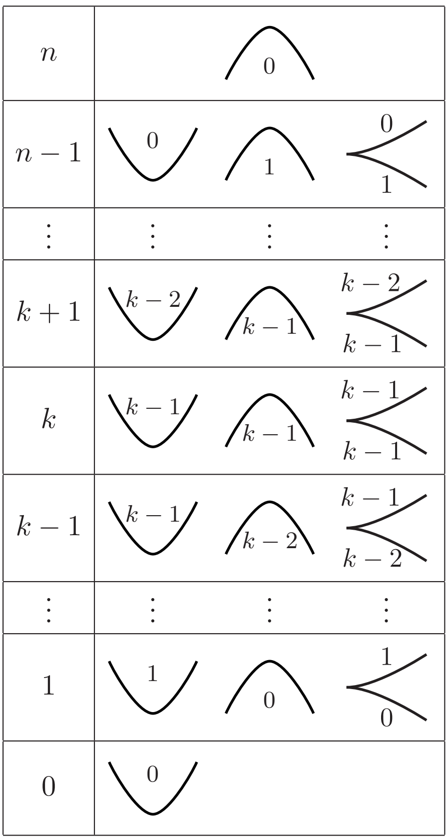
Note that, by these two lemmas, each non-degenerate critical point of G is either a downward convex horizontal point, an upward convex horizontal point or a horizontal cusp point of C_φ.

Lemma 8. *The Morse index of a non-degenerate critical point p of G is related to the behavior of C_φ at p as in Table 1.*

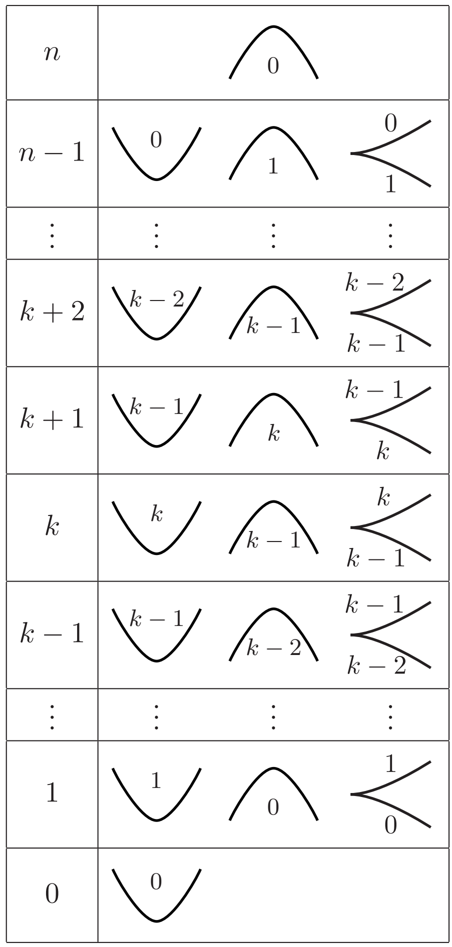
Note that the value of each critical point of G is the g-coordinate of the image of the horizontal point of C_φ. If some critical points of G have the same value, C_φ has a horizontal double tangent line at them. It follows that G is a Morse function if and only if C_φ has neither horizontal inflection points nor horizontal double tangent lines.

Table 1: The relation between the Morse index of p, shown in the first column, and the possibilities of the image of C_φ restricted to a neighborhood of p, shown in the second column. The image of C_φ in $\mathbb{R}^2$ is drawn so that the f-axis is horizontal and the coordinate g increases from bottom to top. Each number in the second column stands for the absolute index of the corresponding subarc of S_φ.

if $n = 2k$ for an integer $k \geq 1$



if $n = 2k+1$ for an integer $k \geq 1$



We consider the induced homotopies of F, G and C_φ by an ambient isotopy of the plane. Let $\{H_t : \mathbb{R}^2 \to \mathbb{R}^2\}_{t\in[-1,1]}$ be a smooth ambient isotopy of $\mathbb{R}^2$. Let $\varphi_t = H_t \circ \varphi$ and $F_t = \pi_f \circ \varphi_t$ and $G_t = \pi_g \circ \varphi_t$ and $C_{\varphi_t} = H_t \circ C_\varphi$ for each $t \in [-1,1]$. Note that, by the definition, $\{\varphi_t\}_{t\in[-1,1]}$ is an isotopy of the stable map φ, and hence consists of stable maps. Note also that φ_t is the product map of F_t and G_t, and C_{φ_t} is the discriminant curve of φ_t for each $t \in [-1, 1]$.

We can read information about the homotopy $\{G_t\}_{t\in[-1,1]}$ of G from the isotopy $\{C_{\varphi_t}\}_{t\in[-1,1]}$ of C_φ as follows. By the above results, $\{G_t\}_{t\in[-1,1]}$ is an isotopy if C_{φ_t} has neither horizontal inflection points nor horizontal double tangent lines for every $t \in [-1, 1]$. A horizontal inflection point appears when $\{C_{\varphi_t}\}_{t\in[-1,1]}$ has a move as in Figures 1, and a horizontal double tangent line appears when $\{C_{\varphi_t}\}_{t\in[-1,1]}$ has a move as in Figure 2. We call these moves a *birth* or a *death of a canceling pair of horizontal points* of $\{C_{\varphi_t}\}_{t\in[-1,1]}$ and a *passing of horizontal points* of $\{C_{\varphi_t}\}_{t\in[-1,1]}$, respectively, if they satisfy certain transversality conditions. We can easily see that $\{G_t\}_{t\in[-1,1]}$ has a passing of critical values when $\{C_{\varphi_t}\}_{t\in[-1,1]}$ has a passing of horizontal points.

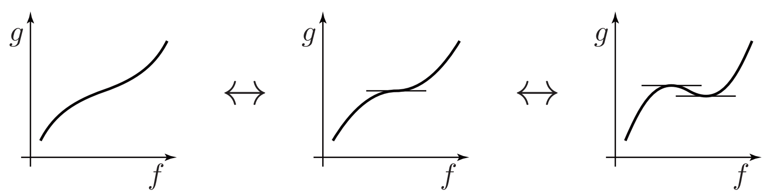


Figure 1: A birth or a death of a canceling pair of horizontal points.

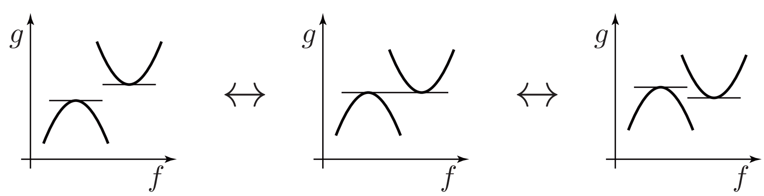


Figure 2: A passing of horizontal points.

Lemma 9. *The homotopy $\{G_t\}_{t\in[-1,1]}$ has a birth (resp. death) of a canceling pair of critical points when $\{C_{\varphi_t}\}_{t\in[-1,1]}$ has a birth (resp. death) of a canceling pair of horizontal points. In particular, if the absolute index of the arc is λ, then the Morse indices of the critical points which are the downward and upward convex horizontal points are either λ and $\lambda + 1$, respectively, or $n - \lambda - 1$ and $n - \lambda$, respectively.*

Proof. By Lemma 6, the homotopy $\{G_t\}_{t\in[-1,1]}$ has a birth (resp. death) of a pair of critical points when $\{C_{\varphi_t}\}_{t\in[-1,1]}$ has a birth (resp. death) of a canceling pair of horizontal points. For the proof of the claim that the pair of critical points is a canceling pair, we refer the reader to the proof of [22, Lemma 16], which dose not depend on the dimension of X. The latter claim of the present lemma almost follows from Lemma 8 since the Morse indices of the critical points of a canceling pair are adjacent integers. It remains to rule out the possibility that $n = 2k+1$ for an integer $k \geq 1$, the absolute index of the arc is $k-1$, and the Morse indices of the critical points are k and $k+1$. By way of contradiction, we assume that such a birth occurs at $t = 0$ and at $p \in X$. Since p is a horizontal inflection point of C_{φ_0} and not a vertical point, p is a regular point of F_0. There exists a local coordinate system $(x_1, x_2, \dots, x_n)$ at $p = (0, 0, \dots, 0)$ such that $F_0(x_1, x_2, \dots, x_n) = x_1 + F_0(p)$. The map φ_0 has the local form $\varphi_0(x_1, x_2, \dots, x_n) = (x_1 + F_0(p),\ G_0(x_1, x_2, \dots, x_n))$ at the fold point p. By Morin's characterization [16, Lemme 1],

$$\boldsymbol{H} = \begin{pmatrix} \left(\frac{\partial^2 G_0}{\partial x_2^2}\right)_p & \left(\frac{\partial^2 G_0}{\partial x_2 \partial x_3}\right)_p & \cdots & \left(\frac{\partial^2 G_0}{\partial x_2 \partial x_n}\right)_p \\ \left(\frac{\partial^2 G_0}{\partial x_3 \partial x_2}\right)_p & \left(\frac{\partial^2 G_0}{\partial x_3^2}\right)_p & \cdots & \left(\frac{\partial^2 G_0}{\partial x_3 \partial x_n}\right)_p \\ \vdots & \vdots & & \vdots \\ \left(\frac{\partial^2 G_0}{\partial x_n \partial x_2}\right)_p & \left(\frac{\partial^2 G_0}{\partial x_n \partial x_3}\right)_p & \cdots & \left(\frac{\partial^2 G_0}{\partial x_n^2}\right)_p \end{pmatrix}$$

is a regular matrix, and the sum of the multiplicities of negative eigenvalues of $\boldsymbol{H}$ is equal to a number κ with which φ_0 has the standard form $\varphi_0(\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_n) = (\mathrm{x}_1,\ \ -\mathrm{x}_2^2 - \cdots - \mathrm{x}_{\kappa+1}^2 + \mathrm{x}_{\kappa+2}^2 + \cdots + \mathrm{x}_n^2)$ at the fold point p. We remark that the coordinate system of $\mathbb{R}^2$ in this form is other than (f, g). Recall that $n = 2k+1$ and that the absolute index $k-1$ of the fold point p is equal to the minimum of $\{\kappa,\ n-\kappa-1\}$. It follows that either $\kappa = k-1$ or $\kappa = k+1$. The sign of the determinant of $\boldsymbol{H}$ is $(-1)^\kappa = (-1)^{k-1}$. Since $\{G_t\}_{t \in [-1,1]}$ has a birth of a canceling pair of critical points of Morse indices k and $k+1$ at $t = 0$ at p, there exists a local coordinate system $(y_1, y_2, \dots, y_n)$ at $p = (0, 0, \dots, 0)$ such that $G_0(y_1, y_2, \dots, y_n) = -y_1^3 - y_2^2 - \cdots - y_{k+1}^2 + y_{k+2}^2 + \cdots + y_n^2$. For $2 \leq i \leq n$ and $2 \leq j \leq n$, the chain rule gives

$$\begin{aligned} \frac{\partial G_0}{\partial x_j} &= \frac{\partial G_0}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \frac{\partial G_0}{\partial y_2}\frac{\partial y_2}{\partial x_j} + \cdots + \frac{\partial G_0}{\partial y_3}\frac{\partial y_3}{\partial x_j} \\ &= -3y_1^2\frac{\partial y_1}{\partial x_j} - 2y_2\frac{\partial y_2}{\partial x_j} - \cdots - 2y_{k+1}\frac{\partial y_{k+1}}{\partial x_j} + 2y_{k+2}\frac{\partial y_{k+2}}{\partial x_j} + \cdots + 2y_n\frac{\partial y_n}{\partial x_j}, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 G_0}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i}\left(-3y_1^2\frac{\partial y_1}{\partial x_j} - 2y_2\frac{\partial y_2}{\partial x_j} - \cdots - 2y_{k+1}\frac{\partial y_{k+1}}{\partial x_j}\right. \\
&\qquad\left. +2y_{k+2}\frac{\partial y_{k+2}}{\partial x_j} + \cdots + 2y_n\frac{\partial y_n}{\partial x_j}\right) \\
&= -\left(6y_1\frac{\partial y_1}{\partial x_i}\frac{\partial y_1}{\partial x_j} + 3y_1^2\frac{\partial^2 y_1}{\partial x_i \partial x_j}\right) - \left(2\frac{\partial y_2}{\partial x_i}\frac{\partial y_2}{\partial x_j} + 2y_2\frac{\partial^2 y_2}{\partial x_i \partial x_j}\right) \\
&\quad - \cdots - \left(2\frac{\partial y_{k+1}}{\partial x_i}\frac{\partial y_{k+1}}{\partial x_j} + 2y_{k+1}\frac{\partial^2 y_{k+1}}{\partial x_i \partial x_j}\right) \\
&\quad + \left(2\frac{\partial y_{k+2}}{\partial x_i}\frac{\partial y_{k+2}}{\partial x_j} + 2y_{k+2}\frac{\partial^2 y_{k+2}}{\partial x_i \partial x_j}\right) \\
&\quad + \cdots + \left(2\frac{\partial y_n}{\partial x_i}\frac{\partial y_n}{\partial x_j} + 2y_n\frac{\partial^2 y_n}{\partial x_i \partial x_j}\right).
\end{aligned}
$$

By substituting $p = (0, 0, \ldots, 0)$,

$$
\begin{aligned}
\left(\frac{\partial^2 G_0}{\partial x_i \partial x_j}\right)_p &= -2\left(\frac{\partial y_2}{\partial x_i}\right)_p\left(\frac{\partial y_2}{\partial x_j}\right)_p - \cdots - 2\left(\frac{\partial y_{k+1}}{\partial x_i}\right)_p\left(\frac{\partial y_{k+1}}{\partial x_j}\right)_p \\
&\quad + 2\left(\frac{\partial y_{k+2}}{\partial x_i}\right)_p\left(\frac{\partial y_{k+2}}{\partial x_j}\right)_p + \cdots + 2\left(\frac{\partial y_n}{\partial x_i}\right)_p\left(\frac{\partial y_n}{\partial x_j}\right)_p.
\end{aligned}
$$

We have

$$
\boldsymbol{H} = \boldsymbol{J}\begin{pmatrix}
-2 & & & & & \\
 & \ddots & & & & \\
 & & -2 & & & \\
 & & & 2 & & \\
 & & & & \ddots & \\
 & & & & & 2
\end{pmatrix}{}^t\boldsymbol{J},
$$

where

$$
\boldsymbol{J} = \begin{pmatrix}
\left(\frac{\partial y_2}{\partial x_2}\right)_p & \left(\frac{\partial y_3}{\partial x_2}\right)_p & \cdots & \left(\frac{\partial y_n}{\partial x_2}\right)_p \\
\left(\frac{\partial y_2}{\partial x_3}\right)_p & \left(\frac{\partial y_3}{\partial x_3}\right)_p & \cdots & \left(\frac{\partial y_n}{\partial x_3}\right)_p \\
\vdots & \vdots & & \vdots \\
\left(\frac{\partial y_2}{\partial x_n}\right)_p & \left(\frac{\partial y_3}{\partial x_n}\right)_p & \cdots & \left(\frac{\partial y_n}{\partial x_n}\right)_p
\end{pmatrix}.
$$

The determinant of $\boldsymbol{H}$ is therefore $|\boldsymbol{J}|(-2)^k 2^{n-k-1}|{}^t\boldsymbol{J}| = (-1)^k 2^{n-1}|\boldsymbol{J}|^2$. Thus, we have a contradiction to the sign of the determinant of $\boldsymbol{H}$. $\square$

We remark that, during the isotopy $\{C_{\varphi_t}\}_{t\in[-1,1]}$, a horizontal cusp point may appear as in Figure 3, but not as in Figure 4. If $\{C_{\varphi_t}\}_{t\in[-1,1]}$ has a local move as in Figure 4, the homotopy $\{G_t\}_{t\in[-1,1]}$ has a birth or a death of a pair of critical points at a cusp point by Lemma 6, but it is impossible by Lemma 7. In general, at an ordinary cusp of a plane curve, the tangent line always separates the two branches [18, Proposition 1.6].

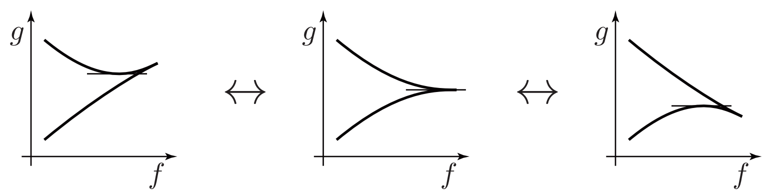


Figure 3: A possible move involving a horizontal cusp point.

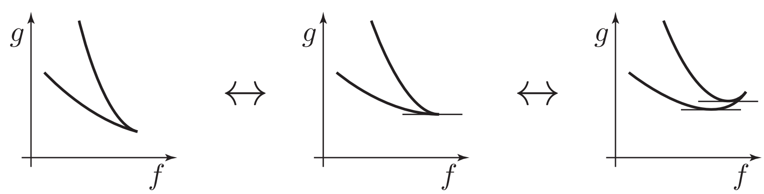


Figure 4: An impossible move involving a horizontal cusp point.

4. Rotating the discriminant curve

In this section, we prove the inequalities (1) and (2) in Theorem 5 and a lemma for later use, by observing a rotation of the discriminant curve in the target plane.

We use Notation 2 and suppose that the two functions F, G are Morse functions and the product map φ is stable. Let $C_\varphi : S_\varphi \to \mathbb{R}^2$ denote the discriminant curve of φ.

We assume some generic conditions. We first assume that C_φ satisfies the conditions (i) and (ii) as in the statement of the theorem. The curve C_φ also satisfies the following conditions after an arbitrarily small isotopy in $\mathbb{R}^2$:

(iii) the first and the third derivatives are linearly independent at each inflection point,

(iv) the tangent lines at inflection points are not double tangent lines,

(v) inflection points and double tangent lines have pairwise distinct slopes.

Note that such a small isotopy can be chosen not to increase the numbers of negative slope inflection points and negative slope double tangent lines. We may therefore assume these conditions by Remark 4.

We consider the induced homotopies by the $(\pi/2)$-rotation of the plane. For each $\theta \in \left[0, \frac{\pi}{2}\right]$, let $H_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote the θ-rotation about the origin, that is, $H_\theta(f, g) = (f\cos\theta - g\sin\theta,\ f\sin\theta + g\cos\theta)$ for $(f, g) \in \mathbb{R}^2$. Let $\varphi_\theta = H_\theta \circ \varphi$ and $F_\theta = \pi_f \circ \varphi_\theta$ and $G_\theta = \pi_g \circ \varphi_\theta$ and $C_{\varphi_\theta} = H_\theta \circ C_\varphi$. Note that φ_θ is a stable map and the product map of F_θ and G_θ, and that C_{φ_θ} is the discriminant curve of φ_θ.

The induced homotopy $\{G_\theta\}_{\theta\in\left[0,\frac{\pi}{2}\right]}$ connects F and G as follows. By the definitions,

$$\begin{aligned} G_\theta(x) &= (\pi_g \circ H_\theta \circ \varphi)(x) = (\pi_g \circ H_\theta)(F(x), G(x)) \\ &= \pi_g(F(x)\cos\theta - G(x)\sin\theta, F(x)\sin\theta + G(x)\cos\theta) \\ &= F(x)\sin\theta + G(x)\cos\theta \end{aligned}$$

for $\theta \in \left[0, \frac{\pi}{2}\right]$ and $x \in X$. In particular, $G_0 = G$ and $G_{\frac{\pi}{2}} = F$.

We count the number of births and deaths of $\{G_\theta\}_{\theta\in\left[0,\frac{\pi}{2}\right]}$. They correspond to births and deaths of canceling pairs of horizontal points of $\{C_{\varphi_\theta}\}_{\theta\in\left[0,\frac{\pi}{2}\right]}$ by Lemma 9. The rotation $\{C_{\varphi_\theta}\}_{\theta\in\left[0,\frac{\pi}{2}\right]}$ of C_φ has such births and deaths only when inflection points of C_φ become horizontal. The condition (iii) guarantees the transversality condition of the births and deaths. Note that only negative slope ones become horizontal during the $(\pi/2)$-rotation. Thus, the number of births and deaths is equal to the number $i^-(\varphi)$ of negative slope inflection points of C_φ.

We count the number of passings of $\{G_\theta\}_{\theta\in\left[0,\frac{\pi}{2}\right]}$. They correspond to passings of horizontal points of $\{C_{\varphi_\theta}\}_{\theta\in\left[0,\frac{\pi}{2}\right]}$. The rotation $\{C_{\varphi_\theta}\}_{\theta\in\left[0,\frac{\pi}{2}\right]}$ of C_φ has such passings only when double tangent lines of C_φ become horizontal. The condition (iv) guarantees the transversality condition of the passings. Note that only negative slope ones become horizontal during the $(\pi/2)$-rotation. By the condition (ii), the number of passings is equal to the number $t^-(\varphi)$ of negative slope double tangent lines of C_φ.

The homotopy $\{G_\theta\}_{\theta\in\left[0,\frac{\pi}{2}\right]}$ is generic and gives the inequalities (1) and (2). The requirement that $\{G_\theta\}_{\theta\in\left[0,\frac{\pi}{2}\right]}$ has at most one of the births, the deaths and the passings at each time is guaranteed by the condition (v).

We can also prove the following lemma by observing the rotation of the discriminant curve.

Lemma 10. *Critical points of F and G which are a vertical point and a horizontal point of C_φ located as in Figure 5 have the same Morse index.*

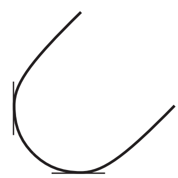


Figure 5: A leftward convex vertical point and a downward convex horizontal point.

Proof. After an appropriate ambient isotopy of $\mathbb{R}^2$, we may assume that Figure 5 shows the image of C_φ in a disk $D \subset \mathbb{R}^2$ centered at the origin, and that C_φ has no inflection points in $\varphi^{-1}(D)$. For each $\theta \in \left[0, \frac{\pi}{2}\right]$, the curve C_{φ_θ} has a unique horizontal point which is not an inflection point in $\varphi^{-1}(D)$, that is, G_θ has a unique non-degenerate critical point in $\varphi^{-1}(D)$. This shows that the critical points of G_0 and $G_{\frac{\pi}{2}}$ in $\varphi^{-1}(D)$ have the same Morse index. $\square$

5. Shearing the discriminant curve

In this section, we prove the inequalities (3), (4) and (5) in Theorem 5, by observing a certain process of shearing the discriminant curve in the target plane.

We use Notation 2 and suppose that the two functions F, G are Morse functions and the product map φ is stable. Let $C_\varphi : S_\varphi \to \mathbb{R}^2$ denote the discriminant curve of φ.

We assume some generic conditions. The curve C_φ satisfies the following conditions after an arbitrarily small isotopy in $\mathbb{R}^2$:

(vi) horizontal points and vertical points are not cusp points and do not map to double points,

(vii) the images of horizontal points, vertical points and cusp points have pairwise distinct g-coordinates.

Note that such a small isotopy can be chosen not to increase the numbers of negative slope cusp points and double points of negative slope subarcs. We may therefore assume these conditions by Remark 4.

We define an ambient isotopy $\{H_t\}_{t\in[g_-,g_+]}$ of the plane as follows. Let g_- be a value below the minimal value of G, and let g_+ be a value above the maximal value of G. Choose δ to be a sufficiently small positive constant and Δ to be a sufficiently large constant. Let $\{h_t : \mathbb{R} \to \mathbb{R}\}_{t\in[g_-,g_+]}$ be a smooth family of monotone increasing smooth functions such that $h_t(g) = \Delta(g-t)$ if $g \le t-\delta$ and $h_t(g) = 0$ if $g \ge t+\delta$. Let $H_t : \mathbb{R}^2 \to \mathbb{R}^2$, $(f,g) \mapsto \big(f + h_t(g) - h_{g_-}(g),\ g\big)$ for each $t \in [g_-, g_+]$.

The isotopy $\{H_t\}_{t\in[g_-,g_+]}$ shears the discriminant curve C_φ by the process illustrated in Figure 6. Let R_t^+, R_t^- and r_t denote the regions $\{(f,g) \in \mathbb{R}^2 \mid g > t+\delta\}$, $\{(f,g) \in \mathbb{R}^2 \mid g < t-\delta\}$ and $\{(f,g) \in \mathbb{R}^2 \mid t-\delta \le g \le t+\delta\}$, respectively, for each $t \in [g_-, g_+]$. Note that the image of C_φ is contained in $\{(f,g) \in \mathbb{R}^2 \mid g_- < g < g_+\}$, and the thin band r_t runs over it from below to above as t ascends from g_- to g_+. In the upper region R_t^+, the curve $H_t \circ C_\varphi$ remains unchanged from C_φ. In the lower region R_t^-, the curve $H_t \circ C_\varphi$ is the result of shearing C_φ so that it has positive slope outside of small neighborhoods of horizontal points.

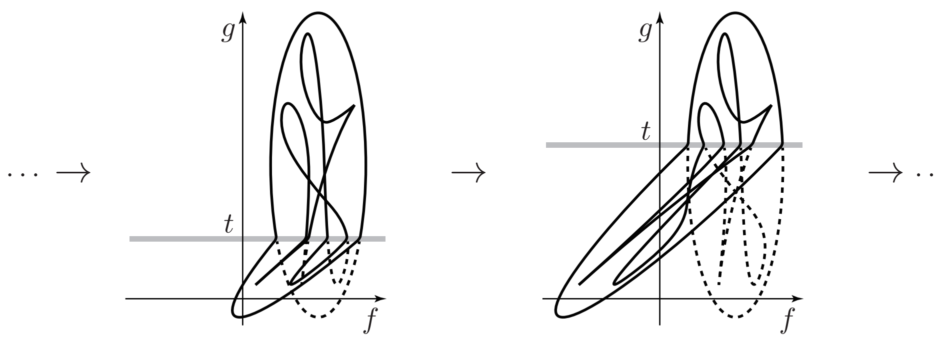


Figure 6: The deformation of $H_t \circ C_\varphi$ as t ascends. The original curve C_φ is shown as a dotted curve, and the thin band r_t is shown in gray.

We consider the induced homotopies by the isotopy $\{H_t\}_{t\in[g_-,g_+]}$. For each $t \in [g_-, g_+]$, let $\varphi_t = H_t \circ \varphi$ and $F_t = \pi_f \circ \varphi_t$ and $G_t = \pi_g \circ \varphi_t$ and $C_{\varphi_t} = H_t \circ C_\varphi$. Note that φ_t is a stable map and the product map of F_t and G_t, and that C_{φ_t} is the discriminant curve of φ_t.

The induced homotopy $\{F_t\}_{t\in[g_-,g_+]}$ connects F and a Morse function isotopic to G as follows. Note that all the inflection points and double tangent lines of the sheared curve $C_{\varphi_{g_+}}$ have positive slopes. The functions F_{g_+} and G_{g_+} are Morse functions by the result in Section 3, and are isotopic by the result in Section 4. By the definitions,

$$\begin{aligned}
F_{g_-}(x) &= (\pi_f \circ H_{g_-} \circ \varphi)(x) = (\pi_f \circ H_{g_-})\,(F(x), G(x)) \\
&= \pi_f\,(F(x), G(x)) \\
&= F(x) \\
G_{g_+}(x) &= (\pi_g \circ H_{g_+} \circ \varphi)(x) = (\pi_g \circ H_{g_+})\,(F(x), G(x)) \\
&= \pi_g \big(F(x) + h_{g_+}\,(G(x)) - h_{g_-}\,(G(x))\,, G(x)\big) \\
&= G(x)
\end{aligned}$$

for $x \in X$. That is, $F_{g_-} = F$ and $G_{g_+} = G$.

In the following subsections, we frequently omit the phrase "the image of" preceding terms of discriminant curves.

5.1. Numbers of births and deaths

In this subsection, we count the numbers of births and deaths of $\{F_t\}_{t\in[g_-,g_+]}$.

What to observe are restricted as follows. Births and deaths of $\{F_t\}_{t\in[g_-,g_+]}$ correspond to births and deaths of canceling pairs of vertical points of $\{C_{\varphi_t}\}_{t\in[g_-,g_+]}$ by Lemma 9. In the upper region R_t^+, the curve C_{φ_t} has no vertical inflection points as well as C_φ. In the lower region R_t^-, all the inflection points of C_{φ_t} has positive slopes. No births and no deaths therefore occur in R_t^+ and R_t^-, but in the thin band r_t. Let α be a component of $S_\varphi \cap \varphi^{-1}(r_t)$ which is an arc apart from horizontal points, vertical points and cusp points of C_φ. In the case where C_φ has positive slope in α, the curve C_{φ_t} does also. In the case where C_φ has negative slope in α, the curve C_{φ_t} has one rightward convex vertical point in α. In both cases, C_{φ_t} has no vertical inflection points in α. No births and no deaths therefore occur when all the components of $S_\varphi \cap \varphi^{-1}(r_t)$ are such arcs, but when r_t passes a horizontal point, a vertical point or a cusp point of C_φ.

What to observe are classified as follows. Each cusp point of C_φ is pointing either northeast, northwest, southeast or southwest, by the condition (vi). Each horizontal (resp. vertical) point of C_φ is not a double point by (vi), and is convex either downward or upward (resp. leftward or rightward) since G (resp. F) is a Morse function. The following figures do not lose generality since δ is sufficiently small, Δ is sufficiently large, and C_φ has only finitely many horizontal points, vertical points and cusp points.

When r_t passes a downward convex horizontal point of C_φ, a birth occurs as in Figure 7. In particular, by combining Lemmas 9 and 10, if the horizontal point of C_φ is a critical point of G of Morse index λ, then the critical points of the canceling pair have Morse indices λ and $\lambda+1$.

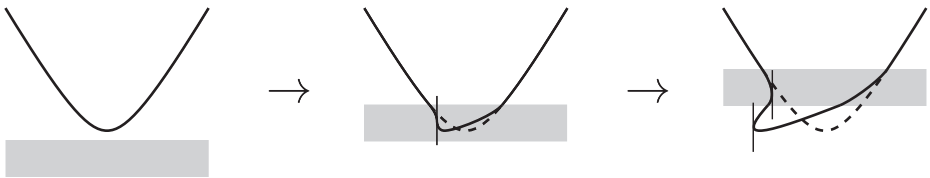


Figure 7: The deformation of C_{φ_t} when r_t passes a downward convex horizontal point of C_φ. The band r_t looks pretty thick because the picture has been greatly enlarged.

When r_t passes a leftward convex vertical point of C_φ, a death occurs as in Figure 8. In particular, by Lemma 9, if the vertical point of C_φ is a critical point of F of Morse index λ, then the critical points of the canceling pair have Morse indices λ and $\lambda+1$.

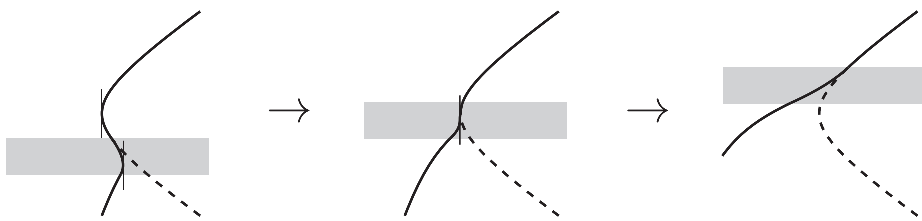


Figure 8: At a leftward convex vertical point of C_φ.

When r_t passes a rightward convex vertical point or an upward convex horizontal point of C_φ, no birth or no death occurs as in Figure 9 or 10, respectively.

When r_t passes a southeast pointing cusp point of C_φ, a birth occurs as in Figure 11. We remark that the tip of the cusp is moved as in Figure 3 but not as in Figure 4, and the birth occurs on the right branch. Suppose that the absolute indices of the left and right branches are either $\lambda-1$ and λ, respectively, or $\lambda-1$ and $\lambda-1$, respectively, or $n-\lambda$ and $n-\lambda-1$, respectively. By Lemma 8, the vertical cusp point in the third picture of Figure 11 has Morse index λ. By Lemma 9, the critical points of the canceling pair have Morse indices λ and $\lambda+1$.

When r_t passes a northwest pointing cusp point of C_φ, a death occurs as in Figure 12. If the absolute indices of the left and right branches are

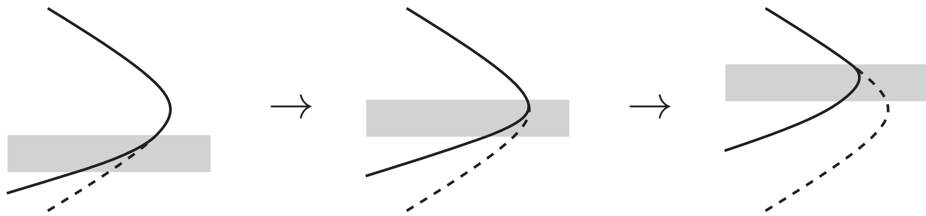


Figure 9: At a rightward convex vertical point of C_φ.

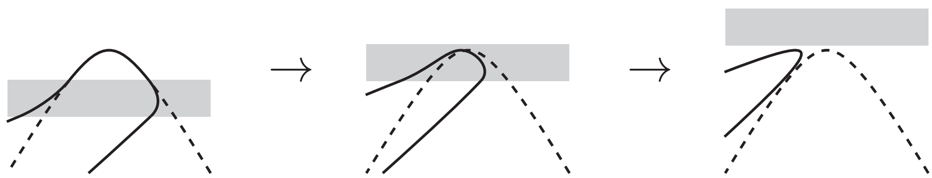


Figure 10: At an upward convex horizontal point of C_φ.

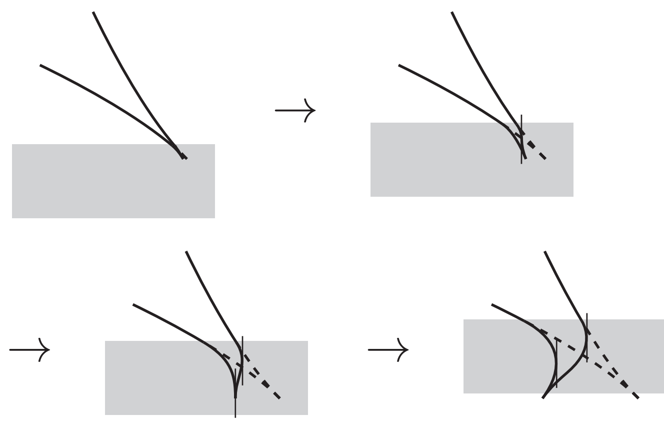


Figure 11: At a southeast pointing cusp point of C_φ.

either $\lambda - 1$ and λ, respectively, or $\lambda - 1$ and $\lambda - 1$, respectively, or $n - \lambda$ and $n - \lambda - 1$, respectively, then the critical points of the canceling pair have Morse indices λ and $\lambda + 1$.

When r_t passes a northeast pointing cusp point or a southwest pointing cusp point of C_φ, no birth or no death occurs.

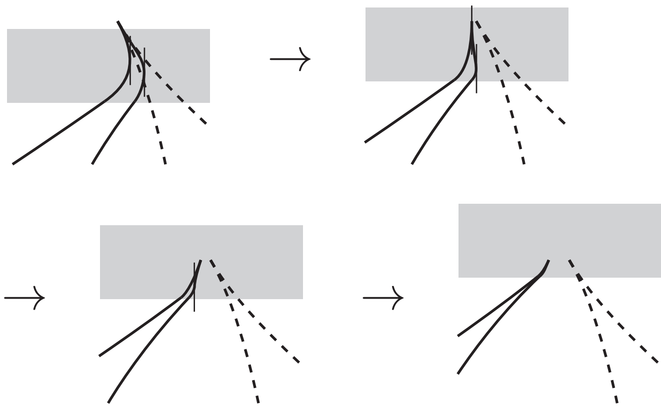


Figure 12: At a northwest pointing cusp point of C_φ.

Putting the above observations together, the homotopy $\{F_t\}_{t\in[g_-,g_+]}$ has

$$\sharp\left\{ \quad , \quad \text{of } C_\varphi \right\} \text{ births and } \sharp\left\{ \quad , \quad \text{of } C_\varphi \right\} \text{ deaths.}$$

In particular, the number of births of canceling pairs of critical points of Morse indices λ and $\lambda+1$ is

$$\begin{cases} \sharp\left\{ \; 0 \; \text{of } C_\varphi \right\} & (\lambda = 0) \\ \sharp\left\{ \; \lambda \; , \; {}^{\lambda}_{\lambda-1} \; \text{of } C_\varphi \right\} & (0<\lambda<\frac{n}{2}) \\ \sharp\left\{ \; \lambda \; , \; {}^{\lambda-1}_{\lambda-1} \; \text{of } C_\varphi \right\} & (\lambda = \frac{n}{2}) \\ \sharp\left\{ \; \lambda \; , \; {}^{n-\lambda-1}_{n-\lambda} \; \text{of } C_\varphi \right\} & (\frac{n}{2}<\lambda<n), \end{cases}$$

and the number of deaths of canceling pairs of critical points of Morse indices λ and $\lambda+1$ is

$$\begin{cases} \sharp\left\{ \textcircled{0} \quad \text{of } C_{\varphi} \right\} & (\lambda=0) \\ \sharp\left\{ \textcircled{\lambda} \;, \quad \begin{matrix} \lambda \\ \lambda-1 \end{matrix} \quad \text{of } C_{\varphi} \right\} & (0<\lambda<\frac{n}{2}) \\ \sharp\left\{ \textcircled{\lambda} \;, \quad \begin{matrix} \lambda-1 \\ \lambda-1 \end{matrix} \quad \text{of } C_{\varphi} \right\} & (\lambda=\frac{n}{2}) \\ \sharp\left\{ \textcircled{\lambda} \;, \quad \begin{matrix} n-\lambda-1 \\ n-\lambda \end{matrix} \quad \text{of } C_{\varphi} \right\} & (\frac{n}{2}<\lambda<n). \end{cases}$$

Here, each circled number stands for the Morse index of the corresponding critical point of F or G, and each non-circled number stands for the absolute index of the corresponding subarc of S_{φ}.

5.2. Number of passings

In this subsection, we estimate the number of passings of $\{F_t\}_{t\in[g_-,g_+]}$.

Passings of $\{F_t\}_{t\in[g_-,g_+]}$ are classified as follows. They correspond to passings of vertical points of $\{C_{\varphi_t}\}_{t\in[g_-,g_+]}$. For each passing, each of the two vertical points belongs to either the upper region R_t^+, the lower region R_t^- or the thin band r_t. It cannot happen that both belong to R_t^+, in which vertical points do not move as t ascends. It also cannot happen that both belong to R_t^-, in which vertical points are translated uniformly as t ascends. The passings are therefore classified into the following four types:

(I) one belongs to R_t^+ and the other belongs to R_t^-,

(II) one belongs to R_t^+ and the other belongs to r_t,

(III) one belongs to r_t and the other belongs to R_t^-,

(IV) both belong to r_t.

We estimate the number of passings of type (I). As t ascends, vertical points of C_{φ_t} in R_t^- go leftward, while those in R_t^+ do not move. Note that each vertical point of C_{φ_t} in R_t^- closely accompanies a horizontal point of C_{φ_t}, which comes from a horizontal point of C_{φ} as in Figure 7 or 10. In this sense, exactly one locus of $\big\{$vertical point of C_{φ_t} in $R_t^- \mid t\in[g_-,g_+]\big\}$ starts from each horizontal point of C_φ. For a horizontal point p of C_φ, the number of passings of type (I) caused by the locus of $\{\text{vertical point of } C_{\varphi_t} \text{ in } R_t^- \mid t \in [g_-, g_+]\}$ starting from p is at most the number of vertical points of C_φ in $R^+(p)$, where $R^+(p)$ denotes the half plane $\{(f,g) \in \mathbb{R}^2 \mid g > G(p)\}$ above $\varphi(p)$. The total number of passings of type (I) is therefore at most

$$\sum \left\{ \sharp \left\{ \langle\, , \,\rangle \ \text{of } C_\varphi \text{ in } R^+(p) \right\} \,\middle|\, p \in \left\{ \smile, \frown \ \text{of } C_\varphi \right\} \right\}.$$

We estimate the number of passings of type (II). As t ascends, vertical points of C_{φ_t} in r_t go leftward, while those in R_t^+ do not move. Note that each vertical point of C_{φ_t} in r_t closely traces a negative slope subarc of C_φ, and that the tracing starts when r_t passes either a downward convex horizontal point, a rightward convex vertical point, or a southeast pointing cusp point of C_φ as in Figure 7, 9 or 11, respectively. In this sense, exactly one locus of $\{\text{vertical point of } C_{\varphi_t} \text{ in } r_t \mid t \in [g_-, g_+]\}$ starts from each downward convex horizontal point or rightward convex vertical point of C_φ, and that exactly two loci of $\{\text{vertical point of } C_{\varphi_t} \text{ in } r_t \mid t \in [g_-, g_+]\}$ start from each southeast pointing cusp point of C_φ. The number of passings of type (II) caused by each locus of $\{\text{vertical point of } C_{\varphi_t} \text{ in } r_t \mid t \in [g_-, g_+]\}$ is at most the number of vertical points of C_φ minus one. Here, the minus one is because no such passing is caused by the rightmost rightward convex vertical point of C_φ. The total number of passings of type (II) is therefore at most

$$\begin{aligned}&\left(\sharp \left\{ \langle\, , \,\rangle \ \text{of } C_\varphi \right\} - 1 \right) \left(\sharp \left\{ \smile, \,\rangle \ \text{of } C_\varphi \right\} + 2\, \sharp \left\{ \lrcorner \ \text{of } C_\varphi \right\} \right)\\ &= (m(F) - 1) \left(\sharp \left\{ \smile, \,\rangle \ \text{of } C_\varphi \right\} + 2\, \sharp \left\{ \lrcorner \ \text{of } C_\varphi \right\} \right).\end{aligned}$$

We estimate the number of passings of type (III). As t ascends, vertical points of C_{φ_t} in R_t^- go leftward faster than those in r_t, since Δ is sufficiently large. We can see that the number of passings of type (III) is at most

$$\begin{aligned}&\left(\sharp \left\{ \smile, \,\rangle \ \text{of } C_\varphi \right\} + 2\, \sharp \left\{ \lrcorner \ \text{of } C_\varphi \right\} \right) \left(\sharp \left\{ \smile, \frown \ \text{of } C_\varphi \right\} - 1 \right)\\ &= \left(\sharp \left\{ \smile, \,\rangle \ \text{of } C_\varphi \right\} + 2\, \sharp \left\{ \lrcorner \ \text{of } C_\varphi \right\} \right) (m(G) - 1).\end{aligned}$$

Here, the minus one is because no passing of type (III) is caused by the locus of $\left\{\text{vertical point of } C_{\varphi_t} \text{ in } R_t^- \mid t \in [g_-, g_+]\right\}$ starting from the uppermost upward convex horizontal point of C_φ.

We count the number of passings of type (IV). Recall that, as t ascends, vertical points of C_{φ_t} in r_t go leftward closely tracing negative slope subarcs of C_φ. Two of them cause a passing when r_t passes a double point of such subarcs as in Figure 13. The number of passings of type (IV) is therefore equal to the number $d^-(\varphi)$ of double points of negative slope subarcs of C_φ.

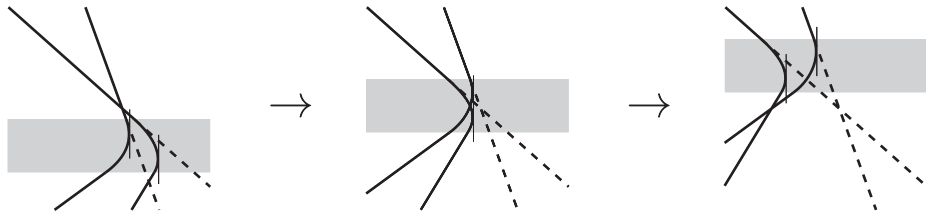
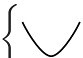


Figure 13: At a double point of negative slope subarcs of C_φ.

Putting the above estimations together, the number of passings of $\{F_t\}_{t\in[g_-,g_+]}$ is at most

$\sum \Big\{ \sharp \Big\{$  ,  of C_φ in $R^+(p) \Big\} \;\Big|\; p \in \Big\{$  ,  of $C_\varphi \Big\} \Big\}$

$+ \Big(\sharp \Big\{$  ,  of $C_\varphi \Big\} + 2\,\sharp \Big\{$  of $C_\varphi \Big\} \Big) (m(F) + m(G) - 2)$

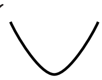
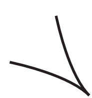
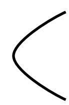
$+ d^-(\varphi).$

5.3. Bounding the distances

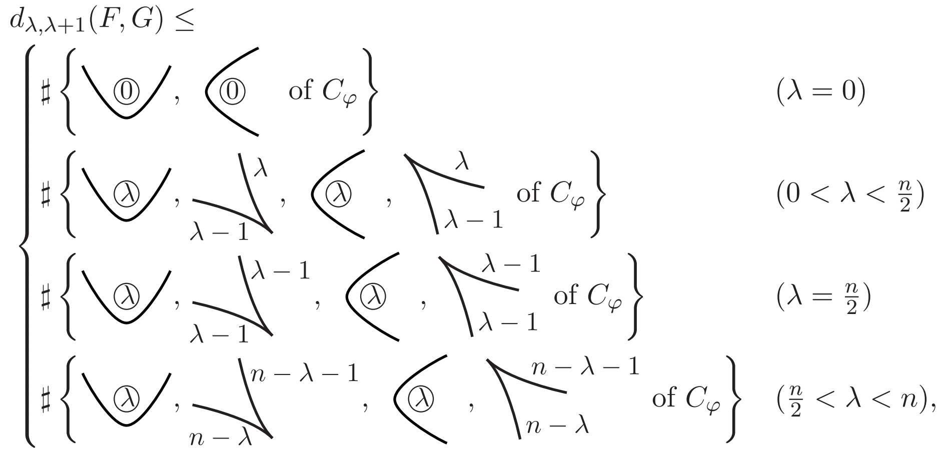
In this subsection, we bound from above the distances between the Morse functions F and G by using the results of the previous subsections and similar observations.

We may assume that $\{F_t\}_{t\in[g_-,g_+]}$ is a generic homotopy as follows. The condition (vii) guarantees that $\{F_t\}_{t\in[g_-,g_+]}$ has either a single birth or a single death, ignoring passings, at each time. We can also arrange the required condition of the passings by certain generic conditions of C_φ and Remark 4.

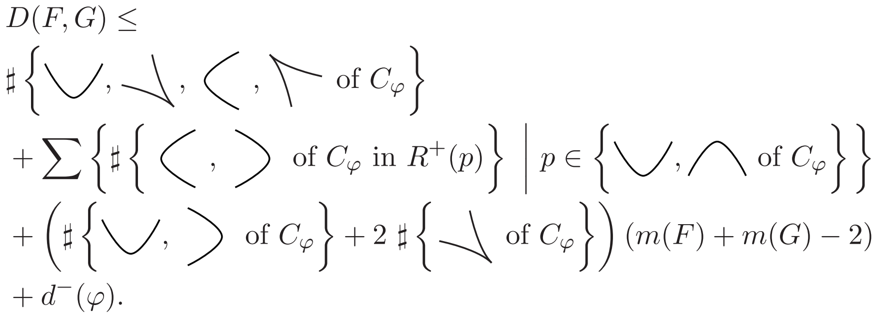
By the results of the previous subsections, the generic homotopy $\{F_t\}_{t\in[g_-,g_+]}$ gives the following bounds:

(6) $d(F, G) \leq \sharp \Big\{$  ,  ,  ,  of $C_\varphi \Big\}$,

(7)

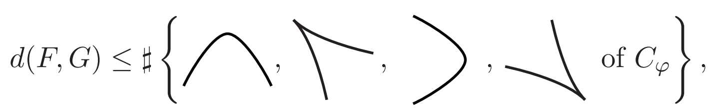


(8)



We can also obtain other bounds by observing another process of shearing the discriminant curve C_φ. Let $H'_t : \mathbb{R}^2 \to \mathbb{R}^2$, $(f,g) \mapsto (f - h_t(-g) + h_{-g_+}(-g),\ g)$ for each $t \in [-g_+, -g_-]$. The ambient isotopy $\{H'_t\}_{t\in[-g_+,-g_-]}$ of $\mathbb{R}^2$ shears C_φ positively as well as $\{H_t\}_{t\in[g_-,g_+]}$ but the process is upside-down. By similar observations, the induced homotopy of F by $\{H'_t\}_{t\in[-g_+,-g_-]}$ gives the following bounds:

(9)



(10)

$$d_{\lambda-1,\lambda}(F,G) \leq \begin{cases} \sharp\left\{ \lambda, \quad {}^{\lambda}_{\lambda-1}, \quad \lambda, \quad {}^{\lambda}_{\lambda-1} \text{ of } C_\varphi \right\} & (0<\lambda<\frac{n}{2}) \\ \sharp\left\{ \lambda, \quad {}^{\lambda-1}_{\lambda-1}, \quad \lambda, \quad {}^{\lambda-1}_{\lambda-1} \text{ of } C_\varphi \right\} & (\lambda=\frac{n}{2}) \\ \sharp\left\{ \lambda, \quad {}^{n-\lambda-1}_{n-\lambda}, \quad \lambda, \quad {}^{n-\lambda-1}_{n-\lambda} \text{ of } C_\varphi \right\} & (\frac{n}{2}<\lambda<n) \\ \sharp\left\{ n, \quad n \quad \text{ of } C_\varphi \right\} & (\lambda=n), \end{cases}$$

(11) $D(F,G) \leq$

$$\begin{aligned} &\sharp\left\{ \quad, \quad, \quad, \quad \text{ of } C_\varphi \right\} \\ &+ \sum \left\{ \sharp\left\{ \quad, \quad \text{ of } C_\varphi \text{ in } R^-(p) \right\} \,\middle|\, p \in \left\{ \quad, \quad \text{ of } C_\varphi \right\} \right\} \\ &+ \left(\sharp\left\{ \quad, \quad \text{ of } C_\varphi \right\} + 2\,\sharp\left\{ \quad \text{ of } C_\varphi \right\} \right) (m(F)+m(G)-2) \\ &+ d^-(\varphi), \end{aligned}$$

where $R^-(p)$ denotes the half plane below $\varphi(p)$.

We combine the above bounds to obtain the desired ones. By combining (6) and (9), we obtain

$$\begin{aligned} d(F,G) &\leq \frac{1}{2}\,\sharp\left\{ \quad, \quad, \quad, \quad \text{ of } C_\varphi \right\} \\ &\quad + \sharp\left\{ \quad, \quad \text{ of } C_\varphi \right\} \\ &= \frac{1}{2}\,(m(F)+m(G)) + c^-(\varphi) \end{aligned}$$

to conclude the proof of (3). By combining (7) and (10), we conclude the proof of (4). For example, in the case where $0 < \lambda < \frac{n}{2}$,

$$\begin{aligned}
d_{\lambda-1,\lambda}(F,G) + d_{\lambda,\lambda+1}(F,G) &\leq \sharp\left\{ \lambda,\ \lambda,\ \lambda \qquad \lambda \quad \text{of } C_\varphi \right\} \\
&\quad + 2\,\sharp\left\{ {}^{\lambda}_{\lambda-1},\ {}^{\lambda}_{\lambda-1} \quad \text{of } C_\varphi \right\} \\
&\leq m_\lambda(F) + m_\lambda(G) + 2c^{-}_{\lambda-1}(\varphi).
\end{aligned}$$

By combining (8) and (11), we obtain

$$\begin{aligned}
&D(F,G) \\
&\leq \frac{1}{2}\,\sharp\left\{ ,\ ,\ ,\ \quad \text{of } C_\varphi \right\} + \sharp\left\{ ,\ \quad \text{of } C_\varphi \right\} \\
&\quad + \frac{1}{2}\sum \left\{ \sharp\left\{ ,\ \quad \text{of } C_\varphi \right\} \,\middle|\, p \in \left\{ ,\ \quad \text{of } C_\varphi \right\} \right\} \\
&\quad + \frac{1}{2}\left(\sharp\left\{ ,\ ,\ ,\ \quad \text{of } C_\varphi \right\} + 2\,\sharp\left\{ ,\ \quad \text{of } C_\varphi \right\} \right) \\
&\quad (m(F) + m(G) - 2) + d^{-}(\varphi) \\
&= \frac{1}{2}\left(m(F) + m(G)\right) + c^{-}(\varphi) + \frac{1}{2}m(F)m(G) \\
&\quad + \frac{1}{2}\big(m(F) + m(G) + 2c^{-}(\varphi)\big)\left(m(F) + m(G) - 2\right) + d^{-}(\varphi) \\
&= \frac{1}{2}\big(m(F) + m(G) + 2c^{-}(\varphi)\big)\left(m(F) + m(G) - 1\right) + \frac{1}{2}m(F)m(G) + d^{-}(\varphi)
\end{aligned}$$

to conclude the proof of (5).

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