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Holomorphic triples and the prescribed curvature problem on S^2

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We prove new results on existence of solutions for the prescribed gaussian curvature problem on the euclidean sphere S^2. Those results are achieved by relating this problem with the holomorphic triples theory on Riemann surfaces. We think this approach might be applied to study some other semi-linear elliptic equations of 2nd order on the sphere.

1. Introduction

Let M be a closed Riemann surface with metric g_0. By a pointwise conformal metric we mean another metric g given by dilation of g_0 by a positive smooth function. Therefore, we can write $g = e^{2u}g_0$ for a a function $u \in C^\infty(M)$. If K_0 and K denote the gaussian curvatures of g_0 and g, respectively, it can be shown [15]

$$\Delta u + Ke^{2u} - K_0 = 0, \tag{1}$$

where Δ denotes the Laplace-Beltrame operator on the metric g_0. Thus, finding a metric pointwise conformal to g_0 with curvature K is equivalent to finding classical solutions to the elliptic equation (1).

This problem has been treated by several authors since the late 1960s [1, 2, 15, 16]. In [15] Kazdan and Warner obtained some general necessary and suficient conditions on the functions K, K_0 to assure existence of solutions to (1). They also found some non-existence conditions mainly in the case of the euclidean sphere.

On the other hand, it has long been known that equations like (1) are a particular case of the theory of holomorphic triples over Kähler manifolds [5,

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11]. This theory grew out of the seminal work of Donaldson and Uhlenbeck-Yau about special metrics on stable vector bundles, which developed into an active area of work since the 1980s [3, 4, 6, 7, 11, 21, 22]. The Vortex equation was introduced in [3] and evolved into the holomorphic triples theory [5, 11], where not only holomorphic vector bundles, but also prescribed cohomology classes on the bundles are considered.

In [12] the study of equation (1) is presented in connection with the vortex and holomorphic triples theory, by means of two distinct though related problems:

$$\Delta u + ||[\phi]||_u^2 - \lambda = 0, \tag{2}$$

$$\Delta u + ||[\eta]||_u^2 - \lambda = 0. \tag{3}$$

Equations (2) and (3) are defined on a closed Riemann surface M for a real parameter $\lambda > 0$ and for *cohomology classes* $[\phi]$ and $[\eta]$ living, respectively, in the cohomology complex of holomorphic line bundles L and L^* over M. The terms $||[\phi]||_u^2$ and $||[\eta]||_u^2$ refer to the pointwise squared norm of representatives of these classes, in a hermitian metric given by dilation of the original metric by a factor e^{2u}. The function u is a real smooth function on M and is meant to be the unknown in the equations.

In the prescribed curvature problem presented by equation (1) one is often interested in the case $K_0 \equiv$constant. Since the work of Kazdan and Warner this is already well known for all surfaces with non-positive Euler characteristic, as well as for the projective plane $\mathbb{PR}^2$. Despite some non-trivial non-existence conditions have been found, the case of $M = S^2$ is where most open questions remain. It amounts to say that up to our knowledge, all results on existence for (1) after [16] play on several suficient conditions for the function K, one of them being $K > 0$ [8, 23]. Existence for (1) is also known when K is symmetric about the origing (considering the cannonical inclusion $S^2 \hookrightarrow \mathbb{R}^3$), after the work [20].

Our results apply for functions K which are the squared modulus of holomorphic sections, typically having some zeros, and not necessarily symmetric about the origin. Most importantly, those results can only be established after we explicitly connect equations (2) and (3), and strongly rely on algebraic-geometric elements of the involved bundles, like their Chern classes. We conjecture that this algebraic fact we use for studying equation (1) might be applied even for more general functions K, and has not been pursued by other authors so far.

A brief description of this work: in section 2 we collect some well-known facts on the theory of line bundles over riemann surfaces, as well as results on metric equations like the vortex equation; in section 3 we prove the main results necessary to understand the cohomology classes of the dual bundle L^* from the analytical viewpoint, contained in Lemmas 3.1, 3.2 and 3.3; and in section 4 we apply those results to show existence or non-existence of solutions for (1), for some conformal curvatures K, which are summarized by Lemma 4.6 and Theorems 4.7 through 4.10.

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2. Basics on the geometry of holomorphic bundles

This section is only meant to set up notation. For a deep study throughout these matters we recommend [14, 18].

2.1. Hermitian bundles and cohomology

Let E be a smooth complex vector bundle over a complex manifold X. Associated to E we have the dual bundle E^*, conjugate bundle $\overline{E}$ and endomorphism bundle $End(E)$. A (hermitian) metric is then a smooth isomorphism $H : E \to \overline{E}^*$ which is positive definite in each fiber. The bundle E together with the structure given by H is a *hermitian bundle*.

Denote by $(T^*X)_{\mathbb{C}}$ the complexified cotangent bundle of X, which splits as $(T^*X)_{\mathbb{C}} = T^{1,0}X \oplus T^{0,1}X$. The bundle $T^{1,0}X$ is the holomorphic cotangent bundle of X (home of the famous "holomorphic differentials"). Let $\Lambda^{p,q}T^*X = \Lambda^p T^{1,0}X \otimes \Lambda^q T^{0,1}X$ for non-negative integers p, q, and let $\Gamma(\cdot)$ be the functor that takes a bundle to its space of smooth sections. We set $\Omega^{p,q}(E) = \Gamma(\Lambda^{p,q}T^*X \otimes E)$. Any $\phi \in \Omega^{p,q}(E)$ is a smooth section of holomorphic type (p, q) and values in E.

A *holomorphic structure* on E is an operator $D'' : \Omega^{p,q}(E) \longrightarrow \Omega^{p,q+1}(E)$ that satisfies $(D'')^2 = 0$ and enjoys some typical properties of a covariant derivative (see [14, 18]). Indeed, a *connection* is a covariant derivative $D : \Omega^m(E) \longrightarrow \Omega^{m+1}(E)$, where $\Omega^m(E) = \oplus_{p+q=m}\Omega^{p,q}(E)$. Any connection decomposes after the splitting of the cotangent bundle $D = D^{1,0} + D^{0,1}$. It is well known that for a given hermitian metric H and holomorphic structure D'' there is only one connection $D = D_{H,D''}$ compatible with both, which means, $D^{0,1} = D''$ and $D(H) = 0$. This connection is called the *Chern connection*.

The *curvature* of a connection D is the compound $F_D = D^2 : \Omega^m(E) \to \Omega^{m+2}(E)$, which is a 2-form section of the bundle $End(E)$. An important topological invariant associated to the bundle E is its *first Chern class* $\frac{i}{2\pi}[tr(F_D)]$ which is a cohomology class on the base manifold (here $tr(\cdot)$ is the trace of the endomorphism coefficient of F_D and $i = \sqrt{-1}$). The curvature $F_D \in \Omega^2(End(E))$ of any Chern connection has only the $(1,1)$ component, so that $F_D = D'' \circ D' + D' \circ D''$ (we denote $D' = D^{1,0}$ from now on).

Since $(D'')^2 = 0$ we get a cochain complex $(\Omega^{p,q}(E), D'')$ whose cohomology we denote

$$\mathcal{H}^{p,q}(E) = \frac{ker\, D'' : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)}{im\, D'' : \Omega^{p,q-1}(E) \to \Omega^{p,q}(E)}\ .$$

A *holomorphic section* is any $\phi \in \Omega^{p,q}(E)$ such that $D''\phi = 0$. Similarly, an *anti-holomorphic section* is any section η solving $D'\eta = 0$, for a given D' operator.

Let $\phi \in \Omega^{p,q}(E)$, we set the H-dual of ϕ as $\phi^{*H} = \overline{H(\phi)} \in \Omega^{q,p}(E^*)$. The H-dual of a section valued form is obtained by conjugating the form part and dualizing, in the usual way, the bundle coefficient. If the connection D is hermitian then a section ϕ is holomorphic if and only if ϕ^{*H} is anti-holomorphic.

2.2. Line bundles, degrees and divisors over surfaces

We now turn our attention to the case of a closed oriented Riemann surface $X = M$.

Recall that a *meromorphic section* on the holomorphic bundle E over M is a holomorphic section ϕ on $M - \{x_1, \ldots, x_t\}$ and such that in a neighborhood of each x_j, $\phi = z_j^{m_j}\zeta_j$, where z_j is a holomorphic local coordinate on M with $z_j(x_j) = 0$ and ζ_j is a regular holomorphic local section. The *divisor* of ϕ is the formal linear combination $\mathrm{Div}(\phi) = \sum_{j=1}^{t} m_j.x_j$, and the degree of ϕ is $\deg(\phi) = \sum_{j=1}^{t} m_j$. The integer m_j is the *order* of ϕ at x_j, $m_j = \mathrm{ord}_{x_j}(\phi)$.

A *line bundle* is a holomorphic bundle L of rank 1 over M. It can be shown that any line bundle has a non-vanishing meromorphic section ϕ ([9]), and we set $\deg(L) = \deg(\phi)$. Since the endomorphism bundle of L is just the trivial bundle $M \times \mathbb{C}$, and the curvature reduces to a closed 2-form on M, we get an analitycal way of computing its degree,

$$\deg(L) = \int_M \frac{i}{2\pi} F_D. \tag{4}$$

Observe that $\mathcal{H}^{1,0}(L)$ is the space of holomorphic sections of the bundle $T^{1,0}(M) \otimes L$, hence to avoid it to be trivial we always assume

$$\deg(L) \geq -\deg(T^{1,0}(M)). \tag{5}$$

Of great interest to us are the cohomologies $\mathcal{H}^{1,0}(L)$ and $\mathcal{H}^{0,1}(L^*)$. Clearly $\mathcal{H}^{1,0}(L)$ is identified with the set of holomorphic sections. On the other hand any section on L^* of holomorphic type $(0,1)$ is D''-closed, and so represents a cohomology class in $\mathcal{H}^{0,1}(L^*)$. By standard Hodge Theory [14] any class on $\mathcal{H}^{0,1}(L^*)$ has exactly one harmonic representative, which must be H-antiholomorphic, hence the map

$$* H : \mathcal{H}^{1,0}(L) \longrightarrow \mathcal{H}^{0,1}(L^*) \tag{6}$$

is an anti-isomorphism between these two vector spaces.

By wedging the 1-forms we define a bilinear operator $\Omega^{1,0}(L) \times \Omega^{0,1}(L^*) \to \Omega^2(\mathbb{C})$ taking sections ϕ and η to $(\phi \wedge \eta)$, and a coupling

$$(\!(\phi, \eta)\!) = \int_M i(\phi \wedge \eta). \tag{7}$$

Because of Stokes' Theorem and integration by parts this coupling descends to cohomology classes, so that $(\!([\phi], [\eta])\!) = (\!(\phi, \eta)\!)$, as long as ϕ and η represent classes $[\phi] \in \mathcal{H}^{1,0}(L)$ and $[\eta] \in \mathcal{H}^{0,1}(L^*)$, respectively. Similarly, we have a coupling given by the metric H by setting $\langle\!\langle \phi, \psi \rangle\!\rangle_H = (\!(\phi, \psi^{*H})\!)$, for any $\phi, \psi \in \Omega^{1,0}(L)$.

Two metrics H and H_0 on L are related by a positive dilation in each fiber, so that $H = H_u = H_0 e^{2u}$ for a smooth function u on M. For a section ϕ on L we have $|\phi(x)|^2_{H_u} = |\phi(x)|^2_u = |\phi(x)|^2_0 e^{2u(x)}$ for any $x \in M$, and for a section η on L^* it holds $|\eta(x)|^2_u = |\eta(x)|^2_0 e^{-2u(x)}$, for the metric on L^* is set by duality. The curvatures F_{H_u} and F_{H_0} associated to the correspondent Chern connections are related by $i\Lambda F_{H_u} = i\Lambda F_{H_0} - \Delta u$, where Λ is the contraction with the volumn element ν on M and Δ is the Laplace-Beltrame operator on functions. Assuming that $|M| = 1$ and H_0 is a metric yielding constant curvature $i\Lambda F_{H_0} = 2\pi \deg(L)$ we obtain

$$i\Lambda F_{H_u} = 2\pi \deg(L) - \Delta u. \tag{8}$$

We restrict to the case of the euclidean sphere $M = S^2$, but we dilate the standard metric by a constant factor so that its gaussian curvature is 4π and $|S^2| = 1$. If $x \in S^2$ is any point we set $z = z_x : S^2 - \{x\} \to \mathbb{C}$ as a stereographic projection with north pole at x.

Two important facts about bundles over the sphere are stated below. The first one comes from a simple computation using a stereographic coordinate, while the proof for the second can be found in [13].

Lemma 2.1. *Let $M = S^2$ be the base manifold. Then*
(a) Line bundles are classified by their degrees.
(b) Any rank 2 holomorphic bundle E splits holomorphically as the sum of line bundles, $E = L_1 \oplus L_2$.

Let $N = (0, 0, 1)$ and $z = z_N$. Any line bundle L has a trivialization over $S^2 - \{N\}$ given by a "cannonical" meromorphic section ζ_L whose singular set is $\{N\}$. By using the coordinate $w = 1/z$ we get a section $\zeta_{L,S}$ holomorphic and regular over $S^2 - \{S\}$, where $S = -N$. The gauge transformation between them is

$$\zeta_L(x) = w(x)^{\deg(L)}\, \zeta_{L,S}(x) \qquad \text{for all } x \in S^2 - \{S, N\}. \tag{9}$$

If $\deg(L) \geq 0$ an arbitrary holomorphic section of L is given by $h\zeta_L$ for some polynomial $h = h(z)$ of degree bounded by $\deg(L)$. The bundle $T^{1,0}S^2$ is spanned by the holomorphic differential $\zeta_{T^{1,0}S^2} = dz$, hence the set $\mathcal{H}^{1,0}(L)$ of holomorphic sections of $\Omega^{1,0}(L) = \Gamma(L \otimes T^{1,0}S^2)$ consists of sections ϕ of the form $\phi = g\, \zeta_L\, dz$, where $g(z)$ is a polynomial with degree less than or equal to $\deg(L) - 2$.

In coordinates, the H-dual of ϕ is the anti-holomophic section $\eta = \overline{g}\, \zeta_L^{*H} d\overline{z}$, and the section ζ_L^{*H} can be written

$$\zeta_L^{*H} = \frac{\zeta_{L^*}}{|\zeta_{L^*}|_H^2} = |\zeta_L|_H^2\, \zeta_{L^*}\ . \tag{10}$$

For later use we express the H-norm of η in the z and w coordinates:

$$|\eta|_H^2 = |g|^2 |\zeta_L|_H^2 |dz|^2 = |g|^2 |w|^{2(\deg(L)-2)} |\zeta_{L,S}|_H^2 |dw|^2. \tag{11}$$

The next Lemma helps us to find an explicit expression for $|\zeta_L|_{H_0}^2$.

Lemma 2.2. *Let ζ be a meromorphic section on L and H be a metric. Then in any open region where ζ is regular the H-curvature of L is given by $i\Lambda F_H = -\Delta \ln |\zeta|_H$.*

Taking the cannonical section ζ_L we get $\Delta \ln |\zeta_L|_{H_0}(x) = -2\pi\deg(L)$ for all $x \in S^2 - \{N\}$. An inspection shows that $\Delta[\frac{\deg(L)}{2} \ln(1+|z|^2)] = 2\pi\deg(L)$, thus if we set

$$|\zeta_L|^2_{H_0}(x) = (1 + |z(x)|^2)^{-\deg(L)} \qquad \text{for } x \neq N \tag{12}$$

we get a prospective function describing the metric H_0. To make sure it works fine we notice that in the other trivialization

$$|\zeta_{L,S}|^2_{H_0}(x) = |w(x)^{-\deg(L)}\zeta_L(x)|^2_{H_0} = \frac{1}{(|w(x)|^2+1)^{\deg(L)}} \qquad \text{for } x \neq S, \tag{13}$$

so H_0 is smooth at each fiber of L. If any other metric H_u yields constant curvature to L then by equation (8) it holds $\Delta u = 0$, so u is a constant and H_u is just a uniform dilation of the given H_0. We set equation (12) (or (13)) as the definition for H_0.

2.3. Holomorphic extensions and stability

Recall that a holomorphic extension of E_2 by E_1 is a short exact sequence of holomorphic bundles and morphisms $e:\ 0 \to E_1 \to E \to E_2 \to 0$ over the same base manifold. There is a natural concept of *isomorphism of extensions*, and we define $Ext(E_2, E_1)$ as the set of classes of isomorphic extensions. Also let $Hom(E_2, E_1) = E_1 \otimes E_2^*$ be the bundle of homomorphisms $E_2 \to E_1$. The proof of the next Lemma can be found in [14, 19].

Lemma 2.3. *There is a natural one-to-one correspondence between* $Ext(E_2, E_1)$ *and* $\mathcal{H}^{0,1}(Hom(E_2, E_1))$.

For a holomorphic bundle E we define

$$\operatorname{div}(E) = \sup\{\deg(J) \mid J \subset E \text{ is a holomorphic line subbundle}\}. \tag{14}$$

It is well known that $\operatorname{div}(E)$ is finite on Riemann Surfaces [13]. Taking $[\eta] \in \mathcal{H}^{0,1}(Hom(E_2, E_1))$ we can define $\operatorname{div}[\eta] = \operatorname{div}(E)$ where E is the middle term of the extension associated to $[\eta]$.

Now fix L_1, L_2 line bundles over S^2 and consider holomorphic extensions $0 \to L_1 \to E \to L_2 \to 0$. Therefore E is a rank 2 vector bundle which is *topologically*, but not *holomorphically* in general, the direct sum of L_1 and L_2. We set from now on the bundle $L = L_2 \otimes L_1^*$. Hence the set of extensions of L_2 by L_1 is just $\mathcal{H}^{0,1}(L^*)$.

Lemma 2.4. *Let* $[\eta] \in \mathcal{H}^{0,1}(L^*)$*. Then* $\operatorname{div}[\eta] \leq \max\{\deg(L_1), \deg(L_2)\}$*. In case* $\deg(L_2) > \deg(L_1)$ *equality holds if and only if* $[\eta] = 0$.

Proof. Let

$$0 \to L_1 \to E \overset{\pi}{\to} L_2 \to 0 \tag{15}$$

be the extension associated to $[\eta]$. Let $J \subset E$ be a holomorphic line subbundle and ϕ be a (non-trivial) meromorphic section of J. Consider first that $\pi(\phi) \equiv 0$. Then the range of ϕ lies within L_1, hence $J = L_1$ and $\deg(J) = \deg(L_1)$.

Now suppose $\pi(\phi) \not\equiv 0$. Then $\pi(\phi)$ is a meromorphic section on L_2. If $x \in M$ let z be a holomorphic coordinate with $z(x) = 0$. Let ζ be a regular holomorphic section on J in a neighborhood of x. Then $\phi(z) = h(z)\zeta(z)$ for z close to 0, where $h(z)$ is some local meromorphic function. Clearly $\mathrm{ord}_x(\phi) = \mathrm{ord}_x(h) \leq \mathrm{ord}_x(h) + \mathrm{ord}_x(\pi(\zeta)) = \mathrm{ord}_x(\pi(\phi))$, since $\mathrm{ord}_x(\pi(\zeta)) \geq 0$. We conclude that

$$\deg(\pi(\phi)) = \sum_{x\in M} \mathrm{ord}_x(\pi(\phi)) \geq \sum_{x\in M} \mathrm{ord}_x(\phi) = \deg(\phi). \tag{16}$$

Therefore $\deg(L_2) \geq \deg(J)$. Considering both cases we arrive at $\deg(J) \leq \max\{\deg(L_1), \deg(L_2)\}$. From this and the arbitrarity of J the first assertion of the Lemma follows.

Now if $[\eta] = 0$ the extension (15) is trivial, hence $L_2 \hookrightarrow E$ holomorphically. Thus $\mathrm{div}(E) \geq \max\{\deg(L_1), \deg(L_2)\}$. The first part of the proof already gave us the reversal inequality, and we obtain $\mathrm{div}[\eta] = \max\{\deg(L_1), \deg(L_2)\}$.

Finally assume $\deg(L_2) > \deg(L_1)$ and $\mathrm{div}[\eta] = \max\{\deg(L_1), \deg(L_2)\} = \deg(L_2)$. Let $J \subset E$ be a holomorphic line subbundle such that $\deg(J) = \deg(L_2)$. Since $J \neq L_1$ the restriction map $\pi|_J : J \to L_2$ is a non-trivial holomorphic morphism. Further $\pi|_J$ is a section of $L_2 \otimes J^*$, and $\deg(L_2 \otimes J^*) = \deg(L_2) - \deg(J) = 0$, hence $\pi|_J$ has no zeros. This is equivalent to saying that $E = L_1 \oplus J$ holomorphically. It is straightforward to check that the trivial extension $0 \to L_1 \to E \to J \to 0$ is isomorphic to extension (15). The consclusion is that $[\eta] = 0$, and the second assertion of the Lemma is proven. □

2.4. The metric equations

Let $0 \to E_1 \to E \to E_2 \to 0$ be an extension, thus $E_1 \subset E$ is subholomorphic and $E_2 = E/E_1$. A metric $H = H_E$ induces metrics H_j on E_j and an identification $E_2 \sim E_1^{\perp H}$. Respect to the orthogonal decomposition $E = E_1 \oplus E_2$

we can write the equation

$$i\Lambda F_H = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \tag{17}$$

where the right-hand-side is a section of $End(E)$, assembled as a weighted combination of the orthogonal projections $E \to E_j$, $j = 1, 2$, for some real constants τ_1, τ_2, and Λ is the contraction with the Kähler form on the surface.

The problem stated by equation (17) is a particular case of the holomorphic and cohomology triples problems, which have been introduced in [11] and [5]. These problems constitute a generalization of the Hermite-Einstein equation over Kähler manifolds [22]. The typical theorem in those theories, known as *the Hitchin-Kobayashi correspondence*, states that a solution exists for the metric equations as long as an algebraic condition called *stability* (or *polystability* in a more general case) is satisfied for the involved bundles and perhaps some other structures, like prescribed sections or cohomology classes.

To properly express this theory, whose details can be found in [5, 11], we would need to elaborate on definitions and notation that go far beyond the line of our article. Instead, we'd rather state in a summary what concern to us. Since we have line bundles L_1, L_2 over the riemann surface M an extension of L_2 by L_1 is some $[\eta] \in \mathcal{H}^{0,1}(L^*)$. If $\alpha \in \mathbb{R}$ we define the α-slope of $[\eta]$,

$$\mu_\alpha([\eta]) = \frac{\deg(L_1) + \deg(L_2) + \alpha}{2}. \tag{18}$$

Then we say that $[\eta]$ is α-stable if

$$\max\{\deg(L_1), \mathrm{div}[\eta] + \alpha\} < \mu_\alpha([\eta]). \tag{19}$$

From inequality (19) and the definition of μ_α we conclude that $[\eta]$ is α-stable if and only if $\deg(L_1) - \deg(L_2) < \alpha < \deg(L_1) + \deg(L_2) - 2\mathrm{div}[\eta]$. A necessary condition for α-stability is then that a strict inequality happen between the first and the third members of the latter. The next theorem replicates the results from Proposition 3.8 and Theorem 3.9 of [5].

Theorem 2.5. *Let τ_1 and τ_2 be real numbers such that $\tau_1 + \tau_2 = 2\pi(\deg(L_1) + \deg(L_2))$. Let $\alpha = \frac{1}{2\pi}(\tau_1 - \tau_2) < 0$ and assume $[\eta] \neq 0$. Then there is a metric H_E satisfying* (17) *if and only if $[\eta]$ is α-stable.*

Remark 2.6. Proposition 3.8 of [5] skips the condition $\alpha < 0$. That is actually necessary to derive the α-stability in case the metric solution H_E

exists. By the way, there is a straightforward example of a solution H_E for (17) in an extension over S^2, where any $\alpha \geq 0$ is allowed, hence outside the admissible range of α-stability as defined by (19).

Let $H = H_E$ be a metric satisfying (17) for an extension $[\eta]$ of line bundles. The metric connection on E is then

$$D_E = \begin{pmatrix} D_1 & A \\ -A^{*H} & D_2 \end{pmatrix} , \tag{20}$$

for D_j being the Chern connections on L_j and A being the second fundamental form of the inclusion $L_1^{\perp H} \hookrightarrow E$. Computing $F_H = D_E^2$ and substituting into equation (17) we find the system

$$\begin{cases} i\Lambda F_1 - i\Lambda A \wedge A^{*H} = \tau_1 \\ i\Lambda F_2 - i\Lambda A^{*H} \wedge A = \tau_2 \\ \qquad\qquad D(A) = 0 \ . \end{cases} \tag{21}$$

The form A has holomorphic type $(0,1)$ since $L_1^{\perp H} \hookrightarrow E$ is antiholomorphic. Indeed, A is D''-closed, and its cohomology class is the one given in the beginning, $[A] = [\eta]$. The third equation in (21) implies that A is antiholomorphic, and we can assume from now on that $A = \eta + D''\xi$ is the H-antiholomorphic representative of the class $[\eta]$.

Making $\lambda = 2\pi \deg(L) + \tau_1 - \tau_2$ and following a computation similar to [12] (equations (3.8)-(3.10)) we obtain from (21)

$$\begin{cases} \Delta u + 2|\eta + D''\xi|_0^2 e^{-2u} - \lambda = 0 \\ D'_u(\eta + D''\xi) = 0 \ , \end{cases} \tag{22}$$

where u is the function associated to the pointwise metric change in L, L^*.

Therefore, a solution H_E for (17) gives us a smooth function u on S^2 and a section ξ of L^* that solve (22). Reciprocally, given a pair $(u, \xi) \in C^\infty(S^2) \times \Omega^0(L^*)$ that solves (22) it is straightforward to obtain the correspondent solution H_E for (17) (see [12]).

The dual problem of (22) is stated as follows: using the H-identification given by (6) we set $\phi = (\eta + D''\xi)^{*H_u} \in \mathcal{H}^{1,0}(L)$. Clearly $|\phi|^2_{H_u} = |\eta + D''\xi|^2_{H_u}$, hence (22) becomes equivalent - for the particular solution u - to system

$$\begin{cases} \Delta u + 2|\phi|_0^2 e^{2u} - \lambda = 0 \\ D''(\phi) = 0. \end{cases} \tag{23}$$

Remark 2.7. The first of equations (22) and (23) carry a factor 2 which had been absorbed in equations (2) and (3) (see also equations 3.6 and 3.8 in [12]).

We take a minute to compare problems (22) and (23). They are not quite the same because varying the real parameter λ and keeping $[\eta]$ fixed will vary the metric, and so will change the correspondent $[\phi] = [\eta]^{*H_u}$. It might sound that (23) is more likely to the taste of the analyst, because a fixed $[\phi]$ has only one representative regardless of the metric and one has a shape for the term $|\phi|_0^2$. On the other hand the same class $[\eta]$ has different representatives for different metrics, making the sight of the term $|\eta + D''\xi|_0^2$ a bit obscure.

Nevertheless, equations (23) lose an important characteristic that is enjoyed by (22): its linearization is not sign definite. This is roughly accounted for the difference in sign of the exponents in e^{2u} and e^{-2u} of either one. Further, system (22) has the results on extensions holding in the range $0 < \lambda < 2\pi\deg(L)$. The translation of results from (22) to (23) for some cases of $[\phi]$ is one of the main targets of this work.

Observe that we can refer to a solution (u, ξ) for (22) simply by u, since there is only one section $\xi = \xi(u)$ satisfying the second of equations (22).

Lemma 2.8. ([12] Theorem 4.7 and Corollary 4.9) *Let $0 < \lambda_0 < 2\pi\deg(L)$. Assume there is a solution $u = u_0$ for* (22) *in the parameters $[\eta] = [\eta_0]$ and $\lambda = \lambda_0$. Then this solution is unique. There is a neighborhood $U \times (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subset \mathcal{H}^{0,1}(L^*) \times \mathbb{R}$ of $([\eta_0], \lambda_0)$ and a smooth map $u : U \times (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to C^\infty(S^2)$ taking parameters to solutions of* (22).

Lemma 2.9. *Let $[\eta] \in \mathcal{H}^{0,1}(L^*) - \{0\}$. Then $0 < 4\pi(\deg(L_2) - \mathrm{div}[\eta]) \leq 2\pi\deg(L)$. System* (22) *has a solution for any $\lambda \in (0, 4\pi(\deg(L_2) - \mathrm{div}[\eta]))$ and no solution for $\lambda \in [4\pi(\deg(L_2) - \mathrm{div}[\eta]), 2\pi\deg(L))$.*

Proof. Inequality $0 < 4\pi(\deg(L_2) - \mathrm{div}[\eta])$ holds because of $[\eta] \neq 0$ and Lemma 2.4. For the other inequality we pick an extension $0 \to L_1 \to E \to L_2 \to 0$ representing $[\eta]$. From lemma 2.1 part (b) we have $E = \tilde{L_1} \oplus \tilde{L_2}$ holomorphically, for some line bundles $\tilde{L_1}, \tilde{L_2}$ with $\deg(\tilde{L_2}) \geq \deg(\tilde{L_1})$. The general theory of Chern classes gives us $\deg(E) = \deg(L_1) + \deg(L_2) = \deg(\tilde{L_1}) + \deg(\tilde{L_2})$. Clearly $\mathrm{div}[\eta] = \deg(\tilde{L_2}) \geq 1/2(\deg(L_1) + \deg(L_2))$, hence $4\pi(\deg(L_2) - \mathrm{div}[\eta]) \leq 2\pi\deg(L)$.

The second assertion of the Lemma comes from the α-stability condition restated for the parameter λ combined with Theorem 2.5 and the subsequent discussion. □

3. The space of extensions over S^2.

After the results in the previous section it becomes relevant to understand the space $\mathcal{H}^{0,1}(L^*)$, which is non-trivial because of (5).

Let $[\eta] \in \mathcal{H}^{0,1}(L^*)$ and H be a metric. Let η be the H-antiholomorphic representative for $[\eta]$. We aim to compute $\mathrm{div}[\eta]$. For that sake consider an extension $0 \to L_1 \to E \to L_2 \to 0$ for $[\eta]$. After Lemma 2.3 we can take $E = L_1 \oplus_{top} L_2$ with holomorphic operator D''_η, where

$$D''_\eta = \begin{pmatrix} D''_1 & \eta \\ 0 & D''_2 \end{pmatrix}. \tag{24}$$

We denote the cannonical meromorphic section of L_j by ζ_j, $j = 1, 2$. The investigation of the meromorphic sections of E starts with the

Lemma 3.1. *There is a smooth $f : S^2 \to \mathbb{C}$ such that $\psi = (f\zeta_1, \zeta_2)$ is a meromorphic section of E. If $J \subset E$ is any line bundle not equal to L_1 then there is a meromorphic function h on S^2 such that $\tilde{\psi} = ((f + h)\zeta_1, \zeta_2)$ is meromorphic and spans J.*

In the sequel we will write $k = \deg(L)$. The proof of Lemma 3.1 will come straight after the next result.

Lemma 3.2. *Write $\eta = \overline{g}\,\zeta_L^{*H} d\overline{z}$. Then there is a single $f \in C^\infty(S^2)$ with values in $\mathbb{C}$, and such that*

$$\overline{\partial}_z f = -\overline{g}\,|\zeta_L|^2_H \tag{25}$$

and $f(N) = 0$, N the north pole. This function can be written as $f = \mathcal{O} - p_f$ in a neighborhood of N, with p_f a polynomial in $w = \frac{1}{z}$ of degree smaller than k and $\mathcal{O}$ is a local smooth function such that $|\mathcal{O}(w)| \leq C|w|^k$, for some constant $C > 0$. Still, $p_f(w) = \sum_{j=1}^{k-1} b_j w^j$ and

$$b_j = \int_{S^2} \frac{|\eta|^2_H}{g}\, w^{-j+1}\, \nu(w), \qquad 1 \leq j \leq k-1. \tag{26}$$

Proof. Let $h \in C^\infty(S^2)$. Define a function $f : S^2 - \{N\} \to \mathbb{C}$ by

$$f(z) = \int_{S^2} \frac{h(z')}{z - z'}\, \nu(z'). \tag{27}$$

We claim that f is actually defined in the whole of S^2, is C^∞ and it holds $\overline{\partial}_z f(z) = h(z)/(1 + |z|^2)^2$ away from the north pole. Indeed, we can rewrite the integral in (27) as an integral in the plane

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} \frac{h(z')}{(1+|z'|^2)^2(z'-z)}\, dz' \wedge d\overline{z}'. \tag{28}$$

Recall that by the $\overline{\partial}$-Poincaré Lemma [14] the integral on the right-hand side of (28) is a function in the parameter z whose $\overline{\partial}_z$-derivative equals $h(z)/(1+|z|^2)^2$. In spite of the integral in [14] be performed in a bounded region of the plane the argument for the derivative requires a local computation and still holds in our case. Finally, changing in (27) the coordinate z' by $w' = 1/z'$ we get

$$f(w) = \int_{S^2} \frac{h(w')}{w'-w} ww'\, \nu(w'), \tag{29}$$

from what we can see f is well defined and smooth in N $(w = 0)$, as well as $f(N) = 0$. If $\tilde{f}$ is any function smooth on S^2 and such that $\overline{\partial}_z \tilde{f} = \overline{\partial}_z f$ we have $\tilde{f} - f$ holomorphic in $S^2 - \{N\}$, and hence constant because $\tilde{f} - f$ is bounded. We conclude there is exactly one f satisfying $\overline{\partial}_z f(z) = h(z)/(1+|z|^2)^2$.

Clearly the first part of the Lemma follows if we take $h(z) = -\overline{g}\,|\zeta_L|_H^2(1+|z|^2)^2$, observing that this choice makes h smooth: $h(z) = -g(z)^{-1}|\eta(z)|_H^2 = -\overline{g(z)}|\zeta_{L\otimes T^{1,0}S^2}(z)|_H^2$. The function f given by (27) becomes

$$f(w) = \int_{S^2} \frac{|\eta|_H^2}{g}\, \frac{ww'}{w-w'}\, \nu(w'). \tag{30}$$

Using identity (11) we see that the integrands on (26) can be written, in each trivialization ζ_L or $\zeta_{L,S}$, as

$$\frac{|\eta|_H^2}{g}\, w^{-j+1} = \overline{g}|\zeta_L|_H^2|dz|^2 w^{-j+1} = (\overline{g}\,\overline{w}^{k-2})w^{k-1-j}|\zeta_{L,S}|_H^2|dw|^2.$$

The second term of the above equation is bounded for $|w| \geq 1$ while the third one is bounded for $|w| \leq 1$. From this we check that the coefficients given by (26) are well-defined.

The last part of the Lemma will be proved using the usual trick on managing expansions for the function $1/(w'-w)$. Fix $w \neq \infty$ and compute from equation (30)

$$
\begin{aligned}
(31)\quad f(w) &= \int\limits_{S^2} \frac{|\eta|_H^2}{g}\,\frac{ww'}{w-w'}\,\nu(w') \\
&= \int\limits_{|w'|<2|w|} \frac{|\eta|_H^2}{g}\,\frac{ww'}{w-w'}\,\nu(w') + \int\limits_{|w'|\geq 2|w|} \frac{|\eta|_H^2}{g}\,\frac{ww'}{w-w'}\,\nu(w') \\
&= \int\limits_{|w'|<2|w|} \frac{|\eta|_H^2}{g}\,\frac{ww'}{w-w'}\,\nu(w') - \int\limits_{|w'|\geq 2|w|} \frac{|\eta|_H^2}{g}\left(\sum_{m=0}^{\infty}\frac{w^{m+1}}{(w')^m}\right)\nu(w').
\end{aligned}
$$

The power series appearing in the above equation converges absolutely, and the respective integral can be written

$$
\begin{aligned}
(32)\qquad & \int\limits_{|w'|\geq 2|w|} \frac{|\eta|_H^2}{g}\left(\sum_{m=0}^{\infty}\frac{w^{m+1}}{(w')^m}\right)\nu(w') \\
&= \sum_{m=0}^{k-2} w^{m+1} \int\limits_{|w'|\geq 2|w|} \frac{|\eta|_H^2}{g}(w')^{-m}\,\nu(w') \\
&\quad + \sum_{m=k-1}^{\infty} w^{m+1} \int\limits_{|w'|\geq 2|w|} \frac{|\eta|_H^2}{g}(w')^{-m}\,\nu(w') \\
&= \sum_{j=1}^{k-1} w^j \int\limits_{S^2} \frac{|\eta|_H^2}{g}(w')^{-j+1}\,\nu(w') \\
&\quad - \sum_{m=0}^{k-2} w^{m+1} \int\limits_{|w'|<2|w|} \frac{|\eta|_H^2}{g}(w')^{-m}\,\nu(w') \\
&\quad + \sum_{m=k-1}^{\infty} w^{m+1} \int\limits_{|w'|\geq 2|w|} \frac{|\eta|_H^2}{g}(w')^{-m}\,\nu(w').
\end{aligned}
$$

Let $p_f(w)$ be the polynomial given by the Lemma, and $\mathcal{O} = f + p_f$. Combining (31) and (32) we obtain

$$\mathcal{O}(w) = f(w) + p_f(w) \tag{33}$$
$$= \int\limits_{|w'|<2|w|} \frac{|\eta|_H^2}{g} \frac{ww'}{w-w'} \nu(w') - -\sum_{j=1}^{k-1} w^j \int\limits_{S^2} \frac{|\eta|_H^2}{g} (w')^{-j+1} \nu(w')$$
$$+ \sum_{m=0}^{k-2} w^{m+1} \int\limits_{|w'|<2|w|} \frac{|\eta|_H^2}{g} (w')^{-m} \nu(w')$$
$$- \sum_{m=k-1}^{\infty} w^{m+1} \int\limits_{|w'|\geq 2|w|} \frac{|\eta|_H^2}{g} (w')^{-m} \nu(w') + p_f(w).$$

The second and fifth terms of the last member of (33) cancell out. We end up with

$$\mathcal{O}(w) = \int\limits_{|w'|<2|w|} \frac{|\eta|_H^2}{g} \frac{ww'}{w-w'} \nu(w') \tag{34}$$
$$+ \sum_{m=0}^{k-2} w^{m+1} \int\limits_{|w'|<2|w|} \frac{|\eta|_H^2}{g} (w')^{-m} \nu(w')$$
$$- \sum_{m=k-1}^{\infty} w^{m+1} \int\limits_{|w'|\geq 2|w|} \frac{|\eta|_H^2}{g} (w')^{-m} \nu(w')$$
$$= T_1 + T_2 - T_3.$$

To finish the proof we will show that an estimate of the form $|T_m| \leq C|w|^k$ holds, for $m = 1, 2, 3$. We can assume $|w| \leq 1$ in the computations. As usual in this kind of argument we denote by $C(\cdot)$ a positive parameter that depends only on the terms inside parenthesis. Different occurrences of C may mean different "constants".

Replacing w by w' in equation (11) we get

$$\frac{|\eta|_H^2}{g} = (w')^{k-2} (\overline{g}\, (\overline{w}')^{k-2}) |\zeta_{L,S}|_H^2 |dw'|^2 = (w')^{k-2} M_H(w'), \tag{35}$$

where $M_H(w')$ remains bounded if $|w'| \le 2$. Thus

$$
\begin{aligned}
(36) \qquad |T_1| &\le \int\limits_{|w'|<2|w|} \frac{|\eta|_H^2}{|g|} \left| \frac{ww'}{w-w'} \right| \nu(w') \\
&\le \int\limits_{|w'|<2|w|} |w'|^{k-2}\, |M_H(w')| \frac{|ww'|}{|w-w'|}\, \nu(w') \\
&< |w|^k 2^{k-1} \|M_H\|_{L^\infty(|w'|<2)} \int\limits_{|w'|<2|w|} \frac{\nu(w')}{|w-w'|}.
\end{aligned}
$$

An easy estimate shows that

$$
(37) \qquad \int\limits_{|w'|<2|w|} \frac{\nu(w')}{|w-w'|} < C \qquad \text{if } |w| \le 1.
$$

Hence from (36) and (37) we obtain

$$
(38) \qquad |T_1| \le C(H)\, |w|^k \qquad \text{if } |w| \le 1.
$$

The estimate for T_2 follows a similar line to T_1:

$$
\begin{aligned}
(39) \qquad |T_2| &= \left| \sum_{m=0}^{k-2} w^{m+1} \int\limits_{|w'|<2|w|} \frac{|\eta|_H^2}{g} (w')^{-m}\, \nu(w') \right| \\
&\le \sum_{m=0}^{k-2} |w|^{m+1} \int\limits_{|w'|<2|w|} |M_H(w')|\, |w'|^{k-2} |w'|^{-m}\, \nu(w') \\
&\le \sum_{m=0}^{k-2} |w|^{m+1} |2w|^{k-2-m} \|M_H\|_{L^\infty(|w'|<2)} \int\limits_{|w'|<2|w|} \nu(w') \\
&\le C(H) |w|^{k-1} \int\limits_{|w'|<2|w|} \nu(w').
\end{aligned}
$$

The integral in the last member of (39) is the area of a geodesic disc of radius R. This disk has area smaller than its image under the conformal mapping $w' : S^2 - \{S\} \to \mathbb{R}^2$, so $\int_{|w'|<2|w|} \nu(w') < C\,|w|^2$. We get

$$|T_2| \le C(H)|w|^{k+1}. \tag{40}$$

Finally we estimate T_3.

$$\begin{aligned} T_3 &= \sum_{m=k-1}^{\infty} w^{m+1} \int_{|w'|\ge 2|w|} \frac{|\eta|_H^2}{g} (w')^{-m}\,\nu(w') \\ &= w \int_{|w'|\ge 2|w|} \frac{|\eta|_H^2}{g} \sum_{m=k-1}^{\infty} \left(\frac{w}{w'}\right)^m \nu(w') \\ &= w \int_{|w'|\ge 2|w|} \frac{|\eta|_H^2}{g(w')^{k-1}}\, w^{k-1} \sum_{m=0}^{\infty} \left(\frac{w}{w'}\right)^m \nu(w'). \end{aligned} \tag{41}$$

Therefore,

$$|T_3| \le 2|w|^k \int_{|w'|\ge 2|w|} \frac{|\eta|_H^2}{|g||w'|^{k-1}}\,\nu(w') < C\,|w|^k \int_{S^2} \frac{|\eta|_H^2}{|g||w'|^{k-1}}\,\nu(w'). \tag{42}$$

The integral in S^2 can be split into integrals in the north and south hemispheres. The first of them satisfies

$$\int_{|w'|\le 1} \frac{|\eta|_H^2}{|g||w'|^{k-1}}\,\nu(w') \;<\; \int_{|w'|\le 1} \frac{|M_H(w')|}{|w'|}\,\nu(w') \le C\,\|M_H\|_{L^\infty(|w'|\le 1)}. \tag{43}$$

The south hemisphere integral is estimated as

$$\begin{aligned} &\int_{|w'|>1} \frac{|\eta|_H^2}{|g||w'|^{k-1}}\,\nu(w') \\ &< \int_{|w'|>1} \frac{|\eta|_H^2}{|g|}\,\nu(w') = \int_{|w'|>1} |g||\zeta_L|_H^2 |dz|^2\,\nu(w') \\ &\le C\,\|g\|_{L^\infty(|w'|>1)} \|\zeta_L\|^2_{L^\infty_H(|w'|>1)} \|dz\|^2_{L^\infty(|w'|>1)}. \end{aligned} \tag{44}$$

From (42), (43) and (44) we obtain

$$|T_3| \le C(H)|w|^k. \tag{45}$$

Altogether inequalities (38), (40) and (45) imply $|\mathcal{O}(w)| \leq C|w|^k$. This completes with the Lemma's proof. □

Proof of Lemma 3.1. Let's first assume *there is* some meromorphic section ψ on E, not contained in L_1. Then $\psi = (\psi_1, \psi_2)$ and up to its set of poles it must satisfy

$$\begin{cases} D_1''(\psi_1) + \eta(\psi_2) = 0 \\ D_2''(\psi_2) = 0. \end{cases} \tag{46}$$

Therefore ψ_2 is meromorphic on L_2. Multiplying ψ by a suitable meromorphic function we can assume that $\psi_2 = \zeta_2$. Similarly, we can write $\psi_1 = f.\zeta_1$ for some function f smooth outside of the singular set of ψ. Recalling that in coordinates we have

$$\eta = \overline{g} \cdot \frac{\zeta_1}{|\zeta_1|_H^2} \cdot \zeta_2^{*H} \cdot d\overline{z},$$

the first of equations (46) holds (using $\psi_2 = \zeta_2$) if and only if

$$\overline{\partial} f + \overline{g}\,\frac{|\zeta_2|_H^2}{|\zeta_1|_H^2} d\overline{z} = 0,$$

or equivalently,

$$\overline{\partial}_z f = -\overline{g}\,|\zeta_L|_H^2.$$

Clearly the above steps can be reversed, and if we start off at the solution f for (25), given by Lemma 3.2, we construct a meromorphic section ψ satisfying the conditions of Lemma 3.1.

Now assume $J \subset E$ is a line subbundle different from L_1. We pick a meromorphic section $\tilde{\psi}$ spanning J, and because of the above argument, can assume $\tilde{\psi} = (\tilde{f}\,\zeta_1, \zeta_2)$. Further $\tilde{f}$ satisfies equation (25) in all but finite many points. We conclude that $\tilde{f} - f = h$ is meromorphic and $\tilde{\psi}$ has the form $\tilde{\psi} = ((f + h)\zeta_1, \zeta_2)$. □

Now let $J \subset E$ be a holomorphic line subbundle, and $\tilde{\psi}$ be a meromorphic section spanning J, in the form given by Lemma 3.1, with function f vanishing at N, as given by Lemma 3.2. At any $x \in S^2 - \{N\}$ ζ_2 is regular, thus x cannot be a zero of $\tilde{\psi}$. And x is a pole of $\tilde{\psi}$ if and only if x is a pole of h of the same order. Hence

$$\operatorname{ord}_x(\tilde{\psi}) = \min\{0, \operatorname{ord}_x(h)\} \qquad \text{for } x \neq N. \tag{47}$$

On a vicinity of the north pole we write

$$\begin{aligned} \tilde{\psi}(w) &= ((f(w) + h(w))\zeta_1(w), \zeta_2(w)) \\ &= ((\mathcal{O}(w) - p_f(w) + h(w))w^{\deg(L_1)}\zeta_{L_1,S}, w^{\deg(L_2)}\zeta_{L_2,S}) \\ &= w^{\deg(L_1)}((\mathcal{O}(w) - p_f(w) + h(w))\zeta_{L_1,S}, w^k\zeta_{L_2,S}). \end{aligned} \tag{48}$$

Observe that $\operatorname{ord}_N(\tilde{\psi}) = m$ if and only if m is the only integer such that $w^{-m}\tilde{\psi}$ is a regular holomorphic section around $w = 0$. Because $\frac{\mathcal{O}(w)}{w^k}$ (for $w \neq 0$) is bounded, a quick study of the cases $\operatorname{ord}_N(h - p_f) < k$ and $\operatorname{ord}_N(h - p_f) \geq k$ (take this order to be infinite if $h - p_f$ is null) leads to

$$\operatorname{ord}_N(\tilde{\psi}) = \deg(L_1) + \min\{\operatorname{ord}_N(h - p_f), k\}. \tag{49}$$

Let s^- denote the number of poles (accounting for multiplicity) of h in $S^2 - \{N\}$. From (47) and (49) we get

$$\deg(\tilde{\psi}) = \sum_{x \in S^2} \operatorname{ord}_x(\tilde{\psi}) = \deg(L_1) + \min\{\operatorname{ord}_N(h - p_f), k\} - s^-. \tag{50}$$

Our aim is to compute $\operatorname{div}E$, which is the maximum among the degrees of $\tilde{\psi}$ for all such meromorphic sections. Thus we need to find an appropriate meromorphic h that maximizes the right-hand-side of (50). Since $p_f(N) = 0$ we should choose h so that $h(N) = 0$, otherwise we would have $\operatorname{ord}_N(h) \leq 0$, so $\operatorname{ord}_N(h - p_f) = \operatorname{ord}_N(h)$ and $\deg(\tilde{\psi}) = \deg(L_1) + \operatorname{ord}_N(h) - s^- \leq \deg(L_1)$. In particular we can write, without loss of generality,

$$h(w) = \frac{y(w)}{1 - v(w)}, \tag{51}$$

where $y(w)$ and $v(w)$ are polynomials, $y(0) = 0 = v(0)$ and y and $1 - v$ have no common zeros. The number of poles s^- of h equals the maximum degree among the polynomials $y(w)$ and $1 - v(w)$, hence to allow $\deg(\tilde{\psi}) > \deg(L_1)$ we can assume both degrees to be less than k.

Lemma 3.3. *Follow the above notation and conditions for $y(w)$, $v(w)$ and s^-, and for any polynomial in w denote by a subindex j the coefficient of*

w^j in it. Let $\{b_j\}$ be the coefficients given by (26)*. Consider the system of equations*

$$\left\{\begin{aligned} b_1 &= y_1 \\ b_2 &= y_2 + (yv)_2 \\ b_3 &= y_3 + (yv)_3 + (yv^2)_3 \\ &\vdots \\ b_{k-1} &= y_{k-1} + (yv)_{k-1} + (yv^2)_{k-1} + \cdots + (yv^{k-2})_{k-1} \end{aligned}\right. \tag{52}$$

Let $j^* \leq k$ be the maximum integer such that all equations in system (52) *with index $j < j^*$ are satisfied. Then*

$$\deg(\tilde{\psi}) = \deg(L_1) + j^* - s^-. \tag{53}$$

Proof. We only need to show that $j^* = \min\{\mathrm{ord}_N(h - p_f), k\}$ and use equation (50). Because h is holomorphic at $w = 0$ we can write $h(w) = p_h(w) + \mathcal{O}_h(w)$ where $p_h(w)$ is a polynomial of degree lower than k and $\mathcal{O}_h = h - p_h$ has order greater than $k - 1$ in $w = 0$. Then $\min\{\mathrm{ord}_N(h - p_f), k\} = \min\{\mathrm{ord}_N(p_h - p_f), k\}$. Expanding h in the polynomials y and v close to $w = 0$ we get

$$h(w) = \frac{y(w)}{1 - v(w)} = \sum_{m=0}^{\infty} y(w)v(w)^m = p_h(w) + \mathcal{O}_h(w). \tag{54}$$

For any order $1 \leq j \leq k - 1$ the only summands in the third member of (54) that add up to the j-th coefficient of p_h are those $y\,v^m$ with $m < j$. Hence,

$$p_{h\,j} = y_j + (yv)_j + (yv^2)_j + \cdots + (yv^{j-1})_j. \tag{55}$$

Therefore the j-th equation of system (52) is nothing but a statement of equality between the j-th coefficients of p_f and p_h. If $j^* < k$ then all such equations for $j < j^*$ are satisfied but the equation for $j = j^*$ is not, thus the first non-vanishing coefficient of $p_h - p_f$ is $(p_h - p_f)_{j^*}$. If $j^* = k$ then $p_h - p_f \equiv 0$. In both cases one has $\min\{\mathrm{ord}_N(p_h - p_f), k\} = j^*$. □

The practical application of Lemma 3.3 will be shown on Section 4. For now it is interesting to notice that div[η] will appear as the maximum right-hand-side value of equation (53). This value depends on the parameters j^*, s^- and ultimately, on the coefficients b_j for $1 \leq j \leq k - 1$. However, the latter seem to depend upon the metric H, besides the very cohomology class

$[\eta]$, after equation (26). Amazingly, it turns out that $\{b_j\}$ *do not depend upon the metric*, as the next result states.

Lemma 3.4. *Let $\beta = \{z^{j-1}\zeta_L dz\}_{1\le j<k}$ be a basis of $\mathcal{H}^{1,0}(L)$, and let β^* be the dual cannonical basis of $\mathcal{H}^{0,1}(L^*)$. Then for a given $[\eta] \in \mathcal{H}^{0,1}(L^*)$ the coefficients $\{b_j\}$ obtained from formula* (26) *using any metric are the coordinates of $[\eta]$ in β^*.*

Proof. Fix a metric H and let η be the H-antiholomorphic representative for $[\eta]$. Set $\phi = \eta^{*H}$, thus $\phi = g\,\zeta_L dz$ for some polynomial g. Then at each $x \in S^2$,

$$|\eta|_H^2 = i\Lambda(\eta^{*H} \wedge \eta) = i\Lambda(\phi \wedge \eta). \tag{56}$$

Equations (26) turn into

$$b_j = \int_{S^2} \frac{i}{g}(\phi \wedge \eta)\, z^{j-1} = \int_{S^2} i(z^{j-1}\zeta_L dz \wedge \eta) = (\!([z^{j-1}\zeta_L dz], [\eta])\!), \tag{57}$$

hence b_j is the coupling of $[\eta]$ with the j-th vector of the basis β. □

4. Some conformal curvatures on S^2

In this section we use the previous theory to show existence of metrics pointwise conformal to the standard metric on S^2 for some non-negative curvatures with zeros.

4.1. Projectivized cohomology as a parameter space

We set one more equivalence to simplify our analysis. Let α be a non-zero complex constant. If $[\eta] \in \mathcal{H}^{0,1}(L^*)$ is non-zero, η represents $[\eta]$, then (u, ξ) solves (22) if and only if $(u + \ln|\alpha|, \alpha\xi)$ solves (22) after replacing η by $\alpha\eta$. Solutions for classes that are multiple of each other differ by a constant. The case $[\eta] = 0$ is of no interest for equation $\Delta u - \lambda = 0$ has no solution at all if $\lambda \neq 0$. This motivates us to work on the projectivization

$$\mathbb{P}^{0,1} = \frac{\mathcal{H}^{0,1}(L^*) - \{0\}}{[\eta] \sim \alpha[\eta]} \simeq \mathbb{CP}^{k-2}. \tag{58}$$

We similarly define $\mathbb{P}^{1,0}$ as the projectivization of $\mathcal{H}^{1,0}(L)$ and the natural home for function parameters for equation (23). For a metric H the function given by (6) is homogeneous and passes to a diffeomorphism $*H : \mathbb{P}^{1,0} \to \mathbb{P}^{0,1}$. To avoid cumbersome notation we will denote the projective class of some $[\eta] \in \mathcal{H}^{0,1}(L^*)$ ($[\phi] \in \mathcal{H}^{1,0}(L)$) by the same symbol $[\eta] \in \mathbb{P}^{0,1}$ ($[\phi] \in \mathbb{P}^{1,0}$). Though we must take care of the scaling when consider equations (22) and (23). Hence we denote by $u = u([\eta], \lambda)$ the *zero mean value component* of a solution for (22). For the given projective $[\eta]$ we choose any smooth section representative $\eta \in [\eta]$: the solution of (22) is given by $u + C$ for a uniquely defined real constant C (as long as λ is in the existence range). This approach seems good to us because allows the definition of the function u given by Lemma 2.8 directly in $\mathbb{P}^{0,1}$ and avoids the necessity of a normalization condition on η.

Let $m \geq 1$. Define

$$\mathbb{P}^{0,1}_m = \{[\eta] \in \mathbb{P}^{0,1} \,|\, \mathrm{div}[\eta] \geq \deg(L_2) - m\}. \tag{59}$$

The interest on the sets $\mathbb{P}^{0,1}_m$ stands for a neat paraphrase of Lemma 2.9:

Corollary 4.1. *If $m > q$ then $\mathbb{P}^{0,1}_m \supset \mathbb{P}^{0,1}_q$. For $[\eta] \in \mathbb{P}^{0,1}$ and $m \leq \deg(L)$, $m \in \mathbb{Z}$, it holds $[\eta] \in \mathbb{P}^{0,1}_m - \mathbb{P}^{0,1}_{m-1}$ if and only if the range of values of $\lambda \in (0, 4\pi \deg(L))$ for which there are solutions of (22) is $(0, 4\pi m)$.*

From Lemma 2.4 and the argument in the proof of Lemma 2.9 we get, for any $[\eta]$, $\deg(L_2) - 1 \geq \mathrm{div}[\eta] \geq \deg(L_2) - \lfloor \frac{k}{2} \rfloor$, and thus the decreasing sequence

$$\mathbb{P}^{0,1} = \mathbb{P}^{0,1}_{\lfloor \frac{k}{2} \rfloor} \supset \mathbb{P}^{0,1}_{\lfloor \frac{k}{2} \rfloor - 1} \supset \mathbb{P}^{0,1}_{\lfloor \frac{k}{2} \rfloor - 2} \supset \cdots \supset \mathbb{P}^{0,1}_2 \supset \mathbb{P}^{0,1}_1. \tag{60}$$

The notation is suggestive in the sense that we conjecture all $\mathbb{P}^{0,1}_m$ are copies of $\mathbb{CP}^r$, for different dimensions r, inside $\mathbb{P}^{0,1} \simeq \mathbb{CP}^{k-2}$. We have not been able to prove it so far, but only for the ending terms of the sequence.

Lemma 4.2. *There is a complex embedding $\mathbb{CP}^1 \to \mathbb{P}^{0,1}$ which is a diffeomorphism onto $\mathbb{P}^{0,1}_1$. For any $[\eta] \in \mathbb{P}^{0,1}_1$ the divisor of the class $[\eta]^{*H_0}$ is $(k-2)x$ for some $x \in S^2$.*

Proof. Let $[\eta] \in \mathbb{P}^{0,1}_1$, thus $\mathrm{div}[\eta] = \deg(L_2) - 1 = \deg(L_1) + k - 1$. Following Lemma 3.3 and equation (53) for a section $\tilde{\psi}$ with maximal degree we find that $j^* - s^- = k - 1$. Because of the bounds $1 \leq j^* \leq k$ and $s^- \geq 0$

we get $s^- \leq 1$. The polynomials $y(w)$ and $v(w)$ are linear or null, and system (52) turns into

$$\left\{\begin{aligned} b_1 &= y_1 \\ b_2 &= y_1 v_1 \\ b_3 &= y_1 v_1^2 \\ &\vdots \\ b_{k-1} &= y_1 v_1^{k-2} \end{aligned}\right. \tag{61}$$

In case $s^- = 0$ and $j^* = k-1$ the meromorphic function h of the Lemma is identically zero, so $y_1 = 0$. We get $b_j = 0$ for $1 \leq j \leq k-2$ and $b_{k-1} \neq 0$. Otherwise, $s^- = 1$ and $j^* = k$. Thus $y_1 \neq 0$ and all equations in (61) are satisfied. With this characterization it is easy to see that the function

$$\Psi[b_1 : b_2] = \begin{cases} \left[1 : \dfrac{b_2}{b_1} : \dfrac{b_2^2}{b_1^2} : \cdots : \dfrac{b_2^{k-2}}{b_1^{k-2}}\right] & \text{if } b_1 \neq 0 \\ \left[\dfrac{b_1^{k-2}}{b_2^{k-2}} : \dfrac{b_1^{k-3}}{b_2^{k-3}} : \cdots : \dfrac{b_1}{b_2} : 1\right] & \text{if } b_2 \neq 0 \end{cases} \tag{62}$$

is a diffeomorphism from $\mathbb{CP}^1$ onto the homogeneous coordinates of the classes $[\eta]$ with $\operatorname{div}[\eta] = \deg(L_2) - 1$.

Now we look for possibilities for the divisor of $[\phi] = [\eta]^{*H_0}$. First consider the case $\phi = \zeta_L dz$ (hence g is a constant). In the computation of b_j in formula (26) we can replace $|\eta|^2_{H_0}$ by $|\phi|^2_{H_0}$. Due to the rotational symmetry for the metric H_0 in (12) and of the holomorphic coordinate, the integrals (26) vanish for $j > 1$ and is non-zero in $j = 1$, therefore the coefficients associated to $[\zeta_L dz]^{*H_0}$ are $b_1 \neq 0$ and $b_j = 0$, $2 \leq j \leq k-1$. We conclude that $[\zeta_L dz]^{*H_0} = \Psi[1 : 0] \in \mathbb{P}_1^{0,1}$.

In general, let $\phi = (z-a)^{k-2}\zeta_L dz$, where $a = z(x_0)$ for some $x_0 \in S^2$. A not so short analytic argument to show that $[\phi]^{*H_0}$ is in $\mathbb{P}_1^{0,1}$ is simply to compute b_j with formula (26) and showing those are in geometric progression. A more direct geometric approach, though, is noticing that the coefficients given by (26) depend on the basis $\{z^j \zeta_L dz\}_{0\leq j\leq k-2}$ of $\mathcal{H}^{1,0}(L)$. Change this basis to $\{\tilde{z}^j \phi\}_{0\leq j\leq k-2}$ where $\tilde{z} = z_{x_0}$ is a stereographic coordinate satisfying $\tilde{z}(-x_0) = 0$, and use $\tilde{w} = 1/\tilde{z}$ to replace w in the integrals (26). Clearly the whole construction of Lemmas 3.1 and 3.2 does not depend on the fact that $N = (0,0,1)$, or rather, on the coordinate chart used. In the new charts given by $\tilde{z}$ (or $\tilde{w}$) and ϕ, the argument follows like in the previous paragraph, so

$[\phi]^{*H_0} \in \mathbb{P}_1^{0,1}$ in this case also. This shows that all classes $[\phi] \in \mathbb{P}^{1,0}$ with a zero of order $k-2$ are the images of classes in $\mathbb{P}_1^{0,1}$ under $*H_0$. The conclusion then follows since both of the set of those classes, as well as $\mathbb{P}_1^{0,1}$, are diffeomorphic to $\mathbb{CP}^1$, and $*H_0$ is a diffeomorphism between them. □

4.2. The isometry group of S^2

Let $\varphi : S^2 \to S^2$ be an isometry. Take points $x, y \in S^2$ with $\varphi(x) = y$. Choose stereographic coordinates z, v around x and y, respectively, such that $z(x) = v(y) = 0$. Since φ is conformal and is an isometry it is not hard to see that $v = \varphi(z) = bz$ for a unitary complex b, if φ preserves orientation, and $v = b\overline{z}$, if φ reverses orientation. Then, for h a complex-valued function on S^2 we set for any $x \in S^2$

$$\varphi^* h(x) = \begin{cases} h(\varphi(x)) & \text{if } \varphi \text{ is orientation preserving} \\ \overline{h(\varphi(x))} & \text{if } \varphi \text{ is orientation reversing} \end{cases} \tag{63}$$

The conjugation in the second case above aims to preserve holomorphicity: h is holomorphic in some open set $U \subset S^2$ if and only if $\varphi^* h$ is holomorphic in $\varphi^{-1}(U)$. This definition is naturally extended to a complex-valued differential form ω: writing locally $\omega = h\,\mu$ for h a function and μ a real-valued form we set $\varphi^*\omega = \varphi^* h \varphi^* \mu$, where $\varphi^*\mu$ is the usual pull-back of forms.

We must define a similar notion for classes in $\mathbb{P}^{1,0}$ and $\mathbb{P}^{0,1}$. This is not that simple because there is no cannonical identification between the fibers L_x and $L_{\varphi(x)}$, for x in S^2. We do that by first defining the pull-back of divisors. If $\mathcal{D} = \sum_j a_j x_j$ we set $\varphi^*\mathcal{D} = \sum_j a_j \varphi^{-1}(x_j)$.

Now fix some holomorphic ζ in L whose divisor is $\mathcal{D}$ and set $\varphi^*\zeta$ as some non-trivial holomorphic section with divisor $\varphi^*\mathcal{D}$. If ψ is an arbitrary smooth section in $\Omega^{p,q}(L)$ then $\psi = \omega \otimes \zeta = \omega\zeta$ for some form ω smooth away of the singular set of ζ. Define

$$\varphi^*\psi = \varphi^*\omega\varphi^*\zeta. \tag{64}$$

Lemma 4.3. *Let $H = H_u$ be a metric. Then:*
(i) The operator D'' commutes with φ^*. In particular, $\psi \in \Omega^{p,q}(L)$ is meromorphic with divisor $\mathcal{D}$ if and only if $\varphi^*\psi$ is meromorphic with divisor $\varphi^*\mathcal{D}$.
(ii) There is a constant $c > 0$ such that for any $\psi, \chi \in \Omega^{p,q}(L)$ it holds

$$\varphi^*\langle \psi \wedge \chi \rangle_{H_u} = c\langle \varphi^*\psi \wedge \varphi^*\chi \rangle_{H_{\varphi^* u}}. \tag{65}$$

(iii) For any section-valued form ψ one has $\varphi^*(D'_u\psi) = D'_{\varphi^*u}(\varphi^*\psi)$. In particular, φ^* commutes with D'_{H_0}.

Proof. (i) Let ψ be a smooth (p,q)-section, then $\psi = \omega\zeta$. Thus

$$D''(\varphi^*\psi) = D''(\varphi^*\omega\varphi^*\zeta) = \overline{\partial}(\varphi^*\omega)\varphi^*\zeta = \varphi^*\overline{\partial}\omega\varphi^*\zeta \\ = \varphi^*(\overline{\partial}\omega\zeta) = \varphi^* D''\psi, \tag{66}$$

since both of ζ and $\varphi^*\zeta$ are holomorphic and $\overline{\partial}$ commutes with φ^* by a property of the pull-back on forms. Therefore φ^* takes meromorphic sections to meromorphic sections. Let $\psi = \omega\zeta$ be meromorphic. Then ω is meromorphic. Tensoring meromorphic sections adds up their divisors, hence

$$\mathcal{D}(\varphi^*\psi) = \mathcal{D}(\varphi^*\omega) + \mathcal{D}(\varphi^*\zeta) = \varphi^*\mathcal{D}(\omega) + \varphi^*\mathcal{D}(\zeta) \\ = \varphi^*\mathcal{D}(\omega\zeta) = \varphi^*\mathcal{D}(\psi). \tag{67}$$

(ii) Consider first that $\psi = \chi = \zeta$ and $u = 0$, so $H = H_0$. Equality (65) turns into $\varphi^*|\zeta|^2_{H_0} = c|\varphi^*\zeta|^2_{H_0}$. Notice that both members of this equation are functions with singularities in the same points, namely the divisor set of $\varphi^*\zeta$. We claim that the function $f = \ln\left(\frac{\varphi^*|\zeta|^2_{H_0}}{|\varphi^*\zeta|^2_{H_0}}\right)$ is actually smooth in S^2. For any x in this singular set, let $y = \varphi(x)$, and take z, v coordinates centered at x, y, and such that $v(z) = \varphi(z) = z$ (assume without loss of generality φ is orientation preserving). Then

$$\zeta(v) = v^m\tilde{\zeta}_y(v), \quad \varphi^*\zeta(z) = z^m\tilde{\zeta}_x(z) \tag{68}$$

where $\tilde{\zeta}_y, \tilde{\zeta}_x$ are regular around $v = 0$ and $z = 0$, respectively, and

$$f(z) = \ln\left(\frac{|\zeta(v(z))|^2_{H_0}}{|\varphi^*\zeta(z)|^2_{H_0}}\right) = \ln\left(\frac{|\tilde{\zeta}_y(v(z))|^2_{H_0}}{|\tilde{\zeta}_x(z)|^2_{H_0}}\right) \tag{69}$$

is clearly smooth at x. Thus $f \in C^\infty(S^2)$.

Now recall Lemma 2.2 and compute

$$\Delta f = \Delta\left(\ln(\varphi^*|\zeta|^2_{H_0}) - \ln|\varphi^*\zeta|^2_{H_0}\right) \\ = \varphi^*(\Delta\ln|\zeta|^2_{H_0}) - \Delta\ln|\varphi^*\zeta|^2_{H_0} \\ = \varphi^*(-4\pi\deg(L)) + 4\pi\deg(L) = 0. \tag{70}$$

In the above we used that ζ, $\varphi^*\zeta$ are holomorphic and that φ^* commutes with Δ. In particular we obtain that f is harmonic in the whole sphere, so

f is constant, and equation (65) follows immediately in this particular case for an appropriate $c > 0$. Now for an arbitrary u,

$$\varphi^*|\zeta|^2_{H_u} = \varphi^*(|\zeta|^2_{H_0}\, e^{2u}) = c|\varphi^*\zeta|^2_{H_0}\, e^{2\varphi^* u} = c|\varphi^*\zeta|^2_{H_{\varphi^* u}}. \tag{71}$$

The general case is a consequence of this one once we write the section-valued forms ψ, χ in components with ζ.
(iii) Again, the general case will follow as routine if we prove it for the very case $\psi = \zeta$. The section $D'_u\zeta$ can be managed implicitly in the equation $\partial|\zeta|^2_{H_u} = \langle D'_u\zeta, \zeta\rangle_{H_u}$. Applying φ^* to it and using part (ii) we derive

$$\varphi^*\partial|\zeta|^2_{H_u} = c\langle\varphi^* D'_u\zeta, \varphi^*\zeta\rangle_{H_{\varphi^* u}}. \tag{72}$$

Interchanging φ^* and ∂ in the above equation yields

$$\partial\varphi^*|\zeta|^2_{H_u} = \partial\left(c|\varphi^*\zeta|^2_{H_{\varphi^* u}}\right) = c\langle D'_{\varphi^* u}\varphi^*\zeta, \varphi^*\zeta\rangle_{H_{\varphi^* u}}, \tag{73}$$

and since ∂ and φ^* commute, the last members of equations (72) and (73) are equal. It follows $\varphi^* D'_u\zeta = D'_{\varphi^* u}\varphi^*\zeta$. This finishes with the Lemma's proof. □

It becomes suitable to define the pull-back of a metric: for $H = H_0 e^{2u}$ we set $\varphi^* H_u = H_{\varphi^* u}$. The definition of φ^* on sections of the bundle L^* is now very natural. For a section ξ on L^* we define $\varphi^*\xi = (\varphi^*(\xi^{*H}))^{*\varphi^* H}$. Clearly one has to show invariance from the metric's choice.

$$\begin{aligned}\varphi^*\xi &= (\varphi^*(\xi^{*H_u}))^{*\varphi^* H_u} = (\varphi^*(\xi^{*H_0})(\varphi^* e^{-2u}))^{*H_{\varphi^* u}} \\ &= (\varphi^*(\xi^{*H_0}))^{*H_0}(e^{-2\varphi^* u})(e^{2\varphi^* u}) = (\varphi^*(\xi^{*H_0}))^{*H_0}.\end{aligned} \tag{74}$$

The proof of the next Lemma will be skipped.

Lemma 4.4. *(i) For any $\xi \in \Omega^{p,q}(L^*)$ it holds $D''\varphi^*\xi = \varphi^*(D''\xi)$. In particular, φ^* descends to the cohomology $\mathcal{H}^{0,1}(L^*)$.*
(ii) If ϕ, η are sections in $\Omega^{1,0}(L)$, $\Omega^{0,1}(L^*)$, respectively, then $\varphi^*(\phi \wedge \eta) = c(\varphi^*\phi \wedge \varphi^*\eta)$, where c is the same constant as in Lemma 4.3 part (ii). In particular $(\!(\phi, \eta)\!) = c(\!(\varphi^*\phi, \varphi^*\eta)\!)$.
(iii) For a section $\eta \in \Omega^{0,1}(L^*)$ it holds $\varphi^*(D'_u\eta) = D'_{\varphi^* u}(\varphi^*\eta)$.

The whole construction of the φ^* pull-back started with a particular holomorphic section of L. Since it is $\mathbb{C}$-linear and up to a constant factor, holomorphic sections are defined by their divisors, we conclude φ^* induces pull-back in a unique way on $\mathbb{P}^{1,0}$ and $\mathbb{P}^{0,1}$. Equivalently, there is a right action of the isometry group $\mathrm{Iso}(S^2)$ on the manifolds $\mathbb{P}^{1,0}$ and $\mathbb{P}^{0,1}$.

The operation of the H-dual given by (6) is defined in the projective cohomology. In view of Corollary 4.1 and the definition of $u([\eta],\lambda)$ we can set the smooth map

$$\begin{aligned}\mathcal{F}: \bigcup_{\lfloor\frac{k}{2}\rfloor\geq m\geq 1} (\mathbb{P}^{0,1}_m - \mathbb{P}^{0,1}_{m-1})\times(0,4\pi m) \to \mathbb{P}^{1,0},\\ \mathcal{F}([\eta],\lambda)=\mathcal{F}_\lambda[\eta]=[\eta]^{*H_{u([\eta],\lambda)}}.\end{aligned} \tag{75}$$

The function $\mathcal{F}$ behaves well respect to the isometries of S^2.

Lemma 4.5. *Let φ be an isometry. Fix $([\eta],\lambda)$ in the domain of $\mathcal{F}$. Then $\varphi^*\mathcal{F}_\lambda[\eta]=\mathcal{F}_\lambda(\varphi^*[\eta])$.*

Proof. Recall that $u([\eta],\lambda)$ designates the zero mean-value component of the actual solution of the first of equations (22). Hence, writing for simplicity $u=u([\eta],\lambda)$ and assuming η is an arbitrary representative of $[\eta]\in\mathbb{P}^{0,1}$, we have

$$\Delta u + 2|\eta + D''\xi|_0^2 e^{-2(u+r)} - \lambda = 0$$

for some $r\in\mathbb{R}$. On the other hand, applying φ^* to this equation yields

$$\begin{aligned}0 &= \varphi^*(\Delta u + 2|\eta + D''\xi|_0^2 e^{-2(u+r)} - \lambda)\\ &= \Delta\varphi^* u + 2ce^{-2r}|\varphi^*(\eta + D''\xi)|_0^2 e^{-2\varphi^* u} - \lambda.\end{aligned} \tag{76}$$

The second term of the third member above is justified by Lemma 4.3 part (ii) together with the observation that φ^* and the contraction operator $i\Lambda$ commute:

$$\begin{aligned}\varphi^*|\eta + D''\xi|_{H_u}^2 &= \varphi^*|\phi|_{H_u}^2 = c\, i\Lambda\langle\varphi^*\phi\wedge\varphi^*\phi\rangle_{H_{\varphi^* u}}\\ &= c|\varphi^*\phi|_{H_{\varphi^* u}}^2 = c|\varphi^*(\eta + D''\xi)|_{H_{\varphi^* u}}^2,\end{aligned} \tag{77}$$

where we write $\phi=(\eta+D''\xi)^{*H_u}$.

Applying φ^* to the second equation in (22) clearly (re)states that $\varphi^*(\eta + D''\xi)$ is $\varphi^* H_u$-antiholomorphic, thanks to Lemma (4.4) part (iii). This shows us that $\varphi^* u + r - \frac{1}{2}\ln(c)$ is a solution to (22) with $\varphi^*(\eta + D''\xi)$ in the place of $\eta + D''\xi$. Since the zero mean value component of this solution is unique we get $\varphi^* u([\eta], \lambda) = u(\varphi^*[\eta], \lambda)$. Thus

$$\varphi^* \mathcal{F}_\lambda[\eta] = \varphi^*([\eta]^{*H_u}) = [\varphi^*\eta]^{*H_{\varphi^* u}} = [\varphi^*\eta]^{*H_{u(\varphi^*[\eta],\lambda)}} = \mathcal{F}_\lambda(\varphi^*[\eta]), \tag{78}$$

and we are done. □

4.3. Applications

For the curvature equation (1), the normalization we adopted in the metric implies $K_0 \equiv 4\pi$. The class of curvatures K is restricted to $|\phi|^2_{H_0}$ for any $[\phi] \in \mathbb{P}^{1,0}$. Thus we are interested in studying problem

$$\Delta u + |\phi|_0^2 e^{2u} - 4\pi = 0. \tag{79}$$

We start with a non-existence lemma which recovers the result in [17].

Lemma 4.6. *Let $[\phi]$ be a class whose divisor set is $(k-2)x_0$, $k > 2$, for some $x_0 \in S^2$. Then there is no radially symmetric solution u for equation* (79)*, respect to the axis of S^2 passing through x_0. In particular, $|\phi|_0^2$ is not the curvature of a rotationally symmetric metric on S^2 pointwise conformal to the standard metric.*

Proof. If a radially symmetric solution u for (79) existed we could set $\eta = \phi^{*H_u}$ and have a solution for system (22) with $\lambda = 4\pi$. On the other hand, computing the coefficients $\{b_j\}$ from (26) in the stereographic coordinate chart w with south pole at x_0 would provide us with $b_1 \neq 0$ and $b_j = 0$, for $2 \leq j \leq k-1$, thanks to the symmetry of both u and $|\phi|_0^2$. Following the same argument as in the proof of Lemma 4.2 we conclude that $[\eta] \in \mathbb{P}_1^{0,1}$. This is an absurd due to Corollary 4.1, and the assumed solution u does not exist. □

Before we go to the existence results on curvatures we first state a nice consequence of a standard differential topology fact. The first part of this result was already known for more general functions [15, 20]. The second part, though, seems to be new before [12].

Theorem 4.7. *Let $[\phi] \in \mathbb{P}^{1,0}$ and $0 < \lambda < 4\pi$. Then there exists at least one solution u for* (23)*. The cardinality of the set os solutions for $[\phi]$ equals the cardinality of the preimage $\mathcal{F}_\lambda^{-1}([\phi])$.*

Proof. Notice that in this range for λ the map $\mathcal{F}_\lambda$ is defined for all $[\eta] \in \mathbb{P}^{0,1}$. Because of Theorem 4.3 part (3) of [12] we have that

$$\lim_{\lambda \to 0^+} \mathcal{F}_\lambda = *H_0. \tag{80}$$

Set $\mathcal{F}_0 = *H_0$. Since the family $\lambda \to \mathcal{F}_\lambda$ is continuous and $\mathcal{F}_0 : \mathbb{P}^{0,1} \to \mathbb{P}^{1,0}$ is a diffeomorphism, the maps $\mathcal{F}_\lambda$ all have topological degree 1. In particular they are surjective. Hence, any $[\phi] \in \mathbb{P}^{1,0}$ is of the form $[\eta]^{*H_{u([\eta],\lambda)}}$, and so has a solution. Because uniqueness of solutions holds for $[\eta]$ each solution for $[\phi]$ corresponds to exactly one element of $\mathcal{F}_\lambda^{-1}([\phi])$. □

Let $S \subset Iso(S^2)$ be a subgroup of the group of isometries of the euclidean sphere. Because of Lemma 4.5 any S-orbit of $\mathbb{P}^{0,1}$ is taken by $\mathcal{F}_\lambda$ onto some S-orbit of $\mathbb{P}^{1,0}$.

We first look at orbits which are unitary and isolated. If $\{[\eta]\}$ is such an orbit, meaning that for any $[\tilde{\eta}]$ sufficiently close to $[\eta]$, $\{[\tilde{\eta}]\}$ is not an S-orbit, then making $[\phi] = \mathcal{F}_0[\eta]$ we get that $\{[\phi]\}$ is also a unitary and isolated S-orbit. The continuity of the family $\mathcal{F}_\lambda$ forces that $\mathcal{F}_\lambda[\eta] = [\phi]$ for all λ in the range of solutions for $[\eta]$. If $[\eta]$ is not in $\mathbb{P}^{0,1}_1$ we get that $|\phi|_0^2$ is the curvature of a conformal metric. The next three theorems explore this idea for some symmetric classes $[\phi]$. In the following we fix some arbitrary point $x_0 \in S^2$ as reference, and denote by l the axis passing through $\{x_0, -x_0\}$.

Theorem 4.8. *For positive integers a, b with $a + b = k - 2$ let $\mathcal{D} = a\,x_0 + b\,(-x_0)$. Then the class $[\phi] \in \mathbb{P}^{1,0}$ with divisor $\mathcal{D}$ admits solution for $\lambda = 4\pi$. $|\phi|_0^2$ is the curvature of a metric conformal to g_0.*

Proof. Let S be the group of rotations about l. The classes in $\mathbb{P}^{1,0}$ with divisors given by $a\,x_0 + b\,(-x_0)$ for $a, b \geq 0$ are the only classes fixed by the S-action. There are finite many of those, hence each one of them is isolated. Finally, in case a and b are strictly positive the corresponding $[\eta] = [\phi]^{*H_0}$ is not in $\mathbb{P}^{0,1}_1$, hence $[\phi]$ admits solution in (23) for $\lambda = 4\pi$. □

Theorem 4.9. *Let $\mathcal{D} = a\,x_0 + a\,(-x_0) + \mathcal{E}$, $a > 0$, where $\mathcal{E}$ is a divisor constructed as follows: choose an integer $n > 2a$ and let $\mathcal{E} = \sum_{j=1}^n x_j$. The points $\{x_j\}_{1 \leq j \leq n}$ lie in the equator respect to l and are evenly separated. The class $[\phi]$ with divisor $\mathcal{D}$ admits solution at $\lambda = 4\pi$, and its H_0-norm squared is the curvature of a conformal metric.*

Proof. Let φ_1 be a rotation of $2\pi/n$ about l. Clearly $[\phi]$ is fixed by the φ_1^* action, but is not isolated. To accomplish that feature we consider φ_2 the reflection respect to a plane β containing l and x_1, and φ_3 the reflection respect to the equatorial plane. Let S be the isometry subgroup generated by $\{\varphi_1, \varphi_2, \varphi_3\}$. If $[\tilde{\phi}]$ is a class fixed by S then any zero of $[\tilde{\phi}]$ that is not x_0 or $-x_0$ repeats itself n times along a parallel, hence the number os those zeros is a multiple of n. Since there are $k-2 = n+2a < 2n$ zeros we conclude that either all zeros are in $\{x_0, -x_0\}$ or else there are n zeros in a parallel and $2a$ zeros in $\{x_0, -x_0\}$. In the second case the φ_3 invariance forces exactly a zeros in each of $x_0, -x_0$ and the parallel to be the equator. Still, there are two ways to inscribe the regular n-edge polygon inside the equatorial circle symmetrically respect to φ_2. In either case we conclude that the set of fixed points of S is discrete and its $\mathcal{F}_0^{-1}$ image is disjoint of $\mathbb{P}_1^{0,1}$. From this the assertion of the lemma follows. □

We now show a less trivial example of existence where the fixed points of the symmetry may not be fixed in the dynamics $\lambda \mapsto \mathcal{F}_\lambda$. Let $S \subset Iso(S^2)$ be a subgroup. Let $Y^{1,0} \subset \mathbb{P}^{1,0}$ be a closed (compact, without boundary) differentiable submanifold invariant for the S-action. Set $Y^{0,1} = (Y^{1,0})^{*H_0}$ its dual. If $\mathcal{F}_\lambda(Y^{0,1}) \subset Y^{1,0}$ for all λ that makes sense and $Y^{0,1} \cap \mathbb{P}_1^{0,1} = \emptyset$ then the same argument of Theorem 4.7 applies since $\mathcal{F}_\lambda : Y^{0,1} \to Y^{1,0}$ has degree 1, and for all $[\phi] \in Y^{1,0}$ there is a solution when $\lambda = 4\pi$.

Theorem 4.10. *Let $\mathcal{D} = a\,x_0 + b\,(-x_0) + \mathcal{E}$ be a divisor with: $n > a > 0$, $n > b > 0$ and $\mathcal{E}$ is a divisor composed by mn zeros (counting multiplicity) evenly distributed in m parallels (the parallels may not be pairwise distinct). Multiple zeros are allowed and the points x_0 and $-x_0$ might contain degenerated parallels. Then the class $[\phi]$ with this divisor has a solution in $\lambda = 4\pi$.*

Proof. Let $Y^{1,0} \subset \mathbb{P}^{1,0}$ be the set of all classes whose divisors are described by the lemma. We first show it is a differentiable submanifold of $\mathbb{P}^{1,0}$, by exhibiting an embedding $i : \mathbb{CP}^m \to \mathbb{P}^{1,0}$ with image $Y^{1,0}$. Let ζ be the canonical section of L with k zeros in $-x_0$, and let z be a stereographic coordinate chart with $z(x_0) = 0$. A representative of $[\phi] \in Y^{1,0}$ is of the form $\phi = z^a g(z)\zeta dz$ for some polynomial $g(z)$ of degree smaller than or equal to $k-2-a-b = mn$. Since the divisor of $g(z)\zeta\, dz$ is $\mathcal{E} + (a+b)(-x_0)$ a closer look at this structure reveals that $g(z) = h(z^n)$ where $h(v)$ is a polynomial in v of degree no greater than m. The vector space of such polynomials is identified with $\mathbb{C}^{m+1}$, and there is an injective homomorphism $h \mapsto z^a h(z^n)\zeta\, dz$, which passes to the projectivizations $\mathbb{CP}^m \to \mathbb{P}^{1,0}$, giving us the above mentioned embedding, and $Y^{1,0} \simeq \mathbb{CP}^m$.

Let $Y^{0,1} = (Y^{1,0})^{*H_0}$. Clearly $Y^{0,1} \cap \mathbb{P}_1^{0,1} = \emptyset$ because $a, b \neq 0$. We claim that $\mathcal{F}_\lambda(Y^{0,1}) \subset Y^{1,0}$ for all λ in an open range containing $(0, 4\pi]$. To see that we'd rather index the space $Y^{1,0} \equiv Y^{1,0}_{a,b}$ and consider all such submanifolds $Y^{1,0}_{a',b'}$ for a' an integer, $0 \le a' < n$ and $b' = (k-2-a') \bmod n$. Notice that in the space $Y^{1,0}_{a',b'}$ the number m' of "parallels" may differ from m, and the cases $a' = 0$ or $b' = 0$ occur, but that does not matter for our argument.

Setting S as the subgroup generated by the rotation of $2\pi/n$ about l it becomes clear that $[\phi] \in \mathbb{P}^{1,0}$ is S-invariant if and only if $[\phi] \in \cup_{0 \le a' < n} Y^{1,0}_{a',b'}$. Hence the image $\mathcal{F}_\lambda(Y^{0,1})$ is contained in the union of the components $Y^{1,0}_{a',b'}$, each of them being a copy of some complex projective space and pairwise disjoint. The continuity of $\lambda \mapsto \mathcal{F}_\lambda$ then precludes the image $\mathcal{F}_\lambda(Y^{0,1})$ from leaving the original copy $Y^{1,0}_{a,b}$. This concludes the lemma. □

Remark 4.11. It is interesting to look at the actual functions $K = 2|\phi|_0^2$ in the previous lemmas. Let $z = z_{-x_0}$ a stereographic coordinate that vanishes at x_0, and ζ a holomorphic section with divisor $k(-x_0)$. A general holomorphic $\phi = g\,\zeta_L dz$ has norm

$$|\phi|^2_{H_0} = 2\pi \frac{|a_0 + a_1 z + \cdots + a_{k-2} z^{k-2}|^2}{(1+|z|^2)^{k-2}} \tag{81}$$

for complex constants a_j, $0 \le j \le k-2$ (the factor 2π is due to the normalization $|S^2| = 1$ on the tangent bundle). Hence, the following functions are curvatures in the conformal structure of g_0:

1) $K(z) = |z|^{2a}(1+|z|^2)^{2-k}$ if $0 < a < k-2$ (Theorem 4.8);
2) $K(z) = |z|^{2a}|z^n - 1|^2(1+|z|^2)^{2-k}$ if $n > 2a > 0$ and $2a + n = k-2$ (Theorem 4.9);
3) $K(z) = |z|^{2a}|z^n - q_1|^2|z^n - q_2|^2 \cdots |z^n - q_m|^2(1+|z|^2)^{2-k}$ for arbitrary complex numbers $q_1, \ldots, q_m$, if $n > a > 0$, $a + mn < k-2$ and $(k - 2 - a) \bmod n > 0$ (Theorem 4.10);

Notice in the above that, except in very few cases, the functions K are not symmetric about the origin, and the existence results of [20] do not apply directly. Since all such functions have zeros one cannot use the results on [8].

Remark 4.12. At this point of the research it seems to us that the holomorphicity of ϕ is not the key to obtaining the existence results on (1), but only the behaviour of $|\phi|_0^2$ around its zeros. In a future work we intend to show existence results for (1) for a larger class of *smooth* functions $K \ge 0$ with finite many zeros with even degrees, and spread over S^2 in a suitable way.

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