Communications in
Analysis and Geometry
Volume 24, Number 3, 521–557, 2016

On complete constant scalar curvature Kähler metrics with Poincaré-Mok-Yau asymptotic property

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Let X be a compact Kähler manifold and S a subvariety of X with higher co-dimension. The aim is to study complete constant scalar curvature Kähler metrics on non-compact Kähler manifold $X - S$ with Poincaré–Mok–Yau asymptotic property (see Definition 1.1). In this paper, the methods of Calabi ansatz and the moment construction are used to provide some special examples of such metrics.

1. Introduction

In Kähler geometry, a basic question is to find on a Kähler manifold a canonical metric in each Kähler class, such as a Kähler–Einstein (K–E) metric, a constant scalar curvature Kähler (cscK) metric, or even an extremal metric. If X is a compact Kähler manifold with the definite first Chern class, the question has been solved thoroughly and there are lots of references on this topic. Among these, the fundamental one [18] is on the Calabi conjecture solved by Yau.

In the non–compact case, Tian and Yau proved in [16, 17] that there exists a complete Ricci–flat metric on $X^* = X - D$, where X is a compact Kähler manifold and D is a neat and almost ample smooth divisor on X; or X is a compact Kähler orbifold and D is a neat, almost ample and admissible divisor on X.

Several years ago, the second named author presented the following question:

Problem 1. Assume that X is a compact Kähler manifold and S is its higher co-dimensional subvariety. Let $X^* = X - S$. How to find a complete canonical metric on such a non-compact Kähler manifold X^*?

Certainly, this problem is equivalent to finding a canonical metric on $\bar{X} - D$, where $\bar{X}$ is a compact Kähler manifold and D is a divisor on $\bar{X}$.

More precisely, blowing up of X along S, one obtains a new compact Kähler manifold $\bar{X} = Bl_S(X)$. Then X^* is bi-holomorphic to $\bar{X} - D$ where D is the exceptional divisor of this blow-up. Hence our problem is transferred to finding a complete canonical metric on $\bar{X} - D$. However, this blowing up process can not make Problem 1 easier since it does not alter the geometric properties of X^*. For example, although $\mathbb{C}P^2 - p$ is bi–holomorphic to $Bl_p(\mathbb{C}P^2) - D$, we can not use Tian–Yau's results mentioned above to get a complete K–E metrics on $\mathbb{C}P^2 - p$ since the exceptional divisor D is not ample.

The basic strategy to solve Problem 1 is to perturb one family of approximate metrics on X^*. This method has been carried out successfully in [2–4, 14, 15] to construct cscK or extremal metrics on blow-up of a Kähler manifold at some points. The key point is that the csck metric of Burns-Simanca [13] on $Bl_0(\mathbb{C}^n)$ is "Asymptotically Locally Euclidean" (ALE) at infinity.

Motivated by this, if one want to solve Problem 1 on $M - \{p_1, \ldots, p_l\}$, one should first construct a canonical metric on $\mathbb{C}^n - 0$ ($n \geq 2$) which is asymptotically Euclidean (AE) at infinity. Fortunately, the metrics in the following theorem admit this asymptotic property. Let r^2 be the Euclidean norm squared function on $\mathbb{C}^n$ with the dimension $n \geq 2$.

Theorem 1.1. *There exist on $\mathbb{C}^n - 0$ a family of complete zero scalar curvature Kähler metrics $\eta_a = \sqrt{-1}\partial\bar{\partial}u_a(r^2)$ $(a > 0)$ with the following asymptotic properties: As $r^2 \to 0$,*

$$u_a(r^2) = a \log r^2 - \frac{2a}{n(n-1)} \log(-\log r^2) + O((\log r^2)^{-1});$$

And as $r^2 \to \infty$,

$$u_a(r^2) = \begin{cases} r^2 + 2a \log r^2 + \frac{a^2}{2r^2} + O(\frac{1}{r^4}), & \text{for} \quad n = 2, \\ r^2 - \frac{na^{n-1}}{(n-1)(n-2)}(r^2)^{2-n} + \frac{a^n}{n}(r^2)^{1-n} + O((r^2)^{-n}) & \text{for} \quad n \geq 3. \end{cases}$$

Here $O(h(r^2))$ is a smooth function whose $k-$th partial derivatives for all $k \geq 0$ are bounded by a constant times $|\partial^k h(r^2)|$.

For the cases of constant scalar curvature $c \neq 0$, we have the following theorem. Denote D^n as the unit disc of $\mathbb{C}^n$.

Theorem 1.2. *1. For any $c < 0$, there exist on $D^n - 0$ a family of complete Kähler metrics $\sqrt{-1}\partial\bar{\partial}u_a(r^2)$ $(a > 0)$ with constant scalar curvature c. As*

$r^2 \to 1$, *these metrics are asymptotic to the Poincaré metric*

$$\sqrt{-1}\frac{n(n+1)}{-c}\partial\bar{\partial}\log(1-r^2).$$

2. For $c>0$ *and* $a>0$ *with* $ac<n(n-1)$, *there exists on* $\mathbb{C}^n-0$ *a Kähler metric* $\sqrt{-1}\partial\bar{\partial}u_a(r^2)$ *with constant scalar curvature* c *which are not complete at infinity and asymptotic to*

$$\sqrt{-1}\partial\bar{\partial}(b\log r^2+\kappa r^{-\frac{2}{\kappa}})$$

for two constants $b(>a)$ *and* $\kappa(>1)$.

In both cases, the metrics have the following asymptotic property: As $r^2\to 0$,

$$u_a(r^2)=a\log r^2-\frac{2a}{n(n-1)-ac}\log(-\log r^2)+O((\log r^2)^{-1}).$$

Naturally one would ask whether there are any complete K–E metrics on $\mathbb{C}^n-0$ or on D^n-0. Using the method in [11], we can prove the following theorem which turns out that in some sense the choice of cscK metrics is optimal.

Theorem 1.3 ([11]). *There do not exist any complete Kähler-Einstein metrics on* $\mathbb{C}^n-0$ *or on* D^n-0.

In fact, we can prove X^* can not admit any complete Kähler-Einstein metrics in our further paper [8]. Theorems 1.1 and 1.2 remind us to recall the Mok–Yau metric in [11]. In 1980s, Mok and Yau introduced on D^n-0 the metric with bounded Ricci curvature

$$\sqrt{-1}\partial\bar{\partial}(\log r^2-\log(-\log r^2)).$$

They used this metric to characterize domains of holomorphy by holomorphic sectional conditions. Comparing the Mok–Yau metric with the metrics in Theorems 1.1 and 1.2 leads to the following definition.

Definition 1.1 (see also [7]). Let X be a compact Kähler manifold with a Kähler metric ω_X and let S be a higher co–dimensional subvariety. A Kähler metric ω on $X-S$ has the Poincaré–Mok–Yau (PMY) asymptotic

property if near the subvariety S

$$\omega = \omega_X + \sqrt{-1}\partial\bar{\partial}(a \log r^2 - b \log(-\log r^2) + O((\log r^2)^{-1})),$$

where r is some distance function to S, and a and b are two positive constants.

It can be directly checked that the sectional curvatures of the metric with PMY asymptotic property are bounded, and volume is finite near the subvariety N.

In the second part of this paper, we generalize Theorems 1.1 and 1.2 to the cases of holomorphic vector bundles. We will use the moment construction to find complete cscK metrics on the complement of the zero section in (the total space of) a holomorphic vector bundle or a projective bundle (i.e. a ruled manifold). There are many references such as [1, 9, 10, 12] which use the method of moment construction to look for canonical metrics on Kähler manifolds. One can consult [9] for construction of cscK metrics on vector bundles and [1] for extremal metrics on ruled manifolds.

Let M be a compact m–dimensional Kähler manifold with a cscK metric ω_M. Let (L, h) be a holomorphic line bundle over M with a hermitian metric h, which is given by local positive functions $h(z)$ defined on the open sets which locally trivialize L. For the technical reason, assume that there exists a constant λ such that

$$\sqrt{-1}\partial\bar{\partial} \log h(z) = \lambda\omega_M. \tag{1}$$

Let (E, π) be the direct sum of n (≥ 2) copies of L with associate hermitian metric h. Let ν be the logarithm of the fibre norm squared function defined by h and consider Calabi ansate

$$\omega = \pi^*\omega_M + \sqrt{-1}\partial\bar{\partial} f(\nu).$$

Denote the zero section of E simply by M. Also denote $\mathbb{U}$ as the set of points p in E such that $\nu(p) < 0$. We first concern about csck metrics with *PMY asymptotic property* on $E - M$ or on $\mathbb{U} - M$.

Theorem 1.4. *Let M be a compact Kähler manifold and ω_M a Kähler metric with constant scalar curvature c_M. Let L be a holomorphic line bundle over M with a hermitian h. Assume that h and ω_M satisfy (1). Let E be the direct sum of n (≥ 2) copies of L.*

1. If $\lambda \geq 0$, there exists a constant c_0 such that for any $c \leq c_0$, there exists on $\mathbb{U} - M$ or on $E - M$ a complete Kähler metric with constant scalar

curvature c. Such metrics admit the Poincaré–Mok–Yau asymptotic property except the case that the metrics are defined on $\mathbb{U} - M$ *with* $c = c_0(< 0)$ *and* $\lambda > 0$.

2. If $\lambda < 0$ *and* $c_M > 0$, *there exists on* $E - M$ *a complete positive constant scalar curvature Kähler metric with Poincaré–Mok–Yau asymptotic property.*

If $\lambda = 0$, this theorem generalizes Theorem 1.1 and the case $c < 0$ of Theorem 1.2.

We then consider Problem 1 on a projective bundle. Denote $\mathcal{O}$ as the structure sheaf of M. The projective bundle $\mathbb{P}(E \oplus \mathcal{O})$ over M has a globally defined section s: for $q \in M$, $s(q)$ is a point corresponding to the line $\mathcal{O}_q$. The following theorem gives some special solutions to Problem 1.

Theorem 1.5. *Under the assumptions of Theorem 1.4, if* $\lambda < 0$ *and* $c_M \in \mathbb{R}$ *or if* $\lambda > 0$ *and* $c_M \in (m(m + 2n - 1)\lambda, +\infty)$, *there exists on* $\mathbb{P}(E \oplus \mathcal{O}) - M$ *a complete constant scalar curvature Kähler metric with Poincaré–Mok–Yau asymptotic property.*

Acknowledgements. Fu is supported in part by NSFC grant 11421061. Yau is supported in part by NSF grant DMS-0804454. Zhou is supported by China Postdoctoral Science Foundation grant No.2015M571479 .

2. Complete cscK metrics on $\mathbb{C}^n - 0$ and $D^n - 0$.

In this section, we construct complete cscK metrics on $\mathbb{C}^{n*}$ or D^{n*} with the dimension $n \geq 2$. Here denote $\mathbb{C}^{n*} = \mathbb{C}^n - 0$ and $D^{n*} = D^n - 0$. We first follow Calabi's method [6] to get an ODE on the Kähler potential. Then we determine the constants of integration appeared in the ODE by discussing completeness of the metric near the punctured point. Afterwards we analyze the asymptotic properties of the Kähler potential. Thus, Theorems 1.1 and 1.2 are proven. In the last subsection, we give some remarks and a simple proof of Theorem 1.3.

2.1. Calabi ansatz.

Let $w = (w_1, w_2, \ldots, w_n)$ be the coordinates of $\mathbb{C}^n$. Assume that the Kähler metric we are seeking for is rotationally symmetric. That is, if we let

$$r^2 = \sum_{\alpha=1}^{n} |w_\alpha|^2 \quad \text{and} \quad t = \log r^2,$$

then the Kähler potential is a function $u(t)$. By a direct calculation,

$$g_{\alpha\bar{\beta}} := \frac{\partial^2 u(t)}{\partial w_\alpha \partial \bar{w}_\beta} = e^{-t}u'(t)\delta_{\alpha\beta} + e^{-2t}\bar{w}_\alpha w_\beta(u''(t) - u'(t)).$$

Hence,

$$\det(g_{\alpha\bar{\beta}}) = e^{-nt}u'(t)^{n-1}u''(t),$$

and $\omega = \sqrt{-1}\partial\bar{\partial}u(t)$ is a Kähler metric if and only if

$$u'(t) > 0 \quad \text{and} \quad u''(t) > 0.$$

For simplicity, let

$$v(t) = -\log\det(g_{\alpha\bar{\beta}}) = nt - (n-1)\log u'(t) - \log u''(t). \tag{2}$$

The components of the Ricci tensor of ω are

$$R_{\alpha\bar{\beta}} = \frac{\partial^2 v(t)}{\partial w_\alpha \partial \bar{w}_\beta} = e^{-t}v'(t)\delta_{\alpha\beta} + e^{-2t}\bar{w}_\alpha w_\beta(v''(t) - v'(t))$$

and then the scalar curvature is

$$c(t) = g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}} = (n-1)\frac{v'(t)}{u'(t)} + \frac{v''(t)}{u''(t)}. \tag{3}$$

Here $(g^{\alpha\bar{\beta}})$ denotes the inverse matrix of $(g_{\alpha\bar{\beta}})$. Explicitly,

$$g^{\alpha\bar{\beta}} = \frac{e^t}{u'(t)}\delta_{\alpha\beta} + w_\alpha\bar{w}_\beta\left(\frac{1}{u''(t)} - \frac{1}{u'(t)}\right).$$

Assume that the scalar curvature of ω is a constant c. Integrating (3) with the integrating factor $u'(t)^{n-1}v'(t)$, we obtain the first order differential relation between $u(t)$ and $v(t)$

$$v'(t)u'(t)^{n-1} = \frac{1}{n}c(u'(t))^n + c_1$$

with an arbitrary constant c_1. Substituting (2) to the above equation and multiplying both sides with $u''(t)$, we get the equation

$$\begin{aligned} &nu'(t)^{n-1}u''(t) - (n-1)u'(t)^{n-2}u''(t)^2 - u'(t)^{n-1}u'''(t) \\ &= \frac{1}{n}cu'(t)^n u''(t) + c_1 u''(t). \end{aligned}$$

Integrating the above equation, we obtain

$$u'(t)^n - u'(t)^{n-1}u''(t) = \frac{c}{n(n+1)}u'(t)^{n+1} + c_1u'(t) + c_2$$

with another arbitrary constant c_2. If we denote $\phi(t) = u'(t)$, then the above equation can be written as the first order differential equation

$$\frac{d\phi}{dt} = \frac{F(\phi)}{\phi^{n-1}} \tag{4}$$

with

$$F(\phi) = -\frac{c}{n(n+1)}\phi^{n+1} + \phi^n - c_1\phi - c_2$$

or rewritten as

$$dt = \frac{\phi^{n-1}d\phi}{F(\phi)}. \tag{5}$$

It follows that $u(t)$ is a Kähler potential if and only if

$$\phi(t) > 0 \qquad \text{and} \qquad F(\phi) > 0.$$

2.2. Completeness.

Assume that $\phi = \phi(t)$ is a solution to ODE (4) or (5) in an interval $(-\infty, t_0)$, where t_0 can be equal to $+\infty$, and assume that it determines a Kähler potential $u = u(t)$ in the punctured disc $D^{n*}(r_0)$ with radius $r_0 = \exp(\frac{t_0}{2})$. In this subsection, for the sake of the completeness of $\omega = i\partial\bar{\partial}u(t)$ near the punctured point, the constants c_1 and c_2 in ODE (4) can be determined. The key point is the following observation.

Lemma 2.1. *Under the above assumption, the metric ω determined by the Kähler potential $u(t)$ is complete near the punctured point if and only if $F(\phi)$ has a factor $(\phi - a)^2$ with $a > 0$ and $\lim_{t\to-\infty}\phi = a$. Hence*

$$c_1 = na^{n-1} - \frac{c}{n}a^n \qquad \text{and} \qquad c_2 = (1-n)a^n + \frac{c}{n+1}a^{n+1}. \tag{6}$$

Proof. Since the metric $\omega = \sqrt{-1}\partial\bar{\partial}u(t)$ is rotationally symmetric, for any point $p \in D^{n*}(r_0)$, the ray $\gamma(s) = sp$, $s \in (0, 1]$, is a geodesic. The tangent

vector of this curve at the point sp is p and its square norm under the metric ω is, if we assume that $r^2(p) = 1$,

$$|p|^2_{sp} = \sum w_\alpha(p)\bar{w}_\beta(p)g_{\alpha\bar{\beta}}(sp) = u''(t)r^{-2}.$$

Then the length of $\gamma(s)$ is

$$l = \int_0^1 |\gamma'(s)|_{sp}ds = \int_0^1 \sqrt{u''(t)}\frac{dr}{r} = \frac{1}{2}\int_{-\infty}^0 \sqrt{u''(t)}dt = \frac{1}{2}\int_{-\infty}^0 \sqrt{\frac{d\phi}{dt}}dt.$$

Under the assumption that $d\phi/dt > 0$ and $\phi(t) > 0$, there is a nonnegative constant a such that

$$\lim_{t\to-\infty} \phi = a.$$

By Equation (5), we have

$$l = \frac{1}{2}\int_a^{\phi(0)} \sqrt{\frac{\phi^{n-1}}{F(\phi)}}d\phi.$$

The completeness requires $l = +\infty$, which is equivalent to the fact that $F(\phi)$ has a factor $(\phi - a)^2$. Hence, we can determine c_1 and c_2 as in (6).

We claim $a > 0$. If $a = 0$, $F(\phi) = \phi^n(-\frac{c\phi}{n(n+1)} + 1)$. Hence, $l < +\infty$, which leads to a contradiction. □

2.3. Discussions of the solutions and proofs of Theorems 1.1 and 1.2.

Because of completeness, in this subsection assume that the constants c_1 and c_2 have been chosen as in (6). Hence, $F(a) = F'(a) = 0$. Since

$$F''(\phi) = (n(n-1) - c\phi)\phi^{n-2},$$

in case $c \le 0$, $F''(\phi) > 0$ and so $F(\phi) > 0$ on domain $(a, +\infty)$. In case $c > 0$, if assume that the constants a and c satisfy

$$ac < n(n-1),$$

then on domain $(a, \frac{n(n-1)}{c})$, $F''(\phi) > 0$ and so $F(\phi) > 0$. Hence, $F(\phi) > 0$ on domain (a, b) for some constant b or $b = +\infty$. Thus, we obtain the solution

of Equation (5), up to a constant:

$$t = t(\phi) = \int_{\phi_0}^{\phi} \frac{x^{n-1}}{F(x)} dx, \qquad \phi \in (a, b), \tag{7}$$

for a given $\phi_0 \in (a, b)$. Since $F(\phi)$ has the factor $(\phi - a)^2$, $\lim_{\phi \to a^+} t(\phi) = -\infty$. In the following we will discuss more details of the solutions (7) for different signs of c and finish the proofs of Theorems 1.1 and 1.2.

1. Case $c = 0$**.** In this case,

$$F(\phi) = \phi^n - na^{n-1}\phi + (n-1)a^n.$$

Since the only root of $F'(\phi) = 0$ is a, $F(\phi)$ obtains its minimum at the point a and $F(\phi) > 0$ for all $\phi > a$. As $\phi \to \infty$, $\frac{\phi^{n-1}}{F(\phi)} \to \frac{1}{\phi}$ and solution (7) has the property that $t \simeq \log \phi$. Hence $\phi = \phi(t)$ is defined on the entire punctured space $\mathbb{C}^{n*}$. Therefore there exist a family (depending on $a > 0$) of zero cscK metrics with Kähler potential $u(t)$ such that $u'(t) = \phi(t)$.

Since $\phi \to a$ as $t \to -\infty$,

$$dt = \frac{\phi^{n-1} d\phi}{F(\phi)} \sim \frac{2a}{n(n-1)} \frac{1}{(\phi - a)^2} d\phi.$$

It turns out that from $\phi(t) = u'(t) = r^2 u'(r^2)$,

$$u(r^2) \sim a \log r^2 - \frac{2a}{n(n-1)} \log(-\log r^2).$$

Moreover, by L'Hôpital's rule we get the more accurate expression of $u(r^2)$:

$$u(r^2) = a \log r^2 - \frac{2a}{n(n-1)} \log(-\log r^2) + O((\log r^2)^{-1}).$$

On the other hand, we divide the case into $n = 2$ and $n \geq 3$ to discuss the approximation of the solution as $r^2 \to \infty$. For $n = 2$,

$$dt = \frac{\phi d\phi}{(\phi - a)^2} = \left(\frac{1}{\phi - a} + \frac{a}{(\phi - a)^2} \right) d\phi$$

and it follows that

$$t = \log(\phi - a) - \frac{a}{\phi - a} \tag{8}$$

and $u(r^2) \simeq a \log r^2 + r^2$. Obviously, the derived metric is complete at entire $\mathbb{C}^{n*}$. Moreover, L'Hôpital's rule can be used to get more accurate estimate

$$u(r^2) = r^2 + 2a \log r^2 + \frac{a^2}{2r^2} + O\left(\frac{1}{r^4}\right).$$

For $n \geq 3$, as $r^2 \to \infty$

$$dt = \frac{1}{\phi}\big(1 + na^{n-1}\phi^{1-n} - (n-1)a^n\phi^{-n} + O(\phi^{-n-1})\big)$$

and then

$$t = \log \phi + \frac{n}{1-n}a^{n-1}\phi^{1-n} + \frac{n-1}{n}a^n\phi^{-n} + O(\phi^{-n-1})$$

which implies

$$\begin{aligned}\phi - e^t &= \phi\left(1 - \exp\left(\frac{n}{1-n}a^{n-1}\phi^{1-n} + \frac{n-1}{n}a^n\phi^{-n} + O(\phi^{-n-1})\right)\right)\\ &= \frac{n}{n-1}a^{n-1}\phi^{2-n} - \frac{n-1}{n}a^n\phi^{1-n}\\ &\quad - \frac{n^2}{(1-n)^2}a^{2n-2}\phi^{2-2n} + O(\phi^{-n-1}).\end{aligned}$$

Replacing ϕ by e^t in the right hand side of the above equality, we have

$$\phi = e^t + \frac{n}{n-1}a^{n-1}(e^t)^{2-n} - \frac{n-1}{n}a^n(e^t)^{1-n} + O((e^t)^{-n-1})$$

or

$$u(r^2) = r^2 - \frac{na^{n-1}}{(n-1)(n-2)}(r^2)^{1-2n} + \frac{a^n}{n}(r^2)^{1-n} + O((r^2)^{-n}).$$

Thus we have finished the proof of theorem 1.1.

We give the picture of $\phi = \phi(t)$ with $n = 2$ and $a = 1$ as Figure 1. Recall that in this situation the function $\phi = \phi(t)$ is defined in Equation (8). Note that we also have

$$\det(g) = e^{-2t}\phi(t)\phi'(t) = \exp\left(-\frac{2}{\phi - 1}\right).$$

We give a rotational picture of the function $\exp(-\frac{2}{\phi-1})$ as Figure 2 which shows the AE and PMY properties of the metric.

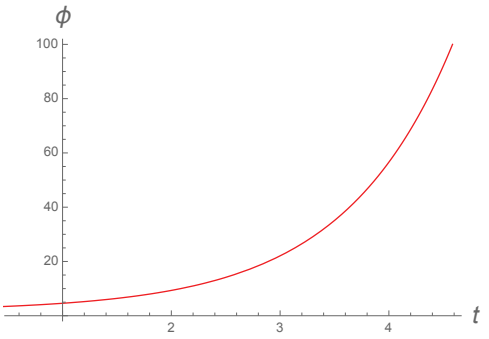


Figure 1: The graph of $\phi(t)$ with $c = 0$ and $a = 1$.

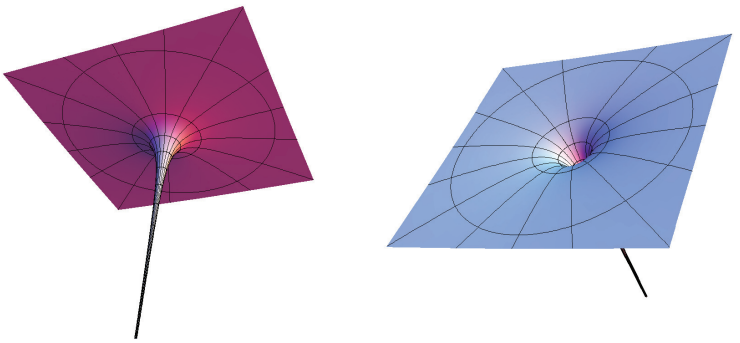


Figure 2: PMY and AE.

2. Case $c < 0$**.** In this case, we have seen that $F(\phi) > 0$ when $\phi \in (a, +\infty)$. From (7), we also see that when $\phi \to +\infty$, the upper bound of $t = t(\phi)$ exists since the degree of F is $n + 1$. For simplicity, we take this upper bound to be zero since the solution (7) is unique up to be a constant. Then $u(r^2)$ is defined on the punctured unit disc D^{n*}.

The analysis of the boundary behavior is as follows. As $\phi \to \infty$,

$$t = \frac{n(n+1)}{-c}\frac{1}{\phi} + O\left(\frac{1}{\phi^2}\right)$$

which implies

$$u(r^2) = \frac{n(n+1)}{-c}\log(-\log r^2) + O(\log r^2)$$

or

$$u(r^2) = \frac{n(n+1)}{-c}\log(1-r^2) + O(\log r^2),$$

where the right hand side is the Kähler potential of the standard Poincaré metric on D^n. Hence, the metric we constructed is also complete near the boundary of D^n.

For the asymptotic behavior of $\phi = \phi(t)$ at the origin, it is the same as for the case $c = 0$: As $r^2 \to 0$,

$$u(r^2) = a\log r^2 - \frac{2a}{n(n-1)-ac}\log(-\log r^2) + O((\log r^2)^{-1}).$$

Then we have finished the proof of Theorem 1.2 for the case $c < 0$.

We give the picture of $\phi = \phi(t)$ as $n = 2$, $a = 1$ and $c = -6$ as Figure 3. Note in this situation,

$$\frac{d\phi}{dt} = \frac{\phi}{(\phi-1)^2(\phi+3)}.$$

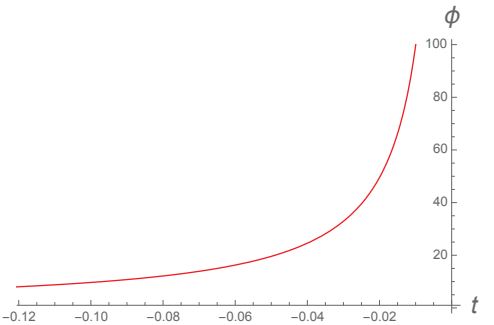


Figure 3: The graph of $\phi(t)$ with $c = -6$.

3. Case $c > 0$**.** We have seen that if the constants c and a satisfy the relation $ac < n(n-1)$, then on domain $(a, \frac{n(n-1)}{c})$, $F(\phi) > 0$. Obviously, when ϕ is big enough, $F(\phi) < 0$. Hence we can let b be the first number in $(a, +\infty)$ such that $F(b) = 0$. It follows that there is a polynomial $G(\phi)$ such

that we can write

$$F(\phi) = \frac{c}{n(n+1)}(\phi - a)^2(b - \phi)G(\phi).$$

We first claim $G(b) > 0$. If $G(b) = 0$, then $F(b) = F'(b) = 0$. Together with $F(a) = F'(a) = 0$, there are at least two different positive roots for the equation $F''(\phi) = 0$. However, equation

$$F''(\phi) = -c\phi^{n-1} + n(n-1)\phi^{n-2} = 0$$

has only one positive root $\phi = \frac{n(n-1)}{c}$, which leads to a contradiction. Hence, as $\phi \to b$,

$$t \sim -\kappa \log(b - \phi), \tag{9}$$

where

$$\kappa = -\frac{b^{n-1}}{F'(b)} > 0.$$

This implies that when $\phi \in (a, b)$, $t \in (-\infty, +\infty)$ and then the Kähler potential $u(r^2)$ is defined on entire $\mathbb{C}^{n*}$.

The approximation of $u(r^2)$ near the zero is the same as for the case $c = 0$:

$$u(r^2) = a \log r^2 - \frac{2a}{n(n-1) - ac} \log(-\log r^2) + O((\log r^2)^{-1}).$$

Whereas when $t \to \infty$, from (9) we can derive

$$u(r^2) = b \log r^2 + \kappa r^{-\frac{2}{\kappa}} + O(r^{-2}).$$

The metric is not complete as $r^2 \to \infty$. In fact as in the proof of Lemma 2.1, the length of the geodesic ray $\gamma(s) = sp$ on domain $(1, +\infty)$ is

$$l = \int_{\phi(\log(r^2(p)))}^{b} \sqrt{\frac{\phi^{n-1}}{F(\phi)}} d\phi \simeq \int_{\phi(\log(r^2(p)))}^{b} \frac{1}{\kappa\sqrt{\phi - b}} d\phi < \infty.$$

Thus we have finished the proof of Theorem 1.2 for the case $c > 0$.

We give the picture of function $\phi = \phi(t)$ in case $n = 2$, $a = 1$ and $c = 1$ as Figure 4. In this situation,

$$\frac{d\phi}{dt} = \frac{6\phi}{(\phi - 1)^2(4 - \phi)}.$$

2.4. Further remarks

1. For $n = 1$, $\partial\bar{\partial}\log r^2 = 0$ and

$$\sqrt{-1}\partial\bar{\partial}(-\log(-\log r^2))$$

is the standard Poincaré metric on D^* with Gauss curvature -1. One can also construct on $\mathbb{C}^*$ a complete metric with zero Gauss curvature

$$\sqrt{-1}\partial\bar{\partial}(\log r^2)^2 = \sqrt{-1}\frac{dz \wedge d\bar{z}}{r^2}.$$

2. It is mentioned in Introduction that the Mok–Yau metric defined on D^{2*} has good properties. One can see that $\sqrt{-1}\partial\bar{\partial}(-\log(-\log r^2))$ is also a Kähler metric on D^{2*}. However, its scalar curvature is infinity as $r^2 \to 0$. In fact, the term $\log r^2$ in the Mok–Yau metric results in the boundedness of the scalar curvature near the punctured point. Hence, the asymptotic property appeared in Definition 1.1 is named as the PMY asymptotic property.

3. Write $\mathbb{C}P^n = \mathbb{C}^n \cup \mathbb{C}P^{n-1}$ and viewed zero as a point p in $\mathbb{P}^n$. One will ask whether the metric on $\mathbb{C}^{n*}$ constructed above with $c > 0$ can be extended across $\mathbb{C}P^{n-1}$. This is impossible. In fact, it can be seen form Lemma 4.1 in Section 4 below that the metric $\sqrt{-1}\partial\bar{\partial}u(t)$ can be extended across $\mathbb{C}P^{n-1}$ if and only if

$$\kappa = -\frac{b^{n-1}}{F'(b)} = 1. \tag{10}$$

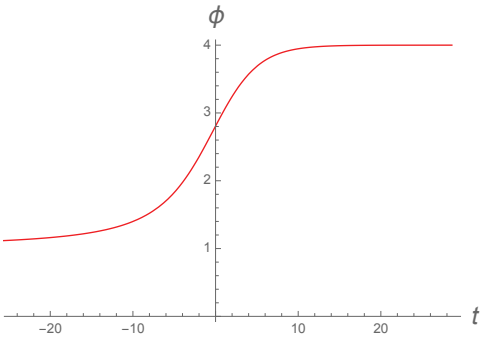


Figure 4: The graph of $\phi(t)$ with $c = 1$.

Replacing ϕ by $a\phi$ and c by $a^{-1}c$, we can assume that $a = 1$ in $F(\phi)$. From $\kappa = -\frac{b^{n-1}}{F'(b)}$ and $F(b) = 0$, we get the relation

$$b = \frac{n^2 - 1 - c + (1 - \frac{1}{\kappa})b^n}{n^2 - c}.$$

Then $b > a = 1$ implies $\kappa \neq 1$, which is a contradiction to (10). So in this way we can not get a complete cscK metric on $\mathbb{C}P^n - p$. In fact, regarding $\mathbb{C}P^{n-1}$ as the divisor at infinity one can show the metrics are of conical type on $\mathbb{C}P^n - 0$ along the divisor $\mathbb{C}P^{n-1}$.

In our another paper [8], it has been proved that there also do not exist any complete cscK metrics on $\mathbb{C}P^n - p$ with PMY asymptotic property. Nevertheless, a family of complete extremal metrics on $\mathbb{C}P^n - p$ have been constructed in [8].

4. At last, we give a simple proof of Theorem 1.3 which states that there are not any complete K–E metrics on $\mathbb{C}^{n*}$ or on D^{n*}.

Proof of Theorem 1.3. Since $\mathbb{C}^{n*}$ and D^{n*} are not compact, by Myers' theorem, we only need to consider the cases $c \leq 0$. In [11], Mok and Yau proved that if a bounded domain Ω admits a complete hermitian metric such that $-C \leq$ Ricci curvature ≤ 0, then Ω is a domain of holomorphy. Since D^{n*} is not a holomorphic domain, it does not admit a complete E–K metric with $c \leq 0$.

For the nonexistence of negative K–E metrics on $\mathbb{C}^{n*}$, we use generalized Yau's Schwarz Lemma.

Lemma 2.2. *[11, 19] Let (M, ω_g) be a complete Kähler manifold with scalar curvature bounded below by $-K_1$ and let (N, ω_h) be a hermitian manifold of the same dimension with Hermitian Ricci curvature $Ric \leq -K_2\omega_h$ for some $K_2 > 0$. If $f : M \to N$ is a holomorphic map and the Jacobian is nonvanishing at one point, then $K_1 > 0$ and*

$$f^*\omega_h^n \leq \left(\frac{K_1}{nK_2}\right)^n \omega_g^n.$$

Now take $M = N = \mathbb{C}^{n*}$ and take the metric ω_g on M as in Theorem 1.1. Then the above lemma leads to the nonexistence of negative K–E metric on $N = \mathbb{C}^{n*}$.

As for the Ricci–flat case, if we let $\omega = \sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j$ be a complete Ricci–flat metric on $\mathbb{C}^{n*}$, then the function $\log\det(g_{i\bar{j}})$ is pluriharmonic on it. Since the de Rham cohomology group $H^1_{dR}(\mathbb{C}^{n*}, \mathbb{R})$ vanishes, $\log\det(g_{i\bar{j}})$

is the real part of a holomorphic function. By Hartogs' extension theorem for holomorphic functions, $\log\det(g_{i\bar{j}})$ is pluriharmonic on the entire space $\mathbb{C}^n$. Hence, $(g_{i\bar{j}}) > C(\delta_{i\bar{j}})$ for some positive constant C_1 near the origin. Choose $z_2 = \cdots = z_n = 0$. It follows that $g_{1\bar{1}}(z_1, 0, \ldots, 0)$ is bounded near the origin. Let $z_1 = x_1 + \sqrt{-1}y_1$ and l be a ray from $(x_1, 0, \ldots, 0)$ to origin. The length of l is

$$\int_0^{x_1} \sqrt{g_{1\bar{1}}}dx < \infty$$

which contradicts the completeness of the metric near the origin of $\mathbb{C}^n$. Then we finish the proof. □

3. A momentum construction of complete cscK metrics

This section is devoted to prove Theorem 1.4. The first subsection almost follows the paper [9]. That is we first use the Calabi ansatz to derive an ODE and then use the moment profile to simplify it. In the second subsection completeness of metrics near zero section and at infinity are used to get constraint conditions. In the third subsection, we then consider the existence of metrics and their asymptotic property.

3.1. The momentum construction

Let M be a compact Kähler manifold with a Kähler metric ω_M. Let $\pi : L \to M$ be a holomorphic line bundle with a hermitian metric h. For any point $q \in M$, there is a holomorphic coordinate system $(U, z = (z_1, \ldots, z_m))$ of q with $z(q) = 0$ such that $L|_U$ is holomorphically trivial. Under this trivialization, the hermitian metric h can be given by a positive function $h(z)$. Assume that ω_M and h satisfy the condition

$$\sqrt{-1}\partial\bar{\partial}\log h = \lambda\omega_M, \qquad \text{for some constant } \lambda. \tag{11}$$

Let E be the direct sum of n (≥ 2) copies of L, i.e. $E = L^{\oplus n}$, with an associated hermitian metric still denoted by h. We also denote $\pi : E \to M$. We have a local trivialization of E induced from one of L and denote the fiber coordinates by $w = (w_1, \ldots, w_n)$. In the following, we denote α and β as the lower index of the components of w and i and j as the lower index of the components of z. There is a fibrewise norm squared function r^2 on the

total space of E defined by h

$$r^2 = h(z)\sum_{\alpha=1}^{n}|w_\alpha|^2.$$

If M is viewed as the zero section of E, i.e. the set defined by $r^2 = 0$, we want to construct complete cscK metrics on $E - M$ under condition (11). In this section denote $E - M$ simply by E^*.

Let

$$\nu = \log r^2 = \log h(z) + t, \qquad \text{with} \qquad t = \log\big(\sum_{\alpha=1}^{n}|w_\alpha|^2\big). \tag{12}$$

Consider Calabi ansatz

$$\omega = \pi^*\omega_M + \sqrt{-1}\partial\bar\partial f(\nu).$$

Using condition (11), we have

$$\begin{aligned}\omega =&(1+\lambda f'(v))\pi^*\omega_M + f''(\nu)\sqrt{-1}\partial\log h\wedge\bar\partial\log h\\ &+ f'(v)\cdot\sqrt{-1}(\partial\log h(z)\wedge\bar\partial t + \partial t\wedge\bar\partial\log h(z))\\ &+ f'(v)\cdot\sqrt{-1}\partial\bar\partial t + f''(v)\cdot\sqrt{-1}\partial t\wedge\bar\partial t\end{aligned}$$

and

$$\begin{aligned}\omega^{m+n} = (1+\lambda f'(v))^m\pi^*\omega_M^m\wedge(f'(v)\cdot\sqrt{-1}\partial\bar\partial t\\ + f''(v)\cdot\sqrt{-1}\partial t\wedge\bar\partial t)^n.\end{aligned} \tag{13}$$

The reason one can derived the above equality is $\det(\frac{\partial^2 t}{\partial w_\alpha\partial\bar w_\beta}) = 0$. In practise, when computing at any point $p\in\pi^{-1}(q)$, one can let $w_\alpha(p) = 0$ for $2\le\alpha\le\beta$. Then ω is (strictly) positive if and only if

$$f'(\nu) > 0, \qquad f''(\nu) > 0, \qquad \text{and} \qquad 1+\lambda f'(\nu) > 0. \tag{14}$$

Definition 3.1. The above constructed metric ω is call a bundle adapted metric.

Next we compute the Ricci curvature and the scalar curvature of the bundle adapted metric ω. From (13), we have

$$\det(\omega) = \det(\omega_M) \cdot (1 + \lambda f'(\nu))^m e^{-nt} f'(\nu)^{n-1} f''(\nu). \tag{15}$$

Let

$$\Psi(\nu) = \log\big((1 + \lambda f'(\nu))^m f'(\nu)^{n-1} f''(\nu)\big).$$

For any $q \in M$, we can assume that the local coordinates $z = (z_1, \ldots, z_n)$ around q also satisfy $\partial h|_q = \bar{\partial} h|_q = 0$. Then under assumption (11), the Ricci form of ω at a point $p \in \pi^{-1}(q)$ is

$$\begin{aligned} Ric(\omega)|_p &= -\sqrt{-1}\partial\bar{\partial}(\log\det(g_M) + \Psi(\nu) - nt)|_p \\ &= Ric(\omega_M)|_q - \lambda\Psi'(\nu)\omega_M|_q \\ &\quad - (\Psi'(\nu) - n)\sqrt{-1}\partial\bar{\partial}t|_p - \Psi''(\nu)\sqrt{-1}\partial t \wedge \bar{\partial}t|_p, \end{aligned}$$

where $Ric(\omega_M)$ is the Ricci form of ω_M on M. The matrix composed by components of metric ω at p is

$$\begin{pmatrix} (1 + \lambda f'(\nu))(g_{i\bar{j}})_{m\times m} & 0 \\ 0 & \big(f'(\nu)\delta_{\alpha\beta} + f''(\nu)\bar{w}_\alpha w_\beta\big)_{n\times n} \end{pmatrix},$$

where $(g_{i\bar{j}})_{m\times m}$ is the coefficients matrix of metric ω_M. Its inverse matrix is

$$\begin{pmatrix} \frac{1}{1+\lambda f'(\nu)}(g^{i\bar{j}})_{n\times n} & 0 \\ 0 & (\frac{e^t}{f'(\nu)}\delta_{\alpha\beta} + w_\alpha\bar{w}_\beta(\frac{1}{f''(\nu)} - \frac{1}{f'(\nu)}))_{n\times n} \end{pmatrix}.$$

If c_M denotes the scalar curvature of ω_M, the scalar curvature of ω at the point p is

$$c = \frac{c_M}{1 + \lambda f'(\nu)} - \frac{\lambda m\Psi'(\nu)}{1 + \lambda f'(\nu)} - (n-1)\frac{\Psi'(\nu) - n}{f'(\nu)} - \frac{\Psi''(\nu)}{f''(\nu)}. \tag{16}$$

The above formula of scalar curvature is obviously globally defined.

Usually it is more suitable to use the Legendre transform to solve scalar curvature Equation (16). From the positivity (14) of ω, $f(\nu)$ must be strictly convex. Then one can take the Legendre transform $\mathcal{F}(\tau)$ of $f(\nu)$. The Legendre transform $\mathcal{F}(\tau)$ is defined in term of the variable $\tau = f'(\nu)$ by the

formula

$$f(\nu) + \mathcal{F}(\tau) = \nu\tau.$$

Let $I \subset \mathbb{R}_+$ be the image of $f'(\nu)$. The *momentum profile* $\varphi(\tau)$ of the metric is defined to be $\varphi : I \to \mathbb{R}$,

$$\varphi = \frac{1}{\mathcal{F}''(\tau)}.$$

Then the following relations can be verified:

$$\varphi(\tau) = f''(\nu) \qquad \text{and} \qquad \frac{d\tau}{d\nu} = \varphi.$$

Also, ν can be viewed as a function of τ, up to a constant,

$$\nu(\tau) = \int \frac{1}{\varphi(\tau)} d\tau. \tag{17}$$

The advantage of the Legendre transform is that the scalar curvature of ω can be described in terms of $\varphi(\tau)$ and the domain I of τ. Especially, the boundary completeness (see the next subsection) and the extendability properties (see the next section) can also be read off from the behaviour of $\phi(\tau)$ near the end points of I.

Using these transformations, we have

$$\Psi(\tau) = \log((1+\lambda\tau)^m \tau^{n-1} \varphi(\tau)).$$

Let

$$Q(\tau) = (1+\lambda\tau)^m \tau^{n-1}.$$

Then $\Psi = \log(Q\varphi)$. By direct computation,

$$\Psi'(\nu) = \frac{1}{Q}\frac{\partial(Q\phi)}{\partial\tau}$$
$$\Psi''(\nu) = -(m\lambda\tau^{n-1} + (n-1)(1+\lambda\tau))\frac{\varphi}{Q^2}\frac{\partial(Q\varphi)}{\partial\tau} + \frac{\varphi}{Q}\frac{\partial^2(Q\varphi)}{\partial\tau^2}.$$

Inserting the above equalities into (16), and replacing $f'(\nu)$ by τ, then simplifying it, at last we obtain

$$c = \frac{c_M}{1+\lambda\tau} + \frac{n(n-1)}{\tau} - \frac{1}{Q}\frac{\partial^2(Q\varphi)}{\partial^2\tau}. \tag{18}$$

Now we assume that c_M is a constant. We want to look for φ such that c is a constant. By integrations, we solve Equation (18) as

$$\begin{aligned} &(\varphi Q)(\tau) = (\varphi Q)(a) + (\varphi Q)'(a)(\tau - a) + P(\tau), \qquad \text{with} \\ &P(\tau) = \int_a^\tau (\tau - x)\left(\frac{c_M}{1+\lambda x} + \frac{n(n-1)}{x} - c\right) Q(x)dx, \end{aligned} \tag{19}$$

where a is the left endpoint of I. Here $(\varphi Q)(a)$ and $(\varphi Q)'(a)$ are constants. $P(\tau)$ is a polynomial and hence $\varphi(\tau)$ is a real rational function.

3.2. Completeness

Assume that $I = (a, b) \subset \mathbb{R}^+$, b may be infinity, is the maximum interval where $\varphi(\tau)$ determined by (19) is defined and positive. Further assume that

$$\lim_{\tau \to a^+} \nu(\tau) = -\infty$$

and

$$1 + \lambda\tau > 0, \qquad \text{when } \tau \in (a, b).$$

Then from the above subsection, the bundle adapt metric ω is well-defined on $\mathbb{U}^*(\nu(b)) = \{p \in E \mid -\infty < \nu(p) < \nu(b)\} \subset E^*$. Here $\nu(b) = \lim_{\tau \to b^-} \nu(\tau)$. If $\nu(b)$ is infinity, then $\mathbb{U}^*(\nu(b)) = E^*$, and if $\nu(b)$ is a constant, we can take an integration constant in (17) such that $\nu(b) = 0$ and hence $\mathbb{U}^*(\nu(b))$ is $\mathbb{U}^* = \mathbb{U} - M$ as defined in Introduction. We first establish the following lemma.

Lemma 3.1. *Under the above assumptions, the bundle adapt metric ω is complete near the zero (punctured) section if and only if $a > 0$ and $\varphi(a) = \varphi'(a) = 0$. Thus, $\varphi(\tau) = \frac{P(\tau)}{Q(\tau)}$.*

Moreover, if b is finite, then ω is defined on E^* and ω is complete if and only if φ also satisfies $\varphi(b) = \varphi'(b) = 0$; Whereas if b is infinity, then ω is defined on E^* or on $\mathbb{U}^*$ and is automatically complete.

Proof. For any point $q \in M$, $p \in \pi^{-1}(q) \cap \mathbb{U}^*(\nu(b))$, we consider the ray starting from q on the fiber $\pi^{-1}(q)$:

$$\gamma(s) = s \cdot p, \qquad s \in (0, 1] \ \text{ or } \ s \in [1, s_0) \tag{20}$$

for $s_0^2 = \exp(\nu(b) - \nu(p))$. Such an s_0 can be derived from the following calculation by (12):

$$\nu(b) = \nu(s_0 \cdot p) = \log h(z(q)) + t(s_0 \cdot p) = \log s_0^2 + \nu(p).$$

Since $\pi^{-1}(q) \cap \mathbb{U}^*(\nu(b))$ is a totally geodesic submanifold of $(\mathbb{U}^*(\nu(b)), \omega)$ and the induced metric on $\pi^{-1}(q) \cap \mathbb{U}^*(\nu(b))$ is $U(n)$-invariant, $\gamma(s)$ is a geodesic on $\pi^{-1}(q) \cap \mathbb{U}^*(\nu(b))$, and hence is also a geodesic on $\mathbb{U}^*(\nu(b))$. Also since M is compact, the metric ω is complete if and only of the lengths of the rays $\gamma(s)$ defined in (20) are infinity.

As done in Lemma 2.1, the length of $\gamma(s)$ on domain $(0, 1]$ is

$$l_1 = \int_0^1 |\gamma'(s)| ds = \frac{1}{2} \int_{-\infty}^{\nu(p)} \sqrt{f''(\nu)} d\nu = \frac{1}{2} \int_a^{\tau(\nu(p))} \frac{1}{\sqrt{\varphi(\tau)}} d\tau.$$

The completeness near the zero section requires $l_1 = +\infty$, which is equivalent to that $\varphi(\tau)$ has a factor $(\tau - a)^2$, i.e. $\varphi(a) = \varphi'(a) = 0$. We claim $a > 0$. If $a = 0$, the lowest degree term of polynomial $P(\tau)$ defined in (19) would be determined as

$$\int_0^\tau (\tau - x) n(n-1) x^{n-2} dx = \tau^n.$$

Hence we can write $P(\tau)$ as $P(\tau) = \tau^n(1 + A(\tau))$ for some polynomial $A(\tau)$ and thus get

$$\varphi(\tau) = \frac{P(\tau)}{Q(\tau)} = \tau \frac{1 + A(\tau)}{(1 + \lambda\tau)^m}.$$

In this way we find $l_1 < +\infty$, which is a contradiction.

Next, we should consider the endpoint b. The length l_2 of $\gamma(s)$ for $s \in [1, s_0)$ is

$$l_2 = \int_1^{s_0} |\gamma'(s)| ds = \frac{1}{2} \int_{\nu(p)}^{\nu(s_0 \cdot p) = \nu(b)} \sqrt{f''(\nu)} d\nu = \frac{1}{2} \int_{\tau(\nu(p))}^{\tau(\nu((b))) = b} \frac{1}{\sqrt{\varphi}} d\tau$$

If b is finite, then $P(\tau)$ has a factor $(\tau - b)$. By (17), $\lim_{\tau \to b^-} \nu(\tau) = +\infty$. Hence, ω is well-defined on E^*. If ω is complete, the above l_2 is also infinity, which is equivalent to say that $\varphi(b) = \varphi'(b) = 0$. If b is infinity, then $\varphi(\tau)$ is defined on $(a, +\infty)$. Since $\wp \triangleq \deg P(\tau) - \deg Q(\tau) = 2$ or 1, from (17) if $\wp = 2$, ω is defined on $\mathbb{U}^*$ and if $\wp = 1$, ω is defined on E^*. Also since $\wp = 2$ or 1, there exists a constant C big enough such that $\varphi(\tau) < C\tau$ as $\tau \to \infty$. Hence, in this situation, l_2 is infinity and ω is automatically complete. □

3.3. Existence of complete cscK metrics

We discuss the solutions in this subsection divided into three cases: $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

1. Case $\lambda > 0$. Given constants c_M, $\lambda > 0$, and $a > 0$, define the set $\mathfrak{C}$ to be of "allowable scalar curvatures" as

$$\mathfrak{C} = \{\ c \in \mathbb{R} \mid \varphi(\tau) > 0 \ \text{ for } \ \tau \in (a, +\infty)\ \}.$$

$\mathfrak{C}$ is not empty since $\varphi(\tau)$ is positive if $c << 0$. $\mathfrak{C}$ has a supermum. In fact, if $c > 0$, $P(\tau)$ and hence $\varphi(\tau)$ will be negative when τ is big enough. Hence, the supermum c_0 of $\mathfrak{C}$ is nonpositive. We can easily get the conclusions: If $c_M \geq 0$, $c_0 = 0 \in \mathfrak{C}$; If $c_M < 0$, two possibilities occur: one is $c_0 \in \mathfrak{C}$, and the other is $c_0 \notin \mathfrak{C}$, which means that there exists a constant b such that $\varphi(b) = 0$ and $\varphi(\tau)$ is positive on (a, b). Hence we should consider the existence of Kähler metrics with constant scalar curvature c as the following four cases:

$$(i)\ c < c_0; \quad (ii)\ c = c_0 = 0 \in \mathfrak{C}; \quad (iii)\ 0 > c = c_0 \in \mathfrak{C}; \quad \text{or } (iv)\ c = c_0 \notin \mathfrak{C}.$$

Proposition 3.1. *Given constants c_M, $\lambda > 0$ and $a > 0$, there exists a constant $c_0 \leq 0$ such that:*

1. For any $c \leq c_0$ in cases (i) and (iii), there exists a complete cscK metric ω on $\mathbb{U}^*$ with constant scalar curvature c; and

2. For $c = c_0$ in cases (ii) and (iv), there exists a complete cscK metric ω on E^* with constant scalar curvature c.

Proof. For cases (i) and (iii), since the degrees of polynomials $P(\tau)$ and $Q(\tau)$ are $m + n + 1$ and $m + n - 1$ respectively, the limit of $\nu(\tau)$ defined by (17) is finite as $\tau \to +\infty$. Set this constant to be zero. Then the metric ω is defined on $\mathbb{U}^*$. According to Lemma 3.1, ω is complete.

For case (ii), $\deg P(\tau) - \deg Q(\tau) = 1$. The limit of $\nu(\tau)$ is infinity as $\tau \to +\infty$. Hence the metric is defined on E^* and is complete by Lemma 3.1.

For case (iv), $\varphi(\tau) \geq 0$ for $\tau \in (a, +\infty)$. In this case there exists a constant b such that $\varphi(\tau) > 0$ in $\tau \in (a, b)$ and $\varphi(b) = 0$. Hence $\varphi'(b) = 0$. According to Lemma 3.1, ω is defined on E^* and is complete. □

We give two examples.

Example 3.1. Consider the case $c_0 \in \mathfrak{C}$ and $c_0 < 0$. Take

$$c = \frac{c_M}{1+\lambda a} + \frac{n(n-1)}{a}.$$

We have

$$\begin{aligned}&\frac{c_M}{1+\lambda x} + \frac{n(n-1)}{x} - c\\ &= \frac{a-x}{a(1+\lambda a)x(1+\lambda x)}\left(\lambda(c_M a + n(n-1)(1+\lambda a))x + n(n-1)(1+\lambda a)\right).\end{aligned}$$

If $c_M < 0$, $\lambda > 0$ and $a > 0$ satisfy

$$c_M = -\frac{n(n-1)(1+\lambda a)^2}{\lambda a^2},$$

we find that $P(\tau) > 0$ when $\tau \in (a, +\infty)$. It is easy to check that

$$c_0 = c = -\frac{n(n-1)}{\lambda a^2}.$$

Example 3.2. Then consider the case $c_0 \notin \mathfrak{C}$. Let $m = 1$, $n = 2$, $\lambda = 1$, $c_M = -4$ and $a = 1$. It follows that

$$\varphi(\tau) = \int_1^\tau (\tau - x)\big(-cx^2 - (c+2)x + 2\big)dx.$$

We can solve the inequality $\varphi(\tau) \geq 0$ to get $c \leq \psi(\tau)$. Here

$$\psi(\tau) = \frac{\frac{1}{3} - \tau + \tau^2 - \frac{\tau^3}{3}}{\frac{7}{12} - \frac{5}{6}\tau + \frac{1}{6}\tau^3 + \frac{1}{12}\tau^4}.$$

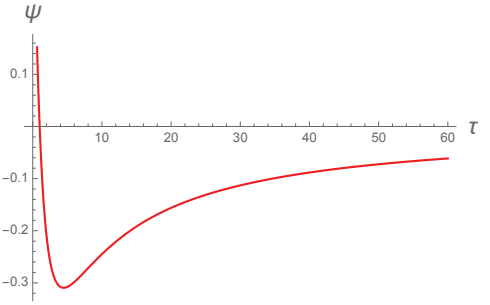
Hence $\varphi(b) = \varphi'(b) = 0$ if and only if $c_0 = \min_{\tau\in(a,+\infty)} \psi(\tau) = \psi(b)$.

The pictures of $\psi(\tau)$ and $\varphi(\tau)$ are given as Figure 5. We find that $\psi(\tau)$ achieves its maximum at $\tau = 4.4641$ with the maximum -0.3094. So $c_0 = -0.3094$ and $b = 4.4641$. Thus, $\varphi(\tau)$ gives a complete cscK metric on E^* with scalar curvature c_0.

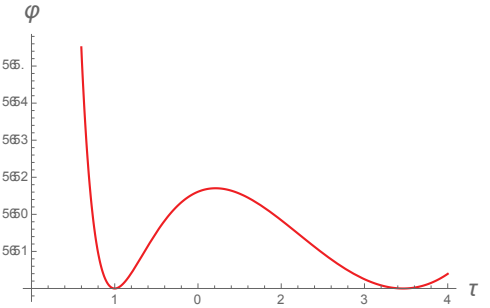
We consider the asymptotic property. Let

$$\kappa(\tau) = \frac{c_M}{1+\lambda\tau} + \frac{n(n-1)}{\tau} - c.$$

Since $\varphi(a) = \varphi'(a) = 0$, $\kappa(a) = \varphi''(a)$.



(a) the graph of $\psi(\tau)$



(b) the graph of $\varphi(\tau)$

Figure 5: The case of $c_0 \notin \mathfrak{C}$.

Proposition 3.2. *For cases (i), (ii) and (iv) the cscK metrics in Proposition 3.1 have the PMY asymptotic property, and for case (iii), the metrics have the asymptotic property: As* $r^2 \to 0$,

$$f(r^2) = a \log r^2 - 2\left(\frac{3}{\kappa'(a)}\right)^{\frac{1}{2}} (-\log r^2)^{\frac{1}{2}} + O(\log(-\log r^2)),$$

or

$$f(r^2) = a \log r^2 - \frac{3}{2}\left(\frac{8}{\kappa''(a)}\right)^{\frac{1}{3}} (-\log r^2)^{\frac{2}{3}} + O((-\log r^2)^{\frac{1}{2}}).$$

Proof. For cases (i), (ii) and (iv), we claim that $\kappa(a) > 0$. If the claim holds, then as $r^2 \to 0$,

$$\frac{d\tau}{d\nu} = \varphi(\tau) = \frac{\kappa(a)}{2}(\tau - a)^2 + O\left(\frac{1}{\tau - a}\right),$$

from which we can get

$$f(r^2) = a\log r^2 - \frac{2}{\kappa(a)}\log(-\log r^2) + O((\log r^2)^{-1}),$$

which means that the metric is with PMY asymptotic property.

We prove the claim. For case (i), since $\frac{c_M}{1+\lambda a} + \frac{n(n-1)}{a} - c_0 \geq 0$, $\kappa(a) = (\kappa(a) + c - c_0) + (c_0 - c) > 0$. For case (ii), if $\kappa(a) = 0$, then $\varphi^{(3)}(a) = \kappa'(a)$ and $\varphi(\tau)$ has the Taylor expansion at $\tau = a$:

$$\varphi(\tau) = \frac{\kappa'(a)}{3!}(\tau - a)^3 + o((\tau - a)^3).$$

The positivity of $\varphi(\tau)$ when $\tau > 0$ implies $\kappa'(a) \geq 0$. On the other hand, if $\kappa(a) = 0$, i.e., $\frac{c_M}{1+\lambda a} + \frac{n(n-1)}{a} = 0$ as $c = 0$, then

$$\kappa'(a) = \frac{-c_M}{(1+\lambda a)^2} - \frac{n(n-1)}{a^2} = -\frac{n(n-1)}{(1+\lambda a)a^2} < 0,$$

which is a contradiction to $\kappa'(a) \geq 0$. Hence $\kappa(a) \neq 0$ and $\kappa(a) > 0$ is deduced from the positivity of $\varphi(\tau)$. We then consider case (iv). In this case, $I = (a, b)$ and

$$\varphi(a) = \varphi'(a) = \varphi(b) = \varphi'(b) = 0.$$

These equalities guarantee that there are already two roots in (a,b) for $\varphi''(\tau) = \kappa(\tau)Q(\tau) = 0$. Hence $\kappa(a) = \varphi''(a) \neq 0$. The positivity of φ then implies $\kappa(a) > 0$.

For case (iii), we first prove that $\kappa(a) = 0$ which means that the metrics for this case is not PMY. Since $c = c_0 < 0$, there exist constants $a_0 \in (a, \infty)$ and $C > 0$ such that $\kappa(\tau) \geq C$ in $[a_0, \infty)$. If $\kappa(a) > 0$, there would exist constants $a_1 > a$ and $C_1 > 0$ such that $\kappa(\tau) > C_1$ in (a, a_1) and $\varphi(\tau) > C_1$ in $(a_1, a_0]$. Then we could choose a positive constant ϵ such that $\varphi(\tau)$ is still positive by replacing $c = c_0$ with $c = c_0 + \epsilon$. Hence $c = c_0 + \epsilon \in \mathfrak{C}$ which contradicts to that c_0 is the supermum of $\mathfrak{C}$.

In this case, there are two subcases which should be considered: $\kappa'(a) > 0$ and $\kappa'(a) = 0$: If $\kappa'(a) > 0$,

$$f''(\nu) = \frac{d\tau}{d\nu} = \varphi(\tau) = \frac{\kappa'(a)}{3!}(\tau - a)^3 + O((\tau - a)^4);$$

If $\kappa'(a) = 0$, $\kappa''(a) \neq 0$ and

$$f''(\nu) = \frac{d\tau}{d\nu} = \varphi(\tau) = \frac{\kappa''(a)}{4!}(\tau - a)^4 + O((\tau - a)^5).$$

The conclusion then follows. □

Proposition 3.3. *The metrics in Proposition 3.1 with constant scalar curvature c have the asymptotic property:*

1. For cases (i) and (iii) (hence defined on $\mathbb{U}^*$), as $r^2 \to 1$,

$$f(r^2) = -\frac{(m+n)(m+n+1)}{c}\log(-\log r^2) + O(\log r^2);$$

2. For case (ii) (hence defined on E^*), as $r^2 \to \infty$,

$$f(r^2) = \frac{1}{\theta_1}(r^2)^{\theta_1} + \theta_2 \log r^2 + O(r^{-2})$$

with $\theta_1 = \frac{c_M + n(n-1)\lambda}{\lambda(m+n)(m+n+1)}$ and $\theta_2 = \frac{(m+n)(c_M(m-1)+n(n-1)m\lambda)}{m(m+n-2)(c_M+n(n-1)\lambda)}$;

3. For case (iv) (hence defined on E^*), as $r^2 \to \infty$,

$$f(r^2) = b\log r^2 - \frac{2}{\kappa(b)}\log(\log r^2) + O((\log r^2)^{-1}) \qquad \text{with} \quad \kappa(b) > 0.$$

Proof. We omit the proof here. It is a calculus exercise. □

2. Case $\lambda = 0$. In this case, $\kappa(\tau) = c_M + \frac{n(n-1)}{\tau} - c$. Hence $c_0 = c_M$.

Proposition 3.4. *Given constants c_M, $\lambda = 0$, and $a > 0$, there exists a complete cscK metrics on $\mathbb{U}^*$ with $c < c_M$ and on E^* with $c = c_M$. All these metrics admit PMY asymptotic property.*

Proof. The proof is the same as Propositions 3.1 and 3.2. □

3. Case $\lambda < 0$. The method of this case is quite different from the cases $\lambda \geq 0$.

Proposition 3.5. *For any $\lambda < 0$ and $c_M > 0$, there exists on E^* a complete positive cscK metric with PMY asymptotic property.*

Proof. We need to prove that there exist two constants a and b with $0 < a < b < -\frac{1}{\lambda}$ such that the function $\varphi(\tau)$ is positive on domain (a, b) and $\varphi(b) = \varphi'(b) = 0$.

On interval $(0, -\frac{1}{\lambda})$, the polynomial $Q(\tau)$ is positive. Hence there is a number $b \in (a, -\frac{1}{\lambda})$ such that $\varphi(b) = \varphi'(b) = 0$ if and only if $P(b) = P'(b) = 0$, and when $\tau \in (a, b)$, $\varphi(\tau)$ is positive if and only if $P(\tau)$ is positive.

From $P(b) = P'(b) = 0$, we can solve c_M and c as

$$c_M = n(n-1)\frac{H_1(a,b)}{H_2(a,b)} \qquad \text{and} \qquad c = n(n-1)\frac{H_3(a,b)}{H_2(a,b)}, \tag{21}$$

where we have defined

$$\begin{aligned} H_1(a,b) &= \int_a^b \frac{Q(x)}{x}dx \int_a^b xQ(x)dx - \left(\int_a^b Q(x)dx\right)^2, \\ H_2(a,b) &= -\frac{1}{\lambda}\int_a^b \frac{Q(x)}{1+\lambda x}dx \int_a^b (1+\lambda x)Q(x)dx + \frac{1}{\lambda}\left(\int_a^b Q(x)dx\right)^2, \\ H_3(a,b) &= \int_a^b \frac{xQ(x)}{1+\lambda x}dx \int_a^b \frac{Q(x)}{x}dx - \int_a^b Q(x)dx \int_a^b \frac{Q(x)}{1+\lambda x}dx. \end{aligned} \tag{22}$$

We first note that when $\lambda < 0$ and $0 < a < b < -\frac{1}{\lambda}$, the functions $H_i(a,b)$ for $i = 1, 2, 3$ are always positive. The proofs for the first and second functions are direct by the Hölder inequality. The proof for the third one is also direct by using the common techniques in calculus. Thus the constant c_M and c defined in (21) are indeed positive.

We need the following.

Claim: For any given positive c_M, there exist two constants a and b with $0 < a < b < -\frac{1}{\lambda}$ such that the first equality in (21) holds.

Proof. Define a function

$$H(\zeta,\tau) = \frac{H_1(\zeta,\tau)}{H_2(\zeta,\tau)}, \qquad \zeta,\tau \in \left(0, -\frac{1}{\lambda}\right),\ \zeta < \tau.$$

By continuity, if we can prove that as $\zeta \to 0$, $H(\zeta, 2\zeta) \to \infty$, and as $\epsilon \to 0$, $H(\frac{1-2\epsilon}{-\lambda}, \frac{1-\epsilon}{-\lambda}) \to 0$, then the claim holds. But as $\zeta \to 0$ and (hence) $(1 + \lambda\zeta) \to 1$, one can easily estimate to get $H_1(\zeta, 2\zeta) = O(\zeta^{2n})$ and $H_2(\zeta, 2\zeta) = O(\zeta^{(2n+1)})$, and hence $H(\zeta, 2\zeta) = O(\zeta^{-1})$. On the other hand, as $\epsilon \to 0$, $H_1(\frac{1-2\epsilon}{-\lambda}, \frac{1-\epsilon}{-\lambda}) = O(\epsilon^{2m+1})$, and $H_2(\frac{1-2\epsilon}{-\lambda}, \frac{1-\epsilon}{-\lambda}) = O(\epsilon^{2m})$ and hence $H(\frac{1-2\epsilon}{-\lambda}, \frac{1-\epsilon}{-\lambda}) = O(\epsilon)$. □

According to the claim, we have $P(b) = P'(b) = 0$. The condition $P(\tau) > 0$ for $\tau \in (a, b)$ is automatically satisfied. For if there exists a point $\xi \in (a, b)$ such that $P(\xi) = 0$, equation $P'''(\tau) = \kappa(\tau)Q(\tau) = 0$ has three roots in (a, b). This is impossible.

From the proof of Proposition 3.2, we see that if $\kappa(a) > 0$, then the metric has the PMY asymptotic property. Since $P(a) = P'(a) = 0$, $\kappa(a) = 0$

is equivalent to $P''(a) = 0$, and hence $P'''(\tau) = 0$ has three roots in (a, b). It is impossible. □

Proof of Theorem 1.4. It follows from Propositions 3.1, 3.2, 3.3, 3.4 and 3.5. □

4. CscK PMY metrics on $\mathbb{P}(E \oplus \mathcal{O}) - M$

Recall that $\mathbb{P}(E \oplus \mathcal{O})$ can be viewed as a compactification of E: E can be imbedded into $\mathbb{P}(E \oplus \mathcal{O})$. In fact, let $(U, z = (z_1, \ldots, z_m))$ be a local holomorphic chart of M such that $E|_U$ is (holomorphically) isomorphic to $U \times \mathbb{C}^n$. If we denote the coordinates of $\mathbb{C}^n$ as $w = (w_1, \ldots, w_n)$, the imbedding map can be defined as follows: for any $p \in E|_q$, $q \in U$,

$$p \mapsto (q, w_1(p), \ldots, w_n(q)) \mapsto (q, [1, w_1(p), \ldots, w_n(p)]).$$

This map is clearly well-defined on E. It defines a section s of $\mathbb{P}(E \oplus \mathcal{O})$:

$$q \mapsto (q, (0, \ldots, 0)) \mapsto (q, [1, 0, \ldots, 0])$$

which is just the zero section of E. Hence we still denote $s(M)$ simply by M. Set $D_\infty = \mathbb{P}(E \oplus \mathcal{O}) - E$. D_∞ is a divisor on $\mathbb{P}(E \oplus \mathcal{O})$ and is called the infinity divisor. By these notations, $E - M$ is bi-holomorphic to $\mathbb{P}(E \oplus \mathcal{O}) - s(M) - D_\infty$. Now the question is when the metric ω defined on $E - M$ as the above section can be extended across D_∞.

First note that if ω can be extended across D_∞, ω must be defined on $E - M$ and is not complete at infinity. Hence according to the proof of Lemma 3.1, the endpoint b of $I = (a, b)$ is finite.

Lemma 4.1. *Let ω be the bundle adapted metric with momentum profile $\varphi(t)$ in (19). Assuming that there is a constant b such that $\varphi(\tau)$ is positive on (a, b) and $\varphi(b) = 0$. Then ω defined on $E - M$ can be extended across D_∞ if and only if $\varphi'(b) = -1$.*

Proof. The proof of this lemma is well-known. One can consult references [1, 6, 9]. Here we write out details.

Since the metric ω on $E - M$ is bundle adapt, we only need to prove that the metric $\omega_0 = i\partial\bar{\partial} f(\tau)$ defined on fiber $E|_q - \pi(q) = \mathbb{C}^n - 0$ can be extended to $\mathbb{C}P^{n-1}$.

First recall that $\mathbb{C}P^n \setminus [1, 0, \ldots, 0]$ is bi-holomorphic to the line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^{n-1}$. Let $[v_0, \ldots, v_n]$ be the homogeneous coordinates of $\mathbb{C}P^n$.

Here the open set $\mathbb{C}^n$ is $v_0 = 1$ and the hyperplane $\mathbb{C}P^{n-1}$ is $v_0 = 0$. Hence $w_\alpha = \frac{v_\alpha}{v_0}$ is the $\alpha-$th coordinate of $\mathbb{C}^n$ and $[v_1, \ldots, v_n]$ is the homogeneous coordinates of $\mathbb{C}P^{n-1}$. Let $U_\alpha = \{[v_1, \ldots, v_n] \mid v_\alpha \neq 0\}$ and define $w_\beta^\alpha = \frac{v_\beta}{v_\alpha}$ with $\beta \neq \alpha$. Let v^α be the coordinate of the trivialization of $\mathcal{O}(1)|_{U_\alpha}$. Then its transition function defined on $U_\alpha \cap U_\beta$ is

$$v^\alpha = \frac{1}{w_\beta^\alpha} v^\beta = w_\alpha^\beta v^\beta.$$

The bi-holomorphic map $\psi : \mathbb{C}P^n - [1, 0, \ldots, 0] \to \mathcal{O}(1)$ is

$$[v_0, \ldots, v_n] \mapsto [w_1^\alpha, \ldots, w_{\beta-1}^\alpha, w_{\beta+1}^\alpha, \ldots, w_n^\alpha, \frac{v_0}{v_\alpha}], \qquad \text{for } v_\alpha \neq 0.$$

Define on $\mathcal{O}(1)$ the function

$$\tilde{r}^2 = \frac{|v^\alpha|^2}{1 + \sum_{\beta \neq \alpha} |w_\beta^\alpha|^2}.$$

We have $\tilde{r}^2 = \frac{1}{r^2}$ on $\mathbb{C}^n - 0$.

By Direct computation, we have

$$\begin{aligned}
\omega_0 &= -\sqrt{-1} f'(t) \partial\bar{\partial} \log \tilde{r}^2 + \sqrt{-1} f''(t) \partial \log \tilde{r}^2 \wedge \bar{\partial} \log \tilde{r}^2 \\
&= (f'(t) + f''(t)) \omega_{FS} + \frac{f''(t^2)}{\tilde{r}^2} (\sqrt{-1} \partial\bar{\partial} \tilde{r}^2) \\
&= (\tau + \varphi(\tau)) \omega_{FS} + \frac{\varphi(\tau)}{\tilde{r}^2} (\sqrt{-1} \partial\bar{\partial} \tilde{r}^2)
\end{aligned}$$

Define the functions

$$f_1(\tilde{r}^2) = \begin{cases} \tau + \varphi(\tau), & \tilde{r}^2 > 0 \\ b & \tilde{r}^2 = 0 \end{cases},$$

and

$$f_2(\tilde{r}^2) = \begin{cases} \frac{\varphi(\tau)}{\tilde{r}^2} & \tilde{r}^2 > 0 \\ 1 & \tilde{r}^2 = 0. \end{cases}$$

Since as $\tilde{r}^2 \to 0$, $\tau \to b$ and $\lim_{\tau \to b} \varphi(\tau) = 0$, the function $f_1(\tilde{r}^2)$ is continuous at $\tilde{r}^2 = 0$. As to f_2, we shall prove that if we take a suitable constant in (17), then it is also continues at $\tilde{r}^2 = 0$.

In fact $f_2(\tilde{r}^2)$ is smooth. Since $\varphi(b) = 0$, $\varphi'(b) = -1$, and φ is rational, we can write

$$\varphi(\tau) = (b - \tau)(1 + (b - \tau)\varphi_1(\tau))$$

for some smooth function $\varphi_1(\tau)$. Then

$$t = \int \frac{1}{\varphi(\tau)} d\tau = -\log(b - \tau) - \varphi_2(\tau).$$

Here $\varphi_2(\tau)$ is a smooth function with $\varphi_2(b) = 0$. Hence

$$\tilde{r}^2 = \frac{1}{r^2} = e^{-t} = e^{\varphi_2(\tau)}(b - \tau) \tag{23}$$

and

$$f_2(\tilde{r}^2) = (1 + (b - \tau)\varphi_1(\tau))e^{-\varphi_2(\tau)}$$

is a smooth function of τ. Moreover, by the implicit function theorem, we can solve (23) to get a smooth function $\tau = \tau(\tilde{r}^2)$. Hence $f_2(\tilde{r}^2)$ is a smooth function of $\tilde{r}^2$. Now we can also see that $f_1(\tilde{r}^2)$ is smooth at $\tilde{r}^2 = 0$ since

$$f_1(\tilde{r}^2) = \tau + f_2(\tau)\tilde{r}^2.$$

The metric ω_0 can be extended across $\mathbb{C}P^{n-1}$ by defining

$$\tilde{\omega}_0 = f_1(\tilde{r}^2)\omega_{FS} + f_2(\tilde{r}^2)(\sqrt{-1}\partial\bar{\partial}\tilde{r}^2)$$

Since $d(\omega_0) = 0$ and $f_1(\tilde{r}^2)$ and $f_2(\tilde{r}^2)$ is smooth at $\tilde{r}^2 = 0$, $\tilde{\omega}_0$ is also Kähler. □

According to Lemma 4.1, the momentum profile φ in (19) gives a complete cscK metric on $\mathbb{P}(E \oplus \mathcal{O}) - M$ if and only if

(i) $\varphi(b) = 0$ and $\varphi'(b) = -1$ with $b > a$,

(ii) $\varphi(\tau)$ is positive on domain (a, b).

However, condition (ii) is satisfied automatically if condition (i) holds. For one can show that under condition (i) a is the unique solution of $P(x) = 0$ for $x \in (0, b)$. In fact, we have the following result.

Lemma 4.2. *If $\varphi(b) = 0$ and $\varphi'(b) = -1$, then $\varphi(\tau) > 0$ on domains $(0, a)$ and (a, b).*

Proof. Since $P''(\tau) = \kappa(\tau) = 0$ has at most two roots and we have already $P(a) = P(b) = P'(a) = 0$, $P(\tau) = 0$ has at most one root ξ in $(0, a)$ or in (a, b). Also since $\varphi'(b) = -1$ and $\varphi(b) = 0$, $P(\tau)$ is positive as $\tau \to b$. Hence if $\xi \in (a, b)$, there are two cases should be considered. One is $\varphi(\tau) > 0$ on $(a, \xi) \cup (\xi, b)$, the other is $\varphi(\tau) < 0$ on (a, ξ). The former is impossible as $\varphi'(\xi) = 0$ which would lead to $\kappa(\tau) = 0$ has at least three roots. The latter is also impossible as from this one can derive $\varphi''(a) = 0$ which would also lead the contradiction. Thus, $\xi \notin (a, b)$.

Since $\varphi(\tau) > 0$ when τ is near zero, as the same reason, $\xi \notin (0, a)$. □

Hence, in the following we only need to find a constant b such that condition (i) is satisfied. We can solve constants c_M and c from $\varphi(b) = 0$ and $\varphi'(b) = -1$ as:

$$c_M = \frac{n(n-1)H_1 + L_1}{H_2} \tag{24}$$

and

$$c = \frac{n(n-1)H_3 + L_2}{H_2}$$

with the definitions (22) and of

$$L_1 = Q(b) \int_a^b xQ(x)dx - bQ(b) \int_a^b Q(x)dx,$$
$$L_2 = bQ(b) \int_a^b Q(x)dx - Q(b) \int_a^b \frac{xQ(x)}{1 + \lambda x} dx.$$

So we should determine the range of c_M such that $\varphi(\tau)$ satisfies (i).

Proposition 4.1. *If $\lambda < 0$, the range of c_M is $\mathbb{R}$.*

Proof. Let $\tilde{H} = \frac{n(n-1)H_1 + L_1}{H_2}$. First, we take $b = 2a$ and estimate $\tilde{H}(a, 2a)$ as $a \to 0^+$. We get

$$(n(n-1)H_1 + L_1)(a, 2a) = \alpha_1 a^{2n} + O(a^{2n+1})$$

with

$$\alpha_1 = \frac{-2^{n-1}(n+1)^2 + n}{n(n+1)} < 0,$$

and $H_2(a, 2a) = O(a^{2n+1})$. Since when $\lambda < 0$, $H_2(a, 2a) > 0$, we have

$$\lim_{a \to 0^+} H(a, 2a) = -\infty.$$

On the other hand, we take $b = \sqrt{a}$ and do estimates. As $a \to 0^+$, we also have

$$(n(n-1)H_1 + L_1)(a, \sqrt{a}) = \alpha_2 a^{n+\frac{1}{2}} + O(a^{n+1})$$
$$\text{with } \alpha_2 = \frac{-2\lambda m}{n(n+1)(n+2)} > 0,$$

and $H_2(a, \sqrt{a}) = O(a^{n+1})$. Hence we have $\lim_{a\to 0^+} \tilde{H}(a, \sqrt{a}) = +\infty$.

Now the result follows from the continuity of $\tilde{H}$. □

Proposition 4.2. *If $\lambda > 0$, the range of c_M is $(m(m+2n-1)\lambda, \infty)$.*

Proof. First we note that when $\lambda > 0$, $H_2(a, 2a) < 0$. Thus by the estimates in the proof of the above lemma, we get $\lim_{a\to 0^+} \tilde{H}_2(a, 2a) \to +\infty$.

Next we do estimates: as $b \to +\infty$

$$(n(n-1)H_1 + L_1)(a, b) \sim \frac{-m(m+2n-1)}{(m+n)^4 - (m+n)^2} b^{2(m+n)}$$
$$H_2 \sim \frac{-\lambda^{2m-1} b^{2m+2n}}{(m+n)^2(m+n-1)(m+n+1)}$$

Hence

$$\lim_{b\to\infty} c_M = m(m+2n-1)\lambda$$

At last, we need to prove

$$c_M > m(m+n-1)\lambda$$

i.e., to prove when $b \geq a$,

$$K(b) = n(n-1)H_1(b) + L_1(b) - m(m+2n-1)\lambda H_2(b) < 0.$$

This is a calculus exercise and we leave to readers.

In summary, the range of c_M is $(m(m+2n-1)\lambda, \infty)$. □

Proposition 4.3. *The metrics in Propositions 4.1 and 4.2 admit the PMY asymptotic property.*

Proof. We need to prove $\kappa(a) > 0$. By Lemma 4.2, a is a minimum of $\varphi(x)$. From the Taylor expansion of $\varphi(x)$ at $x = a$, if $\kappa(a) = 0$, $\kappa'(a) = -\frac{\lambda c_M}{(1+\lambda a)^2} - \frac{n(n-1)}{a^2} = 0$. So if $\lambda c_M \geq 0$, it is impossible. Thus for $\lambda > 0$ (hence $c_M > 0$),

or $\lambda < 0$ and $c_M \leq 0$, the metrics are PMY. For the case $\lambda < 0$ and $c_M > 0$, if $k'(a) = 0$, then

$$\kappa''(x) = \frac{2\lambda^2 c_M}{(1+\lambda x)^3} + \frac{2n(n-1)}{x^3}$$

would have one root in $(0, b)$ when $\varphi(a) = \varphi'(a) = \varphi''(a)(= \kappa(a)) = \varphi(b) = 0$. But it is impossible when $c_M > 0$. Hence $\kappa'(a) \neq 0$ and then $\kappa(a) > 0$. The metrics are also PMY. □

The proof of Theorem 1.5. It follows from Propositions 4.1, 4.2, and 4.3. □

We give an example of cscK metric with $\lambda > 0$.

Example 4.1. Let $m = 1$, $n = 2$, $\lambda = 1$ and $a = 1$. Then

$$\begin{aligned}\varphi(\tau) &= \frac{1}{\tau(\tau+1)} \int_1^\tau (\tau - x)(c_M x + 2(1+x) - cx(1+x))dx \\ &= -\frac{1}{12\tau(\tau+1)}(-1+\tau)^2\Big(c\tau^2 + (4c - 2c_M - 4)\tau + 7c - 4c_M - 20\Big)\end{aligned}$$

We can solve $\varphi(b) = \varphi'(b) = 0$ to get

$$c_M = \frac{2(-13 + 37b + 39b^2 + 7b^3 + 2b^4)}{(-1+b)^2(1+4b+b^2)}.$$

For example, let $b = 2$. Then

$$c_M = \frac{610}{13} \qquad \text{and} \qquad c = \frac{276}{13}$$

which implies

$$\varphi(x) = \frac{-23x^4 + 60x^3 + 13x^2 - 114x + 64}{13x(1+x)}.$$

We give a picture of $\varphi(\tau)$ as Figure 6.

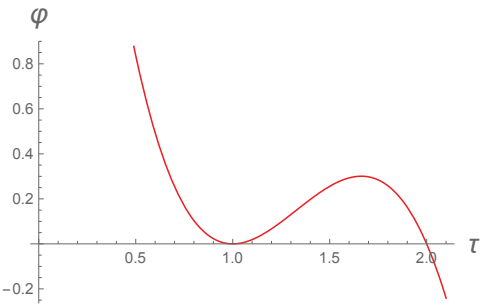


Figure 6: the graph of $\varphi(\tau)$ on $[1,2]$ with $\lambda = 1$, $c_M = \frac{610}{13}$ and $c = \frac{276}{13}$.

We also give two examples of csck metrics with $\lambda < 0$.

Example 4.2. Let $m = 1$, $n = 2$, $\lambda = -1$ and $c_M = 2$. We choose $a = 0.001$, then the graph as Figure 7 shows $b \simeq 0.0893745$ and $c \simeq 68.7366$, or $b \simeq 0.998$ and $c \simeq 11.9761$. We give a picture of $\varphi(\tau)$ with $b = 0.0894$ as Figure 8.

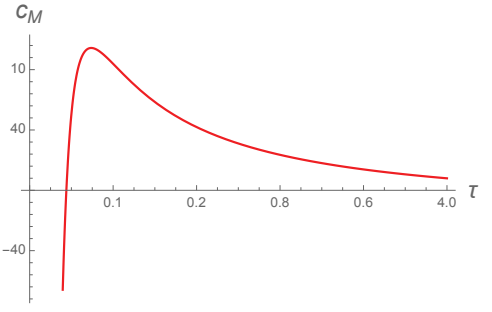


Figure 7: the graph of c_M with $a = 0.001$.

Example 4.3. Let $m = 1$, $n = 2$, $\lambda = -1$ and $c_M = -2$. If $a = 0.1$, then $b \simeq 0.61146$ and $c \simeq 5.02242$. We give a picture of $\varphi(\tau)$ as Figure 9.

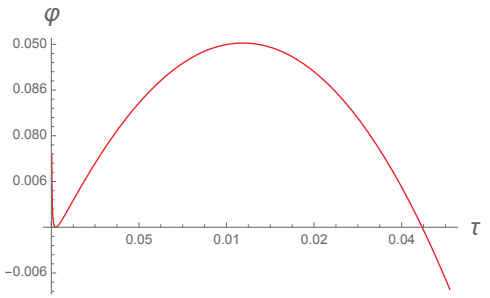


Figure 8: the graph of $\varphi(\tau)$ with $a = 0.001$, $\lambda = -1$ and $c_M = 2$.

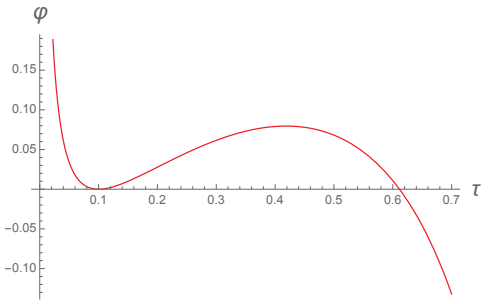


Figure 9: the graph of $\varphi(\tau)$ with $a = 0.1$, $\lambda = -1$ and $c_M = -2$.

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Received December 4, 2015