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A note on center of mass

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We will discuss existence of center of mass on asymptotically Schwarzschild manifold as defined by Huisken-Yau and by Regge-Teitelboim, Beig-Ó Murchadha, Corvino-Schoen. Conditions of existence and examples on non-existence are given.

1. Preliminary

In this note, we will discuss the existence and non-existence of center of mass as defined in [2, 5, 11, 14]. The results are related in particular to Theorem 4.2 in [11].

Let (M^3, g) be an asymptotically Schwarzschild (AS) manifold. That is: M is diffeomorphic to $\mathbb{R}^3$ with metric g such that for some $R_0 > 0$, g is given by

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} + p_{ij} \tag{1}$$

on $\mathbb{R}^3 \setminus B_0(R_0)$, where $p_{ij}(x) = O_4(|x|^{-2})$, and $m > 0$ is the ADM mass of the manifold. Here and below $B_0(r) := \{x \in \mathbb{R}^3 | \ |x| < r\}$. The notation $\phi = O_k(r^\alpha)$ means that there is a constant C such that for all $0 \le i \le k$, and for all multi-index β with $|\beta| = i$, $|\partial^\beta \phi|(x) \le C|x|^{\alpha - i}$ on $\mathbb{R}^3 \setminus B_0(R_0)$.

In [11], Huisken-Yau proved the existence and uniqueness of stable constant mean curvature foliation $\{\Sigma_r\}_{r \ge R_1}$ near infinity on an AS manifold for some $R_1 > R_0$, where each Σ_r is a perturbation of coordinate sphere $S(r) = \partial B_0(r)$. Let $F(r) : \Sigma_r \to \mathbb{R}^3$ be the embedding of Σ_r in M. The Huisken-Yau center of mass is defined as follows: Let $\mathbf{c}_{\text{HY}}(r) = (c^1_{\text{HY}}(r), c^2_{\text{HY}}(r), c^3_{\text{HY}}(r))$,

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where

$$c^\alpha_{\text{HY}}(r) = \frac{\int_{\Sigma_r} x^\alpha d\sigma_0}{\int_{\Sigma_r} d\sigma_0}. \tag{2}$$

for $\alpha = 1, 2, 3$. Here $d\sigma_0$ is the area element induced by the Euclidean metric and x^α are the coordinate functions on Σ_r. The Huisken-Yau center of mass $\mathbf{c}_{\text{HY}}$ is defined as:

$$\mathbf{c}_{\text{HY}} = \lim_{r\to\infty} \mathbf{c}_{\text{HY}}(r) \tag{3}$$

provided the limit exists. Note that $\mathbf{c}_{\text{HY}}(r)$ depends only on Σ_r and the asymptotic coordinates chosen. Consider the foliation $\tilde{\Sigma}_s$ obtained by Ye [16] which is a perturbation of $\{|x - p(s)| = s\}$ for some $p(s)$ with $p(s)$ being uniformly bounded as $s \to \infty$. By the uniqueness result in [11] for s large $\tilde{\Sigma}_s = \Sigma_{r(s)}$ with $r(s) \to \infty$ as $s \to \infty$. Hence if we use the foliation by Ye in (2), then the limit in (3) will give the Huisken-Yau center of mass, provided the limit exists. See [8] for more details.

There is a Hamiltonian formulation of center of mass by Regge-Teitelboim [14], also by Beig-Ó Murchadha [2] and Corvino-Schoen [5]. Following [8], we denote it by $\mathbf{c}_{\text{CS}}$ which is defined as follows. Let

$$c^\alpha_{\text{CS}}(r) = \frac{1}{16\pi m} \int_{|x|=r} \left[x^\alpha (g_{ij,i} - g_{ii,j})\nu^j_g - \left(h_{i\alpha}\nu^i_g - h_{ii}\nu^\alpha_g \right) \right] d\sigma_g \tag{4}$$

where $h_{ij} = g_{ij} - \delta_{ij}$ and ν_g is the unit outward normal of $\{|x| = r\}$ with respect to g. Here and below repeated indices means summation. Let $\mathbf{c}_{\text{CS}}(r) = (c^1_{\text{CS}}(r), c^2_{\text{CS}}(r), c^3_{\text{CS}}(r))$. The Hamiltonian formulation of center of mass is given by

$$\mathbf{c}_{\text{CS}} = \lim_{r\to\infty} \mathbf{c}_{\text{CS}}(r) \tag{5}$$

provided the limit exists.

Let ν_0 be the unit outward normal of $\{|x| = r\}$ with respect to Euclidean metric and let $d\sigma_0$ be the area element induced by the Euclidean metric. Then one can check that on $\{|x| = r\}$, $\nu_g = (1 + \frac{m}{2r})^{-2}\nu_0 + O(r^{-2})$ and $d\sigma_g = (1 + \frac{m}{2r})^4(1 + O(r^{-2}))d\sigma_0$. Hence we have

$$\begin{aligned} c^\alpha_{\text{CS}}(r) = &\frac{1}{16\pi m} \int_{|x|=r} \left[x^\alpha (g_{ij,i} - g_{ii,j})\nu^j_0 - \left(h_{i\alpha}\nu^i_0 - h_{ii}\nu^\alpha_0 \right) \right] d\sigma_0 \\ &+ O(r^{-1}). \end{aligned} \tag{6}$$

In this note we want to discuss the existence of $\mathbf{c}_{\text{HY}}$ and $\mathbf{c}_{\text{CS}}$. As mentioned above, $\mathbf{c}_{\text{HY}}$ may be defined using the foliation constructed by Ye [16]. Let us recall the construction by Ye. For $r > 0$ large enough, we can find a perturbed rescaled center $\tau(r) \in \mathbb{R}^3$ and a real-valued function $\phi^{(r)}(z)$ on the unit sphere $\mathbb{S}^2$ such that the surface of the constant mean curvature in the foliation is given by

$$\tilde{\Sigma}_r = \left\{ r\left(z + \tau(r) + \phi^{(r)}(z)\nu_0(z)\right) \mid z \in \mathbb{S}^2 \right\} \tag{7}$$

which has constant mean curvature $\frac{2}{r} - \frac{4m}{r^2}$. Here ν_0 is the unit outward normal of unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. By [16], $|\tau(r)| \le Cr^{-1}$ and $|\phi^{(r)}|_{C^{2,\frac{1}{2}}(\mathbb{S}^2)} \le Cr^{-2}$. Huang [8] proved the following:

Proposition 1.

$$\lim_{r\to\infty} (\mathbf{c}_{\text{CS}}(r) - \mathbf{c}_{\text{HY}}(r)) = \mathbf{0}.$$

Proof. We sketch the proof here. Let $y = x - r\tau(r)$, and $y = rz$, $z \in \mathbb{S}^2$. So $x = r(z + \tau(r))$. Let $\tilde{\Sigma}_r$ be as in (7). Using the fact that $|\phi^{(r)}|_{C^{2,\frac{1}{2}}(\mathbb{S}^2)} = O(r^{-2})$, one can check that

$$\lim_{r\to\infty} (r\tau(r) - \mathbf{c}_{\text{HY}}(r)) = \mathbf{0}. \tag{8}$$

On the other hand, by [16, (1.14)], for $\alpha = 1, 2, 3$, $\tau = \tau(r)$ satisfies:

$$6mr\tau^\alpha + P_\alpha\left(rf(r,z,\tau) + rb_{ij}(z,\tau)\tau^i\tau^j + w\right) = 0 \tag{9}$$

where b_{ij} is smooth in (z,τ), $z \in \mathbb{S}^2$, $w = w(r,z)$ with $|w| = O(r^{-1})$, P_α is the L^2 projection of functions on $\mathbb{S}^2$ to the linear space spanned by z^α, and f is given by the following relation:

$$H(r,\tau(r),0) = \frac{2}{r} - \frac{4m}{r^2} + \frac{6mz\cdot\tau}{r^2} + \frac{1}{r^2}f(r,z,\tau(r)) + O(r^{-4}). \tag{10}$$

Here $H(r,\tau(r),0)$ is the mean curvature of the surface $\{|x - r\tau| = r\}$ with respect to g. $H(r,\tau(r),0)$ is given by (see [8, (5.1)])

$$\begin{aligned} H(r,\tau(r),0) = {} & \frac{2}{r} - \frac{4m}{r^2} + \frac{6mz\cdot\tau}{r^2} + \frac{9m^2}{r^3} \\ & + \frac{1}{2r^3}q_{ij,k}(y)y^iy^jy^k + \frac{2}{r^3}q_{ij}(y)y^iy^j \\ & - \frac{1}{r}\left(q_{ij,i}(y)y^j + q_{ii}(y) - \frac{1}{2}q_{ii,j}(y)y^j\right) + E \end{aligned} \tag{11}$$

where $E = O(r^{-4})$, $q_{ij} = p_{ij} + \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} - \left(1 + \frac{2m}{r}\right)\delta_{ij}$.

Hence by [8, Lemma 5.1]

$$
\begin{aligned}
(12) \qquad P_\alpha(rf) &= \frac{3}{4\pi}\int_{|z|=1} z^\alpha r f d\sigma_0 \\
&= \frac{3}{4\pi}\int_{|z|=1} z^\alpha r^3 \left(H(r,\tau(r),0) - \frac{2}{r} + \frac{4m}{r^2} - \frac{6mz\cdot\tau}{r^2}\right) d\sigma_0 \\
&\quad + O(r^{-1}) \\
&= -6m\, c^\alpha_{\text{CS}}(r) + O(r^{-1}).
\end{aligned}
$$

Combining this with (8) and (9), the result follows. □

2. A necessary and sufficient condition and an example

In this section we will give a condition so that $\mathbf{c}_{\text{CS}}$ and hence $\mathbf{c}_{\text{HY}}$ exists and give examples of AS manifolds so that the center of mass do not exist. The following result is a direct consequence of the computation in [4, section 5] by Corvino, in [5, p. 215] by Corvino-Schoen and in [2] by Beig-Ó Murchadha. However, we would like to state the result explicitly. We will sketch the proof in the appendix for the convenience for the readers.

Theorem 1. $\mathbf{c}_{\text{CS}}$ *exists if and only if* $\lim_{r\to\infty}\int_{B(r)} x^\alpha R_g dv_g$ *exists for* $\alpha = 1, 2, 3$, *where* R_g *is the scalar curvature of* g.

It was remarked by Huang [10], the theorem is still true for asymptotically flat metric $g_{ij} - \delta_{ij} = O_2(|x|^{-1})$ with AS condition replaced by the following more general Regge-Teitelboim type parity condition: $g^{\text{odd}}_{ij}(x) := g_{ij}(x) - g_{ij}(-x) \in O_2(|x|^{-2})$.

By the theorem, one may expect to find an example of AS metric so that $\mathbf{c}_{\text{CS}}$ and hence $\mathbf{c}_{\text{HY}}$ does not exist. In fact, one may construct such kind of examples in an elementary way.

To motivate the construction, let $\mathbf{b}$ be a nonzero vector in $\mathbb{R}^3$ and let g be the metric given by

$$
(13) \qquad g_{ij} = \left(1 + \frac{m}{2r} + \frac{\mathbf{b}\cdot\mathbf{x}}{r^3}\right)^4 \delta_{ij}
$$

with $m > 0$. Then it is well-known that $\mathbf{c}_{\text{CS}}$ for this metric is given by

$$
\mathbf{c}_{\text{CS}} = \frac{2\mathbf{b}}{m},
$$

see [6]. Let $\phi : [a, \infty) \to \mathbb{R}$ be a smooth bounded function. Consider the metric

$$g_{ij} = \left(1 + \frac{m}{2r} + \frac{\phi(r)\mathbf{b}\cdot\mathbf{x}}{r^3}\right)^4 \delta_{ij} \tag{14}$$

with $m > 0$. If $\phi(t)$ is oscillating near infinity, then one may expect that $\mathbf{c}_{\text{CS}}$ does not exist. More precisely, we have the following:

Theorem 2. *Let $a > 0$. Suppose $\phi : [a, \infty) \to \mathbb{R}$ is a smooth function such that for some constant C the following holds for all $t \geq a$, and $0 \leq l \leq 4$:*

$$|\phi^{(l)}| \leq \frac{C}{(1+t)^l}. \tag{15}$$

Then the metric given by (14) is AS outside $B_0(R)$ for some $R > 0$. Moreover, if $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{c}_{\text{CS}}$ exists if and only if $\lim_{t\to\infty}(3\phi(t) - t\phi'(t))$ exists. If $\lim_{t\to\infty}(3\phi(t) - t\phi'(t)) = \lambda$ exists, then

$$\mathbf{c}_{\text{CS}} = \frac{2\lambda\mathbf{b}}{3m}.$$

Remark 1. It is easy to construct ϕ satisfying (15), but the limit $\lim_{t\to\infty}(3\phi(t) - t\phi'(t))$ does not exist. For example, we may take $\phi(t) = \sin(\log(t))$ or $\phi(t) = \sin(\log(\log(t)))$. Similar examples for the nonexistence of center of mass have also been obtained independently by Cederbaum and Nerz [3, p.13], whom we thank for pointing this out to us.

Proof of Theorem 2. To simplify the notations, let

$$v = \frac{\phi(r)\mathbf{b}\cdot\mathbf{x}}{r^3}$$

and

$$u = 1 + \frac{m}{2r} + v.$$

Then $g_{ij} = u^4\delta_{ij}$. Now $|v| = O(r^{-2})$, and

$$\frac{\partial v}{\partial x^k} = \frac{1}{r^3}\left(\frac{x^k}{r}\phi'(r)\mathbf{b}\cdot\mathbf{x} + \phi(r)b^k - \frac{3x^k\phi(r)\mathbf{b}\cdot\mathbf{x}}{r^2}\right). \tag{16}$$

By the assumption (15), we have $|\partial v| = O(r^{-3})$. Similarly, one can prove that $|\partial^2 v| = O(r^{-4})$, $|\partial^3 v| = O(r^{-5})$, $|\partial^4 v| = O(r^{-6})$. From these, one can see that the metric g is well-defined and is AS.

Next, we want to compute $c^\alpha_{\text{CS}}(r)$. We have

$$\begin{aligned} g_{ij,k} =& 4u^3\frac{\partial u}{\partial x^k}\delta_{ij} \\ =& 4u^3\left[-\frac{mx^k}{2r^3}+\frac{1}{r^3}\left(\frac{x^k}{r}\phi'(r)\mathbf{b}\cdot\mathbf{x}+\phi(r)b^k-\frac{3x^k\phi(r)\mathbf{b}\cdot\mathbf{x}}{r^2}\right)\right]\delta_{ij} \\ =&: f_k\delta_{ij}. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{i,j}(g_{ij,i}-g_{ii,j})x^j \\ &= -2\sum_j f_j x_j \\ &= -8u^3\sum_j x^j\left[-\frac{mx^j}{2r^3}+\frac{1}{r^3}\left(\frac{x^j}{r}\phi'(r)\mathbf{b}\cdot\mathbf{x}+\phi(r)b^j-\frac{3x^j\phi(r)\mathbf{b}\cdot\mathbf{x}}{r^2}\right)\right] \\ &= -8u^3\left[-\frac{m}{2r}+\frac{(r\phi'(r)-2\phi(r))\mathbf{b}\cdot\mathbf{x}}{r^3}\right] \\ &= -8(1+\frac{3m}{2r})\left[-\frac{m}{2r}+\frac{(r\phi'(r)-2\phi(r))\mathbf{b}\cdot\mathbf{x}}{r^3}\right]+O(r^{-3}) \\ &= -8\left[-\frac{m}{2r}-\frac{3m^2}{4r^2}+\frac{(r\phi'(r)-2\phi(r))\mathbf{b}\cdot\mathbf{x}}{r^3}\right]+O(r^{-3}) \\ &= \frac{4m}{r}+\frac{6m^2}{r^2}-\frac{8(r\phi'(r)-2\phi(r))\mathbf{b}\cdot\mathbf{x}}{r^3}+O(r^{-3}). \end{aligned} \tag{17}$$

On the other hand, $h_{ij}=g_{ij}-\delta_{ij}=(u^4-1)\delta_{ij}$. Hence

$$\begin{aligned} \sum_i\left(h_{i\alpha}x^i-h_{ii}x^\alpha\right) &= -2(u^4-1)x^\alpha \\ &= -2x^\alpha\left(\frac{2m}{r}+\frac{3m^2}{2r^2}+\frac{4\phi(r)\mathbf{b}\cdot\mathbf{x}}{r^3}\right)+O(r^{-2}). \end{aligned} \tag{18}$$

So

$$\begin{aligned} & x^\alpha\sum_{i,j}(g_{ij,i}-g_{ii,j})x^j-\sum_i\left(h_{i\alpha}x^i-h_{ii}x^\alpha\right) \\ &= x^\alpha\left[\frac{8m}{r}+\frac{9m^2}{r^2}+\frac{8(3\phi(r)-r\phi'(r))\mathbf{b}\cdot\mathbf{x}}{r^3}\right]+O(r^{-2}). \end{aligned} \tag{19}$$

Hence

$$(20)\qquad \frac{1}{r}\int_{|x|=r}\left[x^\alpha(g_{ij,i}-g_{ii,j})x^j-\left(h_{i\alpha}x^i-h_{ii}x^\alpha\right)\right]d\sigma_0$$
$$=\frac{32\pi b^\alpha}{3}\left[3\phi(r)-r\phi'(r)\right]+O(r^{-1}).$$
$$c^\alpha_{\rm CS}(r)=\frac{2b^\alpha}{3m}\left[3\phi(r)-r\phi'(r)\right]+O(r^{-1}).$$

From this the results follow. □

Remark 2. (i) If $m<0$, the theorem is still true if we use the foliation of Ye [16] to define the center of mass as in (2) and (3).

(ii) One can check the examples in the theorem satisfy the property that $\mathbf{c}_{\rm CS}(r)$ remain bounded for all r. On the other hand, in [9], Huang constructed examples of asymptotically flat manifold so that $\mathbf{c}_{\rm CS}(r)\to\infty$.

3. Examples with nonnegative scalar curvature

The examples constructed in the previous section are very simple. However, there is a drawback. In fact, after the first draft of this work, Huang [10] asked whether there is an example with nonnegative scalar curvature. Wang [15] also pointed out that the above examples do not have nonnegative scalar curvature. For a time symmetric spacelike slice in a spacetime satisfying dominant energy condition, the scalar curvature of the slice must be nonnegative. Hence it is desirable to obtain examples of asymptotically Schwarzschild manifolds with nonnegative scalar curvature and yet the Huisken-Yau center of mass and Hamiltonian formulation of center of mass do not exist.

By [11], the scalar curvature of an asymptotically Schwarzschild manifold must decay like r^{-4}. Given a function f with this decay rate, it is not so difficult to construct a conformally flat and asymptotically flat manifold with scalar curvature being a positive constant times f. However, in order to obtain asymptotically Schwarzschild metric, we need an additional assumption on f. We begin with the following. Let $B_x(r)$ be the Euclidean ball with center at x and with radius r. dv_0 is the Euclidean volume element. As before $B_0(r)$ is the Euclidean ball with center at the origin.

Lemma 1. *Let f be a smooth function on $\mathbb{R}^3$.*

(a) *Suppose*

(i) $f = O(|x|^{-4})$; *and*
(ii) *there is a constant* $C > 0$ *such that*

$$\left|\int_{B_0(r)} x^\alpha f(x) dv_0\right| \le C \tag{21}$$

for $\alpha = 1, 2, 3$ *and for all* $r > 0$.
Then the Newtonian potential

$$\mathcal{V}(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) dv_0(y)$$

is well defined such that $\Delta\mathcal{V} = f$. *Moreover, near infinity* $\mathcal{V}(x) = \frac{c}{|x|} + w(x)$ *with* $w = O(|x|^{-2})$ *where*

$$c = -\frac{1}{4\pi}\int_{\mathbb{R}^3} f(x) dv_0.$$

(b) *Suppose in addition to (i) and (ii) in (a),* $f = O_3(|x|^{-4})$, *then* $w = O_4(|x|^{-2})$.

Proof. To prove (a), by (i) it is easy to see that $\mathcal{V}$ is well-defined and $\Delta\mathcal{V} = f$. We want find the asymptotically behavior of $\mathcal{V}$. For any $x \in \mathbb{R}^3$, let $r = |x|$. Suppose $r = |x| > 1$, we have:

$$\begin{aligned} -4\pi\mathcal{V}(x) &= \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) dv_0(y) \\ &= \left(\int_{B_x(\frac{r}{2})} + \int_{B_0(\frac{r}{2})} + \int_{\mathbb{R}^3\setminus(B_x(\frac{r}{2})\cup B_0(\frac{r}{2}))}\right) \frac{1}{|x-y|} f(y) dv_0(y) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{22}$$

Now

$$|\,\mathrm{I}\,| \le C_1 r^{-4}\int_{B_x(\frac{r}{2})} \frac{1}{|x-y|} dv_0(y) \le C_2 r^{-2}. \tag{23}$$

Here and below, C_i will denote a positive constant which is independent of x and r.

Since outside $B_x(\frac{r}{2})$, $|x-y|\ge\frac{r}{2}$, we have

$$|\mathrm{III}|\le C_3r^{-1}\int_{\mathbb{R}^3\setminus B_0(\frac{r}{2})}|y|^{-4}dv_0(y)\le C_4r^{-2}. \tag{24}$$

To estimate II, let $4\pi c=-\int_{\mathbb{R}^3}fdv_0$ which is well-defined and is finite by (i). For $r=|x|>1$,

$$\begin{aligned}\mathrm{II}+\frac{4\pi c}{r}&=-\frac{1}{r}\int_{\mathbb{R}^3\setminus B_0(\frac{r}{2})}fdv_0+\int_{B_0(\frac{r}{2})}\left(\frac{1}{|x-y|}-\frac{1}{|x|}\right)f(y)dv_0(y)\\&=\mathrm{IV}+\mathrm{V}.\end{aligned} \tag{25}$$

By (i) it is easy to see that

$$|\mathrm{IV}|\le C_5r^{-2}. \tag{26}$$

For $y\in B_0(\frac{r}{2})$, $|x-y|\ge\frac{r}{2}$, and we have

$$\begin{aligned}|\mathrm{V}|&=\left|\int_{B_0(\frac{r}{2})}\left(\frac{2x\cdot y-|y|^2}{|x|\,|x-y|\,(|x|+|x-y|)}\right)f(y)dv_0(y)\right|\\&\le\left|\int_{B_0(\frac{r}{2})}\left(\frac{2x\cdot y}{|x|\,|x-y|\,(|x|+|x-y|)}\right)f(y)dv_0(y)\right|+C_6r^{-2}.\end{aligned} \tag{27}$$

Now for $y\in B_0(\frac{r}{2})$,

$$\left|\frac{1}{|x|}-\frac{1}{|x-y|}\right|\le C_7r^{-2}|y|,$$

and

$$\left|\frac{1}{2|x|}-\frac{1}{|x|+|x-y|}\right|\le C_8r^{-2}|y|.$$

So

$$\begin{aligned}&\left|\int_{B_0(\frac{r}{2})}\left(\frac{2x\cdot y}{|x|\,|x-y|\,(|x|+|x-y|)}\right)f(y)dv_0(y)\right|\\&\le C_9r^{-2}+\sum_\alpha\frac{|x^\alpha|}{r^3}\left|\int_{B_0(\frac{r}{2})}y^\alpha f(y)dv_0(y)\right|\\&\le C_{10}r^{-2}\end{aligned} \tag{28}$$

by assumption (ii).

$$|\mathrm{V}|\le(C_{10}+C_6)r^{-2}. \tag{29}$$

By (22)–(26), (29), one can see that near infinity

$$\mathcal{V}(x) = \frac{c}{|x|} + w(x),$$

for some smooth $w(x)$ so that $|w(x)| = O(|x|^{-2})$. This proves (a).

To prove (b), suppose in addition $f = O_3(|x|^{-4})$. Since $\Delta \frac{1}{|x|} = 0$, we still have

$$\Delta w = f$$

near infinity. By the interior Schauder estimate [7, Theorem 3.9], the assumption that $f = O_3(|x|^{-4})$ and the fact that $w = O(|x|^{-2})$, we conclude that $|\partial w| = O(|x|^{-3})$. Differentiating the equation $\Delta w = f$ and apply the same theorem again, we have $|\partial\partial w| = O(|x|^{-4})$. Continue in this way, using the fact that $f = O_3(|x|^{-4})$, we conclude that $w = O_4(|x|^{-2})$. □

Remark 3. It is well known (see [1, 12] or (28)) that if we only assume that $|f(x)| \le C|x|^{-4}$ near infinity, then we can only have the following estimate of $\mathcal{V}$, see also :

$$\mathcal{V}(x) = \frac{c}{|x|} + O(|x|^{-2} \log |x|).$$

In any case, we still have $\mathcal{V}(x) = O(|x|^{-1})$.

Theorem 3. *Let $K(x) \ge 0$ with $K > 0$ somewhere be a smooth function on $\mathbb{R}^3$ satisfying the following:*

(i) $K(x) = O_3(|x|^{-4})$.

(ii) *There is a constant $C > 0$ such that*

$$\left| \int_{B_0(r)} x^\alpha K(x) dv_0 \right| \le C \tag{30}$$

for $\alpha = 1, 2, 3$ and for all $r > 0$.

Then there is a smooth positive function u such that near infinity $u(x) = 1 + \frac{m}{2|x|} + w(x)$ for some constant $m > 0$ with $w(x) = O_4(|x|^{-2})$ and such that the conformally flat asymptotically Schwarzschild metric

$$g_{ij} = u^4 \delta_{ij}$$

has scalar curvature $R_g(x) = aK(x)$ for some positive constant a. Moreover, $m = \frac{1}{16\pi} \int_{\mathbb{R}^3} aKu^5 dv_0$.

Proof. By [13, Theorem 1.4], there is a smooth positive solution u of

$$8\Delta u + Ku^5 = 0$$

such that $b \le u \le \frac{1}{b}$ for some constant $b > 0$ and such that u tends to a constant $c > 0$ near infinity. Since $|K(x)u^5(x)| \le C|x|^{-4}$ for some constant C for all x, by the proof of Lemma 1(a) and Remark 3 the Newtonian potential $\mathcal{V}$ of $-\frac{1}{8}Ku^5$ satisfies $\mathcal{V} = O(|x|^{-1})$. Hence $u - \mathcal{V}$ is a bounded harmonic function and must be constant which is equal to c because $u(x) \to c$ and $\mathcal{V}(x) \to 0$ as $x \to \infty$. Replacing, u by u/c, still denoted by u, we see that

$$8\Delta u + aKu^5 = 0$$

with $a = c^4$. Moreover, $u = 1 + v$, where $v = \frac{1}{c}\mathcal{V}$.

By [7, Theorem 3.9] and the facts that $aKu^5 = O(|x|^{-4})$ and that u is bounded, we conclude that $|\partial u| = O(|x|^{-1})$. From this we have $\partial(Ku^5) = O(|x|^{-5})$. Differentiating the equation $8\Delta u + aKu^5 = 0$ and apply the same theorem again, we have $|\partial\partial u| = O(|x|^{-2})$. Similarly, one can prove that $|\partial\partial\partial u| = O(|x|^{-3})$. Hence $u^5K = O_3(|x|^{-4})$.

On the other hand, we know that $u(x) = 1 + O(|x|^{-1})$ by Remark 3. Since $|K(x)| \le C(|x|^{-4})$, $|x^\alpha(u^5(x) - 1)K(x)|$ is integrable. By assumption (ii), we conclude that there is a constant C such that

$$\left| \int_{B_0(r)} x^\alpha K(x)u^5(x)dv_0 \right| \le C$$

for all $r > 0$ for $\alpha = 1, 2, 3$. By Lemma 1, we conclude that near infinity

$$u(x) = 1 + v(x) = 1 + \frac{m}{2|x|} + w(x)$$

with $w(x) = O_4(|x|^{-2})$ and $m = \frac{1}{16\pi}\int_{\mathbb{R}^3} aKu^5 dv_0$. Hence $g_{ij} = u^4\delta_{ij}$ is asymptotically Schwarzschild with scalar curvature $R_g(x) = aK(x)$. □

Combining this with Theorem 1, we have:

Corollary 1. *Suppose $K \ge 0$ is a smooth function on $\mathbb{R}^3$ satisfying the following:*

(i) $K(x) = O_3(|x|^{-4})$.

(ii) *There is a constant $C > 0$ such that*

$$\left| \int_{B_0(r)} x^\alpha K(x) dv_0 \right| \le C \tag{31}$$

for $\alpha = 1, 2, 3$ and for all $r > 0$.

(iii) *For some $\alpha = 1, 2, 3$, $\lim_{r\to\infty} \int_{B_0(r)} x^\alpha K(x) dv_0$ does not exist.*

Let u be the positive function obtained in Theorem 3. Then the asymptotically Schwarzschild metric $g_{ij} = u^4 \delta_{ij}$ will have nonnegative scalar curvature so that $\mathbf{c}_{\mathrm{CS}}$ and hence $\mathbf{c}_{\mathrm{HY}}$ does not exist.

By Corollary 1, in order to find an example of conformally flat asymptotically Schwarzschild metric on $\mathbb{R}^3$ with nonnegative scalar curvature so that $\mathbf{c}_{\mathrm{CS}}$ and hence $\mathbf{c}_{\mathrm{HY}}$ does not exist, it is sufficient to find $K(x)$ satisfying the assumptions of the Corollary.

Example: Let us construct $K(x)$ satisfying the conditions (i)–(iii) in Corollary 1. Let ϕ be a smooth function depending only on $r = |x|$ such that $\phi(x) = \frac{\cos(\log|x|)}{|x|^5}$ outside $|x| \ge 1$, say. Let η be another positive function depending only on $r = |x|$ such that $\eta(x) = \frac{1}{|x|^4}$ outside $r \ge 1$. Let $\mathbf{b}$ be a nonzero vector in $\mathbb{R}^3$. Then one can find a positive constant A such that

$$K(x) = \phi(x)\mathbf{b} \cdot x + A\eta(x)$$

is positive on $\mathbb{R}^3$. One can check that K satisfies (i) in Corollary 1. For $\alpha = 1, 2$, or 3, and for any $r > 0$,

$$\int_{B_0(r)} x^\alpha \eta(x) dv_0 = 0$$

because η depends only on r. Hence for r large enough

$$\begin{aligned}
\int_{B_0(r)} x^\alpha K(x) dv_0 &= \int_{B(r)} x^\alpha \phi(x)\mathbf{b} \cdot x dv_0 \\
&= \int_{B_0(1)} x^\alpha \phi(x)\mathbf{b} \cdot x dv_0 + b^\alpha \int_1^r \frac{\cos(\log t)}{t^5} \left(\int_{S(t)} (x^\alpha)^2 d\sigma_t \right) dt \\
&= \int_{B_0(1)} x^\alpha \phi(x)\mathbf{b} \cdot x dv_0 + b^\alpha \int_1^r \frac{\cos(\log t)}{t^5} \cdot \frac{4\pi t^4}{3} dt \\
&= \int_{B_0(1)} x^\alpha \phi(x)\mathbf{b} \cdot x dv_0 + \frac{4\pi b^\alpha}{3} \sin(\log r)
\end{aligned}$$

where $S(t) = \{x \in \mathbb{R}^3 | \ |x| = t\}$ and $d\sigma_t$ is the area element of $S(t)$. Hence K satisfies (ii) and (iii) in Corollary 1.

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Appendix A.

In this appendix we give a proof of Theorem 1:

Theorem 4. $\mathbf{c}_{\text{CS}}$ *exists if and only if* $\lim_{r\to\infty}\int_{B(r)} x^\alpha R_g dv_g$ *exists for* $\alpha = 1, 2, 3$.

Proof.

$$R_{ij} = \partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki}.$$

On an AS manifold, outside $B_0(R_0)$:

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} + p_{ij}$$

with $p_{ij} = O_4(r^{-2})$. Let

$$\bar{g}_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}$$

and let $\Gamma^k_{ij}, \bar{\Gamma}^k_{ij}$ be the Christoffel symbols for g and $\bar{g}$ respectively. Smoothly extend $(1 + \frac{2m}{r})$ as a positive function up to the origin, and denote it by u. By [11] and direct computations, we have

$$\begin{aligned} \Gamma^k_{ij} - \bar{\Gamma}^k_{ij} &= \frac{1}{2}\bar{g}^{sk}\left(p_{is,j} + p_{sj,i} - p_{ij,s}\right) \\ &\quad + \frac{1}{2}\left(g^{sk} - \bar{g}^{sk}\right)\left(g_{is,j} + g_{sj,i} - g_{ij,s}\right) \\ &= \frac{1}{2}\left(p_{ik,j} + p_{kj,i} - p_{ij,k}\right) + O(r^{-4}). \end{aligned} \tag{A.1}$$

Hence

$$\begin{aligned} |\Gamma^k_{ij} - \bar{\Gamma}^k_{ij}| &= O(r^{-3}); \\ |\partial(\Gamma^k_{ij} - \bar{\Gamma}^k_{ij})| + |R_{ij} - \bar{R}_{ij}| &= O(r^{-4}). \end{aligned} \tag{A.2}$$

In particular, $|R_g| = O(r^{-4})$ because the scalar curvature of $R_{\bar g}$ of $\bar g$ is zero near infinity. Let $B(R) = B_0(R)$.

$$\begin{aligned}\int_{B(R)} x^\alpha R_g dv_g &= \int_{B(R)} x^\alpha R_g dv_0 + \int_{B(R)} E dv_0\\&= \int_{B(R)} x^\alpha g^{ij} R_{ij} dv_0 + \int_{B(R)} E dv_0\\&= \int_{B(R)} x^\alpha \left(g^{ij} R_{ij} - \bar g^{ij} \bar R_{ij}\right) dv_0 + C + \int_{B(R)} E dv_0\\&= \int_{B(R)} x^\alpha \bar g^{ij} \left(R_{ij} - \bar R_{ij}\right) dv_0 + C + \int_{B(R)} E dv_0\\&= \int_{B(R)} x^\alpha u^4 \sum_i (R_{ii} - \bar R_{ii}) dv_0 + C + \int_{B(R)} E dv_0\end{aligned}$$

if R is large, where $C = \int_{\mathbb{R}^3} x^\alpha R_{\bar g} dv_g$ is a constant which may be nonzero and E denotes an error term with $|E|(x) = O(|x|^{-4})$ near infinity. Now

$$\left(\Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki}\right) - \left(\bar\Gamma^k_{kl}\bar\Gamma^l_{ji} - \bar\Gamma^k_{jl}\bar\Gamma^l_{ki}\right) = O(r^{-5}).$$

Hence

$$\begin{aligned}&\int_{B(R)} x^\alpha R_g dv_g \qquad \text{(A.3)}\\&= \int_{B(R)} x^\alpha \sum_i \left[\partial_k\left(\Gamma^k_{ii} - \bar\Gamma^k_{ii}\right) - \partial_i\left(\Gamma^k_{ki} - \bar\Gamma^k_{ki}\right)\right] dv_0\\&\quad + C + \int_{B(R)} E dv_0\\&= \int_{\partial B(R)} x^\alpha \left[\sum_{i,k} (\Gamma^k_{ii} - \bar\Gamma^k_{ii})\nu_0^k - \sum_{i,k} (\Gamma^k_{ki} - \bar\Gamma^k_{ki})\nu_0^i\right] d\sigma_0\\&\quad - \int_{B(R)} \left[\sum_i (\Gamma^\alpha_{ii} - \bar\Gamma^\alpha_{ii}) - \sum_k (\Gamma^k_{k\alpha} - \bar\Gamma^k_{k\alpha})\right] dv_0 + C + \int_{B(R)} E dv_0\\&= \int_{\partial B(R)} \left(x^\alpha \sum_{i,k} \left(p_{ik,i} - p_{ii,k}\right)\nu_0^k\right) d\sigma_0 - \int_{B(R)} \sum_i \left(p_{i\alpha,i} - p_{ii,\alpha}\right) dv_0\\&\quad + C + \int_{B(R)} E dv_0\end{aligned}$$

$$= \int_{\partial B(R)} \left[x^\alpha \sum_{i,k} (p_{ik,i} - p_{ii,k}) \nu_0^k - \sum_i \left(p_{i\alpha}\nu_0^i - p_{ii}\nu_0^\alpha \right) \right] d\sigma_0 + C + \int_{B(R)} E dv_0$$

$$= \int_{\partial B(R)} \left[x^\alpha \sum_{i,k} (g_{ik,i} - g_{ii,k}) \nu_0^k - \sum_i \left(h_{i\alpha}\nu_0^i - h_{ii}\nu_0^\alpha \right) \right] d\sigma_0 + C + \int_{B(R)} E dv_0,$$

where $d\sigma_0$ is the Euclidean area element of $\partial B(R)$. From this it is easy to see the theorem is true. □

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