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Some knots in $S^1 \times S^2$ with lens space surgeries

Kenneth L. Baker, Dorothy Buck, and Ana G. Lecuona

We propose a classification of knots in $S^1 \times S^2$ that admit a longitudinal surgery to a lens space. Any lens space obtainable by longitudinal surgery on some knot in $S^1 \times S^2$ may be obtained from a Berge-Gabai knot in a Heegaard solid torus of $S^1 \times S^2$, as observed by Rasmussen. We show that there are yet two other families of knots: those that lie on the fiber of a genus one fibered knot and the 'sporadic' knots. Assuming results of Cebanu, we are able to further conclude that these three families constitute all the doubly primitive knots in $S^1 \times S^2$. Thus we bring the classification of lens space surgeries on knots in $S^1 \times S^2$ in line with the Berge Conjecture about lens space surgeries on knots in S^3.

1. Introduction

A knot K in a 3–manifold M is *doubly primitive* if it may be embedded in a genus 2 Heegaard surface of M so that it represents a generator of each handlebody, i.e. in each handlebody there is a compressing disk that K transversally intersects exactly once. With such a doubly primitive presentation, surgery on K along the slope induced by the Heegaard surface yields a lens space. Berge introduced this concept of doubly primitive and provided twelve families (which partition into three coarser families) of knots in S^3 that are doubly primitive [Ber]. The Berge Conjecture asserts that if longitudinal surgery on a knot in S^3 produces a lens space, then that knot admits a presentation as a doubly primitive knot in a genus 2 Heegaard surface in S^3 in which the slope induced by the Heegaard surface is the surgery slope. This conjecture is often regarded as implicit in [Ber].

This conjecture has a prehistory fueled by the classification of lens space surgeries on torus knots [Mos71], notable examples of longitudinal lens space surgeries on non-torus knots [BR77, FS80], the Cyclic Surgery Theorem [CGLS87], the resolution of the Knot Complement problem [GL89a], several treatments of lens space surgeries on satellite knots [BL89, Wan89, Wu90], and the classification of surgeries on knots in solid tori producing solid tori [Gab89, Gab90, Ber91] to name a few. The modern techniques of Heegaard Floer homology [OS04b, OS04a] opened new approaches that reinvigorated the community's interest and gave way to deeper insights of positivity [Hed10], fiberedness [Ni07], and simplicity [OS05, Ras07, Hed11].

One remarkable turn is Greene's solution to the Lens Space Realization Problem [Gre13]. Utilizing the correction terms of Heegaard Floer homology [OS03], Greene adapts and enhances Lisca's lattice embedding ideas [Lis07] to determine not only which lens spaces may be obtained by surgery on a knot in S^3 but also the homology classes of the corresponding dual knots in those lens spaces. This gives the pleasant corollary that Berge's listing of doubly primitive knots in S^3 is complete.

Our present interest lies in the results of Lisca's work [Lis07] which, with an observation by Rasmussen [Gre13, Section 1.5], solves the $S^1 \times S^2$ version of the Lens Space Realization Problem. That is, the lens spaces which bound rational homology 4–balls as determined by Lisca may each be obtained by longitudinal surgery on some knot in $S^1 \times S^2$. (Note that if a lens space results from longitudinal surgery on a knot in $S^1 \times S^2$ then it necessarily bounds a rational homology 4–ball.) In fact, as Rasmussen observed, the standard embeddings into $S^1 \times S^2$ of the Berge-Gabai knots in solid tori with longitudinal surgeries yielding solid tori suffice. Due to the uniqueness of lattice embeddings in Lisca's situation versus the flexibility of lattice embeddings in his situation, Greene had initially conjectured that these accounted for all knots in $S^1 \times S^2$ with lens space surgeries [Gre13]. In this article we show that there are yet two more families of knots. Blending our work with the thesis and further work in progress of Cebanu [Ceb13, Ceb] we bring the status of the classification of knots in $S^1 \times S^2$ with lens space surgeries in line with the present state of the Berge Conjecture. The main purpose of this article is to propose such a classification of knots and provide the foundation that, together with Cebanu's work, shows our knots constitute all the doubly-primitive knots in $S^1 \times S^2$. (Please refer to Section 1.6 for basic definitions and notation.)

Conjecture 1.1 (Cf. Conjecture 1.9 [Gre13]). *The knots in $S^1 \times S^2$ with a longitudinal surgery producing a lens space are all doubly-primitive.*

Theorem 1.13. *A doubly-primitive knot in $S^1 \times S^2$ is either a Berge-Gabai knot, a knot that embeds in the fiber of a genus one fibered knot, or a sporadic knot.*

The three families of knots in Theorem 1.13 are analogous to the three coarse families of Berge's doubly primitive knots in S^3 and will be described below.

1.1. Lens spaces obtained by surgery on knots in $S^1 \times S^2$

Lisca determines whether a 3–dimensional lens space bounds a 4–dimensional rational homology ball by studying the embeddings into the standard diagonal intersection lattice of the intersection lattice of the canonical plumbing manifold bounding that lens space, [Lis07]. From this and that lens spaces are the double branched covers of two-bridge links, Lisca obtains a classification of which two-bridge *knots* in $S^3 = \partial B^4$ bound smooth disks in B^4 (i.e. are slice) and which two-component two-bridge links bound a smooth disjoint union of a disk and a Möbius band in B^4. As part of doing so, he demonstrates that in the projection to S^3 these surfaces may be taken to have only ribbon singularities. Indeed he shows this by using a single banding to transform these two-bridge links into the unlink, except for two families: one for which he uses two bandings and another which was overlooked.

Via double branched covers and the Montesinos Trick, the operation of a banding lifts to the operation of a longitudinal surgery on a knot in the double branched cover of the original link. Since the double branched cover of the two component unlink is $S^1 \times S^2$, Lisca's work shows in many cases that the lens spaces bounding rational homology balls contain a knot on which longitudinal surgery produces $S^1 \times S^2$. As mentioned above, the lens spaces bounding rational homology balls are precisely those that contain a knot on which longitudinal surgery produces $S^1 \times S^2$: Greene notes Rasmussen had observed that Lisca's list of lens spaces corresponds to those that may be obtained from considering the Berge-Gabai knots in solid tori with a solid torus surgery [Ber91, Gab89] as residing in a Heegaard solid torus of $S^1 \times S^2$, [Gre13, Section 1.5]. By appealing to the classification of lens spaces up to homeomorphisms we may condense the statement as follows:

Theorem 1.2 (Rasmussen via [Gre13, Section 1.5] + Lisca [Lis07]). *The lens space L may be transformed into $S^1 \times S^2$ by longitudinal surgery on a knot if and only if there are integers m and d such that L is homeomorphic to one of the four lens spaces:*

1) $L(m^2, md+1)$ *such that* $\gcd(m,d) = 1$*;*
2) $L(m^2, md+1)$ *such that* $\gcd(m,d) = 2$*;*
3) $L(m^2, d(m-1))$ *such that d is odd and divides $m-1$; or*
4) $L(m^2, d(m-1))$ *such that d divides* $2m+1$*.*

Note that we do permit m and d to be negative integers. We will augment this theorem in Theorem 1.7 with the homology classes known to contain the knots dual to these longitudinal surgeries from $S^1 \times S^2$.

Since the Berge-Gabai knots in solid tori all have tunnel number one, the corresponding knots in $S^1 \times S^2$ are strongly invertible. Quotienting by this strong inversion gives the analogous result for bandings of two-bridge links to the two component unlink.

Corollary 1.3. *The two-bridge link K may be transformed into the two component unlink by a single banding if and only if there are integers m and d such that K is homeomorphic to one of the four two-bridge links:*

1) $K(m^2, md+1)$ *such that* $\gcd(m,d) = 1$*;*
2) $K(m^2, md+1)$ *such that* $\gcd(m,d) = 2$*;*
3) $K(m^2, d(m-1))$ *such that d is odd and divides $m-1$; or*
4) $K(m^2, d(m-1))$ *such that d divides* $2m+1$*,*

In section 2.1, using tangle versions of the Berge-Gabai knots, we offer a direct proof of this corollary from which one may obtain Theorem 1.2 through double branched covers. Of course the content is tantamount to Rasmussen's observation.

Remark 1.4. By an oversight in the statement of [Lis07, Lemma,7.2], a family of strings of integers was left out though they are produced by the proof (cf. [Lec12, Footnote p. 247]). The use of this lemma in [Lis07, Lemma 9.3] causes the second family in Theorem 1.2 above to be missing from [Lis07, Definition 1.1]. Consequentially, the associated two-bridge links (these necessarily have two components) were also not shown to bound a disjoint union of a disk and a Möbius band in B^4 in that article.

We next consider which knots in $S^1 \times S^2$ admit longitudinal surgeries producing the lens spaces of Theorem 1.2. Prompted by the uniqueness of Lisca's lattice embeddings (which follows from [Lis07, Lemma 2.4 and Theorem 6.4]) and seemingly justified by Rasmussen's observation, Greene had originally conjectured that if a knot in $S^1 \times S^2$ admits a longitudinal lens space surgery, then it arises from a Berge-Gabai knot in a Heegaard solid torus of $S^1 \times S^2$, [Gre13, Conjecture 1.8]. These knots belong to five families which we call BGI, BGII, BGIII, BGIV, and BGV and refer to collectively as the family BG of *Berge-Gabai knots*. They are all doubly primitive knots. We show Greene's original conjecture is false, Theorem 1.12, by exhibiting two new families of doubly primitive knots in $S^1 \times S^2$: GOFK of knots that embed in the fiber of a genus one fibered knot and SPOR of 'sporadic' knots that generically don't belong to either of the other two families. Our classification of homology classes of the surgery duals to these three families, Theorem 1.7, shows that indeed the GOFK and SPOR families contain knots that do not belong to the BG family, yielding Theorem 1.12. Let us note that Yamada had previously observed the GOFK family of knots and their lens space surgeries [Yam07]. Using the thesis and further work in progress of Cebanu, Theorem 1.13 shows that these three families constitute all the doubly primitive knots. Conjecture 1.1 (also presented as Conjecture 1.9 in [Gre13]) updates Greene's original conjecture with these knots.

In order to fully state Theorem 1.7 we must first discuss simple knots in lens spaces. This is also the setting for a key ingredient, Theorem 1.6 in our characterization of doubly primitive knots in $S^1 \times S^2$.

1.2. Simple knots

A $(1,1)$–*knot* is a knot K that admits a presentation as a 1–bridge knot with respect to a genus 1 Heegaard splitting of the manifold M that contains it. That is, M may be presented as the union of two solid tori V_α and V_β in which each $K \cap V_\alpha$ and $K \cap V_\beta$ is a boundary parallel arc in the respective solid torus. We say K is *simple* if furthermore there are meridional disks of V_α and V_β whose boundaries intersect minimally in the common torus $V_\alpha \cap V_\beta$ in M such that each arc $K \cap V_\alpha$ and $K \cap V_\beta$ is disjoint from these meridional disks. One may show there is a unique (oriented) simple knot in each (torsion) homology class of a lens space. Let us write $K(p, q, k)$ for the simple knot in $M = L(p,q)$ oriented so that it represents the homology class $k\mu$ where (for a choice of orientation) μ is the homology class of the core curve of one of the Heegaard solid tori and $q\mu$ is the homology class of the other. Observe that trivial knots are simple knots and, as such, permits both S^3 and $S^1 \times S^2$ to have a simple knot. There are no simple knots representing the non-torsion homology classes of $S^1 \times S^2$. Non-trivial simple knots have also been called *grid number one* knots, e.g. in [BGH08, BG09] among others.

The Homma-Ochiai-Takahashi recognition algorithm for S^3 among genus 2 Heegaard diagrams [HOT80] says that a genus 2 Heegaard diagram of S^3 is either the standard one or contains what is called a *wave*, see Section 4. A wave in a Heegaard diagram indicates the existence of a handle slide that will produce a new Heegaard diagram for the same manifold with fewer crossings. As employed by Berge [Ber], the existence of waves ultimately tells us that any $(1,1)$–knot in a lens space with a longitudinal S^3 surgery is isotopic to a simple knot. As the dual to a doubly primitive knot is necessarily a $(1,1)$–knot, it follows that the dual to a doubly primitive knot in S^3 is a simple knot in the resulting lens space.

Negami-Okita's study of reductions of diagrams of 3–bridge links gives insights to the existence of wave moves on genus 2 Heegaard diagrams.

Theorem 1.5 (Negami-Okita [NO85]). *Every Heegaard diagram of genus 2 for $S^1 \times S^2 \# L(p,q)$ may be transformed into one of the standard ones by a finite sequence of wave moves.*

Here, a *standard* genus 2 Heegaard diagram $H = (\Sigma, \{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\})$ for $S^1 \times S^2 \# L(p,q)$ is one for which α_1 and β_1 are parallel and disjoint from $\alpha_2 \cup \beta_2$, and $\alpha_2 \cap \beta_2$ consists of exactly p points. If $p \neq 1$, then the standard diagrams are not unique. For our case at hand however, $p = 1$ and the standard diagram is unique (up to homeomorphism). This enables a proof of a result analogous to Berge's.

Theorem 1.6.

1) *A $(1,1)$–knot in a lens space with a longitudinal $S^1 \times S^2$ surgery is a simple knot.*
2) *The dual to a doubly primitive knot in $S^1 \times S^2$ is a simple knot in the corresponding lens space.*

A proof of this follows similarly to Berge's proof for doubly primitive knots in S^3 and their duals, though there is a technical issue one ought mind. We will highlight this as we sketch the argument of a more general result, Theorem 4.1, in section 4 following Saito's treatment of Berge's work in the appendix of [Sai08].

1.3. The known knots in lens spaces with longitudinal $S^1 \times S^2$ surgeries.

Since our knots in families BG, GOFK, and SPOR are all doubly primitive, then Theorem 1.6 implies their lens space surgery duals are simple knots. In particular, this means these duals are all at most 1–bridge with respect to the Heegaard torus of the lens space, and thus they admit a nice presentation in terms of linear chain link surgery descriptions of the lens space. This surgery description (which we first obtained by simplifying ones suggested by the lattice embeddings) facilitates the calculation of the homology classes of these dual knots and hence their descriptions as simple knots.

Given the lens space $L(p,q)$, let μ and μ' be homology classes of the core curves of the Heegaard solid tori oriented so that $\mu' = q\mu$. The homology class of a knot in $L(p,q)$ is given as its multiple of μ.

Theorem 1.7. *The lens spaces $L(m^2,q)$ of Theorem 1.2 may be obtained by longitudinal surgeries on the following simple knots $K = K(m^2,q,k)$ listed below.*

1) $q = md+1$ *such that* $\gcd(m,d) = 1$ *and either*
   - $k = \pm m$ *so that K is the dual to a* BGI *knot or*
   - $k = \pm dm$ *so that K is the dual to a* GOFK *knot;*
2) $q = md+1$ *such that* $\gcd(m,d) = 2$ *and*
   - $k = \pm m$ *so that K is the dual to a* BGII *knot;*
3) $q = d(m-1)$ *such that d is odd and divides $m-1$ and either*
   - $k = \pm dm$ *so that K is the dual to a* BGIII *knot,*
   - $k = \pm m$ *so that K is the dual to a* BGV *knot, or*
   - $k = \pm 2m = \pm 4m$ *so that K is the dual to a* SPOR *knot if* $m = 1-2d$*; or*
4) $q = d(m-1)$ *such that d divides $2m+1$ and*
   - $k = \pm m$ *or* $k = \pm dm$ *so that K (in each case) is the dual to a* BGIV *knot.*

Theorem 1.8 (Cebanu [Ceb13, Ceb]). *If a knot in a lens space L represents the homology class $\kappa \in H_1(L)$ and admits a longitudinal surgery to*

$S^1 \times S^2$ then, up to homeomorphism, $L = L(m^2, q)$ and $\kappa = k\mu$ are as in some case of Theorem 1.7.

Remark 1.9. Let us outline an argument for the proof of Theorem 1.8 that flows slightly differently than what's in [Ceb13]. Assume K is a knot in a lens space with a longitudinal surgery to $S^1 \times S^2$. Up to homeomorphism, K is in one of Lisca's lens spaces $L(m^2, q)$ listed in Theorem 1.2, and homological arguments show that K has order m in homology. The problem is then to determine which of the homology classes of order m could contain K.

Cebanu shows in his thesis that K must have genus 0 [Ceb13, Theorem 3.4.1 and Section 3.5]. Since a simple knot in a lens space minimizes genus in its homology class [NW14], it is sufficient to identify the simple knots of order m in Lisca's lens spaces $L(m^2, q)$ that have genus 0 and then observe that their homology classes coincide with the classes listed in Theorem 1.7.

Calculating the genus of a simple knot in any given homology class in a lens space is a straightforward application of a formula of Ni [Ni09]. However, determining which homology classes have simple knots of genus 0 for infinite families of lens spaces becomes a tricky elementary number theory problem. By ad hoc means, Cebanu solves this problem for the homology classes of order m in the first two families of Lisca's lens spaces in his thesis [Ceb13, Theorem 4.8.5] and has claimed to have since done this for the remaining two families [Ceb].

Remark 1.10. While Greene's work on lattice embeddings produced a classification of the homology classes of knots in lens spaces with a longitudinal S^3 surgery [Gre13], Lisca's work on lattice embeddings does not appear to produce information about the classification of homology classes of knots in lens spaces with longitudinal $S^1 \times S^2$ surgeries [Lis07].

Remark 1.11. In [DIMS12], the authors classify the strongly invertible knots in $L((2n-1)^2, 2n)$ with longitudinal $S^1 \times S^2$ surgeries. These belong to the first case of Theorem 1.7 with $m = 2n - 1$ and $d = 1$. These knots may be realized as both BGI knots and as GOFK knots.

Theorem 1.12. *The two families* GOFK *and* SPOR *of knots in $S^1 \times S^2$ that admit a longitudinal lens space surgery, generically do not arise from Berge-Gabai knots.*

Proof. As stated earlier, this follows from the listing of the homology classes in Theorem 1.7 of the surgery duals to the knots in our three coarse families of doubly primitive knots. This is examined explicitly for the SPOR family in Proposition 3.5. Alternatively, for the GOFK family, Proposition 2.2 shows that generically these knots are hyperbolic and "most" have volume greater than the hyperbolic BG knots. □

Theorem 1.13. *A doubly-primitive knot in $S^1 \times S^2$ is either a Berge-Gabai knot, a knot that embeds in the fiber of a genus one fibered knot, or a sporadic knot.*

Proof. Theorem 1.8 confirms that the only homology classes in lens spaces that may be realized by surgery duals to knots in $S^1 \times S^2$ are those described in Theorem 1.7. In particular, each of these homology classes are represented by simple knots in lens spaces that are surgery dual to a doubly-primitive knot in family BG, GOFK, or SPOR. Theorem 1.6 implies that the surgery dual to a doubly-primitive knot in $S^1 \times S^2$ (with its doubly-primitive surgery) is a simple knot in the resulting lens space. Since each first homology class in a lens space has a unique simple knot, the result follows. □

Remark 1.14. Conjecture 1.1 then asserts that a knot in $S^1 \times S^2$ with a lens space surgery belongs to family BG, GOFK, or SPOR.

Furthermore, Conjecture 1.1 may be rephrased as saying if a knot K in a lens space L admits a longitudinal $S^1 \times S^2$ surgery, then up to homeomorphism $L = L(p, q)$ and $[K] = \kappa = k\mu$ in some case of Theorem 1.7 and moreover $K = K(p, q, k)$.

1.4. Fibered knots and spherical braids

Ni shows that knots in S^3 with a lens space surgery have fibered exterior [Ni07]. One expects the same to be true for any knot in $S^1 \times S^2$ with a lens space surgery. Using knot Floer Homology, Cebanu shows this is indeed the case.

Theorem 1.15 ([Ceb13, Theorem 3.7.1]). *A knot in $S^1 \times S^2$ with a longitudinal lens space surgery has fibered exterior.*

Prior to learning of Cebanu's results, we had confirmed this for all our doubly primitive knots by explicitly showing they are spherical braids. A link in $S^1 \times S^2$ is a *(closed) spherical braid* if it is transverse to $\{\theta\} \times S^2$ for each $\theta \in S^1$. Braiding characterizes fiberedness for non-null homologous knots in $S^1 \times S^2$.

Lemma 1.16. *A non-null homologous knot in $S^1 \times S^2$ has fibered exterior if and only if it is isotopic to a spherical braid.*

Proof. If K is a spherical braid, then the punctured spheres $(\{\theta\} \times S^2) - N(K)$ for $\theta \in S^1$ fiber the exterior of K.

Now let $X(K) = S^1 \times S^2 - N(K)$ denote the exterior of a knot K in $S^1 \times S^2$. If K is non-null homologous in $S^1 \times S^2$, then the algebraic intersection number of K with a level sphere (its winding number) must be non-zero. Hence the kernel of the map $H_1(\partial X(K)) \to H_1(X(K))$ induced by inclusion is generated by a non-zero multiple of the meridian of K. Therefore, if $X(K)$ is fibered, then the image of $H_2(X(K), \partial X(K)) \to H_1(\partial X(K))$, which is generated by the boundary of a fiber, is a collection of oriented meridional curves. Moreover, these meridional curves are coherently oriented since the fibration of $X(K)$ must restrict to a fibration of $\partial X(K)$. Therefore the trivial (meridional) filling of $X(K)$ which returns $S^1 \times S^2$ must also produce a fibration over S^1 by closed surfaces, the capped off fibers of $X(K)$. Hence the fibers of $X(K)$ must be punctured spheres, and so K is isotopic to a spherical braid. □

Remark 1.17. Together Theorem 1.15 and Lemma 1.16 suggest the study of surgery on spherical braids.

One may care to compare these results with Gabai's resolution of Property R [Gab87]: The only knot in $S^1 \times S^2$ with a surgery yielding S^3 is a fiber $S^1 \times *$.

1.5. Geometries of knots and lens space surgeries

For completeness, here we address the classification of lens space Dehn surgeries on non-hyperbolic knots in $S^1 \times S^2$.

Theorem 1.18. *A non-hyperbolic knot in $S^1 \times S^2$ with a non-trivial lens space surgery is either a (p, q)–torus knot or a $(2, \pm 1)$–cable of a (p, q)–torus knot.*

Proof. We say a knot K in a 3–manifold M is either spherical, toroidal, Seifert fibered, or hyperbolic if its exterior $M - N(K)$ contains an essential embedded sphere, contains an essential embedded torus, admits a Seifert fibration, or is hyperbolic respectively. By Geometrization for Haken manifolds, a knot K in $S^1 \times S^2$ is (at least) one of these. By the Cyclic Surgery Theorem [CGLS87], if a non-trivial knot K admits a non-trivial lens space surgery, then either the surgery is longitudinal or K is Seifert fibered.

If K is spherical, then one may find a separating essential sphere in the exterior of K. Since $S^1 \times S^2$ is irreducible, K is contained in a ball. Therefore K only admits a lens space surgery (in fact an $S^1 \times S^2$ surgery) if K is the trivial knot, [Gab87, GL89b].

If K is Seifert fibered then K is a torus knot. This follows from the classification of generalized Seifert fibrations of $S^1 \times S^2$, [JN83]. Note that the exceptional generalized fiber of $M(-1;(0,1))$ is a regular fiber of $M(0;(2,1),(2,-1))$, and its exterior is homeomorphic to both the twisted circle bundle over the Möbius band and the twisted interval bundle over the Klein bottle. For relatively prime integers p, q with $p \geq 0$, we define a (p,q)–torus knot $T_{p,q}$ in $S^1 \times S^2$ to be a regular fiber of the generalized Seifert fibration $M(0;(p,q),(p,-q))$. (The exceptional fibers may be identified with $S^1 \times$ N and $S^1 \times$ S for antipodal points N, S $\in S^2$.) Equivalently, we may regard $T_{p,q}$ as a curve on $\partial N(S^1 \times *)$, $* \in S^2$, that is homologous to $q\mu + p\lambda \in H_1(\partial N(S^1 \times *))$ for meridian-longitude classes μ, λ and an appropriate choice of orientation on $T_{p,q}$. Observe $[T_{p,q}] = p[S^1 \times *] \in H_1(S^1 \times S^2)$ and $T_{p,q} = T_{p,q+Np}$ for any integer N. Following [Mos71, Gor83] (though note that on the boundary of a solid torus, a (p,q) curve for them is a (q,p) curve for us) any non-trivial lens space surgery on a (p,q)–torus knot with $p \geq 2$ has surgery slope $1/n$, taken with respect to the framing induced by the Heegaard torus, and yields the lens space $L(np^2, npq+1)$.

If K is toroidal (and not spherical) then there is an essential torus T in the exterior of K which compresses in $S^1 \times S^2$. Thus in $S^1 \times S^2$, this torus must either be non-separating or bound a ∂–reducible manifold M that contains K. If T is a non-separating torus in $S^1 \times S^2$, then any lens space obtained by surgery on K must also contain a non-separating torus and hence also be $S^1 \times S^2$. Gabai shows that this implies K is a trivial knot [Gab87], but then T would have a compression disjoint from K. So let us assume T bounds a ∂–reducible manifold M that contains K. Furthermore, since T is essential and K is not contained in a ball, $M - K$ is irreducible and ∂–irreducible. If surgery on K produces the lens space Y with surgery dual knot K', then the image of T in Y bounds a ∂–reducible manifold M' that contains K' (because T must compress in Y' while being essential in the exterior of K').

If M is a solid torus, then the proof in [BL89] applies basically unaltered (because Seifert fibered knots in $S^1 \times S^2$ are torus knots and lens spaces are atoroidal) to show K must be a $(2, \pm 1)$–cable of a torus knot. Here the cable is taken with respect to the framing on the torus knot induced by the Heegaard torus. In this situation ± 1 surgery on K with respect to its framing as a cable is equivalent to $\pm 1/4$ surgery on the (p,q)–torus knot and thus yields $L(4p^2, 4pq \pm 1)$ or its mirror.

If M is not a solid torus then $M = S^1 \times S^2 \# S^1 \times D^2$ and T bounds the exterior X of a non-trivial knot in S^3 for which a meridian in $T = \partial X$ bounds a compressing disk D in M. Gabai shows that if M' were a solid torus then K' would have to be disjoint from a compressing disk for T [Gab89] and hence T would not be essential in the exterior of K; hence $M' = Y \# S^1 \times D^2$. Scharlemann then implies that K must be isotopic into T [Sch90], but this now contradicts that K is not contained in a ball. □

Corollary 1.19.

- *The smallest order lens space obtained by surgery on a toroidal knot in $S^1 \times S^2$ is homeomorphic to $L(4,1)$. The surgery dual is the simple knot $K(4,1,2)$.*
- *The smallest order lens space obtained by surgery on a non-torus, toroidal knot in $S^1 \times S^2$ is homeomorphic to $L(16,9)$. The surgery dual is the (unoriented) simple knot $K(16,9,4)$.*

Note that the orientation of a knot has no bearing upon its surgeries. Ignoring orientations, $K(16,9,4)$ is equivalent to $K(16,9,12)$. The simple knot $K(4,1,2)$ is isotopic to its own orientation reverse.

Also note, because the $(2,1)$–torus knot in $S^1 \times S^2$ contains an essential Klein bottle in its exterior, it is toroidal.

Bleiler-Litherland conjecture that the smallest order lens space obtained by surgery on a hyperbolic knot in S^3 is homeomorphic to $L(18,5)$ [BL89]. Among our list of doubly primitive knots in $S^1 \times S^2$, up to homeomorphism, we find three hyperbolic knots of order 5 in families BGIII, BGV, and GOFK with integral lens space surgeries; all the doubly primitive knots with smaller order are non-hyperbolic.

Conjecture 1.20. *Up to homeomorphism, $L(25,7)$ and $L(25,9)$ are the smallest order lens spaces obtained by surgery on a hyperbolic knot in $S^1 \times S^2$. Moreover, the surgeries occur on the knots shown with their corresponding doubly primitive family in Figure 1. From left to right, their surgery duals are the (unoriented) simple knots $K(25,7,5)$, $K(25,7,10)$, and $K(25,9,10)$ respectively.*

Remark 1.21. The knots on the right and left of Figure 1 are actually isotopic. Kadokami-Yamada show that among the non-torus GOFK knots this is the only knot (up to homeomorphism) that admits two non-trivial lens space surgeries [KY14]. Along these lines, Berge shows there is a unique hyperbolic knot in the solid torus with two non-trivial lens space surgeries [Ber91], and this gives rise to a single BGIV knot (up to homeomorphism) having surgeries to both orientations of $L(49, 18)$ [BHW99].

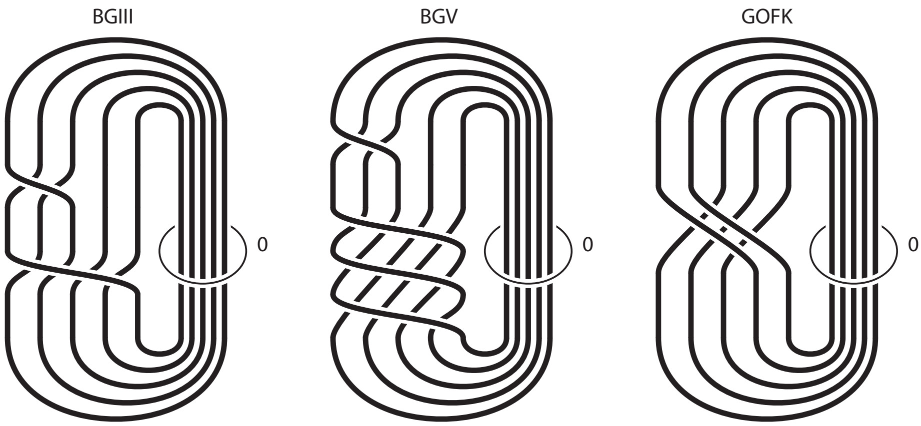

Figure 1: Three presentations of hyperbolic knots in $S^1 \times S^2$ with surgeries to lens spaces of order 25. The ones on the left and right are actually isotopic and have two distinct lens space surgeries.

1.6. Basic definitions and some notation

1.6.1. Dehn surgery. Consider a knot K in a closed 3–manifold M with regular solid torus neighborhood $N(K)$. The isotopy classes of essential simple closed curves on $\partial N(K) \cong \partial(M - N(K))$ are called *slopes*. The *meridian* of K is the slope that bounds a disk in $N(K)$ while the slopes that algebraically intersect the meridian once (and hence are isotopic to K in $N(K)$) are *longitudes*. Given a slope γ, the manifold obtained by removing the solid torus $N(K)$ from M and attaching another solid torus so that γ is its meridian is the result of γ*–Dehn surgery on* K. The core of the attached solid torus is a new knot in the resulting manifold and is the *surgery dual* to K. If γ is a longitude, then γ–Dehn surgery is a *longitudinal surgery* or simply a *surgery*. Fixing a choice of longitude and orienting both the meridian and this longitude so that they represent homology classes μ and λ in $H_1(\partial N(K))$ with $\mu \cdot \lambda = +1$ enables a parametrization associating the slope γ to the extended rational number $\frac{p}{q} \in \mathbb{Q} \cup \{1/0\}$, $\gcd(p,q) = 1$, if for some orientation $[\gamma] = p\lambda + q\mu$. Then γ–Dehn surgery may also be denoted $\frac{p}{q}$–Dehn surgery. Consequentially longitudinal surgery is also called *integral surgery.*

1.6.2. Tangles and bandings. The knot K in M is said to be *strongly invertible* if there is an involution u on M that set-wise fixes K and whose fixed set intersects K exactly twice (and thus the fixed set is a non-empty link), and the involution is said to be a *strong involution*. The quotient of M by u is a 3–manifold $M/\sim$, where $x \sim u(x)$, in which the fixed set of u descends to a link J and the knot K descends to an embedded arc α such that $J \cap \alpha = \partial\alpha$. A small ball neighborhood $B = N(\alpha)$ of α intersects J in a pair of arcs t so that (B,t) is a *rational tangle*, i.e. a tangle in a ball homeomorphic to $(D^2 \times I, \{\pm\frac{1}{2}\} \times I)$ where D^2 is the unit disk in the complex plane and I is the interval $[-1,1]$. The solid torus neighborhood $N(K)$ of K may be chosen so that the image of its quotient under u is B, and equivalently so that it is the double cover of B branched along t. The *Montesinos Trick* refers to the correspondence through branched double covers and quotients by strong involutions between replacing the rational tangle (B,t) with another and Dehn surgery on K. In particular, a *banding* of J along the arc α corresponds to longitudinal surgery on K. A banding of J is the act of embedding of a rectangle $I \times I$ in $M/\sim$ to meet J in the pair of opposite edges $I \times \partial I$ and exchanging those sub-arcs of J for the other pair of opposite edges $\partial I \times I$. The banding occurs along an arc $\alpha = \{0\} \times I$ and the banding produces the dual arc $I \times \{0\}$. Figure 2 illustrates the banding operation and both its framed arc and literal band depictions that we use in this article. Figure 3 shows how a rectangular box labeled with an integer n denotes a sequence of $|n|$ twists in the longer direction. The twists are right handed if $n > 0$ and left handed if $n < 0$. To highlight cancellations, a pair of twist boxes will be colored the same if their labels have opposite sign.

1.6.3. Lens spaces, two bridge links, plumbing manifolds. The lens space $L(p,q)$ is defined as the result of $-p/q$–Dehn surgery on the unknot in S^3. The lens space $L(p,q)$ may be obtained by surgery on the linear chain link as shown at the top of Figure 4 with integral surgery coefficients $-a_1, \ldots, -a_n$ that are the negatives of the coefficients of a continued fraction

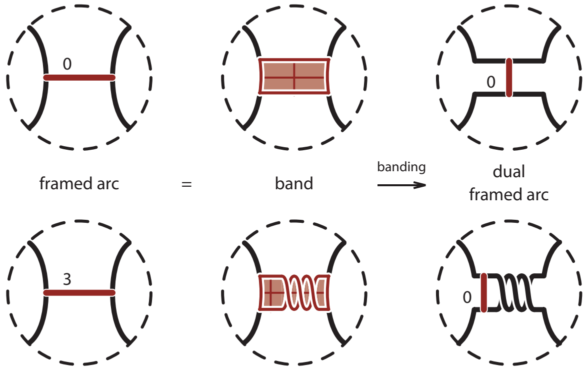

Figure 2: Two examples of bandings along a framed arc.

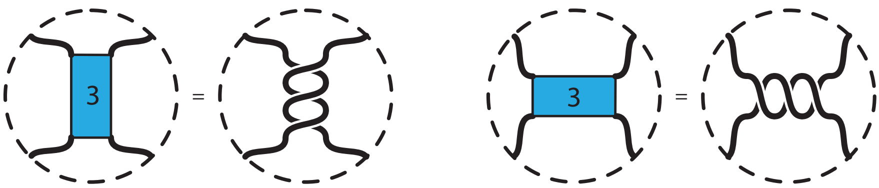

Figure 3: Sequences of twists are represented with oblong rectangles and integers.

expansion

$$\frac{p}{q} = [a_1, \dots, a_n]^- = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots \, a_{n-1} - \cfrac{1}{a_n}}}.$$

The picture of this chain link also shows the axis of a strong involution u that extends through the surgery to an involution of the lens space. The quotient of this involution of the lens space, via the involution of this surgery diagram, is S^3 in which the axis descends to the two-bridge link $K(p, q)$ with the diagram $L(-a_1, \dots, -a_n)$ as shown at the bottom of Figure 4. The orientation preserving double cover of S^3 branched over $K(p, q)$ is the lens space $L(p, q)$. Observe that the two-component unlink is $K(0, 1)$, $0/1 = [0]^-$, and we regard $S^1 \times S^2$ as the lens space $L(0, 1)$.

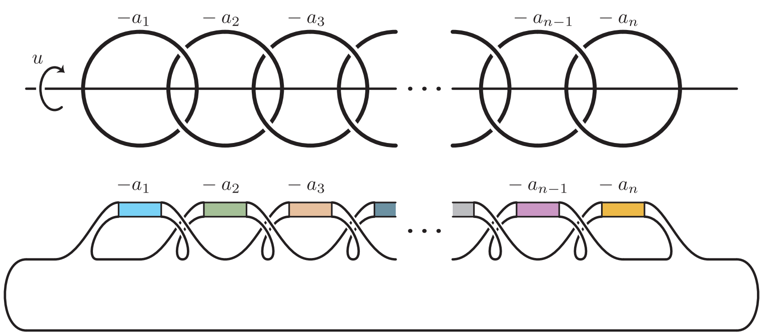

Figure 4: Integer surgery on a linear chain link (above) admits an involution along an axis whose quotient is a two-bridge link S^3 (below).

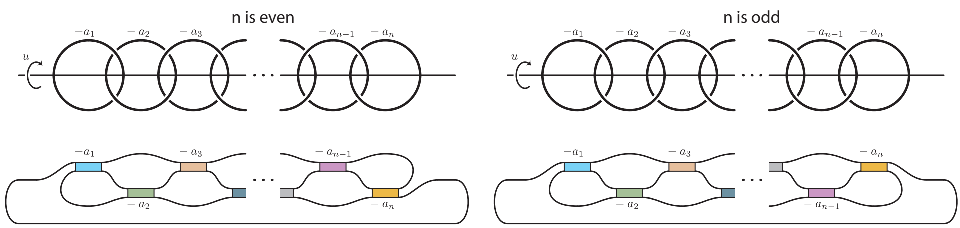

Figure 5: Alternative versions of Figure 4.

From a 4–manifold perspective, the top of Figure 4 is a Kirby diagram for a plumbing manifold whose boundary is $L(p, q)$. By an orientation preserving homeomorphism, we may take $p > q > 0$ and restrict the continued fraction coefficients to be integers $a_i \geq 2$ so that $L(p, q)$ is the oriented boundary of the negative definite plumbing manifold $P(p, q)$ associated to the tuple $(-a_1, \ldots, -a_n)$.

Figure 5 shows alternative (and isotopic) versions of this chain link and two-bridge link diagrams for the two cases of n even and n odd.

1.7. Acknowledgements

The authors would like to thank both John Berge and Joshua Greene for their input, interest, and invaluable conversations. We are especially indebted to Radu Cebanu for sharing his results with us.

This work is partially supported by grant #209184 to Kenneth L. Baker from the Simons Foundation, by the Spanish GEOR MTM2011-22435 to Ana G. Lecuona, and Leverhulme Trust grant RP2013-K-017 and EPSRC grants G039585/1 and H031367/1 to Dorothy Buck.

2. Generalizing Berge's doubly primitive knots

Berge describes twelve families of doubly primitive knots in S^3, [Ber]. Greene confirms that this list is complete, [Gre13]. The first author gives surgery descriptions of these knots and tangle descriptions of the quotients by their strong involutions, [Bak08a, Bak08b]. (These knots admit unique strong involutions, [WZ92].)

We partition Berge's twelve families into three coarse families: BG, the Berge-Gabai knots, arising from knots in solid tori with longitudinal surgeries producing solid tori; GOFK, the knots that embed in the fiber of a genus one fibered knot (the figure eight knot or a trefoil); and SPOR, the so-called sporadic knots, which may be seen to embed in a genus one Seifert surface of a banding of a $(2, \pm 1)$–cable of a trefoil. (The framing of this cabling is with respect to the Heegaard torus containing the trefoil.)

Here, we generalize these three families of doubly primitive knots in S^3 to obtain three analogous families of doubly primitive knots in $S^1 \times S^2$ that we also call BG, GOFK, and SPOR. Since they are strongly invertible, we explicitly describe them through their quotient tangles.

2.1. The BG knots

Let us say a *strong involution* of a knot in a solid torus is an involution of a solid torus whose fixed set is two properly embedded arcs such that the knot intersects this fixed set twice and is invariant under the involution. The quotient of the pair of the solid torus and fixed set under this involution is a rational tangle (B, t) where B is a 3–ball and t is a pair of properly embedded arcs together isotopic into ∂B. The image of the knot in this quotient is an arc α embedded in B with $\partial\alpha = \alpha \cap t$. A knot with a strong involution is *strongly invertible*.

The Berge-Gabai knots in solid tori are all strongly invertible as evidenced by them being 1–bridge braids (or torus knots) and hence tunnel number 1. We say an arc α in a rational tangle is a *Berge-Gabai arc* if it is the image of a Berge-Gabai knot under the quotient by a strong involution. By virtue of the Berge-Gabai knots admitting longitudinal solid torus surgeries, there are bandings along these arcs that produce rational tangles. The dual arcs to these bandings in the new rational tangles are also Berge-Gabai arcs. Using these we obtain a proof of Theorem 1.2 (through a proof of Corollary 1.3 and taking double branched covers) along the lines of Rasmussen's observation.

Direct proof of Corollary 1.3. Lisca's work constrains which two-bridge links admit a single banding to the unlink [Lis07]. Therefore, it remains to demonstrate that each of these lens spaces indeed do admit a single banding. We do so here, and in the process show that each banding may be done with a Berge-Gabai arc.

Doubling a rational tangle (by gluing it to its mirror) produces the two-component unlink $K(0,1)$. Therefore, if (B,t) is a rational tangle and α is a Berge-Gabai arc in (B,t) along which a banding produces the rational tangle (B,t'), then banding the double $(B,t) \cup -(B,t) = (S^3, K(0,1))$ along α produces the two-bridge link $(B,t') \cup -(B,t)$.

In [BB13], descriptions of the Berge-Gabai arcs are derived from the quotient tangle descriptions in [Bak08b] of the knots in Berge's doubly primitive families I – VI [Ber]. As done in [BB13], one may then explicitly observe that family VI is contained within family V, and (with mirroring) family V is dual to family III. Families I, II, and IV are each self-dual. (One family of Berge-Gabai arcs is dual to another if the arc dual to the banding along any arc in the first family, together with the resulting tangle, may be isotoped while fixing the boundary of the tangle into the form of a member of the second family.) It is also shown in [BB13] that these knots in solid tori admit a unique strong involution in which the solid torus quotients to a ball, and hence up to homeomorphism there is a unique arc in a rational tangle corresponding to each Berge-Gabai knot in the solid torus. The resulting classification of bandings between rational tangles up to homeomorphism from [BB13] is shown in Figure 6. Figures 7, 8, and 9 show the result of doubling these families (with their duals) to obtain $K(0,1)$ and then banding along a Berge-Gabai arc to obtain a two-bridge link. Observe that the two-bridge links produced by the dual pairs in families III and V in Figures 7 are equivalent as are the two-bridge links produced by the dual pairs in family IV. For the purposes of proving Corollary 1.3 only one among each of these dual pairs is required.

We now observe that the two-bridge links produced match with those of Lisca's Section 8 [Lis07]. He shows that up to homeomorphism these links may be presented as in the first of one of his Figure 2, Figure 3, or Figure 4. We redraw these three in Figure 10 for the reader's convenience,

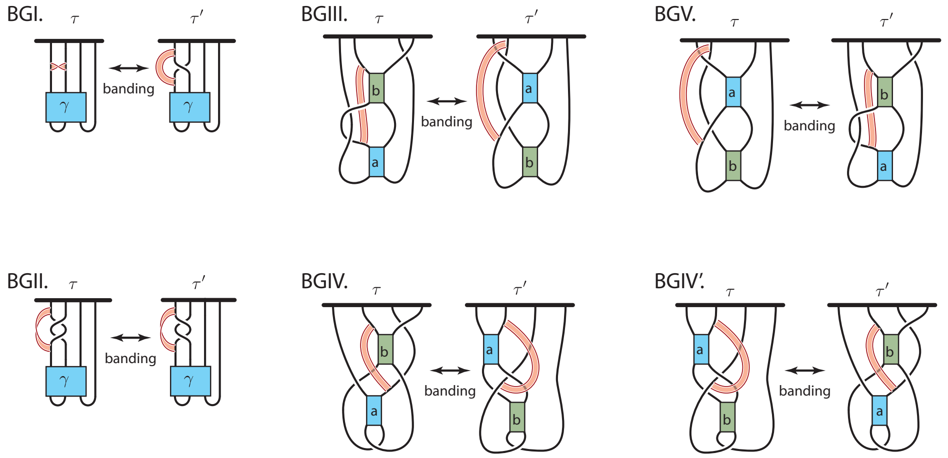

Figure 6: The bandings between rational tangles up to homeomorphism, corresponding to the double branched covers of the Berge-Gabai knots in solid tori. For BGI and BGII γ is a 3–braid.

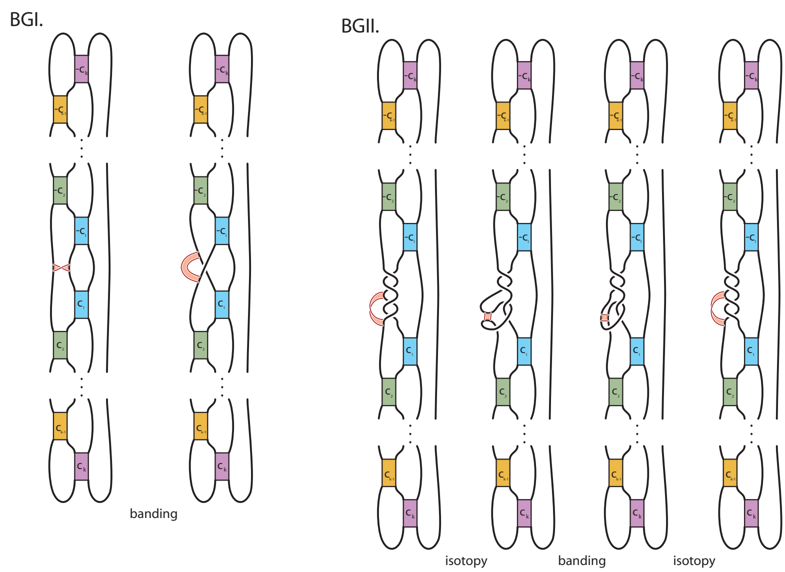

Figure 7: Left, a banding from the unlink to $K(p,q)$ where $\frac{p}{q} = [c_k, \ldots, c_1, -1, -c_1, \ldots, -c_k]^-$ corresponding to family BGI. Right, a banding from the unlink to $K(p,q)$ where $\frac{p}{q} = [c_k, \ldots, c_1, 4, -c_1, \ldots, -c_k]^-$ corresponding to family BGII.

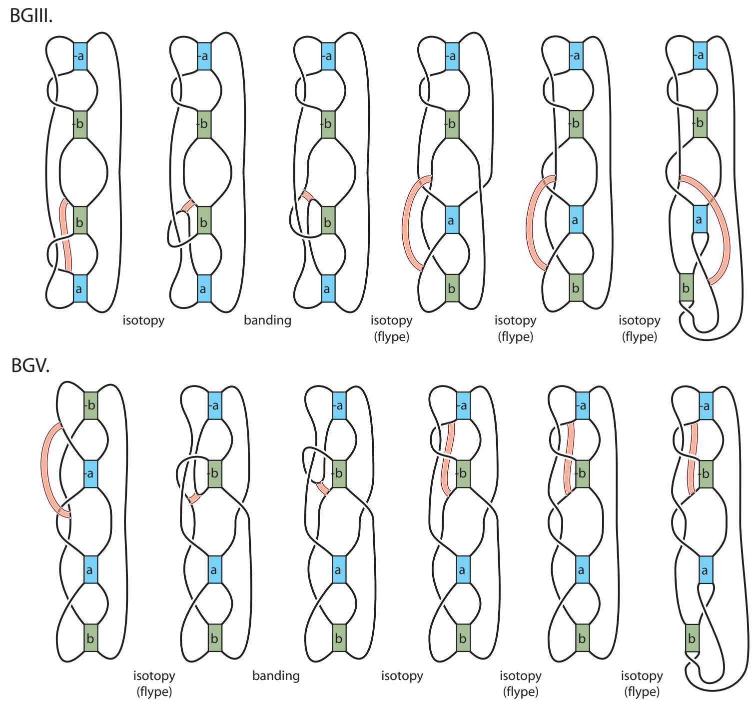

Figure 8: Two bandings between the unlink and $K(p,q)$ where $\frac{p}{q} = [a, 2, b, 2, -a+1, -b+1]^-$ corresponding to families BGIII and BGV.

isotoping his Figure 2. In each of his Figure 2 and Figure 4 Lisca exhibits a single banding as shown in our Figure 10 that transforms those two-bridge links to the unlink. In his Figure 3 he uses two bandings as also shown in our Figure 10. As one may now observe, the two bridge links of Lisca's Figures 2, 3, and 4 correspond (with mirroring and reparametrizations as needed) to those produced respectively in families I, IV, and III of Figures 7, 9, and 8. Note that family II produces two-bridge links not accounted for in Lisca's pictures. Nevertheless the corresponding lens spaces are accounted for in his proof of [Lis07, Lemma 7.2], as discussed in Remark 1.4. □

Remark 2.1. In $S^1 \times S^2$, as in S^3, family I consists of the torus knots while family II consists of the $(2, \pm 1)$–cables of torus knots. (This cable is taken with respect to the framing induced by the Heegaard torus containing the torus knot.) Families III, IV, and V contain hyperbolic knots.

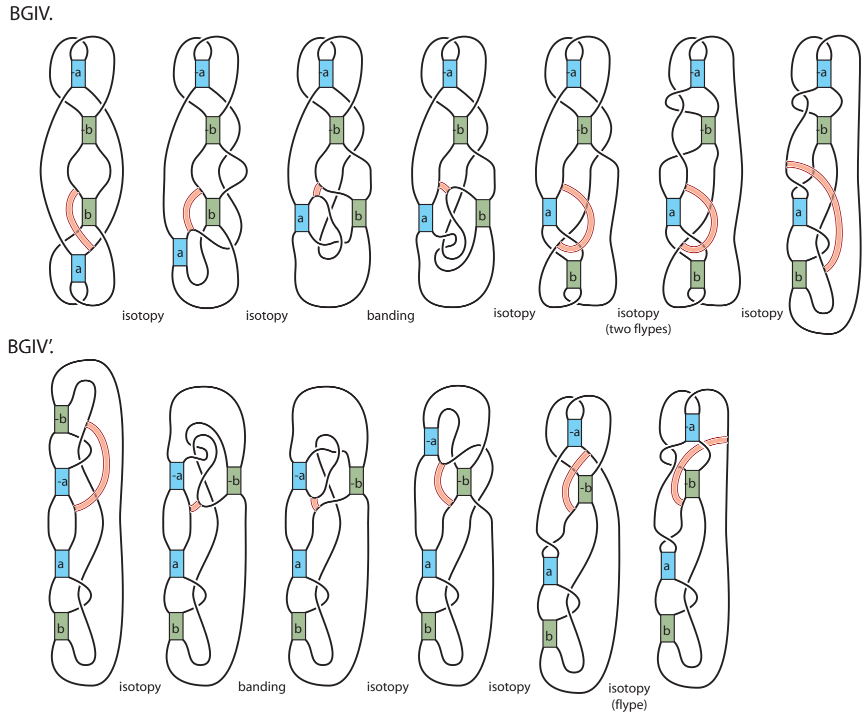

Figure 9: Two bandings between the unlink and $K(p,q)$ where $\frac{p}{q} = [a-1, -2, b, -a+1, 2, -b]^-$ both corresponding to family BGIV.

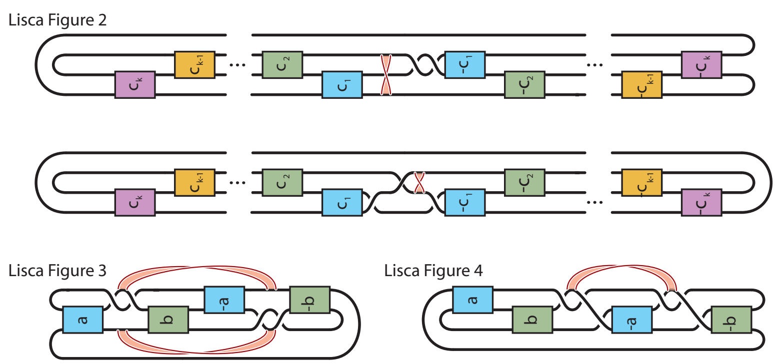

Figure 10:

2.2. The GOFK knots

Conjecture 1.8 of [Gre13] proposes that the knot surgeries corresponding to the double branched covers of the above bandings are, up to homeomorphisms, the only way that integral surgery on a knot in $S^1 \times S^2$ may yield a lens space. However since $S^1 \times S^2$ contains a genus one fibered knot, we may form the family GOFK of knots that embed in the fiber of genus one fibered knots in $S^1 \times S^2$ and then mimic [Bak08a] to produce our first infinite family of counterexamples.

The annulus together with the identity monodromy gives an open book for $S^1 \times S^2$. Plumbing on a positive Hopf band along a spanning arc produces a once-punctured torus open book, i.e. a (null-homologous) genus one fibered knot. One may show (e.g. [Bak14]) that this and its mirror are the only two genus one fibered knots in $S^1 \times S^2$. Any essential simple closed curve in one of these fibers is then a doubly primitive knot in $S^1 \times S^2$ and thus admits a lens space surgery along the slope of its page framing. We call the family of these essential simple closed curves the GOFK. These knots are analogous to the knots in Berge's families VII and VIII, [Ber]. Yamada had previously developed this family of knots with these lens space surgeries [Yam07], though he constructs them from a different viewpoint.

Proposition 2.2. *There are* GOFK *knots that are not Berge-Gabai knots. Moreover the* GOFK *knots contain hyperbolic knots of arbitrarily large volume.*

Proof. Following [Bak08a] each GOFK knot admits a surgery description on the Minimally Twisted $2n+1$ Chain link (the MT$(2n+1)$C for short) for some $n \in \mathbb{Z}$. Furthermore, for each positive integer n and any value N, there is a doubly primitive knot on a once-punctured torus page of this open book with a surgery description on the MT$(2n+1)$C whose surgery coefficients all have magnitude greater than N. Therefore, as in [Bak08a], since MT$(2n+1)$C is hyperbolic for $n \geq 2$ we may conclude using Thurston's Hyperbolic Dehn Surgery Theorem [Thu97] and the lower bound on the volume of a hyperbolic manifold with n cusps [Ada92] that the set of volumes of hyperbolic knots on this once-punctured torus page is unbounded.

The Berge-Gabai knots in $S^1 \times S^2$ of [Gre13, Conjecture 1.8] however all admit surgery descriptions on the MT5C (as apparent from [Bak08b]) and thus have volume less than $\mathrm{vol}(MT5C) < 11$. Thus they are often distinct from the GOFK knots. Alternatively, this conclusion follows from examining the homology classes of their surgery duals as presented in Theorem 1.7 (1). □

Any genus one fibered knot may be viewed as the lift of the braid axis in the double cover of S^3 branched over a closed 3–braid. This enables a pleasant interpretation of the GOFK knots and their lens space surgeries as corresponding to bandings from a closed 3–braid presentation of the unlink to two-bridge links. Because Lisca's list of two-bridge links that admit bandings to the unlink is complete, these bandings must give different bandings to the unlink for many of these two-bridge links; these bandings correspond to surgeries on knots in different homology classes as presented in Theorem 1.7 (1). Indeed Figure 11 shows the two different bandings between the unlink and a two-bridge link corresponding to the GOFK knots on the left and BGI knots on the right.

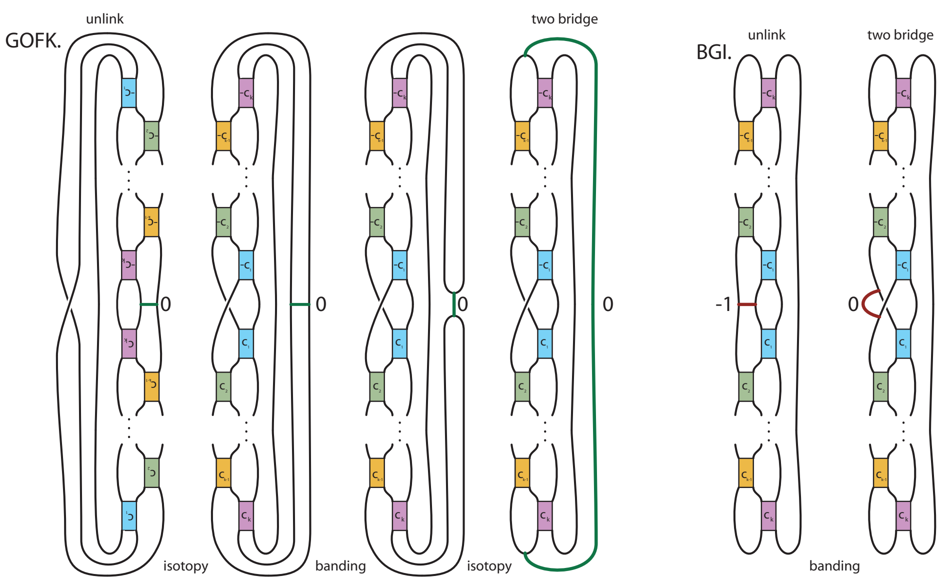

Figure 11: Left, a banding from the unlink to a two-bridge link corresponding to family GOFK. Right, a banding from the unlink to the same two-bridge link corresponding to family BGI.

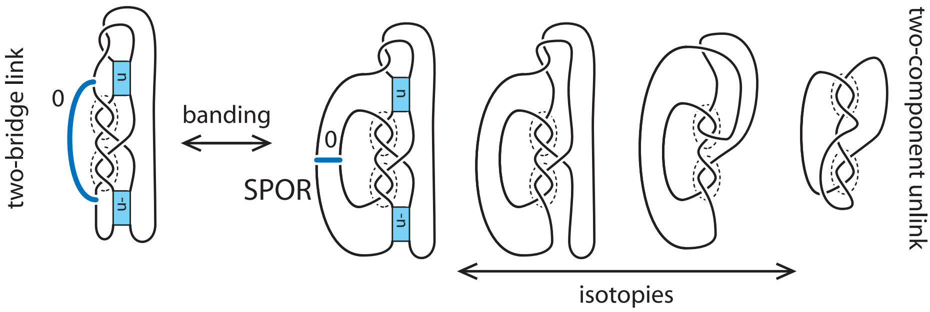

Figure 12: A banding between a two-bridge link and the unlink corresponding to the sporadic family of knots. Replace either of the -2 twists in the dashed ovals, but not both, with -3 twists to obtain bandings corresponding to Berge's families of sporadic knots in S^3.

2.3. The SPOR knots

Berge's families IX–XII of doubly primitive knots in S^3 condense to two families and are collectively referred to as the sporadic knots. In the double branched cover, the family of blue arcs ($n \in \mathbb{Z}$) in the second link of Figure 12 lifts to the analogous sporadic knots in $S^1 \times S^2$, the $S^1 \times S^2$–SPOR knots. (As one may confirm by examining the tangle descriptions in [Bak08b], Berge's two sporadic knot families are obtained by placing -3 instead of -2 twists in either the top or bottom dashed oval, but not both.) This second link is the two-component unlink as illustrated by the subsequent isotopies. The link at the beginning of Figure 12 results from banding as shown. It is a two-bridge link and coincides with the two bridge link in family III of Figure 8 with $a = n$ and $b = -1$ and, after mirroring, with the two-bridge link of Lisca's Figure 4 in our Figure 10 with $a = 2$ and $b = -n + 1$. In Section 3 we show these knots are generically distinct from the Berge-Gabai knots by examining the homology classes of the corresponding knots in the lens spaces.

3. Homology classes of the dual knots in lens spaces

Figure 13 gives, up to homeomorphism, strongly invertible surgery descriptions of the lens space duals to the BG, GOFK, and SPOR knots with their $S^1 \times S^2$ surgery coefficient. Figures 14 and 15 show how quotients of these surgery descriptions produce the tangles in Figures 7, 8, 9, 11, and 12 that

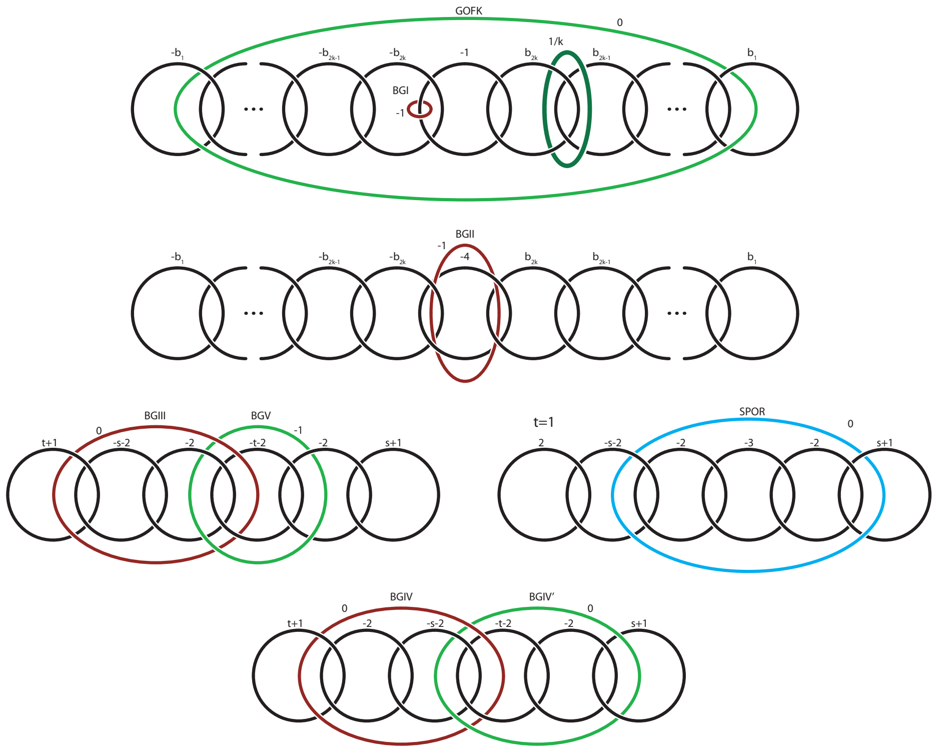

Figure 13: Chain link surgery description of the lens space duals to $S^1 \times S^2$ doubly primitive knots.

defined these knots. We will use these surgery descriptions to determine the homology classes of these knots.

3.1. Continued fractions

First we establish a few basic results about continued fractions. These appear throughout the literature in various forms, but it is useful to set notation and collect them here.

Given the continued fraction $[a_1, \ldots, a_k]^-$, define the numerators and denominators of the "forward" convergents as follows:

$$\begin{array}{lll} P_{-1} = 0 & P_0 = 1 & P_i = a_i P_{i-1} - P_{i-2} \\ Q_{-1} = -1 & Q_0 = 0 & Q_i = a_i Q_{i-1} - Q_{i-2} \end{array}$$

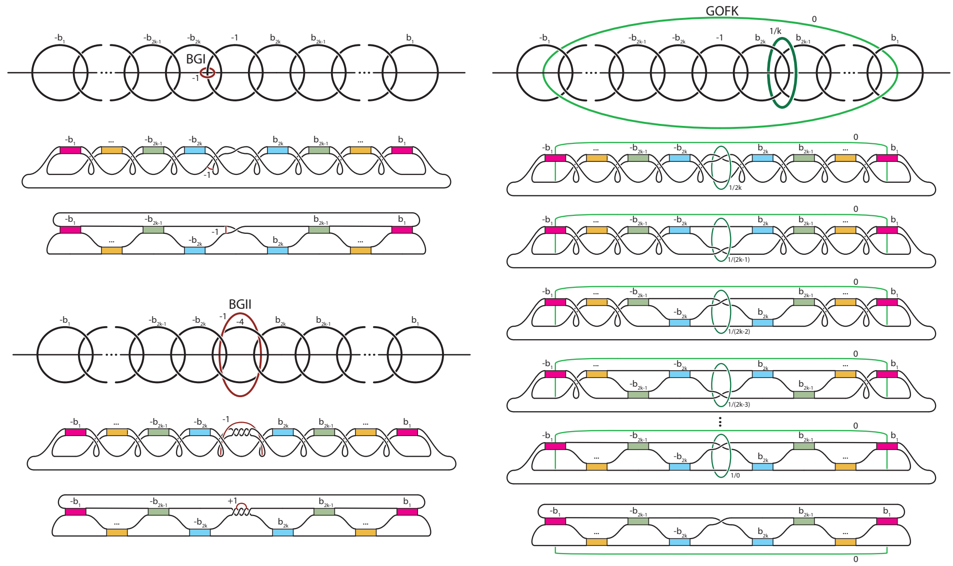

Figure 14: Quotients of the chain link surgery descriptions of the lens space duals to the BGI, BGII, and GOFK knots.

Claim 3.1. *For* $i = 1, \ldots, k$, $\frac{P_i}{Q_i} = [a_1, \ldots, a_i]^-$ *and* $P_{i-1}Q_i - P_iQ_{i-1} = 1$.

Proof. These are immediate when $i = 1$, so assume they are true for continued fractions of length up to i. Writing

$$[a_1, \ldots, a_{i-1}, a_i, a_{i+1}]^- = [a_1, \ldots, a_{i-1}, a_i']^-$$

where $a_i' = a_i - \frac{1}{a_{i+1}}$, the numerator of the forward convergent of the continued fraction on the right hand side is

$$\begin{aligned} P_i' &= a_i'P_{i-1}' - P_{i-2}' = (a_i - \frac{1}{a_{i+1}})P_{i-1} - P_{i-2} \\ &= -\frac{1}{a_{i+1}}P_{i-1} + a_iP_{i-1} - P_{i-2} = -\frac{1}{a_{i+1}}P_{i-1} + P_i. \end{aligned}$$

Similarly the denominator is $Q_i' = -\frac{1}{a_{i+1}}Q_{i-1} + Q_i$. Then

$$\frac{P_{i+1}}{Q_{i+1}} = \frac{a_{i+1}P_i - P_{i-1}}{a_{i+1}Q_i - Q_{i-1}} = \frac{P_i - \frac{1}{a_{i+1}}P_{i-1}}{Q_i - \frac{1}{a_{i+1}}Q_{i-1}} = \frac{P_i'}{Q_i'} = [a_1, \ldots, a_{i+1}]^-$$

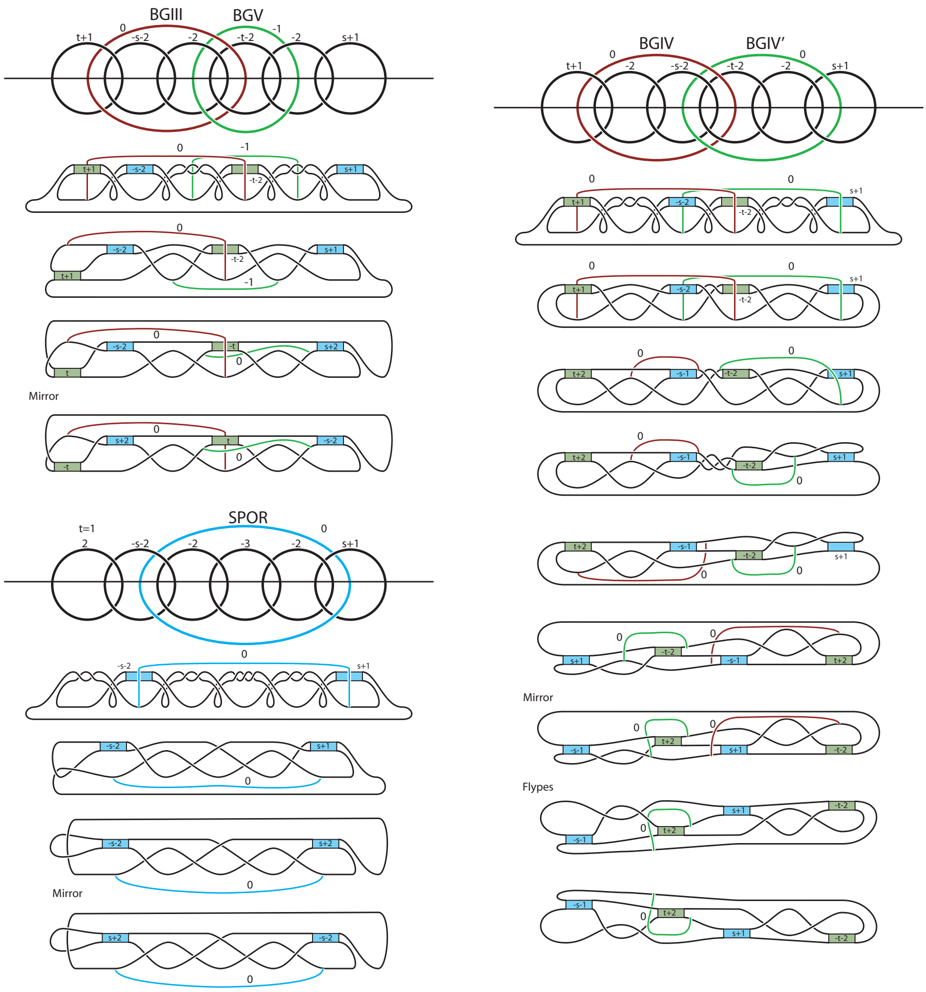

Figure 15: Quotients of the chain link surgery descriptions of the lens space duals to the BGIII, BGV, and BGIV knots. (Isotopies of the arcs are done independently even when shown on the same link. To correspond with Figure 9, the end position for the BGIV arc is third from the bottom on the right while further flypes and an overall rotation are still needed for the BGIV' arc.)

as desired. Also

$$
\begin{aligned}
& P_{i-1}Q_i - P_iQ_{i-1} \\
&= P_{i-1}(a_iQ_{i-1} - Q_{i-2}) - (a_iP_{i-1} - P_{i-2})Q_{i-1} \\
&= P_{i-2}Q_{i-1} - P_{i-1}Q_{i-2} = \cdots = P_{-1}Q_0 - P_0Q_{-1} = 1.
\end{aligned}
$$

□

Given the continued fraction $[a_k, \ldots, a_1]^-$, define the numerators and denominators of the "backward" convergents as follows:

$$\begin{array}{lll} p_{-1} = 0 & p_0 = 1 & p_i = a_i p_{i-1} - p_{i-2} \\ q_{-1} = -1 & q_0 = 0 & q_i = a_{i-1} q_{i-1} - q_{i-2} \end{array}$$

Claim 3.2. *For* $i = 1, \ldots, k$, $\frac{p_i}{q_i} = [a_i, \ldots, a_1]^-$ *and* $q_i = p_{i-1}$.

Proof. These are immediate when $i = 1$, so assume they are true for continued fractions of length up to i. First $q_{i+1} = a_i q_i - q_{i-1} = a_i p_{i-1} - p_{i-2} = p_i$. Then

$$\begin{aligned}[a_{i+1}, a_i, \ldots, a_1]^- &= a_{i+1} - \frac{1}{[a_i, \ldots, a_1]^-} = a_{i+1} - \frac{1}{p_i/p_{i-1}} \\ &= \frac{a_{i+1} p_i - p_{i-1}}{p_i} = \frac{p_{i+1}}{q_{i+1}}.\end{aligned}$$

□

Lemma 3.3. $[b_1, \ldots, b_k, c, -b_k, \ldots, -b_1]^- = \dfrac{cP_k^2}{cP_k Q_k + 1}$

Proof. Notice that for this continued fraction we have $\frac{P_i}{Q_i} = [b_1, \ldots, b_i]^-$ and $\frac{p_i}{q_i} = [-b_i, \ldots, -b_1]^-$ for $i = 1, \ldots, k$. Using the definitions of P_i and p_i one can show that $p_i = (-1)^i P_i$ for $i = 1, \ldots, k$. Then we have:

$$\begin{aligned} &[b_1, \ldots, b_k, c, -b_k, \ldots, -b_1]^- \\ &= [b_1, \ldots, b_k, c - \frac{1}{[-b_k, \ldots, -b_1]^-}]^- \\ &= [b_1, \ldots, b_k, c - \frac{p_{k-1}}{p_k}]^- = \frac{(c - \frac{p_{k-1}}{p_k})P_k - P_{k-1}}{(c - \frac{p_{k-1}}{p_k})Q_k - Q_{k-1}} \\ &= \frac{cp_k P_k - p_{k-1} P_k - p_k P_{k-1}}{cp_k Q_k - p_{k-1} Q_k - p_k Q_{k-1}} \\ &= \frac{(-1)^k cP_k P_k - (-1)^{k-1} P_{k-1} P_k - (-1)^k P_k P_{k-1}}{(-1)^k cP_k Q_k - (-1)^{k-1} P_{k-1} Q_k - (-1)^k P_k Q_{k-1}} \\ &= \frac{(-1)^k cP_k^2}{(-1)^k cP_k Q_k + (-1)^k (P_{k-1} Q_k - P_k Q_{k-1})} = \frac{cP_k^2}{cP_k Q_k + 1} \end{aligned}$$

□

3.2. Homology classes of the duals to the BG, GOFK, and SPOR knots

We now calculate the homology classes of the knots indicated in Figure 13. To do so, orient and index each linear chain link $L = L_1 \cup \cdots \cup L_n$ of n components from right to left as in Figure 16. Denote the exterior of this link by $X(L) = S^3 - N(L)$. Let $\{\mu_i, \lambda_i\}$ be the standard oriented meridian, longitude pair giving a basis for the homology of the boundary of a regular neighborhood of the ith component, $H_1(\partial N(L_i))$. Then $H_1(X(L)) = \langle \mu_1, \dots, \mu_n \rangle \cong \mathbb{Z}^n$. Take λ_i so that it is represented by the boundary of a meridional disk in $H_1(S^3 - N(L_i))$. Then in $H_1(X(L))$ we have $\lambda_i = \mu_{i-1} + \mu_{i+1}$ for $i \in \{1, \dots, n\}$ where $\mu_0 = \mu_{n+1} = 0$. Let $L(-a_1, \dots, -a_n)$ denote the lens space obtained by this surgery description on the chain link L with $-a_i$ surgery on the ith component. The surgery induces the relation $\lambda_i = a_i\mu_i$ for each i and hence the relation $0 = \mu_{i-1} - a_i\mu_i + \mu_{i+1}$ in $H_1(X(L))$. Thus $H_1(L(-a_1, \dots, -a_n)) = \langle \mu_1, \dots, \mu_n \ : \ \mu_{i-1} - a_i\mu_i + \mu_{i+1}, i \in \{1, \dots, n\} \rangle$.

Lemma 3.4. *Let $L(p, q) = L(-a_1, \dots, -a_n)$ be the lens space described by surgery on the n component chain link with surgery coefficient $-a_i$ on the ith component so that $\frac{p}{q} = [a_1, \dots, a_n]^-$.*

Then $\mu_i = P_{i-1}\mu_1$ in $H_1(L(p, q))$ for each $i = 1, \dots, n$. In particular, $p = P_n$ and $q^{-1} = P_{n-1}$ so that $q\mu_n = \mu_1$.

Proof. Since $\mu_2 = a_1\mu_1 = P_1\mu_1$ and $\mu_{i+1} = a_i\mu_i - \mu_{i-1}$, the result follows from the definition of P_i and a simple induction argument. Assuming this statement is true up through i,

$$\mu_{i+1} = a_i\mu_i - \mu_{i-1} = a_iP_{i-1}\mu_1 - P_{i-2}\mu_1 = (a_iP_{i-1} - P_{i-2})\mu_1 = P_i\mu_1.$$

The last statement follows since $P_{i-1} = Q_i^{-1} \bmod P_i$ by Claim 3.1. □

Proof of Theorem 1.7. Figure 13 shows linear chain link surgery descriptions of Lisca's lens spaces with additional unknotted components that describe knots in these lens spaces. Orient these knots in Figure 13 counter-clockwise. The homology class of each such knot K may be determined in terms of the meridians of the chain link by counting μ_i for each time K runs under the ith component to the left and counting $-\mu_i$ for each time K runs under the ith component to the right. Applying Lemma 3.4 allows us to write the homology class of K in terms of μ_1. For the four families of Theorem 1.2, and using its notation, we obtain the following:

(1) With $(-a_1, \dots, -a_{4k+1}) = (-b_1, \dots, -b_{2k}, -1, b_{2k}, \dots, b_1)$, $\frac{p}{q} = [b_1, \dots, b_{2k}, 1, -b_{2k}, \dots, b_1]^- = \frac{m^2}{q}$ where $m = P_{2k}$, $d = Q_{2k}$ and $q = P_{2k}Q_{2k} + 1 = md + 1$. This gives both that $1 - q = -dm$ and that $qm = m \mod p$. Furthermore $\mu_1 = q\mu_{4k+1}$.

$$[K_{\text{BGI}}] = -\mu_{2k+1} = -P_{2k}\mu_1 = -m\mu_1 = -qm\mu_{4k+1} = -m\mu_{4k+1}$$
$$[K_{\text{GOFK}}] = \mu_1 - \mu_{4k+1} = (1-q)\mu_1 = -dm\mu_1 = -dqm\mu_{4k+1} = -dm\mu_{4k+1}$$

(2) With $(-a_1, \dots, -a_{2k+1}) = (-b_1, -b_2, \dots, -b_k, -4, b_k, \dots, b_k)$, $\frac{p}{q} = [b_1, \dots, b_k, 4, -b_k, \dots, -b_1] = \frac{m^2}{q}$ where $m = 2P_k$, $d = 2Q_k$, and $q = 4P_kQ_k + 1 = md + 1$. This gives that $qm = m \mod p$. Furthermore $\mu_1 = q\mu_{4k+1}$.

$$\begin{aligned}[K_{\text{BGII}}] &= \mu_k - 2\mu_{k+1} + \mu_{k+2} = (P_{k-1} - 2P_k + P_{k+1})\mu_1 \\ &= (P_{k-1} - 2P_k + (4P_k - P_{k-1}))\mu_1 = 2P_k\mu_1 = m\mu_1 \\ &= qm\mu_{4k+1} = m\mu_{4k+1}\end{aligned}$$

(3) With $(-a_1, \dots, -a_6) = (t+1, -s-2, -2, -t-2, -2, s+1)$, $\frac{p}{q} = [-t-1, s+2, 2, t+2, 2, -s-1]^- = \frac{(4+3t+2s+2st)^2}{-(3+2s)^2(1+t)}$. Then take $m = 4 + 3t + 2s + 2st$ so that $p = m^2$ and $d = -(3+2s)$ so that $q = d(m-1)$. This gives that d is odd, that $(m-1) = -d(1+t)$ from which $d^{-1}m = (1+t)m \mod p$, and that $qm = -dm \mod p$. Furthermore $\mu_1 = q\mu_6$.

$$\begin{aligned}[K_{\text{BGIII}}] &= \mu_1 + \mu_4 = (1 + P_3)\mu_1 = -(4 + 3t + 2s + 2st)\mu_1 \\ &= -m\mu_1 = -qm\mu_6 = dm\mu_6 \\ [K_{\text{BGV}}] &= \mu_3 - \mu_5 = (P_2 - P_4)\mu_1 = (1+t)(4 + 3t + 2s + 2st)\mu_1 \\ &= (1+t)m\mu_1 = d^{-1}m\mu_1 = d^{-1}qm\mu_6 = -m\mu_6\end{aligned}$$

When $t = 1$ we have a third knot. Then $(-a_1, \dots, -a_6) = (2, -s-2, -2, -3, -2, s+1)$, $\frac{p}{q} = [-2, s+2, 2, 3, 2, -s-1]^- = \frac{(7+4s)^2}{-2(3+2s)^2}$, and $m = 1 - 2d$.

$$\begin{aligned}[K_{\text{BGIII}}] &= -(7+4s)\mu_1 = -m\mu_1 = dm\mu_6, \\ [K_{\text{BGV}}] &= 2(7+4s)\mu_1 = 2m\mu_1 = -2dm\mu_6 = (m-1)m\mu_6 = -m\mu_6, \\ [K_{\text{SPOR}}] &= \mu_2 - \mu_6 = (P_1 - P_5)\mu_1 = 4(7+4s)\mu_1 = 4m\mu_1 = -2m\mu_6.\end{aligned}$$

(4) With $(-a_1, \dots, -a_6) = (t+1, -2, -s-2, -t-2, -2, s+1)$, $\frac{p}{q} = [-t-1, 2, s+2, t+2, 2, -s-1]^- = \frac{(4+3s+3t+2st)^2}{-(3+2s)(3+3s+3t+2st)}$. Then take

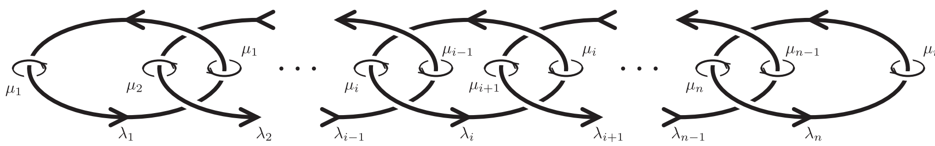

Figure 16.

$m = 4 + 3s + 3t + 2st$ so that $p = m^2$ and $d = -(3 + 2s)$ so that $q = d(m - 1)$. This gives both that $(2m + 1) = -d(3 + 2t)$ from which $d^{-1}m = -(3 + 2t)m \mod p$ and that $qm = -dm \mod p$. Furthermore $\mu_1 = q\mu_6$.

$$\begin{aligned}[K_{\mathrm{BGIV}}] &= \mu_1 + \mu_4 = (1 + P_3)\mu_1 = -(4 + 3s + 3t + 2st)\mu_1 = -m\mu_1 \\ &= -qm\mu_6 = dm\mu_6 \\ [K'_{\mathrm{BGIV}}] &= \mu_3 + \mu_6 = (P_2 + P_5)\mu_1 = -(3 + 2t)(4 + 3s + 3t + 2st)\mu_1 \\ &= d^{-1}m\mu_1 = d^{-1}qm\mu_6 = -m\mu_6.\end{aligned}$$

Let μ and μ' be the homology classes of the two cores of the Heegaard solid tori of $L(p, q)$ suitably oriented so that $q\mu = \mu'$. Then, we have $\mu = \mu_{4k+1}$ for (1) and (2) and $\mu = \mu_6$ for (3) and (4) in the calculations above. Since a knot's orientation does not effect its Dehn surgeries, taking both signs of the homology classes above competes the proof of Theorem 1.7. □

Proposition 3.5. *The* SPOR *knots generically are not Berge-Gabai knots.*

In particular, when $t = 1$ and $n = s + 2 \neq -1, 0$ the surgery duals to SPOR, BGIII, BGV *are mutually distinct. When $t = 1$ and $n = s + 2 = 0$, these knots are all the unknot in S^3. When $t = 1$ and $n = s + 2 = -1$ the knots* SPOR *and* BGIII *are isotopic to but distinct from* BGV.

Proof. Up to mirroring, the lens space obtained by longitudinal surgery on a sporadic knot is $L(p, q) = L((7 + 4s)^2, -2(3 + 2s)^2)$. Let us reparametrize by $s = n - 2$ so that $L(p, q) = L((4n - 1)^2, -2(2n - 1)^2) = L((4n - 1)^2, 8n^2 - 1)$. Again, we take μ and μ' to be the homology classes of the oriented cores of the Heegaard solid tori so that $q\mu = \mu'$. Then by Theorem 1.7 the unoriented knot dual to the sporadic knot represents the homology classes $\pm 2(4n - 1)\mu$ while the duals to the BGIII and BGV knots in this lens space represent the homology classes $\pm(2n - 1)(4n - 1)\mu$ and $\pm(4n - 1)\mu$. Since $q^2 \not\equiv \pm 1 \pmod p$ for $n \neq -1, 0$, the group of isotopy classes of diffeomorphisms of our lens space is $\mathbb{Z}/2\mathbb{Z}$, generated by the involution whose quotient

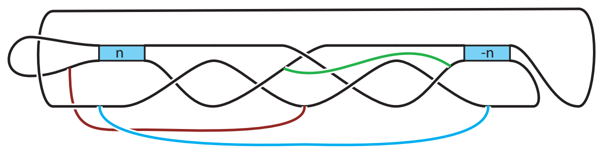

Figure 17: The 0–framed bandings from Figure 15 corresponding to the surgery duals to the BGIII and BGV knots when $t = 1$ and the SPOR knots are shown simultaneously. Each banding produces the unlink.

is the two bridge link, [Bon83, HR85]. This involution acts on $H_1(L(p, q))$ as multiplication by -1. Therefore when $n \neq -1, 0$ (and when $s \neq -3, -2$) the duals to BGIII, BGV, and SPOR are mutually non-isotopic.

For $n = 0$, $L(p, q) = S^3$ so that the knot dual to SPOR, BGIII, and BGV are all the unknot.

For $n = -1$, $L(p, q) = L(25, 7)$ the knots dual to SPOR and BGIII represent the homology classes $\pm 10\mu$. One may directly observe that the corresponding knots are isotopic. The knot dual to BGV represents the homology classes $\pm 5\mu$. The knots in $S^1 \times S^2$ with integral surgeries yielding $L(25, 7)$ are those shown to the left and center in Figure 1. □

Corollary 3.6. *For each integer $n \neq 0, -1$, the three bandings of the two-bridge links in Figure 17 are distinct up to homeomorphism of the two-bridge link.*

4. Doubly primitive knots, waves, and simple knots

We now generalize Berge's results that the duals to doubly primitive knots in S^3 (under the associated lens space surgery) are simple knots and that $(1, 1)$–knots with longitudinal S^3 surgeries are simple knots. We will adapt Saito's proofs given in the appendix of [Sai08].

A *wave* of a genus 2 Heegaard diagram $(S, \bar{x} = \{x_1, x_2\}, \bar{y} = \{y_1, y_2\})$ is an arc α embedded in S so that (up to swapping x's and y's) $\alpha \cap \bar{x} = \partial\alpha \subset x_i$ for $i = 1$ or 2, at each endpoint α encounters x_i from the same side, $\alpha \cap \bar{y} = \emptyset$, and each component of $x_i - \alpha$ intersects $\bar{y}$. A regular neighborhood of $\alpha \cup x_i$ is a thrice-punctured sphere of which one boundary component is not isotopic to a member of $\bar{x}$. A *wave move* along α is the replacement of x_i by this component.

Let us say two simple closed curves on an orientable surface *coherently intersect* if they may be oriented so that every intersection occurs with the same sign. (This includes the possibility that the two curves are disjoint.) We then say a Heegaard diagram is *coherent* if every pair of curves in the diagram coherently intersect.

Say a 3–manifold W of Heegaard genus at most 2 is *wave-coherent* if any genus 2 Heegaard diagram $(S, \bar{x} = \{x_1, x_2\}, \bar{y} = \{y_1, y_2\})$ of W either admits a wave move or is coherent.

Theorem 4.1.

1) *If longitudinal surgery on a* $(1,1)$*–knot in a lens space produces a wave-coherent manifold, then the knot is simple.*
2) *Given a doubly primitive knot in a wave-coherent manifold of Heegaard genus at most* 2*, the surgery dual to the associated lens space surgery is a simple knot.*

Proof. The proof of the first follows exactly the same as that of Saito's Theorem A.5 (with Lemma A.6) in [Sai08] except that we use Proposition 4.3 below in the stead of his Proposition A.1.

The second item then follows because the surgery dual to a doubly primitive knot is a $(1,1)$–knot. See Theorem A.4 [Sai08] for example. □

Corollary 4.2. *A* 3*–manifold of genus at most* 2 *obtained by longitudinal surgery on a non-trivial* $(1,1)$*–knot in* S^3 *or* $S^1 \times S^2$ *is not wave-coherent.*

Proof. Theorem 4.1 applies even if the $(1,1)$–knot is in S^3 or in $S^1 \times S^2$. The trivial knot is the only simple knot in these two manifolds. □

Proof of Theorem 1.6. $S^1 \times S^2$ is wave-coherent by Theorem 1.5 ([NO85]) so the result follows from Theorem 4.1. □

Proposition 4.3 (Cf. Proposition A.1 [Sai08]). *Let* $(S; \bar{x} = \{x_1, x_2\}, \bar{y} = \{y_1, y_2\})$ *be a Heegaard diagram of a* 3*–manifold* W *in which* $\bar{x}$ *and* $\bar{y}$ *intersect essentially. Assume* z *is a simple closed curve in* S *such that* z *intersects each* x_1 *and* y_1 *once and is disjoint from both* x_2 *and* y_2*. If* W *is wave-coherent, then* x_2 *and* y_2 *coherently intersect.*

Sketch of Proof. Saito's proof of the analogous theorem for $W = S^3$ applies to any wave-coherent manifold of genus at most 2 whose genus 2 Heegaard diagrams enjoy the NEI Property: A Heegaard diagram $(S; \bar{x}, \bar{y})$ is said to

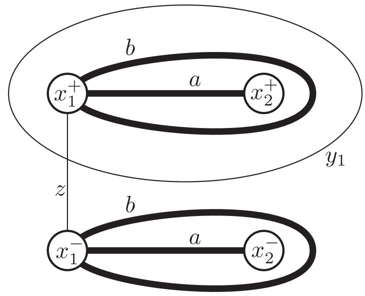

Figure 18.

have the *Non-Empty Intersecting (NEI) Property* if every $x_i \in \bar{x}$ intersects some $y_j \in \bar{y}$ and every $y_j \in \bar{y}$ intersects some $x_i \in \bar{x}$. Any genus 2 Heegaard diagram for S^3 (or any homology sphere) enjoys the NEI Property by [Och79, Lemma 1], and S^3 is wave-coherent by [HOT80]. The main tool is Ochiai's structure theorem for Whitehead graphs of genus 2 Heegaard diagrams with the NEI Property, [Och79, Theorem 1].

Assume $(S, \bar{x}, \bar{y})$ does not enjoy the NEI Property. Then the manifold W contains a non-separating sphere and hence an $S^1 \times S^2$ summand. It follows that W is homeomorphic to $S^1 \times S^2 \# L(p,q)$ for some integers p, q. All such manifolds are all wave-coherent by [NO85, Theorem 1-4].

If $(S, \bar{x}, \bar{y})$ is a standard Heegaard diagram for $W \cong S^1 \times S^2 \# L(p,q)$, then it is coherent and the proposition is satisfied, so further assume the diagram is not standard. Assume $y_0 \in \{y_1, y_2\}$ does not intersect $x_1 \cup x_2$. Then since the diagram is not standard, y_0 cannot be parallel to either x_1 or x_2. Because y_0 is non-separating, $y_0 \cup x_1 \cup x_2$ must be the boundary of thrice-punctured sphere in S. Since z intersects x_1 just once and is disjoint from x_2, it must also intersect y_0. Therefore $y_0 = y_1$. Hence the Heegaard diagram with z must appear as in Figure 18 after gluing x_i^+ to x_i^- for each $i = 1, 2$ to reform S. The thick arcs labeled a and b represent sets of a or b parallel arcs of $y_2 - (x_1 \cup x_2)$. Because z intersects x_1 once, it dictates how the ends of the rest of the arcs encountering x_1 must match up. Since these other arcs all together constitute the single curve y_2, we must have either $b = 1$ and $a = 0$ or $b = 0$ and $a > 0$. In either case the conclusion of the proposition holds. $\square$

Question 4.4. *Which 3–manifolds are wave-coherent? Homma-Ochiai-Takahashi show S^3 is wave-coherent [HOT80], and Negami-Ochiai show the manifolds $S^1 \times S^2 \# L(p,q)$ are wave-coherent [NO85]. In each of these cases,* *wave moves reduce genus 2 Heegaard diagrams into a standard one. On the other hand, note that Osborne shows the lens spaces $L(173, 78)$ and $L(85, 32)$ admit genus 2 diagrams with fewer crossings than the standard stabilization of a genus 1 diagram [Osb82], and hence wave moves alone will not necessarily transform any genus 2 diagram of these lens spaces into the standard stabilized diagram. Nevertheless these minimal diagrams of Osborne are coherent. Are all lens spaces wave-coherent?*

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Department of Mathematics, University of Miami
Coral Gables, FL 33124-4250, USA
E-mail address: k.baker@math.miami.edu

Department of Mathematics, Imperial College London
South Kensington, London SW7 2AZ, UK
E-mail address: d.buck@imperial.ac.uk

Aix Marseille Université, CNRS
Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France
E-mail address: ana.lecuona@univ-amu.fr

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