

# Gromov-Witten theory of product stacks

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Let  $\mathcal{X}_1$  and  $\mathcal{X}_2$  be smooth proper Deligne-Mumford stacks with projective coarse moduli spaces. We prove a formula expressing Gromov-Witten invariants of the product stack  $\mathcal{X}_1 \times \mathcal{X}_2$  in terms of Gromov-Witten invariants of the factors  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . As an application, we deduce a decomposition result for Gromov-Witten theory of trivial gerbes.

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## 1. Introduction

Given two smooth complex projective varieties  $X_1, X_2$ , the natural question of determining the Gromov-Witten theory of the product  $X_1 \times X_2$  in terms of the Gromov-Witten theory of the factors  $X_1$  and  $X_2$  was raised and studied fairly early on in [KM96], [Kau99], and [Beh99b]. In [Beh99b] K. Behrend gave a complete answer, which is a product formula expressing Gromov-Witten invariants of  $X_1 \times X_2$  in terms of those of  $X_1$  and  $X_2$ .

Gromov-Witten theory was extended to symplectic orbifolds by Chen and Ruan [CR02] and to smooth Deligne-Mumford stacks by Abramovich, Graber and Vistoli [AGV02], [AGV08]. Given two smooth proper Deligne-Mumford stacks  $\mathcal{X}_1$  and  $\mathcal{X}_2$  with projective coarse moduli spaces, it is natural to ask for a relationship between the Gromov-Witten invariants  $I^{\mathcal{X}_1 \times \mathcal{X}_2}$  of  $\mathcal{X}_1 \times \mathcal{X}_2$  and the Gromov-Witten invariants  $I^{\mathcal{X}_1}, I^{\mathcal{X}_2}$  of  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . The main result of this paper provides such a relationship, generalizing the results in [Beh99b] to Deligne-Mumford stacks. More precisely, we prove a formula which expresses orbifold Gromov-Witten invariants of  $\mathcal{X}_1 \times \mathcal{X}_2$  in terms of orbifold Gromov-Witten invariants of  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . Roughly speaking this formula takes the following shape:

$$I^{\mathcal{X}_1 \times \mathcal{X}_2}(\omega_1 \otimes \omega_2) = I^{\mathcal{X}_1}(\omega_1) \cup I^{\mathcal{X}_2}(\omega_2).$$

The precise statement is a bit more involved and is explained in Theorem 4.6 below.

Gromov-Witten invariants of  $\mathcal{X}_1 \times \mathcal{X}_2$  are by definition integrals against virtual fundamental classes of moduli spaces of twisted stable maps to  $\mathcal{X}_1 \times \mathcal{X}_2$ . Therefore to prove a product formula for Gromov-Witten invariants of  $\mathcal{X}_1 \times \mathcal{X}_2$  one needs to study the relations between virtual fundamental classes of moduli spaces of twisted stable maps to  $\mathcal{X}_1 \times \mathcal{X}_2$  and virtual fundamental classes of moduli spaces of twisted stable maps to  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . This is what we do: we prove in Theorems 3.13 and 4.3 such comparison results for virtual fundamental classes. These results are the technical core of this paper. Our product formula Theorem 4.6 is obtained as a consequence of these comparison results.

Our approach to the product formula in orbifold Gromov-Witten theory closely follows that of [Beh99b], with one notable new ingredient: we make extensive use of logarithmic geometry, especially Olsson's work on log twisted curves [Ols07], in order to handle various new phenomena which occur due to the presence of stack structures.

Our study of product formula in orbifold Gromov-Witten theory is partly motivated by an interesting conjecture on the structure of Gromov-Witten theory of *étale gerbes*. Let  $G$  be a finite group and  $BG$  the classifying stack of  $G$ . Roughly speaking a  $G$ -gerbe over a Deligne-Mumford stack  $\mathcal{X}$  is a morphism  $\mathcal{Y} \rightarrow \mathcal{X}$  which is a  $BG$ -bundle. The simplest example of a  $G$ -gerbe over  $\mathcal{X}$  is the *trivial  $G$ -gerbe*, given by the projection to the first factor,

$$(1) \quad \mathcal{X} \times BG \rightarrow \mathcal{X}.$$

The Gromov-Witten theoretic decomposition conjecture, formulated in [HHP07] and developed in [TT10] (where it is called gerbe duality), asserts that for the trivial  $G$ -gerbe (1) the Gromov-Witten theory of  $\mathcal{X} \times BG$  is equivalent to the Gromov-Witten theory of the union of several copies of  $\mathcal{X}$ . The number of copies needed is equal to the number of conjugacy classes of  $G$ . Since the trivial  $G$ -gerbe is the simplest, in order for the decomposition conjecture to be plausible it is important that it holds true for the trivial  $G$ -gerbe.

Since  $\mathcal{X} \times BG$  is a product, we can apply our product formula to study Gromov-Witten theory of  $\mathcal{X} \times BG$ . The Gromov-Witten theory of  $BG$  is solved in the work of Jarvis-Kimura [JK02]. In Section 5 we combine these results to calculate Gromov-Witten invariants of  $\mathcal{X} \times BG$ . Our calculation, as formulated in Equations (24) and (25), verifies the Gromov-Witten theoretic decomposition conjecture in this most basic case.

The rest of this paper is organized as follows. Some of the basic notations and conventions are described in Section 1.1. Section 2 contains many preparatory definitions and results. Some of these (e.g. the notion of gerby modular graphs) are new. Section 3, which is the most technical, is devoted to prove the main result Theorem 3.13 on virtual fundamental classes. In Section 4 we present the product formula Theorem 4.6 in orbifold Gromov-Witten theory. Section 5 is devoted to the aforementioned application to trivial gerbes.

### 1.1. Notations and Conventions

We work over the field  $\mathbb{C}$  of complex numbers. By an *algebraic stack* we mean an algebraic stack over  $\mathbb{C}$  in the sense of [Art74]. By a *Deligne-Mumford stack* we mean an algebraic stack over  $\mathbb{C}$  in the sense of [DM69]. We assume moreover all stacks and schemes are quasi-separated, locally noetherian, locally of finite type. Following [Kat89], logarithmic structures are over the étale site of schemes. For the extension of the theory to stacks, which will be assumed, see [Ols03a]. According to [Mil80] we denote the stalk of a sheaf  $\mathcal{F}$  over  $X$  at the geometric point  $\bar{x}$  by  $\mathcal{F}_{\bar{x}}$ . We usually denote by curly capital letters twisted curves, and by capital letters prestable curves. E.g. if  $\mathcal{C}$  is a twisted curve, we denote by  $C$  its coarse moduli space. We use greek letters to denote dual graphs, and greek letters with tilde for gerby dual graphs. Unless otherwise explicitly stated, the order of the isotropy groups of marking gerbes of twisted curves will be denoted by the letter  $b$ , while the order of the stabilizer groups of nodes will be denoted by the greek letter  $\gamma$  (cf. Definition 2.14).

Let  $\mathcal{X}$  be a proper smooth Deligne-Mumford stack with projective coarse moduli space  $\pi : \mathcal{X} \rightarrow X$ . The *inertia stack* of  $\mathcal{X}$  may be defined as the fiber product over the diagonal

$$I(\mathcal{X}) := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}.$$

Since we work over  $\mathbb{C}$ , we can fix an identification between the group  $\mu_r$  of  $r$ -th roots of unity and the cyclic group  $\mathbb{Z}_r$  of order  $r$ . Given this  $I(\mathcal{X})$  can be alternatively described as the stack of representable morphisms from constant cyclotomic gerbes to  $\mathcal{X}$ ,

$$I(\mathcal{X}) = \cup_{r \in \mathbb{N}} \text{HomRep}(B\mu_r, \mathcal{X}),$$

see for instance [AGV08]. This description yields a natural decomposition

$$I(\mathcal{X}) = \cup_{r \in \mathbb{N}} I_{\mu_r}(\mathcal{X}), \quad I_{\mu_r}(\mathcal{X}) := \text{HomRep}(B\mu_r, \mathcal{X}).$$

Clearly the stack  $I_{\mu_r}(\mathcal{X})$  is non-empty only for finitely many  $r \in \mathbb{N}$ .

The group  $\mu_r$  acts on every object of  $I_{\mu_r}(\mathcal{X})$ . Removing this  $\mu_r$  by rigidification ([AOV11]) yields another stack

$$\bar{I}(\mathcal{X}) := \cup_{r \in \mathbb{N}} \bar{I}_{\mu_r}(\mathcal{X}),$$

which is called the *rigidified inertia stack of  $\mathcal{X}$* . There are natural maps  $I_{\mu_r}(\mathcal{X}) \rightarrow \bar{I}_{\mu_r}(\mathcal{X})$  and  $I(\mathcal{X}) \rightarrow \bar{I}(\mathcal{X})$ . Alternatively  $\bar{I}(\mathcal{X})$  may be described as the stack of representable maps from (not necessarily trivial) cyclotomic gerbes to  $\mathcal{X}$ .

There are two naturally defined locally constant functions on  $I(\mathcal{X})$ :

$$\text{ord} : I(\mathcal{X}) \rightarrow \mathbb{N}, \quad \text{age} : I(\mathcal{X}) \rightarrow \mathbb{Q},$$

where  $\text{ord}$  sends an object  $[B\mu_r \rightarrow \mathcal{X}]$  to the order  $r \in \mathbb{N}$ , and  $\text{age}$  is the “age” function. These two functions are also defined for  $\bar{I}(\mathcal{X})$ , and by abuse of notation we denote them also by  $\text{ord}$  and  $\text{age}$ .

More details about (rigidified) inertia stacks, age function and their properties can be found in e.g. [AGV08].

We denote by  $H_2^+(X, \mathbb{Z})$  the semi-group of effective curve classes of  $X$ .

We refer the readers to [AGV02] and [AGV08] for the constructions of orbifold Gromov-Witten theory. We will write  $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$  for the moduli stack of twisted stable maps to  $\mathcal{X}$  with specified discrete data  $g, n, \beta$ . When the discrete data is clear from the context or irrelevant for the discussion, we simply denote the moduli stack by  $\mathcal{K}(\mathcal{X})$ .

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## 2. Preparatory results

### 2.1. Prestable curves

We start by recalling the notion of *prestable curve* and of *morphism of prestable curves*.

**Definition 2.1** ([BM96], **Definition 2.1**). A *prestable curve* over a scheme  $T$  is a flat proper morphism  $\pi : C \rightarrow T$  of schemes such that the geometric fibers of  $\pi$  are reduced, connected and 1-dimensional and have at most ordinary double points (nodes) as singularities. The *genus* of a prestable curve  $C \rightarrow T$  is the map  $\bar{t} \mapsto H^1(C_{\bar{t}}, \mathcal{O}_{C_{\bar{t}}})$ , which is a locally constant function  $g : T \rightarrow \mathbb{Z}_{\geq 0}$ .

**Definition 2.2** ([BM96], **Definition 2.1**). A *morphism of prestable curves*  $p : C \rightarrow D$  over  $T$  is a  $T$ -morphism of schemes such that for every geometric point  $\bar{t}$  of  $T$  we have

- 1) if  $\eta$  is the generic point of an irreducible component of  $D_{\bar{t}}$ , then the fiber of  $p_{\bar{t}}$  over  $\eta$  is a finite  $\eta$ -scheme of degree at most one;
- 2) if  $C'$  is the normalization of an irreducible component of  $C_{\bar{t}}$ , then  $p_{\bar{t}}(C')$  is a single point only if  $C'$  is rational.

Prestable curves admit infinitesimal automorphisms. Therefore their moduli stack is an Artin stack. The algebraic stack of  $n$ -pointed, genus  $g$  prestable curves (see [Beh97]), usually denoted by  $\mathfrak{M}_{g,n}$ , is a smooth algebraic stack of dimension  $3g - 3 + n$ . It is not separated nor finite type.

Imposing *stability conditions* ([DM69]), which amounts to requiring that every irreducible rational component of the curve has at least three special points, and that any irreducible genus 1 component has at least one special point, gives objects with finite isomorphisms groups. Given a prestable curve, there is a construction called *stabilization* which produces a stable curve, see [Knu83] for details. A relative version of the construction also exists, which we describe in *Section 2.6*.

### 2.2. Modular Graphs

We recall the definition of *dual graph*<sup>1</sup> of a prestable curve given in [BM96].

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<sup>1</sup>In this paper we do not use the full categorical treatment of graphs developed in [BM96] (cfr. [Beh99a]).

**Definition 2.3.** A graph  $\tau$  is a quadruple  $(F_\tau, V_\tau, j_\tau, \partial_\tau)$ , where  $F_\tau$  and  $V_\tau$  are finite sets,  $\partial_\tau : F_\tau \rightarrow V_\tau$  is a map and  $j_\tau : F_\tau \rightarrow F_\tau$  an involution. We call  $F_\tau$  the set of *flags*,  $V_\tau$  the set of *vertices*,  $S_\tau = \{f \in F_\tau \mid j_\tau f = f\}$  the set of *tails* and  $E_\tau = \{\{f_1, f_2\} \subset F_\tau \mid f_2 = j_\tau f_1, f_1 \neq f_2\}$  the set of *edges* of  $\tau$ . For  $v \in V_\tau$ , let  $F_\tau(v) = \partial_\tau^{-1}(v)$  and  $|v| = \#F_\tau(v)$ , the *valence* of  $v$ .

Graphs are drawn by representing vertices by dots, edges by curves connecting vertices, tails by half open curves, connected only at their closed end to a vertex. In such a way we obtain a topological space which is called the *geometric realization* of the graph and is denoted by  $|\tau|$ . We will always assume that  $|\tau|$  is connected.

**Definition 2.4 ([BM96], Definition 1.5).** A *modular graph* is a graph  $\tau$  endowed with a map  $g_\tau : V_\tau \rightarrow \mathbb{Z}_{\geq 0}$ ;  $v \mapsto g(v)$ . The number  $g(v)$  is called the *genus* of  $v$ .

**Definition 2.5.** The *Euler characteristic* of a graph  $\tau$  is defined as

$$\chi(\tau) = \chi(|\tau|) - \sum_{v \in V_\tau} g(v),$$

where  $|\tau|$  is the geometric realization of the graph.

**Definition 2.6 ([BM96], Definition 1.6).** Let  $A$  be a semigroup with indecomposable zero. An  $A$ -*structure* on  $\tau$  is a map  $\alpha : V_\tau \rightarrow A$ . A pair  $(\tau, \alpha)$  is called a *modular graph with A-structure* or an  $A$ -*graph*. A *marked graph* is a pair  $(A, \tau)$  where  $A$  is a semigroup with indecomposable zero and  $\tau$  is an  $A$ -graph.

There is a notion of *stability* for graphs, namely

**Definition 2.7 ([BM96], Definition 1.9).** A vertex  $v$  of an  $A$ -graph  $(\tau, \alpha)$  is *stable* if either  $\alpha(v) \neq 0$  or  $\alpha(v) = 0$  and  $2g(v) + |v| \geq 3$ . The  $A$ -graph  $\tau$  is called *stable* if all of its vertices are stable.

Whenever an  $A$ -graph is not stable there is a canonical procedure to “stabilize” it, which we describe below. Before that we need to recall some notions of morphisms of graphs introduced in [BM96].

**Definition 2.8 ([BM96], Definition 1.7).** Let  $(\sigma, \alpha)$  and  $(\tau, \beta)$  be  $A$ -graphs. A *combinatorial morphism*  $a : (\sigma, \alpha) \rightarrow (\tau, \beta)$  is a pair of maps  $a_F : F_\sigma \rightarrow F_\tau$  and  $a_V : V_\sigma \rightarrow V_\tau$  satisfying the following conditions:

1) the diagram

$$\begin{array}{ccc} F_\sigma & \xrightarrow{\partial_\sigma} & V_\sigma \\ a_F \downarrow & & \downarrow a_V \\ F_\tau & \xrightarrow{\partial_\tau} & V_\tau \end{array}$$

is commutative.

- 2) For every  $v \in V_\sigma$  let  $w = a_V(v)$ . The induced map  $a_{V,v} : F_\sigma(v) \rightarrow F_\tau(w)$  is injective;
- 3) Let  $f \in F_\sigma$  and  $\bar{f} = j_\sigma(f)$ . If  $f \neq \bar{f}$ , there exists  $n \geq 1$  and  $2n$  not necessarily distinct flags  $f_1, \dots, f_n, \bar{f}_1, \dots, \bar{f}_n \in F_\tau$  such that
- $f_1 = a_F(f)$ ,  $\bar{f}_n = a_F(\bar{f})$ ;
  - $j_\tau(f_i) = \bar{f}_i$ ;
  - $\partial_\tau(\bar{f}_i) = \partial_\tau(f_{i+1})$  for  $i = 1, \dots, n-1$ ;
  - for all  $i = 1, \dots, n-1$  we have

$$\bar{f}_i \neq f_{i+1} \Rightarrow g(v_i) = 0 \quad \text{and} \quad \beta(v_i) = 0;$$

- 4) for every  $v \in V_\sigma$  we have  $\alpha(v) = \beta(a_V(v))$  and  $g(v) = g(a_V(v))$ .

A *combinatorial morphism of marked graphs*  $(B, \sigma, \alpha) \rightarrow (A, \tau, \beta)$  is a pair  $(\xi, a)$ , where  $\xi : A \rightarrow B$  is a homomorphism of semigroups and  $a : (\sigma, \alpha) \rightarrow (\tau, \xi \circ \beta)$  is a combinatorial morphism of  $B$ -graphs.

According to [BM96], Proposition 1.13, given an  $A$ -graph  $\tau$  which is not stable, there exists a stable  $A$ -graph  $\tau^s$ , together with a combinatorial morphism  $a : \tau^s \rightarrow \tau$ , such that every combinatorial morphism  $\sigma \rightarrow \tau$ , where  $\sigma$  is a stable  $A$ -graph, factors uniquely through  $\tau^s$ . In [Ibid.], Section 5 the following characterization of the stabilization morphism is provided. For every edge of  $\tau^s$   $\{f, \bar{f}\}$ , there exists a unique sequence of distinct edges  $\{f_1, \bar{f}_1\}, \dots, \{f_n, \bar{f}_n\}$  in  $E_\tau$  such that  $a(f) = f_1$ ,  $a(\bar{f}) = \bar{f}_n$ , and  $\partial \bar{f}_i = \partial f_{i+1}$  for all  $i = 1, \dots, n-1$ . All vertices  $v_i = \partial \bar{f}_i = \partial f_{i+1}$  have valence two. We call  $\{f_1, \bar{f}_1, \dots, f_n, \bar{f}_n\}$  the *long edge* associated to  $\{f, \bar{f}\}$ . The edges  $\{f_i, \bar{f}_i\}$  are called the *factors* of the long edge.

For every tail  $f$  of  $\tau^s$  there exists a unique sequence of flags  $f_1, \dots, f_n$  of  $\tau$  such that

- if  $n$  is odd, then  $\{f_1, f_2\}, \dots, \{f_{n-2}, f_{n-1}\}$  are edges of  $\tau$ ,  $a(f_1) = f_1$ ,  $\partial f_{2i} = \partial f_{2i+1}$ , for  $i = 1, \dots, \frac{1}{2}(n-1)$ , and  $f_n$  is a tail of  $\tau$ ;



2) if  $n$  is even, then  $\{f_1, f_2\}, \dots, \{f_{n-1}, f_n\}$  are edges of  $\tau$ ,  $a(f) = f_1$ ,  $\partial f_{2i} = \partial f_{2i+1}$ , for  $i = 1, \dots, n/2 - 1$ , and  $\partial f_n$  has valence one.

All vertices  $v_i = \partial f_{2i} = \partial f_{2i+1}$ , for  $i = 1, \dots, \lfloor \frac{1}{2}(n - 1) \rfloor$  have valence two. We call  $\{f_1, \dots, f_n\}$  the *long tail* associated to  $f$ . If  $n$  is odd, we call the edges  $\{f_{2i-1}, f_{2i}\}$ , for  $i = 1, \dots, \frac{1}{2}(n - 1)$ , and the tail  $f_n$  the *factors* of this long tail. If  $n$  is even, we call the edges  $\{f_{2i-1}, f_{2i}\}$  for  $i = 1, \dots, n/2$  *factors* of the long tail.

Let  $\mathcal{X}$  be a smooth proper Deligne-Mumford stack with projective coarse moduli space. Orbifold Gromov-Witten theory of  $\mathcal{X}$  is based on the notion of twisted stable map developed in [AV02], [AGV02] and [AGV08]. The source curves of such maps are *twisted curves*, which we describe in the following.

### 2.3. Twisted curves

In this section we recall the definition and some useful results about twisted curves which explain why the definition of gerby dual graphs is well posed.

**Definition 2.9 ([AV02], Definition 4.1.2).** Let  $S$  be a scheme. A (*balanced*) *twisted curve* over  $S$  is a proper flat Deligne-Mumford stack  $\mathcal{C} \rightarrow S$  whose fibers are pure one-dimensional and geometrically connected with at most nodal singularities. Let  $\mathcal{C} \rightarrow C$  be the coarse moduli space of  $\mathcal{C}$ , and  $C^{sm} \subset C$  the open subset where  $C \rightarrow S$  is smooth. Then the inverse image  $\mathcal{C} \times_C C^{sm} \subset \mathcal{C}$  is equal to the open substack of  $\mathcal{C}$  where  $\mathcal{C} \rightarrow S$  is smooth. Moreover for any geometric point  $\bar{s} \rightarrow S$  the map  $\mathcal{C}_{\bar{s}} \rightarrow C_{\bar{s}}$  is an isomorphism over some dense open subset of  $C_{\bar{s}}$ . As a consequence the coarse moduli space is also a nodal curve. For any geometric point mapping to a node  $\bar{p} \rightarrow C$ , there exists an étale neighborhood  $\text{Spec}(A) \rightarrow C_{\bar{s}}$  of  $\bar{p}$ , where  $A$  is a strictly henselian local ring, and an étale morphism

$$\text{Spec } A \rightarrow \text{Spec} (\mathcal{O}_{S, \bar{s}}[x, y]/xy - t)$$

for some  $t \in \mathcal{O}_S$  such that the pullback  $\mathcal{C} \times_C \text{Spec } A$  is isomorphic to

$$[\text{Spec} (A[z, w]/zw = t', z^n = x, w^n = y)/\Gamma]$$

for some  $t' \in \mathcal{O}_{S, \bar{s}}$ , where  $\Gamma$  is a finite cyclic group of order  $n$  such that if  $\gamma \in \Gamma$  is a generator then  $\gamma(z) = \zeta \cdot z$  and  $\gamma(w) = \zeta^{-1} \cdot w$ . This kind of action is called *balanced*. A twisted curve has *genus*  $g$  if the genus of  $C_{\bar{s}}$  is  $g$  for any geometric point  $\bar{s} \rightarrow S$ . An *n-pointed twisted curve* is a twisted curve together with a collection of disjoint closed substacks  $\{\Sigma_i\}_{i=1}^n$  such that

- 1) each  $\Sigma_i$  is contained in the smooth locus of  $\mathcal{C} \rightarrow S$ ;
- 2) the stacks  $\Sigma_i$  are étale gerbes over  $S$ ;
- 3) if  $\mathcal{C}_{gen}$  denotes the complement of the  $\Sigma_i$  in the smooth locus of  $\mathcal{C} \rightarrow S$  then  $\mathcal{C}_{gen}$  is a scheme.

A smooth  $n$ -pointed prestable twisted curve  $\mathcal{C}$  over an arbitrary scheme  $S$  is determined up to a unique isomorphism by its coarse moduli space  $C$  and by the order of the isotropy groups of its marked sections, see [AGV08], *Theorem 4.2.1*.

**Theorem 2.10.** *Let  $b_1, \dots, b_n \geq 1$  be integers numbers. Let  $C$  be an  $n$ -pointed smooth prestable curve over  $S$ . Let  $S_i \subset C$  denote the image of the sections of  $C \rightarrow S$ . Let  $L_i := \mathcal{O}_C(-S_i)$  and let  $\sigma_i$  be the canonical sections vanishing on  $S_i$ . Then there is, up to isomorphisms, a unique twisted curve  $\mathcal{C}$  over  $S$  with coarse moduli space  $C$  such that the  $i$ -th marked gerbe is banded by  $\mu_{b_i}$ . It is obtained as*

$$\mathcal{C} \simeq \sqrt[b_1]{(C, L_1, \sigma_1)} \times_C \cdots \times_C \sqrt[b_n]{(C, L_n, \sigma_n)}$$

where  $\sqrt[b_i]{(C, L_i, \sigma_i)}$  denote the construction known as “root of a line bundle with section” [Cad07], [AGV08].

For families of nodal twisted curves there is no such a simple description in terms of the coarse moduli space and of the order of the isotropy groups of the singular points. As we will describe in *Section 2.7.2*, logarithmic geometry is needed to properly describe singular twisted curves.

In case the nodes are gerbes over the base of the curve, it is still possible to describe a nodal twisted curve in terms of smooth twisted curves by means of a construction analogous to the *clutching construction* of [Knu83], extended to stacks in [AGV08], *Section 5* and *Appendix A*. However in general the singular locus of a twisted curve is not a gerbe over the whole base. If we restrict to twisted curves over the spectrum of an algebraically closed field, we can always apply the construction of [AGV08] we alluded to above. In fact the singular curve is isomorphic to the pushout along smooth substacks of smooth twisted curves. Over  $\text{Spec } \mathbb{C}$ , given a prestable curve  $C$ , and given a set of integer numbers specifying the order of the isotropy groups of the nodes, there is only one twisted curve  $\mathcal{C} \rightarrow C$ . Uniqueness can be seen locally. Indeed, the singular locus can be described étale locally as a quotient stack as follows. For simplicity assume that  $\mathcal{C}$  has a single node of order  $r$ , and assume that étale locally near this node the coarse moduli

space  $C$  is isomorphic to  $\text{Spec } \mathbb{C}[x, y]/(xy)$ . According to [Ols07], an étale local description for  $\mathcal{C}$  near this node is

$$[\text{Spec } \mathbb{C}[z, w]/(zw)/\mu_r],$$

where  $\mu_r$  acts via  $z \mapsto \zeta_r z, w \mapsto \zeta_r^{-1} w, \zeta_r := \exp(2\pi\sqrt{-1}/r)$ .

**Remark 2.11.** If the base has a non reduced structure, there can be non isomorphic twisted curves over  $C$ , even when the order of the stabilizer group of the singular locus is assigned. Let  $C$  be the trivial family of nodal curves, étale locally isomorphic to

$$\text{Spec } \mathbb{C}[x, y]/xy \times \text{Spec } \mathbb{C}[\epsilon]/\epsilon^r.$$

There are two families of twisted curves over  $C$  with node of order  $r$ . One is the trivial family, the other can be described étale locally as

$$[\text{Spec } (\mathbb{C}[\epsilon]/\epsilon^r)[z, w]/(zw - \epsilon)/\mu_r],$$

where  $z^r = x, y^r = y, \epsilon^r = 0$ , and the action of the cyclic group given by the usual balanced action. This stack has a closed substack corresponding to a  $\mu_r$ -gerbe over the closed point, but it does not contain a  $\mu_r$  gerbe over the base.

We generalize Definitions 2.3 and 2.4 to the notion of *gerby modular graph*, which allows us to suitably stratify stacks parametrizing twisted curves and twisted stable maps.

**Notation 2.12.** Let  $\mathcal{U}$  be a finite set and  $\mathfrak{o} : \mathcal{U} \rightarrow \mathbb{N}, \mathfrak{a} : \mathcal{U} \rightarrow \mathbb{N} \cup \{0\}$  two set maps.

The following example is important.

**Example 2.13.** Let  $\mathcal{X}$  be a smooth proper Deligne-Mumford stack. A triple  $(\mathcal{U}, \mathfrak{o}, \mathfrak{a})$  is obtained as follows. Let  $\mathcal{U} = \mathcal{U}(\mathcal{X})$  be the set of connected components of the rigidified cyclotomic inertia stack of  $\mathcal{X}$ . Let  $\mathfrak{o}$  be the set map  $\mathcal{U}(\mathcal{X}) \rightarrow \mathbb{N}$  taking  $U \in \mathcal{U}(\mathcal{X})$  to the integer  $r$  such that  $U \subseteq \bar{I}_{\mu_r}(\mathcal{X})$ . Let  $\mathfrak{a} : \mathcal{U}(\mathcal{X}) \rightarrow \mathbb{N} \cup \{0\}$  be the set map taking  $U$  to its age.

**Definition 2.14.** Let  $(\mathcal{U}, \mathfrak{o}, \mathfrak{a})$  be as in Notation 2.12. A *gerby modular graph*  $\tilde{\tau}$  associated to  $(\mathcal{U}, \mathfrak{o}, \mathfrak{a})$  is the data of an underlying modular graph  $\tau$

with a map  $\mathfrak{g} : F_{\tilde{\tau}} \rightarrow \mathcal{U}$  such that  $\mathfrak{g}(f) = \mathfrak{g}(f')$  whenever the flags  $f, f'$  form an edge  $\{f, f'\} \in E_{\tilde{\tau}}$ . We define  $\gamma := \mathfrak{o} \circ \mathfrak{g}$ .

Let  $A$  be a semigroup with indecomposable zero. A *gerby  $A$ -graph* is a gerby modular graph  $\tilde{\tau}$  whose underlying modular graph  $\tau$  is endowed with an  $A$ -structure.

**Remark 2.15.** We abuse the notation slightly: for a gerby modular graph  $\tilde{\tau}$  we denote the data  $F_{\tilde{\tau}}, E_{\tilde{\tau}}$ , e.t.c. associated to its underlying graph  $\tau$  also by  $F_{\tilde{\tau}}, E_{\tilde{\tau}}$ , etc.

**Definition 2.16.** Let  $\pi : \mathcal{X} \rightarrow X$  be a smooth proper Deligne-Mumford stack with projective coarse moduli space  $X$ , and let  $(\mathcal{U} = \mathcal{U}(\mathcal{X}), \mathfrak{o}, \mathfrak{a})$  be the triple obtained from  $\mathcal{X}$  as in Example 2.13. A gerby modular graph  $\tilde{\tau}$  associated to the triple  $(\mathcal{U}, \mathfrak{o}, \mathfrak{a})$ , with  $\mathfrak{g} : F_{\tilde{\tau}} \rightarrow \mathcal{U}(\mathcal{X})$  and an  $H_2^+(X, \mathbb{Z})$ -structure  $\alpha : V_{\tilde{\tau}} \rightarrow H_2^+(X, \mathbb{Z})$ , is called a *gerby  $\mathcal{X}$ -graph*.

The idea behind the definition of gerby  $\mathcal{X}$ -graphs is as follows. Given a representable morphism  $f : \mathcal{C} \rightarrow \mathcal{X}$  from a twisted curve  $\mathcal{C}$  over  $\text{Spec } \mathbb{C}$  to a smooth Deligne-Mumford stack  $\mathcal{X}$ , we would like to encode the structure of the domain curve  $\mathcal{C}$  combinatorially. The structure of the underlying coarse curve  $C$  is encoded in a modular graph  $\tau$  as in [BM96]. The stack structures on  $\mathcal{C}$  are recorded as follows. If  $p \simeq B\mu_r \in \mathcal{C}$  is a stack point, then the restriction  $f|_p : B\mu_r \rightarrow \mathcal{X}$  is a  $\mathbb{C}$ -point of the rigidified inertia stack  $\bar{I}(\mathcal{X})$ . The order of the stabilizer group of the stack point  $p$  can be recovered from the value at the point  $f|_p$  of the locally constant function  $\text{ord} : \bar{I}(\mathcal{X}) \rightarrow \mathbb{N}$ .

**Definition 2.17.** The *dimension* of a gerby  $\mathcal{X}$ -graph  $\tilde{\tau}$  is

$$\dim(\tilde{\tau}, \mathcal{X}) := \chi(\tilde{\tau})(\dim \mathcal{X} - 3) - \int_{\beta(\tilde{\tau})} \omega_{\mathcal{X}} + \#S_{\tilde{\tau}} - \#E_{\tilde{\tau}} - \sum_{f \in S_{\tilde{\tau}}} \mathfrak{a}(\mathfrak{g}(f)),$$

where  $\beta(\tilde{\tau}) := \sum_{v \in V_{\tilde{\tau}}} \alpha(v) \in H_2^+(X, \mathbb{Z})$ .

### 2.4. Prestable curves and stable maps associated to gerby graphs

As mentioned above, gerby graphs will allow us to stratify moduli spaces of curves and of maps according to the topological type and the stacky structure of the curves. As in [Beh99b] the idea is that Gromov-Witten classes, which are classes in  $H_*(\overline{M}_{g,n})$ , are determined by their restriction to the boundary strata in  $\overline{M}_{g,n}$ , which are labelled by modular graphs  $\tau$ .

This restriction can be computed in terms of intersection theory over moduli spaces of maps whose source curves have generically the topological type and the stack structure specified by a dual gerby graph  $\tilde{\tau}$  with underlying modular graph  $\tau$ . We will compare the virtual fundamental classes of those moduli stacks of maps to  $\mathcal{X}_1 \times \mathcal{X}_2$  and to  $\mathcal{X}_1, \mathcal{X}_2$ .

Let  $\tilde{\tau}$  be a gerby modular graph. We define  $\tilde{\tau}$ -marked prestable curve, which generalizes Definition 2.6 of [BM96].

**Definition 2.18.** A  $\tilde{\tau}$ -marked prestable curve over  $T$  is a pair  $(\mathcal{C}, \Sigma)$ , where  $\mathcal{C} = (\mathcal{C}_v)_{v \in V_{\tilde{\tau}}}$  is a family of twisted curves over  $T$ ,  $\Sigma = (\Sigma_i)_{i \in F_{\tilde{\tau}}}$  is a collection of disjoint étale gerbes banded by cyclic groups over  $T$  which are closed substacks  $\Sigma_i \hookrightarrow \mathcal{C}_v$  such that for every geometric point  $t \in T$ ,

- 1)  $\Sigma_i|_t$  is contained in the smooth locus of  $\mathcal{C}_{\partial(i),t}$  for any  $i \in F_{\tilde{\tau}}$ ;
- 2)  $\Sigma_i|_t$  and  $\Sigma_j|_t$  are disjoint for  $i \neq j$ ;
- 3) the coarse curve  $C_{v,t}$  has genus  $g(C_{v,t}) = g(v)$  for all  $v \in V_{\tilde{\tau}}$ ;
- 4)  $\Sigma_i|_t \simeq B\mu_r$  where  $r = \gamma(i)$  for  $i \in F_{\tilde{\tau}}$ .
- 5) for any  $\{i, \bar{i}\} \in E_{\tilde{\tau}}$  a band-inverting isomorphism  $\xi_{i\bar{i}} : \Sigma_i \xrightarrow{\sim} \Sigma_{\bar{i}}$  over  $T$  is assigned,  $\mathcal{C}_{\partial(i)}$  and  $\mathcal{C}_{\partial(\bar{i})}$  are glued along  $\Sigma_i$  and  $\Sigma_{\bar{i}}$  via  $\xi_{i\bar{i}}$ . The twisted curve  $\mathcal{C}$  is obtained by gluing the various  $\mathcal{C}_{\partial(i)}$  and  $\mathcal{C}_{\partial(\bar{i})}$  together.

The gerby graph  $\tilde{\tau}$  specifies how to glue the curves corresponding to the vertices in  $V_{\tilde{\tau}}$  along gerbes indexed by edges  $\{i, \bar{i}\}$  in  $E_{\tilde{\tau}}$  via the isomorphisms inverting the band  $\xi_{i\bar{i}}$ . The twisted curves can be glued by doing a pushout in the category of Deligne-Mumford stacks. The gluing procedure associates to the node obtained after gluing a cyclic group of automorphisms fixing the coarse curve. This is shown in the following Lemma.

**Lemma 2.19 (see [ACV01], Proposition 7.1.1).** *Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two smooth twisted curves and let  $i_1 : \Sigma \hookrightarrow \mathcal{C}_1$  and  $i_2 : \Sigma \hookrightarrow \mathcal{C}_2$  be two  $\mu_r$ -gerbes. Let  $\mathcal{C}$  be the pushout along  $\Sigma$ . Then  $\text{Aut}(\mathcal{C})_C \simeq \mu_r$ , where  $\text{Aut}(\mathcal{C})_C$  is the group of automorphisms of  $\mathcal{C}$  leaving  $C$  fixed.*

For a gerby modular graph  $\tilde{\tau}$ , let  $\mathfrak{M}^{tw}(\tilde{\tau})$  denote the moduli stack of  $\tilde{\tau}$ -marked prestable curves.

Let  $\mathcal{X}$  be a smooth proper Deligne-Mumford stack with projective coarse moduli space  $X$ . Let  $\tilde{\tau}$  be a gerby  $\mathcal{X}$ -graph. Denote by  $\alpha : V_{\tilde{\tau}} \rightarrow H_2^+(X, \mathbb{Z})$  the  $H_2^+(X, \mathbb{Z})$ -structure on the underlying modular graph  $\tau$ . For the notion of twisted stable map we refer to [AV02], Definition 4.3.1.

**Definition 2.20.** A  $\tilde{\tau}$ -marked twisted stable map over  $T$  is a morphism  $f : (\mathcal{C}, \Sigma) \rightarrow \mathcal{X}$  such that

- 1)  $(\mathcal{C} = (\mathcal{C}_v)_{v \in V_{\tilde{\tau}}}, \Sigma = (\Sigma_i)_{i \in F_{\tilde{\tau}}})$  is a  $\tilde{\tau}$ -marked prestable twisted curve over  $T$ ;
- 2)  $f : (\mathcal{C}, \Sigma) \rightarrow \mathcal{X}$  is a twisted stable map;
- 3) For  $i \in F_{\tilde{\tau}}$ , the restriction  $f|_{\Sigma_i}$  is an object over  $T$  of the component of  $\bar{I}_\mu(\mathcal{X})$  indexed by  $\mathfrak{g}(i) \in \mathcal{U}(\mathcal{X})$ ;
- 4) Let  $(C = (C_v)_{v \in V_{\tilde{\tau}}}, \bar{\Sigma} = (\bar{\Sigma}_i)_{i \in F_{\tilde{\tau}}})$  be the coarse curve and  $\bar{f} : C \rightarrow X$  the induced map, then the data  $(C, \bar{\Sigma}, \bar{f})$  are a stable  $(X, \tau, \alpha)$ -map over  $T$  in the sense of [BM96], Definition 3.2.

In particular, this means that  $\bar{f}_*[C_v] = \alpha(v)$  for  $v \in V_{\tilde{\tau}}$ .

Let

$$\mathcal{K}(\mathcal{X}, \tilde{\tau}) \subset \mathcal{K}_{1-\chi(\tau), \#S_\tau}(\mathcal{X}, \beta(\tilde{\tau}))$$

be the moduli stack of  $\tilde{\tau}$ -marked twisted stable maps to  $\mathcal{X}$ .

### 2.5. Relative coarse moduli space

We discuss the construction of relative coarse moduli spaces.

Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of algebraic stacks locally of finite presentation. Let  $I(\mathcal{X}/\mathcal{Y}) = Ker(I(\mathcal{X}) \rightarrow f^*I(\mathcal{Y})) = \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$  denote the relative inertia stack, which is assumed to be finite. In ([AOV11], Theorem 3.1) the authors prove that there exists a relative coarse moduli space, namely

**Theorem 2.21.** *There exists an algebraic stack  $X$ , and morphisms  $\mathcal{X} \xrightarrow{\pi} X \xrightarrow{\bar{f}} \mathcal{Y}$  such that  $f = \bar{f} \circ \pi$ , satisfying the following properties:*

- 1)  $\bar{f} : X \rightarrow \mathcal{Y}$  is representable;
- 2) if there is  $\mathcal{X} \xrightarrow{\pi'} X' \xrightarrow{\bar{f}'} \mathcal{Y}$  with  $\bar{f}'$  representable, then there is a unique  $h : X \rightarrow X'$  such that  $\pi' = h \circ \pi$  and  $\bar{f}' = \bar{f} \circ h$ ;
- 3)  $\pi$  is proper and quasi finite;
- 4)  $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\mathcal{X}}$  is an isomorphism;
- 5) if  $X'$  is a stack and  $X' \rightarrow X$  is a representable flat morphism, then  $\mathcal{X} \times_X X' \rightarrow X' \rightarrow \mathcal{Y}$  satisfies (1) – (4) for the morphism  $\mathcal{X} \times_X X' \rightarrow \mathcal{Y}$ .

We give an explanation of the above definition here. If  $\mathcal{X}$  and  $\mathcal{Y}$  are tame stacks over a scheme  $T$ , the formation of the relative coarse moduli space commutes with base change on  $T$ . Deligne-Mumford stacks over the complex numbers are tame. For twisted curves over  $\mathbb{C}$  the local description of the relative coarse moduli space is as follows. Étale locally, a twisted curve is isomorphic to  $\mathcal{X} = [V/\Gamma]$ , where  $V$  is the spectrum of a henselian local ring and  $\Gamma$  is a finite group acting on  $V$  fixing a geometric point  $\bar{s} \rightarrow V$ . Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of Deligne-Mumford type. Let  $K \subset \Gamma$  be the subgroup fixing the composite  $\bar{s} \rightarrow \mathcal{X} \rightarrow \mathcal{Y}$ , namely  $K$  is the kernel of the map  $\Gamma \rightarrow \text{Aut}_{\mathcal{Y}}(f(\bar{s}))$ . Let  $U = V/K$  be the geometric quotient, and  $Q = \Gamma/K$ . Let  $\bar{\mathcal{X}} = [U/Q]$ . Then there exist a natural projection  $q : \mathcal{X} \rightarrow \bar{\mathcal{X}}$  and a unique factorization  $g : \bar{\mathcal{X}} \rightarrow \mathcal{Y}$  such that  $f = g \circ q$  identifying  $\bar{\mathcal{X}}$  as the relative coarse moduli space of  $f$  ([AOV11], *Proposition 3.4*).

## 2.6. Partial stabilization

Let  $\pi : C \rightarrow T$  be a prestable curve and let  $\{x_i\}_{i=1}^n$  be a set of sections through the smooth locus of  $T$ . Let  $f : C \rightarrow M$  be a morphism to a smooth projective variety  $M$ . There is a canonical way to associate to it a stable map. This is proved in [BM96]. Let  $N$  be a very ample invertible sheaf on  $M$ . Let  $L = \omega_{C/T}(\sum x_i) \otimes f^*N^{\otimes 3}$ , where  $\omega_{C/T}$  is the relative dualizing sheaf. The relative stabilization is defined as

$$C' = \text{Proj}(\oplus_{k \geq 0} \pi_* L^{\otimes k}).$$

The absolute stabilization is recovered from this construction by considering the case where  $M$  is a point. The *partial stabilization* is obtained in the following way. Given a prestable curve  $C$ , suppose that it has a dual graph with some unstable vertices. Suppose we want to contract the components corresponding to some of those vertices. It is possible to choose a set of sections making the curve stable. Leaving out the sections which make stable the components to be contracted and stabilizing gives the partial stabilization.

Given a morphism  $f : \mathcal{C} \rightarrow \mathcal{M}$  from a twisted curve over  $T$  to a smooth Deligne Mumford stack  $\mathcal{M}$ , [AV02], *Section 9* provides a construction which associates to  $f$  a twisted stable map. If  $f$  is not representable, one reduces to the representable case by taking the relative coarse moduli space as described above. Then one considers the morphism  $f : C \rightarrow M$  induced by passing to the coarse moduli spaces. Let  $C' \rightarrow M$  be the stable map obtained by taking the relative stabilization. Let  $L'$  be a “sufficiently relatively ample”<sup>2</sup>

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<sup>2</sup>See [AV02] for what this means.

line bundle with respect to  $C' \rightarrow M$ . Let  $\mathcal{L}$  be its pullback to  $\mathcal{C}$ . Define  $\mathcal{A} := \bigoplus_{k \geq 0} \pi_* \mathcal{L}^{\otimes k}$ , where  $\pi : \mathcal{C} \rightarrow \mathcal{M} \times T$ . The relative stabilization in the stack case is defined as

$$\mathcal{C}' := \text{Proj}_{\mathcal{M}} \mathcal{A} \rightarrow \mathcal{M}.$$

One verifies that  $\mathcal{C}'$  is flat over  $T$  and  $\mathcal{C}' \rightarrow \mathcal{M}$  is a twisted stable map.

**2.6.1. Dual graphs and stabilization.** Recall that, as explained on page 230, in [BM96], *Proposition 1.13*, the authors prove that, given an  $A$ -graph, where  $A$  is a semigroup with indecomposable zero, there exists a stable  $A$ -graph  $\tau^s$  and a combinatorial morphism  $\tau^s \rightarrow \tau$  with the following universal property: any combinatorial morphism  $\sigma \rightarrow \tau$ , where  $\sigma$  is a stable  $A$ -graph, factors uniquely through  $\tau^s$ .

We adapt the notion of combinatorial morphism for gerby graphs.

**Definition 2.22.** A combinatorial morphism of marked gerby graphs

$$(B, \mathcal{U}_B, \sigma) \rightarrow (A, \mathcal{U}_A, \tau)$$

is a combinatorial morphism of the underlying marked graphs  $(a, \xi) : (B, \sigma) \rightarrow (A, \tau)$ , together with a set map  $\omega : \mathcal{U}_A \rightarrow \mathcal{U}_B$  such that for all  $f \in F_\sigma$ ,  $\mathfrak{g}(f) = \omega \circ \mathfrak{g}(a_F(f))$ .

**Remark 2.23.** In this paper we will be concerned with the case where  $A = H_2^+(X_1 \times X_2, \mathbb{Z})$ ,  $B = H_2^+(X_i, \mathbb{Z})$ , and  $\mathcal{U}_A = I(\mathcal{X}_1 \times \mathcal{X}_2)$ ,  $\mathcal{U}_B = I(\mathcal{X}_i)$ .

**Example 2.24.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be two smooth proper Deligne-Mumford stacks with projective coarse moduli spaces  $X$  and  $Y$ . Let  $(\mathcal{U}(\mathcal{X}), \mathfrak{o}_{\mathcal{X}}, \mathfrak{a}_{\mathcal{X}})$  and  $(\mathcal{U}(\mathcal{Y}), \mathfrak{o}_{\mathcal{Y}}, \mathfrak{a}_{\mathcal{Y}})$  be the triples obtained from  $\mathcal{X}$  and  $\mathcal{Y}$  as in Example 2.13. Let  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism. We can define part of the data constituting a combinatorial morphism of marked gerby graphs from  $F$ . More precisely we have a semi-group homomorphism  $\overline{F}_* : H_2^+(X, \mathbb{Z}) \rightarrow H_2^+(Y, \mathbb{Z})$ , where  $\overline{F} : X \rightarrow Y$  is the induced map between coarse moduli spaces.

We define a set map  $F_* : \mathcal{U}(\mathcal{X}) \rightarrow \mathcal{U}(\mathcal{Y})$  as follows. The morphism  $F$  induces a morphism between the respective cyclotomic inertia stacks  $I(\mathcal{X}) \rightarrow I(\mathcal{Y})$  which maps connected components to connected components. Indeed, let  $\mathcal{G} \rightarrow \mathcal{X}$  be an object of  $I_{\mu_r}(\mathcal{X})$  (or  $\overline{I}_{\mu_r}(\mathcal{X})$ ). The morphism  $\mathcal{G} \rightarrow \mathcal{Y}$  obtained by composing with  $F$  factors through the relative coarse moduli space gerbe (see Section 2.5), which is isomorphic to  $\mathcal{G}$  if  $F$  is representable. This defines canonically an object of  $I_{\mu_s}(\mathcal{Y})$  (or  $\overline{I}_{\mu_s}(\mathcal{Y})$ ) for some  $s \in \mathbb{N}$  such that  $s|r$ .



Indeed the order of the cyclic group banding the gerbe obtained as relative coarse moduli space divides the order of the cyclic group banding  $\mathcal{G}$ .

### 2.7. Logarithmic geometry

We give a brief introduction to *logarithmic geometry*, which provides a characterization of twisted curves we will heavily use to prove some technical lemmas in what follows. Logarithmic structures were introduced by Fontaine-Illusie and further studied by Kato [Kat89]. Given a scheme  $X$ , a *pre-logarithmic structure*, often called *pre-log structure*, consists of a constructible sheaf of monoids  $M$  endowed with a morphism of monoids  $\alpha : M \rightarrow \mathcal{O}_X$ , where the structure sheaf is considered as a monoid with the multiplicative structure. Given a monoid or a sheaf of monoids  $M$ , we denote by  $M^*$  the submonoid or the subsheaf of invertible elements. When the natural morphism  $\alpha^{-1}(\mathcal{O}_X^*) \rightarrow M^*$  is an isomorphism a pre-log structure is called a *log structure*. In this case we will indicate by  $\lambda : \mathcal{O}_X^* \hookrightarrow M$  the natural inclusion of the subsheaf  $\mathcal{O}_X^* \subset \mathcal{O}_X$  in the log structure  $M$  over  $X$ . The quotient  $M/\alpha^{-1}(\mathcal{O}_X^*)$  is usually denoted by  $\bar{M}$ , and called the *characteristic* or the *ghost sheaf*. There is a canonical way to associate a log structure to a pre-log structure. Given a pre-log structure  $\alpha : M \rightarrow \mathcal{O}_X$ , the associated log structure, denoted by  $M^a$ , is defined as the pushout in the category of sheaves of monoids as in the following diagram

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^*) & \longrightarrow & M \\ \alpha \downarrow & & \downarrow \\ \mathcal{O}_X^* & \longrightarrow & M^a. \end{array}$$

The morphism to the sucture sheaf  $\alpha^a : M^a \rightarrow \mathcal{O}_X$  is induced by the pair of morphisms  $(\alpha, \iota)$ , where  $\iota : \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$  is the canonical inclusion. A scheme endowed with a log structure  $(X, M_X)$  is called a *log scheme*. Log schemes form a category. A morphism between two log schemes  $(f, f^b) : (X, M_X) \rightarrow (Y, M_Y)$  is a pair consisting of a morphism of schemes  $f : X \rightarrow Y$  and a morphism of sheaves of monoids  $f^b : f^*M_Y \rightarrow M_X$  compatible with the morphisms to the structure sheaves. The pullback of a log structure is defined as the log structure associated to the pre-log structure obtained by taking the inverse image.

An important class of log structures are *fine* log structures. By definition this means that étale locally there exists a morphism of sheaves of monoids

$\beta : P \rightarrow M$ , where  $P$  is the constant sheaf corresponding to a finitely generated *integral* monoid, such that  $M$  is isomorphic to the log structure associated to the pre-log structure defined by  $\alpha \circ \beta : P \rightarrow \mathcal{O}_X$ . A monoid  $P$  is called *integral* if the cancellation law holds, namely, if the equality between elements of  $P$   $x + y = x + y'$  implies  $y = y'$ . In this paper we will work with *locally free* log structures, namely fine log structures such that for any geometric point  $\bar{x} \rightarrow X$  the stalk  $\overline{M}_{\bar{x}}$  is isomorphic to a free monoid  $\mathbb{N}^r$  for some integer  $r$ . This turns out to be equivalent to the existence étale locally of a chart given by a free monoid.

Many basic constructions and definitions used in this paper are based on the notion of *simple morphism* of log structures. A morphism  $\phi : P_1 \rightarrow P_2$  between free monoids is called *simple* if  $P_1$  and  $P_2$  have the same rank,  $\phi$  is injective, and for every irreducible element of  $p_1 \in P_1$  there exists a unique irreducible element  $p_2 \in P_2$  and an integer  $b$  such that  $b \cdot p_2 = \phi(p_1)$ . A morphism of locally free log structures is called *simple* if it induces simple morphisms on the stalks.

The simplest examples of log spaces are points. By [Kat89], *Example 2.5 (2)*, isomorphism classes of fine log structures over the spectrum of an algebraically closed field are in bijection with isomorphism classes of sharp integral monoids. A monoid is called *sharp* if it has no invertible element other than the unit element. Any integral log structure over  $X = \text{Spec } k$  is of the form

$$(2) \quad M = \mathcal{O}_X^* \oplus P \mapsto \mathcal{O}_X, \quad (z, p) \mapsto \begin{cases} z & \text{if } p = 1 \\ 0 & \text{if } p \neq 1. \end{cases}$$

**2.7.1. Automorphisms of logarithmic structures.** Throughout the paper we will often deal with automorphisms of locally free log structures. Here we make some remarks which will be useful in what follows. Let  $M$  be a locally free log structure over  $X$ , where  $X$  is the spectrum of a henselian local ring. Denote by  $x_0$  the closed point of  $X$ . We assume that  $M$  admits a global chart  $\mathbb{N}^r \rightarrow M$ , inducing a bijection  $\mathbb{N}^r \rightarrow \overline{M}_{x_0}$ . Denote by  $e_1, \dots, e_r$  the global sections of  $M$  defined by the chart. Let  $\phi : M \xrightarrow{\sim} M$  be an automorphism inducing the identity on  $\overline{M}$ . Then  $\phi$  corresponds to a collection of units  $\vec{\zeta} = (\zeta_1, \dots, \zeta_r)$  such that  $\alpha(e_i)\zeta_i = \alpha(\phi(e_i))$ , where  $\alpha$  is the map  $M \rightarrow \mathcal{O}_X$ . We first associate  $\vec{\zeta}$  to  $\phi$ . For any  $i = 1, \dots, r$ ,  $\phi(e_i) = e_i + \lambda(\zeta_i)$ , with  $\zeta_i \in \mathcal{O}_X^*$ . Indeed  $\phi$  induces the identity on  $\overline{M}$ , therefore  $\phi(e_i)$  and  $e_i$  can only differ by an unit. Since  $M$  has integral stalks, the units  $\vec{\zeta}$  are uniquely determined. On the other hand, a collection of units  $\vec{\zeta}$  such that for any  $i = 1, \dots, r$ ,  $\alpha(\phi(e_i)) = \zeta_i \alpha(e_i)$  determines an isomorphism of  $M$ . This

is induced via the universal property of the pushout by the map  $\mathbb{N}^r \rightarrow M$  mapping the  $i$ -th standard generator to  $e_i + \zeta_i$  for  $i = 1, \dots, r$ . Such an isomorphism induces the identity on  $\overline{M}$  and is compatible with  $\alpha$ .

We will use this equivalence between automorphisms and collections of units for locally free log structures over spectra of henselian rings, like discrete valuation rings, Artinian rings and algebraically closed fields.

**2.7.2. Log twisted curves.** It follows from Theorem 2.10 that the stack structure of the marking gerbes of a twisted curve is determined by taking the root of a line bundle with section. As discussed thereafter, the stack structure at the nodes cannot be described in this way, unless the nodes are gerbes over the base of the curve. In general, the natural language to describe twisted nodes is log geometry [Ols07].

It is possible to show that given a family of prestable curves  $C$  over  $S$ , there is a pair of log structures  $M_C$  over  $C$  and  $N_S$  over  $S$ , unique up to unique isomorphism, which make the morphism of log schemes  $(C, M_C) \rightarrow (S, N_S)$  *special*, in the sense of Definition 2.6 of [Ols03b] (see also Definition 2.28 below). This is proven in [Ols03b], Theorems 2.7 and 3.18. Let  $\mathcal{C}$  be a twisted curve over  $C$ . The stack structure at the special points of  $\mathcal{C}$  is determined by taking a “root” of the log structure over the base of  $C$  in a sense that we will sketch below. Let us start by recalling how the canonical log structures are defined. A prestable curve  $C \rightarrow S$  is an example of *semistable variety* in the sense of the following:

**Definition 2.25.** A scheme  $X$  over a separably closed field  $k$  is a *semistable variety* if for each closed point  $x$  of  $X$  there exists an étale neighborhood  $(U, x')$  of  $x$ , integers  $r \geq l$ , and an étale morphism

$$U \rightarrow \text{Spec } k[X_1, \dots, X_r]/(X_1 \cdots X_l).$$

We need some preliminary definitions in order to describe special morphisms of log schemes.

**Definition 2.26.** A log smooth morphism  $f : (X, M_X) \rightarrow (S, M_S)$  is *essentially semistable* if for each geometric point  $\bar{x}$  of  $X$ , the monoids  $(f^{-1}\overline{M}_S)_{\bar{x}}$  and  $\overline{M}_{X, \bar{x}}$  are free monoids, and if for suitable isomorphisms  $(f^{-1}\overline{M}_S)_{\bar{x}} \simeq \mathbb{N}^r$ ,  $\overline{M}_{X, \bar{x}} \simeq \mathbb{N}^{r+s}$  the map

$$(f^{-1}\overline{M}_S)_{\bar{x}} \rightarrow \overline{M}_{X, \bar{x}}$$

is of the form

$$(3) \quad e_i \mapsto \begin{cases} e_i & \text{if } i \neq r, \\ e_r + \cdots + e_{r+s} & \text{if } i = r. \end{cases}$$

If a log morphism  $(X, M_X) \rightarrow (S, M_S)$  is essentially semistable locally in the smooth topology, it admits the following description:

**Lemma 2.27.** *Let  $f : (X, M_X) \rightarrow (S, M_S)$  be an essentially semistable morphism of log schemes. Then étale locally on  $X$  and  $S$  there exists charts  $\mathbb{N}^r \rightarrow M_S, \mathbb{N}^{r+s} \rightarrow M_X$  such that the map  $\mathbb{N}^r \rightarrow \mathbb{N}^{r+s}$  given by (3) is a chart for  $f$ , and such that the map*

$$(4) \quad \mathcal{O}_S \otimes_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[\mathbb{N}^{r+s}] \rightarrow \mathcal{O}_X$$

is smooth.

Let  $f : (X, M_X) \rightarrow (S, M_S)$  be an essentially semistable morphism. Let  $x \in X$  be a singular point. Then from (4) we deduce that in an étale neighborhood of  $x$

$$\mathcal{O}_S \otimes_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[\mathbb{N}^{r+s}] \simeq \mathcal{O}_S[x_r, \dots, x_{r+s}]/(x_r \cdots x_{r+s}) \rightarrow \mathcal{O}_X$$

is smooth. Therefore the morphism

$$\overline{M}_{S, \overline{s}} \rightarrow \overline{M}_{X, x}$$

is of the form  $\mathbb{N}^{r'} \rightarrow \mathbb{N}^{r'+s}$  with the map as described in (3) for some  $r' \leq r$ . This implies that there is only one irreducible element in the monoid  $\overline{M}_{S, \overline{s}}$  whose image in  $\overline{M}_{X, x}$  is not irreducible. This defines a canonical map

$$s_X : \{\text{singular points of } X\} \rightarrow \text{Irr}(\overline{M}_{S, \overline{s}})$$

where we denote by  $\text{Irr}(\overline{M}_{S, \overline{s}})$  the set of irreducible elements of  $\overline{M}_{S, \overline{s}}$ .

**Definition 2.28.** An essentially semistable morphism of log schemes  $f : X \rightarrow S$  is *special* at a geometric point  $\overline{s}$  of  $S$  if the map described above

$$s_{X_{\overline{s}}} : \{\text{singular points of } X_{\overline{s}}\} \rightarrow \text{Irr}(\overline{M}_{S, \overline{s}})$$

induces a bijection between the set of connected components of the singular locus of  $X_{\overline{s}}$  and  $\text{Irr}(\overline{M}_{S, \overline{s}})$ .

In [Ols03b], *Theorem 2.7* it is proven that if  $(f, f^b) : (X, M'_X) \rightarrow (S, M'_S)$  is a log smooth, proper, integral<sup>3</sup>, vertical<sup>4</sup> morphism of log schemes and  $X$  is fiberwise a semistable variety, then there exists a pair of log structures  $(M_X, M_S)$  and a morphism  $f^b : f^*M_S \rightarrow M_X$  making  $(f, f^b) : (X, M_X) \rightarrow (S, N_S)$  a special morphism. Moreover there are morphisms of log structures

$$\phi : M_S \rightarrow M'_S, \quad \psi : M_X \rightarrow M'_X$$

and a cartesian diagram

$$\begin{CD} (X, M_X) @>>> (X, M'_X) \\ @V{(f, f^b)}VV @VV{(f', f'^b)}V \\ (S, M_S) @>>> (S, M'_S). \end{CD}$$

The universal property of the log structures  $(M_X, M_S)$  implies that, given another pair  $(\tilde{M}_X, \tilde{N}_S)$  of log structures making of  $f : X \rightarrow S$  a special morphism, there are unique isomorphisms  $\tilde{M}_X \xrightarrow{\sim} M_X, \tilde{N}_S \xrightarrow{\sim} N_S$ .

From the theorem cited above it follows that the canonical log structures associated to a prestable curve or to a twisted curve are essentially unique (up to unique isomorphism). Hence, to prove the existence, it is enough to define them étale locally. Moreover, as observed in [Ols03b] in the proof of *Proposition 2.17*, due to [LMB00], *Proposition 4.18*, there is an equivalence of categories

$$\varinjlim_{U \rightarrow S} (\text{fine log structures on } X \times_S U) \rightarrow (\text{fine log structures on } X \times_S S_s)$$

where  $s$  is a point of  $S$ ,  $S_s = \text{Spec } \mathcal{O}_{S, \bar{s}}^{sh}$  is the spectrum of the (strict) henselization of the local ring of  $s$ ,  $U \rightarrow S$  are étale neighborhoods of  $s$ . Then, by a standard limit argument, it is enough to restrict to curves over strictly henselian local rings. Indeed, there exists some étale neighborhood  $U$  such that a given log structure over  $S_s$  is obtained via pullback from the corresponding log structure over  $U$ . In this paper we work with stacks defined over fields of characteristics zero, therefore usually we make no distinction between strictly henselian and henselian rings.

---

<sup>3</sup>See [Kat89], *Definition 4.1*. This implies flatness of the underlying morphism of schemes.

<sup>4</sup>This property means that the cokernel of  $f^b$  in the category of sheaves of monoids is a group.

In the following we will denote by  $C$ , resp.  $\mathcal{C}$ , a prestable curve, resp. a twisted curve, over the spectrum of a henselian ring  $S$ . The canonical log structures associated to curves as above are the amalgamated sum (i.e. the pushout along  $\mathcal{O}_S^*$ ) of two log structures determined by the nodes and by the marked points. Let us start by describing the log structure associated to the nodes. We need some further definitions. Let  $f : X \rightarrow S$  be a morphism of schemes. Let  $t$  be a global section of  $\mathcal{O}_S$ . Let  $M_S$  be the log structure on  $S$  associated to the pre-log structure  $\mathbb{N} \rightarrow \mathcal{O}_S$  mapping 1 to  $t$ .

**Definition 2.29 ([Ols03b]).** A log smooth morphism  $f : (X, M_X) \rightarrow (S, M_S)$  is *semi-stable* if for every geometric point  $\bar{x} \rightarrow X$  the stalk  $\overline{M}_{X, \bar{x}}$  is a free monoid and the map

$$\mathbb{N} \rightarrow \overline{M}_{S, f(\bar{x})} \rightarrow \overline{M}_{X, \bar{x}}$$

is the diagonal map.

The obstruction to the existence of a semistable log structure can be computed by means of *Theorem 3.18* of [Ols03b]. The obstruction space is given by the first étale cohomology group of a quotient of a suitable subsheaf of  $\mathcal{O}_X^*$ . The canonical pair of log structures defined by the singular loci of  $C$  and  $\mathcal{C}$  are in fact semi-stable relative to  $S$ .

Let us come to the definition for  $f : \mathcal{C} \rightarrow S$ , following [Ols07]. We will denote the coarse moduli space curve by  $C$ . Let  $e_1, \dots, e_n$  be the nodes in the closed fiber of  $C$ . For  $i = 1, \dots, n$  let us choose open subsets  $U_i$  containing only the node  $e_i$ . Let  $\mathcal{U}_i$  be the preimage in  $\mathcal{C}$ . Let  $t_i \in \mathcal{O}_S$  be an element such that étale locally  $\mathcal{U}_i$  is isomorphic to

$$[\text{Spec } \mathcal{O}_S[z, w]/(zw - t_i)]/\mu_{\gamma_i},$$

where  $\gamma_i$  is the order of the preimage of  $e_i$  in  $\mathcal{U}_i$ . Let  $N_S^i$  be the log structure associated to the pre-log structure  $\mathbb{N} \rightarrow \mathcal{O}_S$  mapping 1 to  $t_i$ . It turns out that there exists a semi-stable log structure  $M_{\mathcal{U}_i}^i$  over  $\mathcal{U}_i$  relative to  $(S, N_S^i)$ . This follows from an explicit computation of the obstruction group. This group vanishes only if the nodes are balanced. Once a semi-stable log structure is given over each  $\mathcal{U}_i$ , we have a global semi-stable log structure over  $\mathcal{C}$ . Indeed, since  $f^*N_S^i|_{\mathcal{U}_i} \rightarrow M_{\mathcal{U}_i}^i$  is semi-stable, it is an isomorphism out of the singular locus of  $\mathcal{U}_i$ . If we denote by  $\mathcal{U}_i^-$  the complement of  $e_i$  in  $\mathcal{C}$ ,  $\{\mathcal{U}_i, \mathcal{U}_i^-\}$  is an étale cover of  $\mathcal{C}$ . Over the intersection of the elements of the cover  $M_{\mathcal{U}_i}^i$  and  $f^*N_i$  glue to a global log structure  $M_{\mathcal{C}}^i$ . Indeed the restrictions are isomorphic and

semi-stable log structures are unobstructed. We define  $N_S := \bigoplus_{\mathcal{O}_S^*} N_S^i$  and  $M_{\mathcal{C}} := \bigoplus_{\mathcal{O}_{\mathcal{C}}^*} M_{\mathcal{C}}^i$ . The morphism  $\mathcal{C} \rightarrow S$  extends to a semi-stable morphism

$$(\mathcal{C}, M_{\mathcal{C}}) \rightarrow (S, N_S).$$

As anticipated, there is also another log structure that we have to construct, which depends on the marked points of the curve. Let  $\Sigma_i$  be the  $i$ -th marking gerbe. It corresponds to an invertible sheaf  $\mathcal{J}_i \rightarrow \mathcal{O}_{\mathcal{C}}$ . From [Kat89], *Complement 1* it follows that it defines a log structure  $\mathcal{N}_i$  over  $\mathcal{C}$ . Étale locally it is possible to choose a generator  $f \in \mathcal{J}_i$  for the ideal, and to define  $\mathcal{N}_i$  as the log structure associated to the pre-log structure  $\mathbb{N} \rightarrow \mathcal{O}_{\mathcal{C}}$  mapping 1 to  $f$ . If  $f'$  is another generator of  $\mathcal{J}_i$ , corresponding to another log structure  $\mathcal{N}'_i$ , there exists a unique element  $u \in \mathcal{O}_{\mathcal{C}}^*$  such that  $uf' = f$ . This unit defines an isomorphism  $\mathcal{N}_i \xrightarrow{\sim} \mathcal{N}'_i$ . Taking the amalgamated sum  $\widetilde{M}_{\mathcal{C}} := M_{\mathcal{C}} \oplus_{\mathcal{O}_{\mathcal{C}}^*} \mathcal{N}_i$  (note that the notation we use here differs from that in [Ols07]) gives a global log structure such that the morphism  $N_S \rightarrow \widetilde{M}_{\mathcal{C}}$  is essentially semistable.

The constructions above can certainly be done for a prestable curve with no stack structure. Consider in particular the coarse moduli space  $C$  of  $\mathcal{C}$ . Let us compare the canonical log structures associated to  $C$  with the log structures associated to  $\mathcal{C}$ . This can be done by looking at the étale local description. If the étale local description around a node of the coarse moduli space is

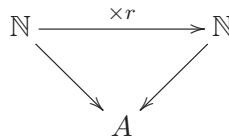
$$\text{Spec } A[x, y]/(xy - t),$$

where  $A$  is an henselian ring, then  $\mathcal{C}$  is isomorphic to

$$[\text{Spec } A[z, w]/(zw - t')/\mu_r],$$

where  $x = z^r$  and  $y = w^r$ , for some  $t' \in A$  such that  $t'^r = t$ .

Denote the log structure over  $S = \text{Spec } A$  associated to  $\mathcal{C}$  by  $N_S^{\mathcal{C}}$  and the log structure associated to  $C$  by  $N_S^C$ . We can see that  $N_S^{\mathcal{C}}$  can be thought of as a root of  $N_S^C$ . Indeed,  $N_S^C$  is associated to the pre-log structure  $\mathbb{N} \rightarrow A$  mapping 1 to  $t$ , while  $N_S^{\mathcal{C}}$  is associated to the pre-log structure  $\mathbb{N} \rightarrow A$  mapping 1 to  $t'$ . Because of that we see that there is a morphism of pre-log structures



which induces a morphism of log structures

$$N_S^C \rightarrow N_S^e.$$

Taking the characteristic and the stalk at the closed point  $s_0$  of  $\text{Spec } A$  we get

$$\overline{N}_{S,s_0}^C \rightarrow \overline{N}_{S,s_0}^e,$$

which is a simple morphism of monoids given again by multiplication by  $r$ . The log structure  $N_S^e$  can be in fact identified with the log structure  $N'_S$  which is part of the data defining a log twisted curve over a prestable curve according to Definition 2.30.

The comparison of the log structures associated to the marked points of  $C$  and to the marking gerbes of  $\mathcal{C}$  is analogous. However, here it is worth noting that the log structures  $\mathcal{N}_i$  associated to the marked points or to the marking gerbes are equivalent to sets of line bundles with sections. This is proven in [Kat89], where these kinds of log structures are called *Deligne-Faltings* log structures. This is of course what one expects, since by Theorem 2.10 the stack structure of the marking gerbes is obtained by taking roots of line bundles with sections over the coarse moduli space curve.

We now give the notion of *n-pointed log twisted curve* introduced in [Ols07], Definition 1.7:

**Definition 2.30.** An *n-pointed log twisted curve* over a scheme  $S$  is a collection of data

$$(C/S, \{\sigma_i, a_i\}, l : N_S \rightarrow N'_S),$$

where  $C/S$  is an  $n$ -pointed prestable curve,  $\sigma_i : S \rightarrow C$  are sections,  $a_i, i = 1, \dots, n$  are integer-valued locally constant functions on  $S$  such that for each  $s \in S$  the integer  $a_i(s)$  is positive and invertible over  $k(s)$ , and  $l : N_S \hookrightarrow N'_S$  is a simple morphism of log structures over  $S$ , where  $N_S$  is the canonical log structure associated to  $C/S$ .

Log twisted curves over  $S$  form a groupoid. Let  $(C^1/S, \{\sigma_i^1, a_i^1\}, l_1 : N_S^1 \hookrightarrow N_S'^1), (C^2/S, \{\sigma_i^2, a_i^2\}, l_2 : N_S^2 \hookrightarrow N_S'^2)$  be two  $n$ -pointed log twisted curves. There are no morphisms between them unless  $a_i^1 = a_i^2 \forall i$ . In this case an isomorphism is given by an isomorphism  $\rho : (C^1/S, \{\sigma_i^1\}) \xrightarrow{\sim} (C^2/S, \{\sigma_i^2\})$  of pointed prestable curves with an isomorphism  $\epsilon : N_S^1 \xrightarrow{\sim} N_S^2$  such that



the diagram

$$\begin{array}{ccc}
 N_S^1 & \xrightarrow{l_1} & N_S'^1 \\
 \simeq \downarrow \psi(\rho) & & \downarrow \epsilon \\
 N_S^2 & \xrightarrow{l_2} & N_S'^2
 \end{array}$$

commutes. Here  $\psi(\rho)$  is the isomorphism induced by the uniqueness of the canonical log structures on  $S$  and  $C$  (stated in [Ols07], *Theorem 3.6* and proved in [Ols03b], *Proposition 2.7*) and by the isomorphism  $\rho$ , by means of which  $f_1^*N_S^2 \rightarrow \rho^*M_{C_2}$  makes  $f_1 : C_1 \rightarrow S$  a special morphism, whence the existence of a pair of isomorphisms  $N_S^1 \xrightarrow{\sim} N_S^2$  and  $M_{C_1}^1 \xrightarrow{\sim} \rho^*M_{C_2}^2$ .

**Remark 2.31.** Let  $f : C \rightarrow S$  be a prestable curve and let  $N_S$  be the associated canonical log structure over the base. Let  $\rho : C \xrightarrow{\sim} C$  be an automorphism of prestable curves. By definition  $\rho$  commutes with the projection  $f$ . Therefore the induced automorphism  $\rho^*N_S \xrightarrow{\sim} N_S$  is the identity.

**Remark 2.32.** Let  $N_S$  be a canonical log structure over the base of a prestable curve or a twisted curve. Assume the curve has one node. There is a chart  $\mathbb{N} \rightarrow N_S$ , which defines a section that we denote by  $e$ . According to the discussion in *Section 2.7.1*, an automorphism of  $N_S$  corresponds by the universal property of the pushout to a map  $e \mapsto e + \lambda(u)$ , where  $\lambda : \mathcal{O}_S^* \hookrightarrow N_S$  denotes the canonical inclusion. Let  $\epsilon : N_S^1 \xrightarrow{\sim} N_S^2$  be a morphism between two log structures as above induced by charts  $\mathbb{N} \rightarrow S$  mapping 1 to  $t_1$ , resp.  $t_2$ . Let  $e_1, e_2$  denote the sections of  $N_S^1, N_S^2$  defined by the respective charts, then  $\epsilon(e_1) = e_2$ . Again, since stalks are integral, and isomorphic to  $\mathbb{N} \oplus \mathcal{O}_{S,s_0}^*$ , a unit  $u$  such that  $t_1u = t_2$  is uniquely associated to  $\epsilon$ .

### 3. Identity for virtual fundamental classes

The purpose of this Section is to prove Theorem 3.13, which is an identity relating the virtual fundamental class for moduli stack of twisted stable maps to  $\mathcal{X}_1 \times \mathcal{X}_2$  with virtual fundamental classes for moduli stacks of twisted stable maps to  $\mathcal{X}_1$  and to  $\mathcal{X}_2$ .

#### 3.1. Set-up

Let  $\tilde{\tau}$  be a gerby  $(\mathcal{X}_1 \times \mathcal{X}_2)$ -graph with underlying modular graph  $\tau$  and gerby structure  $\mathfrak{g} : F_{\tilde{\tau}} = F_{\tau} \rightarrow \mathcal{U}(\mathcal{X}_1 \times \mathcal{X}_2)$ . The projections  $p_i : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow$

$\mathcal{X}_i, i = 1, 2$  define maps  $p_{i*} : H_2^+(X_1 \times X_2, \mathbb{Z}) \rightarrow H_2^+(X_i, \mathbb{Z})$ . Let  $\tau_i, i = 1, 2$  be the modular  $H_2^+(X_i, \mathbb{Z})$ -graphs obtained as the stabilization of  $\tau$  with respect to  $p_{i*}$  (see [BM96], Remark 1.15). By construction there are combinatorial morphisms

$$(H_2^+(X_i, \mathbb{Z}), \tau_i) \rightarrow (H_2^+(X_1 \times X_2, \mathbb{Z}), \tau), \quad i = 1, 2.$$

Consider the maps  $F_{\tau_i} \rightarrow F_\tau$  between the sets of flags. The projections  $p_i$  also define set maps  $\mathcal{U}(X_1 \times X_2) \rightarrow \mathcal{U}(X_i), i = 1, 2$  (see Example 2.24). The compositions

$$F_{\tau_i} \rightarrow F_\tau \xrightarrow{g} \mathcal{U}(X_1 \times X_2) \rightarrow \mathcal{U}(X_i), \quad i = 1, 2,$$

define gerby structures on  $\tau_i, i = 1, 2$ . Denote by  $\tilde{\tau}_i, i = 1, 2$  the resulting  $\mathcal{X}_i$ -graphs<sup>5</sup>.

Let  $\tau^s$  be the common absolute stabilization of  $\tau, \tau_1, \tau_2$ .

Let  $[f : (\mathcal{C}/T, \Sigma) \rightarrow \mathcal{X}_1 \times \mathcal{X}_2] \in \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})(T)$  be an object. For  $i = 1, 2$ , applying relative coarse moduli space and relative stabilization constructions to the composite  $\mathcal{C} \rightarrow \mathcal{X}_1 \times \mathcal{X}_2 \times T \rightarrow \mathcal{X}_i \times T$  yields representable morphisms  $(\mathcal{C}_i, \Sigma_i) \rightarrow \mathcal{X}_i \times T$  fitting in the following diagram

$$(5) \quad \begin{array}{ccc} (\mathcal{C}, \Sigma) & & \\ \downarrow & \searrow & \\ (\mathcal{C}_i, \Sigma_i) & \longrightarrow & \mathcal{X}_i \times T. \end{array}$$

It is easy to check that  $(\mathcal{C}_i, \Sigma_i) \rightarrow \mathcal{X}_i \times T$  in the above diagram is an object in  $\mathcal{K}(\mathcal{X}_i, \tilde{\tau}_i)$ . This defines a morphism

$$p = (p_1, p_2) : \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) \rightarrow \mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1) \times \mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2).$$

---

<sup>5</sup>It follows from the construction that if there exists a  $\tilde{\tau}$ -marked twisted stable map to  $\mathcal{X}_1 \times \mathcal{X}_2$ , then there exists  $\tilde{\tau}_i$ -marked twisted stable maps to  $\mathcal{X}_i$ , and there are combinatorial morphisms  $\tilde{\tau}_i \rightarrow \tilde{\tau}$ .

Throughout the rest this paper we will need the following diagram

$$\begin{array}{ccccccc}
 & & & & p & & \\
 & & & & \curvearrowright & & \\
 (6) & \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) & \xrightarrow{h} & P' & \xrightarrow{\quad} & P & \xrightarrow{q} & \mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1) \times \mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2) \\
 & \downarrow & & \downarrow & \square & \downarrow & \square & \downarrow a \\
 & \mathfrak{D}^{tw}(\tilde{\tau}) & \xrightarrow{l_1} & \mathfrak{D}' & \xrightarrow{l_2} & \mathfrak{B}^{tw} & \xrightarrow{\phi} & \mathfrak{M}^{tw}(\tilde{\tau}_1) \times \mathfrak{M}^{tw}(\tilde{\tau}_2) \\
 & \downarrow e & \curvearrowright & \downarrow & \square & \downarrow & \square & \downarrow s \times s \\
 & \mathfrak{M}^{tw}(\tilde{\tau}) & & & & \mathfrak{B}^{tw} & & \\
 & & & & & \downarrow & & \\
 & & & & & \overline{M}(\tau^s) & \xrightarrow{\Delta} & \overline{M}(\tau^s) \times \overline{M}(\tau^s).
 \end{array}$$

Some explanations are in order:

- 1)  $\overline{M}(\tau^s)$  is the moduli stack of  $\tau^s$ -marked stable curves.  $\Delta$  is the diagonal morphism.
- 2) As indicated in the diagram,  $\mathfrak{B}^{tw}$  is the fiber product of  $\overline{M}(\tau^s)$  and  $\mathfrak{M}^{tw}(\tilde{\tau}_1) \times \mathfrak{M}^{tw}(\tilde{\tau}_2)$  over  $\overline{M}(\tau^s) \times \overline{M}(\tau^s)$ .
- 3) The morphism  $\mathcal{K}(\mathcal{X}_i, \tilde{\tau}_i) \rightarrow \mathfrak{M}^{tw}(\tilde{\tau}_i)$  is obtained by forgetting the twisted stable maps but retaining the twisted curves.
- 4) The stack  $\mathfrak{D}'$  is defined to be  $\mathfrak{D}(\tau) \times_{\mathfrak{B}} \mathfrak{B}^{tw}$  where  $\mathfrak{D}(\tau)$  and  $\mathfrak{B}$  are defined respectively in [Beh99b] diagrams (3) and (2) (Note that Behrend used different notation). The map  $l_2$  is the natural projection. These notations will be explained later.
- 5) The stacks  $P$  and  $P'$  are defined to be fiber products indicated in the diagram.
- 6) The leftmost square will later be shown to be cartesian in Proposition 3.11.

Diagram (6) is similar to [Beh99b], diagram (2). Other ingredients of (6) are explained in the rest of the section.

### 3.2. The stack $\mathfrak{D}^{tw}(\tilde{\tau})$

We define  $\mathfrak{D}^{tw}(\tilde{\tau})$  as the stack whose objects over  $T$  are the following diagrams

$$(7) \quad \begin{array}{ccc} (\mathcal{C}, \Sigma) & \xrightarrow{\phi_1} & (\mathcal{C}_1, \Sigma_1) \\ & \downarrow \phi_2 & \\ & (\mathcal{C}_2, \Sigma_2) & \end{array}$$

where

- 1)  $\mathcal{C}$ , resp.  $\mathcal{C}_i$ , is a  $\tilde{\tau}$ -marked, resp.  $\tilde{\tau}_i$ -marked, twisted curve and  $\{\Sigma\}$ , resp.  $\{\Sigma_i\}$ , denote a collection of disjoint étale gerbes in the smooth locus over the base, of order specified by the gerby graph;
- 2) the morphisms  $\phi_1$  and  $\phi_2$  induce morphisms of prestackable curves between the coarse moduli spaces;
- 3) the product morphism  $\phi_1 \times \phi_2 : \mathcal{C} \rightarrow \mathcal{C}_1 \times \mathcal{C}_2$  is *representable*, while  $\phi_1$  and  $\phi_2$  in general are not; the morphisms  $\phi_i : \mathcal{C} \rightarrow \mathcal{C}_i$  are *faithful*;
- 4) the product morphism  $\phi_1 \times \phi_2$  does not admit infinitesimal automorphisms. This implies that no unstable rational component of a fiber of  $\mathcal{C}$  is contracted in both  $\mathcal{C}_1$  and  $\mathcal{C}_2$ ;
- 5) the orders of the isotropy group of any stack point of the curves is bounded from above by the largest order of the inertia groups of objects of  $\mathcal{X}_1 \times \mathcal{X}_2$ ;
- 6) The stack  $\mathfrak{D}^{tw}(\tilde{\tau})$  is an algebraic stack, which is from the similar argument as in [Beh99b].

**Remark 3.1.** The bound on the orders of the cyclic groups banding the marking gerbes ensures that the stack defined above is of finite type over the stack  $\mathfrak{D}'$ . This follows from [Ols07], *Corollary 1.12* and [Ols06], *Theorem 1.7* by applying an argument similar to [Ols07], *Theorem 1.16*. Moreover, again by [Ols07], *Corollary 1.12*,  $\mathfrak{D}'$  is of finite type over  $\mathfrak{D}$  defined in [Beh99b]. As a consequence, the moduli stack in this paper has a fundamental class (cfr. [AGV08]).

**Remark 3.2.** Note that even though  $\mathfrak{D}^{tw}(\tilde{\tau})$  parametrizes morphisms between stacks, it is in fact equivalent to a 1-category. Indeed the morphisms  $\phi_1 \times \phi_2 : \mathcal{C} \rightarrow \mathcal{C}_1 \times \mathcal{C}_2$  are representable and map a dense open representable substack of  $\mathcal{C}$  to a dense open representable substack of  $\mathcal{C}_1 \times \mathcal{C}_2$ . According to e.g. [AV02], *Lemma 4.2.3*, any automorphism of such a morphism is trivial.

The vertical arrow in diagram (5) defines the morphism

$$\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) \rightarrow \mathfrak{D}^{tw}(\tilde{\tau})$$

appearing in diagram (6).

There is a morphism  $e : \mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{M}^{tw}(\tilde{\tau})$  which takes an object (7) to the  $\tilde{\tau}$ -marked prestable twisted curve  $(\mathcal{C}, \Sigma)$ .

**Remark 3.3.** In what follows we will use the valuative criterion for properness and the infinitesimal lifting criterion for étaleness to prove properties of morphisms between stacks whose objects involve twisted curves. In those contexts we will not worry about the log structures associated to the marking gerbes. Indeed, as far as those are concerned, the criteria are satisfied because they are satisfied for the morphisms induced by passing to the coarse moduli spaces, which follows from [Beh99b].

**Proposition 3.4.** *The morphism  $e : \mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{M}^{tw}(\tilde{\tau})$  is étale.*

*Proof.* Since it is easy to see that the stacks involved are algebraic stacks locally of finite type, it is enough to show that it is formally étale. We use the infinitesimal lifting criterion for étaleness ([Gro67], *Proposition 17.5.3* and *Remark 17.5.4*, and *Definition 4.14* of [LMB00]). We assume our stacks are locally noetherian, therefore to test the criterion we can restrict to artinian local rings. Let  $B$  be a local artinian ring,  $I \subset B$  a square zero ideal and  $A := B/I$  the quotient ring. Consider an object in  $\mathfrak{D}^{tw}(\tilde{\tau})(A)$ :

$$(8) \quad \begin{array}{ccc} \mathcal{C}_A & \xrightarrow{\phi_{1A}} & \mathcal{C}_{1A} \\ \downarrow \phi_{2A} & & \\ \mathcal{C}_{2A} & & \end{array}$$

We will show that (8) extends uniquely to  $B$  once an extension  $\mathcal{C}_A \hookrightarrow \mathcal{C}_B$  of  $\mathcal{C}_A$  to a twisted curve  $\mathcal{C}_B$  over  $B$  is given.

According to [Beh99b], *Lemma 4* we know that the coarse curves  $C_{1A}$  and  $C_{2A}$  extend uniquely to curves  $C_{1B}$  and  $C_{2B}$  over  $B$ . Recall that we have combinatorial morphisms  $\tau_i \rightarrow \tau$ , which include injective maps  $a_V : V_{\tau_i} \rightarrow V_{\tau}$  for  $i = 1, 2$ . In order to show that  $\phi_{iA} : C_A \rightarrow C_{iA}$  extend to  $B$  we choose two sets of sections  $S_i := \{s_{iA}^k\}, i = 1, 2$  such that for any  $v \in V_{\tau}$  which is in the image of  $a_V : V_{\tau_i} \rightarrow V_{\tau}$ ,  $S_i$  stabilizes the irreducible components of the curve  $C_{v,0}$  not contracted in  $C_{v_i,0}$ , where  $C_{v,0}, C_{v_i,0}$  denote the closed fibers. This is possible because for any  $\tau$ -marked prestable curve  $C$  over  $\text{Spec } \mathbb{C}$ , there exists

a graph  $\tau'$  obtained from  $\tau$  by adding some tails, such that  $C$  is in the image of some point  $C'$  of  $\overline{M}(\tau')$  along the morphism  $\overline{M}(\tau') \rightarrow \mathfrak{M}(\tau)$ . Moreover this morphism is smooth, therefore, given an extension of  $C$  to a prestable curve  $C_A$  over some local artinian ring  $A$ , also  $C'$  extends to a stable curve  $C'_A$  such that  $C_A$  is obtained from  $C'_A$  by forgetting the sections corresponding to the additional tails of  $\tau'$ . By smoothness of  $\overline{M}(\tau') \rightarrow \mathfrak{M}(\tau)$ , one can extend the set of sections  $S_i$  to  $B$ . To obtain the extensions  $\phi_{iB} : C_B \rightarrow C_{iB}$  it is enough to forget the set of sections  $S_j, j \neq i$ , to stabilize and then forget the set of sections obtained as image of the sections  $S_i$ .

To show that twisted curves also extend, we use the equivalence between twisted curves and log twisted curves (Definition 2.30). We observe that the morphisms  $\mathcal{C}_A \rightarrow \mathcal{C}_{iA}$  correspond to morphisms of the coarse moduli spaces together with morphisms of the canonical locally free log structures over the base  $A$  fitting in the following diagram:

$$\begin{array}{ccccc} N_{iA} & \hookrightarrow & N_A & \xrightarrow{\alpha} & A \\ l_i \downarrow & & \downarrow l & & \\ N'_{iA} & \hookrightarrow & N'_A & \xrightarrow{\alpha'} & A. \end{array}$$

We denote by  $e_m$  the sections of  $N, N'$  induced by the  $m$ -th standard generator of  $\mathbb{N}^r$  via the charts  $\mathbb{N}^r \rightarrow N, \mathbb{N}^r \rightarrow N'$ , or equivalently the corresponding edges in  $E_\tau, E_{\tilde{\tau}}$ . (We assume here that the curve over the closed point has generic singularity type in order to use in the rest of the proof the language of dual graphs. The proof is identical if the curve has non generic singularity type). With this convention in the diagram above  $\alpha(e_m) = t_m$  and  $\alpha'(e_m) = t'_m$ , where  $t_m, t'_m \in A$  and  $t_m^{\gamma_m} = t'_m$ , with  $\gamma_m = \gamma(e_m)$  being the order of the  $m$ -th node of  $\mathcal{C}$ . Since we assume that we have an extension of  $\mathcal{C}$  to  $B$ , an extension of the morphism  $l : N_A \rightarrow N'_A$  to  $B$  is given. This means that there are given liftings of  $t_m, t'_m$  to  $B$ , that we denote by  $\tilde{t}_m, \tilde{t}'_m$ . We now show that also  $l_i : N_{iA} \rightarrow N'_{iA}$  extend to  $B$  together with the morphisms  $\phi_{iA}^b : N'_{iA} \rightarrow N'_A$ . Consider the diagram of morphisms of pre-log structures over  $\text{Spec } A$

$$\begin{array}{ccc} \mathbb{N}_A^{r_i} & \longrightarrow & \mathbb{N}_A^r \\ (\tilde{\gamma}_i) \downarrow & & \downarrow (\tilde{\gamma}) \\ \mathbb{N}_A^{r_i} & \longrightarrow & \mathbb{N}_A^r, \end{array}$$

where the vertical arrows are diagonal matrices with entries the order of the isotropy groups of the nodes of the closed fiber corresponding to the

irreducible elements of the monoids. (By construction there are bijections  $\mathbb{N}^r \simeq \overline{N}_0$  and  $\mathbb{N}^{r_i} \simeq \overline{N}'_{i,0}$ , where  $\overline{N}_0, \overline{N}'_{i,0}$  are the stalks at the closed point. The correspondence with generic nodes follows from Definition 2.28). The coarse moduli space morphisms  $C \rightarrow C_i$  determine in fact the extensions of  $l_i$  to  $B$  and compatible morphisms  $\phi_{iB}^b : N'_{iB} \rightarrow N'_B$  as follows. Let  $J_{i,k} \subset \mathbb{N}$  be the set of indices such that for  $m \in J_{i,k}$ ,  $e_m \in E_\tau$  partakes in the long edge associated to  $e_k \in E_{\tau_i}$ . The log structures  $N'_{iB}$  determining the stack structure of the nodes of any extension  $\mathcal{C}_{iB}$  compatible with  $\mathcal{C}_B$  must be a sub log structure of  $N'_B$ . Therefore they must be the log structures associated to the pre-log structures defined as follows

$$\bigoplus_{e_k \in E_{\tilde{\tau}_i}} \mathbb{N} \rightarrow B, \quad e_k \mapsto \prod_{m \in J_{i,k}} t_m^{\frac{\gamma_m}{m^{\gamma_{ik}}}}$$

where  $\gamma_m := \gamma(e_m)$ ,  $e_m \in E_\tau = E_{\tilde{\tau}}$ , and  $\gamma_{ik} := \gamma(e_k)$ ,  $e_k \in E_{\tau_i} = E_{\tilde{\tau}_i}$ . This is the unique locally free log structure over  $B$  which satisfies the required compatibility conditions. □

### 3.3. The morphism $l$

There is a morphism

$$\mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{M}^{tw}(\tilde{\tau}_1) \times \mathfrak{M}^{tw}(\tilde{\tau}_2)$$

which takes an object (7) to the pair  $((\mathcal{C}_1, \Sigma_1), (\mathcal{C}_2, \Sigma_2))$ . There is another morphism

$$\mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \overline{M}(\tau^s)$$

which takes an object (7) to the stabilization of the coarse curve  $(C, \overline{\Sigma})$  of  $(\mathcal{C}, \Sigma)$ .

Since  $\tau, \tau_1, \tau_2$  have the same absolute stabilization  $\tau^s$ , the following diagram is 2-commutative,

$$\begin{array}{ccc} \mathfrak{D}^{tw}(\tilde{\tau}) & \longrightarrow & \mathfrak{M}^{tw}(\tilde{\tau}_1) \times \mathfrak{M}^{tw}(\tilde{\tau}_2) \\ \downarrow & & \downarrow s \times s \\ \overline{M}(\tau^s) & \xrightarrow{\Delta} & \overline{M}(\tau^s) \times \overline{M}(\tau^s), \end{array}$$

which implies that there is a morphism

$$l : \mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{B}^{tw},$$

as in (6).

It follows easily from definitions that the morphism  $l : \mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{B}^{tw}$  factors through  $l_1 : \mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{D}'$ , with the notation of diagram (6), i.e.

$$l = l_2 \circ l_1.$$

Recall that the stack  $\mathfrak{D}'$  is defined as the fiber product

$$\mathfrak{D}' = \mathfrak{D}(\tau) \times_{\mathfrak{B}} \mathfrak{B}^{tw}.$$

Unpacking the definitions of  $\mathfrak{B}$  and  $\mathfrak{D}(\tau)$  in [Beh99b], we find that

$$\mathfrak{D}' = \mathfrak{D}(\tau) \times_{\mathfrak{M}(\tau_1) \times \mathfrak{M}(\tau_2)} \mathfrak{M}^{tw}(\tilde{\tau}_1) \times \mathfrak{M}^{tw}(\tilde{\tau}_2).$$

**Remark 3.5.** The morphism  $l$  induces a morphism

$$\bar{l} : \mathfrak{D}(\tau) \rightarrow \mathfrak{B}$$

between the stacks parametrizing the coarse moduli spaces curves. This is the morphism  $l$  in [Beh99b], diagram (2). We recall the proof in [Beh99b] that  $\bar{l}$  is proper. Let  $S_1, S_2$  be finite sets and  $S := S_1 \amalg S_2$ . Let  $\tau'_1$  be a stable modular graph obtained by adding  $S_1$  to  $S_{\tau_1}$  and  $\tau'_2$  a stable modular graph obtained by adding  $S_2$  to  $S_{\tau_2}$ . Let  $\tau'$  be the graph obtained by adding  $S$  to  $S_{\tau}$  in the unique compatible way in order for  $\tau_i \rightarrow \tau$  to give  $\tau'_i \rightarrow \tau'$  inducing the inclusions  $S_i \subset S$ . We then have a cartesian diagram

$$\begin{array}{ccc} \overline{M}(\tau') & \xrightarrow{\delta} & \overline{M}(\tau'_1) \times \overline{M}(\tau'_2) \\ \downarrow & \square & \downarrow \chi \\ \mathfrak{D}(\tau) & \xrightarrow{\tilde{\Delta}} & \mathfrak{M}(\tau_1) \times \mathfrak{M}(\tau_2) \end{array}$$

where  $\overline{M}(\tau')$ ,  $\overline{M}(\tau'_1)$  and  $\overline{M}(\tau'_2)$  are stacks of stable curves and the top row is a local presentation of  $\tilde{\Delta}$ . Notice that partial stabilization is used to define the top row. Properness of  $\bar{l}$  follows from properness of the induced morphism  $\delta$  of the presentation.

In the rest of this Subsection we establish some properties of the morphism  $l_1$ , which will be important in proving Theorem 3.13.

**3.3.1. Properness.**

**Proposition 3.6.** *The morphism  $l_1$  is proper.*



*Proof.* To prove that  $l_1$  is proper, we apply the valuative criterion for properness and using logarithmic geometry to do the explanation.

We work with stacks which are locally noetherian. Moreover the morphism  $l_1$  is of finite type. By [LMB00], Proposition 7.8 and Theorem 7.10 (iii) we can apply the valuative criterion restricting to complete DVR's with algebraically closed residue field. Let  $R$  be such a DVR and let  $K$  be its field of fractions. Put  $V = \text{Spec } R$  and  $U = \text{Spec } K$ .

We first prove that  $l_1$  is separated. Given two objects  $\eta_1$  and  $\eta_2$  in  $\mathfrak{D}^{tw}(\tilde{\tau})(V)$ , an isomorphism  $\beta : l_1(\eta_1) \xrightarrow{\sim} l_1(\eta_2)$  in  $\mathfrak{D}'(V)$  and an isomorphism  $\alpha : \eta_{1,U} \xrightarrow{\sim} \eta_{2,U}$  in  $\mathfrak{D}^{tw}(\tilde{\tau})(U)$ , we want to show that there exists a unique isomorphism  $\tilde{\alpha} : \eta_1 \xrightarrow{\sim} \eta_2$  restricting to  $\alpha$  and such that  $l_{1*}\tilde{\alpha} = \beta$ . This is not hard to see. We only have to worry about isomorphisms associated to the stacky structure of  $\mathcal{C}$ . We can equivalently prove the statement for automorphisms. Since  $R$  is integral, a twisted curve carries automorphisms leaving fixed the coarse moduli space only if a node is a gerbe over  $V$ . In other words the pre-log structure associated to it is pulled back from a point. Then  $\mathcal{C}_U$  and  $\mathcal{C}_V$  have the same automorphism group. By definition of  $\mathfrak{D}^{tw}(\tilde{\tau})$ ,  $\phi_1 \times \phi_2$  induces an isomorphism between automorphisms of a node of  $\mathcal{C}$  and a subgroup of the automorphism group of its image in  $\mathcal{C}_1 \times \mathcal{C}_2$ . Under our hypotheses,  $\beta$  encodes an isomorphism in this subgroup, hence  $\alpha$  is determined uniquely.

We now translate this argument into the language of logarithmic geometry. Consider two objects of  $\mathfrak{D}^{tw}$ , given by diagrams

$$\begin{array}{ccc} & \mathcal{C}_1^{(\phi_1^j, \phi_1^{b,j})} & \mathcal{C}_1^j \\ & \downarrow (\phi_2^j, \phi_2^{b,j}) & \\ & \mathcal{C}_2^j & \end{array}$$

where  $j = 1, 2$ . Here twisted curves are viewed as log twisted curves, and in particular as log schemes. In the following we denote by  $\phi_i^{b,j}$  the morphisms induced between the canonical log structures over the bases of the curves, while we will not mention explicitly the log structures over the curves themselves.

An isomorphism between two objects of  $\mathfrak{D}^{tw}(\tilde{\tau})(U)$  is a triple of compatible isomorphisms  $\vec{\rho} := (\rho, \rho_1, \rho_2)$  of prestable curves, where  $\rho : C^1 \xrightarrow{\sim} C^2$ ,  $\rho_i : C_i^1 \xrightarrow{\sim} C_i^2$ ,  $i = 1, 2$ , plus three compatible isomorphisms over  $\vec{\rho}$  of the simple morphisms of log structures defining the log twisted curves  $\mathcal{C}^j$ ,  $\mathcal{C}_1^j$ ,  $\mathcal{C}_2^j$ ,  $j = 1, 2$ . The isomorphisms  $\vec{\rho}$  between coarse curves have to satisfy the

compatibility conditions given by the commutativity of the following diagrams

$$(9) \quad \begin{array}{ccc} C^1 & \xrightarrow{\phi_i} & C_i^1 \\ \sim \downarrow \rho & \curvearrowright & \rho_i \downarrow \sim \\ C^2 & \xrightarrow{\phi_i} & C_i^2 \end{array}$$

for  $i = 1, 2$ .

Let  $\mathbb{N}^r \rightarrow N^j$  and  $\mathbb{N}^{r_i} \rightarrow N_i^j$ ,  $j = 1, 2$ , be the canonical log structures associated to  $C^j, C_i^j$ , with charts given by the pre-log structures defining them. The isomorphisms of simple morphisms of log structures  $\vec{\psi} = (\psi, \psi_1, \psi_2)$  have to satisfy the condition imposed by the commutativity of the diagrams

$$\begin{array}{ccccc} & & & N^1 - \xrightarrow{l^1} & N'^1 \\ & & & \downarrow \psi(\rho) & \downarrow \psi \\ N_i^1 & \xrightarrow{l_i^1} & N_i'^1 & & \\ \downarrow \psi_i(\rho) & & \downarrow \psi_i & & \\ N_i^2 & \xrightarrow{l_i^2} & N_i'^2 & & \end{array}$$

for  $i = 1, 2$ , where the isomorphism  $\psi(\rho), \psi_i(\rho)$  are as in (5) (see page 247). In what follows we will denote by  $e_m$  (resp.  $e_{ik}$ ) edges in  $E_\tau$  (resp.  $E_{\tau_i}$ ). Edges of the dual graph  $\tau$  (resp.  $\tau_i$ ) correspond to irreducible elements of  $\mathbb{N}^r$  (resp.  $\mathbb{N}^{r_i}$ ) which are mapped to zero in the structure sheaf. We will use the symbols defined above decorated by a superscript  $j$  to denote the sections of  $N^j, N_i^j$  induced by those irreducible elements. (cf. Definition 2.28). From Remark 2.32 it follows that the automorphisms  $\psi(\rho)$  and  $\psi_i(\rho_i)$  correspond to collections of units in  $K^{*r}$  and  $K^{*r_i}$ , that we denote by  $\vec{\eta}$  and by  $\vec{\eta}_i, i = 1, 2$ , such that  $\psi(\rho)(e_m^1) = e_m^2 + \lambda(\eta_m), \psi_i(\rho_i)(e_{ik}^1) = e_{ik}^2 + \lambda_i(\eta_{ik})$ . Moreover,  $\vec{\eta}, \vec{\eta}_i$  satisfy the following compatibility condition

$$\prod_{m \in J_{i,k}} \eta_m = \eta_{ik},$$

for all  $k$  such that  $e_{ik} \in E_{\tau_i}$ . Here  $J_{i,k}$  is the set of indices such that  $m \in J_{i,k}$  if  $e_m \in E_\tau$  belongs to the long edge associated to  $e_{ik} \in E_{\tau_i}$ . The isomorphisms  $\psi, \psi_i$  correspond to collections  $\vec{\zeta}, \vec{\zeta}_i$  in  $K^{*r}$  and  $K^{*r_i}$  which are roots of elements of  $\vec{\eta}, \vec{\eta}_i$ . More precisely  $\zeta_m$  is a  $\gamma(e_m)$ -th root of  $\eta_m$ ,  $\zeta_{ik}$  is a  $\gamma(e_{ik})$ -th root of  $\eta_{ik}$ . For all  $k$  such that  $e_{ik} \in E_{\tau_i} = E_{\tilde{\tau}_i}$  they also satisfy the conditions

$$(10) \quad \prod_{m \in J_{i,k}} \zeta_m^{\epsilon_m^{ik}} = \zeta_{i,k},$$

where  $\epsilon_m^{ik} = \gamma(e_m)/\gamma(e_{ik})$ ,  $e_m \in E_{\tilde{\tau}}$ .

The extension of the isomorphisms  $\vec{\rho}$  to  $V$  is encoded in  $\beta$ , meaning in particular that  $\vec{\eta}$  is in fact in  $R^{*r}$ , hence so is  $\vec{\zeta}$  because  $R$  is integrally closed in  $K$ . The  $\vec{\zeta}_i$  are also in  $R^{*r_i}$  by hypothesis. The compatibility conditions (10) are satisfied in  $R^{*r} \times R^{*r_i}$  since they are in  $K^{*r} \times K^{*r_i}$ . This concludes the proof that the morphism  $l_1$  is separated.

Since  $l_1$  is separated and of finite type, we can apply the valuative criterion for properness in the following form. Given an object of  $\mathcal{D}^{tw}(\tilde{\tau})(U)$ ,

$$(11) \quad \begin{array}{ccc} \mathcal{C}_U & \longrightarrow & \mathcal{C}_{1U} \\ \downarrow & & \\ \mathcal{C}_{2U} & & \end{array}$$

and an extension to  $V$  of its image under  $l_1$

$$(12) \quad \mathcal{C}_{1U} \hookrightarrow \mathcal{C}_{1V}, \quad \mathcal{C}_{2U} \hookrightarrow \mathcal{C}_{2V},$$

we have to show that diagram (11) extends uniquely to  $V$  up to possible finite base extension.

The diagram induced by (11) by passing to the coarse moduli spaces extends to  $V$ . The twisted curves over  $V$  in (12) are the data of the coarse curves  $C_{iV}$  plus simple morphisms of locally free log structures  $l_i : N_{iV} \rightarrow N'_{iV}$ . We recall that by construction those are the log structures associated to the pre-log structures defined in the following way. Let  $C_{i,0}$  be the closed fiber over  $\text{Spec } R$ . We will denote by the same symbol  $e_m$  a section of  $N$  defined by an irreducible element of  $\mathbb{N}^r$  via the chart  $\mathbb{N}^r \rightarrow N$  and the corresponding node in the closed fiber of  $C$ . Étale locally near every node  $e_m$  of  $C_{i,0}$  the curve is isomorphic to  $\text{Spec } R[x, y]/(xy - t^{k_i^m} \gamma_m)$ . Here  $k_i^m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ , and  $\gamma_m := \gamma(e_m) \in \mathbb{N}$  is the order of the stabilizer group of the node  $e_m$  in  $\mathcal{C}_{i0}$  and  $t$  is the uniformizing parameter. If  $k_i^m = \infty$  we put  $t^{k_i^m} = 0$ . For any

node  $e_m$  let  $N_{iV}^m$  be the log structure associated to the pre-log structure

$$(13) \quad \mathbb{N} \rightarrow R, \quad e \mapsto t^{k_i^m} \gamma_m,$$

and analogously let  $N_{iV}^m$  be the log structure associated to

$$(14) \quad \mathbb{N} \rightarrow R, \quad e \mapsto t^{k_i^m}.$$

The canonical log structures  $N_{iV}$  and  $N'_{iV}$  are defined as the amalgamated sums  $\oplus_{R^*} N_{iV}^m$  and  $\oplus_{R^*} N'^m_{iV}$ . The simple morphisms  $l_i$  are induced by the morphisms of pre-log structures

$$\mathbb{N}_V^{r_i} \xrightarrow{(\vec{\gamma}_i)} \mathbb{N}_V^{r_i}$$

where  $(\vec{\gamma}_i)$  is the matrix  $diag(\gamma(e_1), \dots, \gamma(e_{r_i}))$  with  $e_j \in E_{\vec{\tau}_i}$ ,  $i = 1, \dots, r_i$ .

We now show that an extension  $\mathcal{C}_V$ ,  $\phi_{i,V}$ ,  $i = 1, 2$ , exists uniquely. We assume that the curves are not generically nodal because in this case the claim is trivial. Indeed, if  $\mathcal{C}_U$  has a generic node, the pre-log structure associated to that node is of the form  $\mathbb{N} \rightarrow K$  mapping  $e$  to 0, which extends obviously to  $R$ .

For  $i = 1, 2$  we have the morphisms of log structures

$$\begin{array}{ccc} N_{i,V} & \longrightarrow & N'_{i,V} \\ \downarrow & & \\ N_V & & \end{array}$$

The vertical arrow corresponds to the morphism of pre-log structures

$$\mathbb{N}^{r_i} \rightarrow \mathbb{N}^r, \quad e_k \mapsto \sum_{m \in J_{i,k}} e_m,$$

where  $J_{i,k}$  is as above the set of indices such that  $\forall m \in J_{i,k}$ ,  $e_m \in E_\tau$  is in the long edge associated to  $e_k \in E_{\tau_i}$ . For any  $k \in \mathbb{N}$  such that  $e_k \in E_{\tau_i}$ , and for  $i = 1, 2$ , we have a constraint given by

$$\sum_{m \in J_{i,k}} k^m = k_i^k \cdot \gamma_{i,k},$$

where  $k^m, k_i^k \in \mathbb{N}_{\geq 0} \cup \{\infty\}$  are as in (13) and (14), and  $\gamma_{i,k} := \gamma(e_k)$  for  $e_k \in E_{\vec{\tau}_i} = E_{\vec{\tau}_i}$ . Analogously, let  $\gamma_j := \gamma(e_j)$  for  $e_j \in E_{\vec{\tau}}$ . For all  $j = 1, \dots, |E_{\vec{\tau}}|$

put  $d_j := \gcd(k^j, \gamma_j)$  and  $q_j := \gamma_j/d_j$ . Let  $Q = \max_j\{q_j\}$ . Consider the finite extension  $K'$  of  $K$  obtained by adding the  $Q$ -th root of the uniformizing parameter, that we denote by  $t'$ . Let  $R'$  be the integral closure of  $R$  in  $K'$ . Over  $V' = \text{Spec } R'$  we can define the pre-log structure

$$\mathbb{N}_{V'}^r \rightarrow R', \quad e_j \mapsto t'^{Qk^j/\gamma_j}.$$

Note that the  $t'^{k^j/\gamma_j}$  is in  $R'$ . We denote by  $\tilde{N}'_{V'}$ , the associated log structure. Let  $p : V' \rightarrow V$  denote the morphism induced by the extension. We have a simple morphism of pre-log structures

$$\tilde{l} : p^*N_V \rightarrow \tilde{N}'_{V'}$$

giving  $p^*C_V$  the structure of a log twisted curve with the correct order of the stabilizers of the nodes, in a compatible way with the morphisms  $p^*C_V \rightarrow p^*C_{iV}$ . Moreover morphisms of log structures  $p^*N'_{iV} \rightarrow \tilde{N}'_{V'}$ , are also defined. This is the unique structure of log twisted curve which respects the prescription given by the fixed morphisms of gerby dual graphs.  $\square$

**Remark 3.7.** In the above proof the degenerate case corresponding to an unstable genus 1 component without marked points in  $\mathcal{C}$ , contracted in either  $\mathcal{C}_1$  or  $\mathcal{C}_2$  does not appear. In this case the valuative criterion is satisfied because of properness of  $\bar{l}$ .

**3.3.2. Orientation and degree calculation.**

**Lemma 3.8.** *The morphism*

$$\phi \circ l : \mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{M}^{tw}(\tilde{\tau}_1) \times \mathfrak{M}^{tw}(\tilde{\tau}_2)$$

*defines a natural orientation  $[\phi \circ l]$ .*

*Proof.* The morphism  $\phi \circ l$  is of relative Deligne-Mumford type between smooth algebraic stacks of pure dimension. It follows as observed in [Kre99], footnote on page 529 that it is an l.c.i. morphism, hence defines a natural orientation.  $\square$

The following is a *Proposition* in [Beh99b]:

**Proposition 3.9.** *The morphism  $\Delta$  is a proper regular local immersion. The orientations  $[\Delta]$  and  $[\phi]$  defined respectively by  $\Delta$  and  $\phi$  satisfy*

$$[\phi] = (s \times s)^*[\Delta].$$

*Proof.* The morphism  $(s \times s)$  is flat, therefore  $\phi$  is also a regular local immersion. The statement on orientations is obvious.  $\square$

**Proposition 3.10.** *The morphism  $l$  has degree  $\mathfrak{c}$ , where*

$$(15) \quad \mathfrak{c} := \frac{\prod_{e_1 \in E_{\tilde{\tau}_1}} \gamma(e_1) \cdot \prod_{e_2 \in E_{\tilde{\tau}_2}} \gamma(e_2)}{\prod_{e \in E_{\tilde{\tau}}} \gamma(e)}.$$

*Proof.* We briefly recall the definition of the stack of prestable  $\tilde{\tau}$ -marked twisted curves. The objects are collections of prestable twisted curves  $\mathcal{C}_v$  for any  $v \in V_{\tilde{\tau}}$ , with  $|F_{\tilde{\tau}}(v)|$  marked gerbes, such that the  $i$ -th gerbe corresponding to the flag  $f_i \in F_{\tilde{\tau}}(v)$  has order  $\gamma(f_i)$ . Moreover, as part of the defining data, there are given  $|E_{\tilde{\tau}}|$  band-inverting isomorphisms of gerbes, prescribing how to glue gerbes corresponding to flags  $\{f, \bar{f}\} \in E_{\tilde{\tau}}$ . While the order of the stabilizers of marking gerbes and of generic nodes, which are obtained by gluing marking gerbes corresponding to flags which partake in an edge, is fixed by the dual gerby graph, this is not true for the order of the stabilizers of non-generic nodes. Moreover, since we are dealing with prestable curves, more nodes than those specified by the dual graph are allowed and those can have arbitrary cyclic groups as stabilizers. In order to get stacks of twisted curves of finite type over the corresponding stacks of prestable (coarse) curves, we constrain from above the order of the stabilizer of any special point.

As before, we will consider gerby graphs  $\tilde{\tau}$  whose underlying modular graph admits a non-empty absolute stabilization.

Generic curves are obtained by gluing smooth twisted curves  $\mathcal{C}_v$ ,  $v \in E_{\tilde{\tau}}$  along marking gerbes as prescribed by the dual gerby graph  $\tilde{\tau}$ . We can therefore assume that, given a curve over  $\text{Spec } \mathbb{C}$  corresponding to a generic point, no rational unstable component of  $\mathcal{C}$  is ever contracted in both  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . An automorphism of an object  $(\vec{\mathcal{C}}, \vec{\phi}) := (\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \phi_1, \phi_2)$  of  $\mathfrak{D}^{tw}(\tilde{\tau})$  is given by arrows and 2-arrows fitting in the following 2-commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi_1 \times \phi_2} & \mathcal{C}_1 \times \mathcal{C}_2 \\ \tilde{\xi}_e \downarrow & \eta & \downarrow \xi_{e_1}, \tilde{\xi}_{e_2} \\ \mathcal{C} & \xrightarrow{\phi_1 \times \phi_2} & \mathcal{C}_1 \times \mathcal{C}_2. \end{array}$$

Notice however that the 2-arrow  $\eta$  is trivial in this case, because it is an isomorphism between two representable morphisms of Deligne-Mumford stacks with trivial generic stabilizer (cfr. [AV02], Lemma 4.2.3). Therefore automorphisms of  $(\vec{\mathcal{C}}, \vec{\phi})$  are given by compatible triples of automorphisms of

twisted curves  $(\xi, \xi_1, \xi_2)$ . We will see in the following that for our purposes it is convenient to consider the inertia of  $\mathfrak{D}^{tw}(\tilde{\tau})$  relative to  $\mathfrak{D}$ . We only consider automorphisms which fix the coarse moduli space. Since we required in the definition (7) of  $\mathfrak{D}^{tw}(\tilde{\tau})$  that the morphisms  $\phi_i$  are faithful, we easily deduce that automorphisms of  $(\vec{\mathcal{C}}, \vec{\phi})$  are determined by  $\xi$ . For a twisted curve over  $\text{Spec } \mathbb{C}$  automorphisms which fix the coarse moduli space are as in Lemma 2.19. As a consequence, the generic stabilizer of  $\mathfrak{D}^{tw}(\tilde{\tau})$  is the following group

$$\prod_{i=1}^r \mu_{\gamma_i},$$

where  $\gamma_i$  is the order of the  $i$ -th generic node and  $r = |E_{\tilde{\tau}}|$ .

The morphism  $l_2$  in (6) is proper, since it is the base change of the morphism  $\bar{l}$  which is proven to be proper in [Beh99b]. The morphism  $l_1$  is proper as proved in Proposition 3.6. It is quasi-finite because e.g. it fits in the commutative diagram

$$\begin{array}{ccc} \mathfrak{D}^{tw}(\tilde{\tau}) & \xrightarrow{l_1} & \mathfrak{D}' \\ \downarrow e & & \downarrow \\ \mathfrak{M}^{tw}(\tilde{\tau}) & \longrightarrow & \mathfrak{M}(\tau), \end{array}$$

where the left vertical arrow is étale and the bottom horizontal arrow is quasi-finite (see. e.g. [AOV11], Corollary A.8). The stacks  $\mathfrak{D}^{tw}(\tilde{\tau})$  and  $\mathfrak{D}'$  are smooth stacks and  $l_1$  induces a bijection on geometric points. We have that the pushforward along  $l_1$  of the orientation  $[\mathfrak{D}^{tw}(\tilde{\tau})]$  is a multiple by the degree of  $l_1$  of the orientation  $[\mathfrak{D}']$ . The degree of a relative Deligne-Mumford type morphism between Artin stacks can be computed locally in the smooth topology of the base: Let  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be a generically quasi finite relative Deligne-Mumford morphism between Artin stacks, with  $\mathcal{Y}$  integral. Let  $U \rightarrow \mathcal{Y}$  be a smooth surjective morphism from a scheme, let  $V$  the base change along  $F$ . The degree of  $F$  is the degree of  $V \rightarrow U$ .

Degrees of morphisms between Deligne-Mumford stacks can be computed according to [Vis89]. Since the degree is multiplicative we can work relative to  $\mathfrak{D}(\tau)$ . Both  $\mathfrak{D}^{tw}(\tilde{\tau})$  and  $\mathfrak{D}'$  are gerbes over the open (and dense) locus of  $\mathfrak{D}(\tau)$  consisting of curves with only generic nodes. These gerbe structures correspond to the generic nodes of  $\mathcal{C}$  and respectively of  $(\mathcal{C}_1, \mathcal{C}_2)$ . This follows from the description of the automorphisms of objects of  $\mathfrak{D}^{tw}(\tilde{\tau})$ . Being a gerbe is preserved by base-change. The following cartesian diagram

is obtained as a base change of a smooth cover  $D \rightarrow \mathfrak{D}(\tau)$

$$\begin{array}{ccccc}
 D^{tw} & \longrightarrow & D' & \longrightarrow & D \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathfrak{D}^{tw}(\tilde{\tau}) & \longrightarrow & \mathfrak{D}' & \longrightarrow & \mathfrak{D}(\tau).
 \end{array}$$

Both  $D^{tw}$  and  $D'$  are generically Deligne-Mumford gerbes over  $D$ . The degree of the structure morphisms of these generic gerbes can be computed from the relative automorphisms groups of objects of  $\mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{D}(\tau)$ ,  $\mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{D}'$ , and  $\mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{D}(\tau)$ . We deduce

$$\text{deg}(l_1) = \frac{\text{order}(\delta(\mathfrak{D}' \rightarrow \mathfrak{D}(\tau)))}{\text{order}(\delta(\mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{D}(\tau)))},$$

where  $\delta$  denotes the relative inertia. This proves the desired result, because  $\mathfrak{D}'$  is birational to  $\mathfrak{B}^{tw}$  (this follows from the corresponding statement for  $\mathfrak{D}(\tau)$  and  $\mathfrak{B}$  proven in [Beh99b]).  $\square$

### 3.4. Relative obstruction theories

Using properties of the diagram (6) we compare relative obstruction theories on the stacks  $\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})$ ,  $\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)$ , and  $\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)$ . Consequently we prove Theorem 3.13.

**Proposition 3.11.** *The following square<sup>6</sup> is cartesian*

$$(16) \quad \begin{array}{ccccc}
 \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) & \xrightarrow{h} & P & \xrightarrow{q} & \mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1) \times \mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2) \\
 \downarrow & & \downarrow & & \downarrow a \\
 \mathfrak{D}^{tw}(\tilde{\tau}) & \xrightarrow{l} & \mathfrak{B}^{tw} & \xrightarrow{\phi} & \mathfrak{M}^{tw}(\tilde{\tau}_1) \times \mathfrak{M}^{tw}(\tilde{\tau}_2).
 \end{array}$$

*Proof.* The morphism  $h : \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) \rightarrow P$  is defined. Indeed we have a morphism  $p_i : \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) \rightarrow \mathcal{K}(\mathcal{X}_i, \tilde{\tau}_i)$ ,  $i = 1, 2$ , which is given by composing with the projection to  $\mathcal{X}_i$  and by taking first the relative coarse moduli space and then the partial relative stabilization with respect to the morphism to  $\mathcal{X}_i$ . The morphism  $\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) \rightarrow \mathfrak{D}^{tw}(\tilde{\tau})$  is given by constructing the induced morphisms to  $\mathcal{K}(\mathcal{X}_i, \tilde{\tau}_i)$  as described above and then

---

<sup>6</sup>Note that this is part of the diagram (6).



forgetting all the maps to the target stacks and retaining only the morphisms among the source curves. This gives a morphism

$$\Theta : \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) \rightarrow \mathfrak{D}^{tw}(\tilde{\tau}) \times_{\mathfrak{B}^{tw}} P.$$

We now construct a morphism  $\Xi : \mathfrak{D}^{tw}(\tilde{\tau}) \times_{\mathfrak{B}^{tw}} P \rightarrow \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})$ . Given  $f_1 : \mathcal{D}_1 \rightarrow \mathcal{X}_1$ ,  $f_2 : \mathcal{D}_2 \rightarrow \mathcal{X}_2$ , an object  $(\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \phi_1, \phi_2)$  of  $\mathfrak{D}^{tw}(\tilde{\tau})$ , and a pair of isomorphisms  $\alpha_i : \mathcal{C}_i \xrightarrow{\sim} \mathcal{D}_i, i = 1, 2$ , we want to construct a representable morphism  $f : \mathcal{C} \rightarrow \mathcal{X}_1 \times \mathcal{X}_2$ . We claim that

$$(17) \quad \mathcal{C} \xrightarrow{(\phi_1, \phi_2)} \mathcal{C}_1 \times \mathcal{C}_2 \xrightarrow{(f_1 \circ \alpha_1 \circ pr_1, f_2 \circ \alpha_2 \circ pr_2)} \mathcal{X}_1 \times \mathcal{X}_2$$

is a twisted stable map, namely the composed morphism in Equation (17) is representable and stable. This follows immediately from the definition of  $\mathfrak{D}^{tw}(\tilde{\tau})$ , which parametrizes morphisms  $[\phi_1 : \mathcal{C} \rightarrow \mathcal{C}_1, \phi_2 : \mathcal{C} \rightarrow \mathcal{C}_2]$  such that  $\phi_1 \times \phi_2$  is representable and does not admit infinitesimal automorphisms relative to  $\mathcal{C}_1 \times \mathcal{C}_2$ .

We want to prove that  $\Theta \circ \Xi$  is isomorphic to the identity functor of the fiber product and that  $\Xi \circ \Theta$  is isomorphic to the identity functor of  $\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})$ . It is clear that  $\Xi \circ \Theta$  is the identity. It is also easy to check that  $\Theta \circ \Xi$  is isomorphic to the identity. The key point is that due to the universal property of the relative stabilization we have morphisms  $\mathcal{C}_i^s \rightarrow \mathcal{C}_i$ , where  $\mathcal{C}_i^s$  is the curve obtained by applying the relative coarse moduli space and the relative stabilization construction to the morphism  $\mathcal{C} \rightarrow \mathcal{X}_i$  given by the composition of (17) with the projection. Since stable maps form a groupoid this morphism is in fact an isomorphism. This allows to define a natural transformation with the identity functor by composing arrows and 2-arrows arising canonically from the relative stabilization process.  $\square$

Consider the universal family over  $\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})$ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X}_1 \times \mathcal{X}_2 \\ \downarrow \pi & & \\ \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}), & & \end{array}$$

and the universal families over  $\mathcal{K}(\mathcal{X}_i, \tilde{\tau}_i), i = 1, 2$ :

$$\begin{array}{ccc} \mathcal{C}_i & \xrightarrow{f_i} & \mathcal{X}_i \\ \downarrow \pi_i & & \\ \mathcal{K}(\mathcal{X}_i, \tilde{\tau}_i). & & \end{array}$$

According to [AGV08] there are perfect relative obstruction theories

$$(18) \quad \begin{aligned} E_{\mathcal{X}_1 \times \mathcal{X}_2}^{\bullet \vee} &:= R\pi_* (f^* \Omega_{\mathcal{X}_1 \times \mathcal{X}_2} \otimes \omega_\pi) \rightarrow L_{\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) / \mathfrak{M}^{tw}(\tilde{\tau})} \\ E_{\mathcal{X}_i}^{\bullet \vee} &:= R\pi_{i*} (f_i^* \Omega_{\mathcal{X}_i} \otimes \omega_{\pi_i}) \rightarrow L_{\mathcal{K}(\mathcal{X}_i, \tilde{\tau}_i) / \mathfrak{M}^{tw}(\tilde{\tau}_i)}, \quad i = 1, 2. \end{aligned}$$

By Proposition 3.4 the morphism  $e : \mathfrak{D}^{tw}(\tilde{\tau}) \rightarrow \mathfrak{M}^{tw}(\tilde{\tau})$  is étale. Hence  $E_{\mathcal{X}_1 \times \mathcal{X}_2}^{\bullet \vee}$  can be viewed as a relative perfect obstruction theory over  $\mathfrak{D}^{tw}(\tilde{\tau})$ .

**Proposition 3.12.** *The two obstruction theories  $l^* \phi^*(E_{\mathcal{X}_1}^{\bullet} \boxplus E_{\mathcal{X}_2}^{\bullet})$  and  $E_{\mathcal{X}_1 \times \mathcal{X}_2}^{\bullet}$  are naturally isomorphic.*

*Proof.* Consider the morphism

$$p = (p_1, p_2) : \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) \rightarrow \mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1) \times \mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2).$$

Denote by  $p_1^* \mathcal{C}_1$  and  $p_2^* \mathcal{C}_2$  the pullback of the universal families over  $\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)$  and  $\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)$  to  $\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})$ . This yields the following cartesian diagrams

$$\begin{array}{ccccc} & & & \xrightarrow{f'_i} & \\ p_i^* \mathcal{C}_i & \xrightarrow{\quad} & \mathcal{C}_i & \xrightarrow{f_i} & \mathcal{X}_i \\ \downarrow \pi'_i & & \downarrow \pi_i & & \\ \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) & \xrightarrow{p_i} & \mathcal{K}(\mathcal{X}_i, \tilde{\tau}_i). & & \end{array}$$

Since this square is cartesian and the twisted curves are flat over their bases, we have (see e.g. [HRBB09], Proposition A.85)

$$p^*(R\pi_{1*} f_1^* T_{\mathcal{X}_1} \oplus R\pi_{2*} f_2^* T_{\mathcal{X}_2}) \simeq R\pi'_{1*} (f'_1)^* T_{\mathcal{X}_1} \oplus R\pi'_{2*} (f'_2)^* T_{\mathcal{X}_2}.$$

Moreover we have over  $\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})$  morphisms  $q_i : \mathcal{C} \rightarrow p_i^* \mathcal{C}_i$  which correspond to the relative stabilization morphism. Since for such morphisms we

have  $q_{i*}\mathcal{O}_e \simeq \mathcal{O}_{p_i^*e_i}$ , we see that the functor  $Rq_{i*}q_i^*$  is the identity. Therefore

$$\begin{aligned} & R\pi_{1*}'(f_1')^*T_{\mathcal{X}_1} \oplus R\pi_{2*}'(f_2')^*T_{\mathcal{X}_2} \\ & \simeq R\pi_{1*}'Rq_{1*}'q_1'^*(f_1')^*T_{\mathcal{X}_1} \oplus R\pi_{2*}'Rq_{2*}'q_2'^*(f_2')^*T_{\mathcal{X}_2} \\ & \simeq R\pi_*f^*(T_{\mathcal{X}_1} \boxplus T_{\mathcal{X}_2}). \end{aligned}$$

The result follows. □

By the general construction of [BF97], the obstruction theories (18) define virtual fundamental classes for the moduli spaces  $\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})$ ,  $\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)$ , and  $\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)$ . Let  $[\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^{vir}$  and  $[\mathcal{K}(\mathcal{X}_i, \tilde{\tau}_i)]^{vir}$ ,  $i = 1, 2$ , be these virtual fundamental classes. The following Theorem, which relates these virtual fundamental classes, is the main technical heart of this paper.

**Theorem 3.13.** *The following equality holds:*

$$(19) \quad h_*[\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^{vir} = \mathbf{c}\Delta^!([\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)]^{vir} \times [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)]^{vir}).$$

*Proof.* Consider the diagram (6) and its portion (16). The morphism  $l$  is the composition of the morphisms  $l_1$  and  $l_2$ . The morphism  $\phi$  in Diagram (6) is a regular local immersion since  $\Delta$  is. As explained in the later arguments after the main diagram (6),  $\mathfrak{D}'$  is defined to be  $\mathfrak{D}(\tau) \times_{\mathfrak{B}} \mathfrak{B}^{tw}$  where  $\mathfrak{D}(\tau)$  and  $\mathfrak{B}$  are defined respectively in [Beh99b] diagrams (3) and (2) (Note that Behrend used different notation). The map  $l_2$  is the natural projection, which is a regular local immersion since  $\mathfrak{D}(\tau) \rightarrow \mathfrak{B}$  is a regular local immersion by Proposition 3 in [Beh99b]. From Proposition 3.10,  $l_1$  is of relative Deligne-Mumford type and is bijection on geometric points. Hence it is also a regular local immersion. Hence the composition  $\phi \circ l$  is a regular local immersion.

We calculate

$$\begin{aligned} & [\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^{vir} \\ & = [\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}), E_{\mathcal{X}_1 \times \mathcal{X}_2}^\bullet] \\ & = [\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}), l^*\phi^*(E_{\mathcal{X}_1}^\bullet \boxplus E_{\mathcal{X}_2}^\bullet)] \quad (\text{by Proposition 3.12}) \\ & = (\phi \circ l)^![\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1) \times \mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2), E_{\mathcal{X}_1}^\bullet \boxplus E_{\mathcal{X}_2}^\bullet] \quad (\text{by [BF97], Proposition 7.2}) \\ & = (\phi \circ l)^!([\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1), E_{\mathcal{X}_1}^\bullet] \times [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2), E_{\mathcal{X}_2}^\bullet]) \quad (\text{by [BF97], Proposition 7.4}) \\ & = (\phi \circ l)^!([\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)]^{vir} \times [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)]^{vir}). \end{aligned}$$

One therefore has

$$\begin{aligned}
 & \mathbf{c} \cdot \Delta^!([\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)]^{vir} \times [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)]^{vir}) \\
 &= \mathbf{c} \cdot a^*(s \times s)^*[\Delta]([\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)]^{vir} \times [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)]^{vir}) \\
 &= \mathbf{c} \cdot a^*[\phi]([\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)]^{vir} \times [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)]^{vir}) \\
 &= a^*l_*[\phi \circ l]([\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)]^{vir} \times [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)]^{vir}) \quad (\text{by Proposition 3.10}) \\
 &= h_*(\phi \circ l)^!([\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)]^{vir} \times [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)]^{vir}) \\
 &= h_*[\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^{vir}. \quad \square
 \end{aligned}$$

### 4. Gromov-Witten classes

In this Section we present consequences of Theorem 3.13 in Gromov-Witten invariants.

#### 4.1. Weighted virtual classes

We begin with an easy Lemma.

**Lemma 4.1.** *Let  $\mathcal{X}_1$  and  $\mathcal{X}_2$  be two Deligne-Mumford stacks. There is a natural isomorphism*

$$I(\mathcal{X}_1 \times \mathcal{X}_2) \simeq I(\mathcal{X}_1) \times I(\mathcal{X}_2).$$

*Proof.* This follows easily from the definition of the inertia stacks. □

Gromov-Witten theory of a Deligne-Mumford stack  $\mathcal{X}$  may be defined using the moduli stacks  $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ . Gromov-Witten classes are defined by intersecting the virtual fundamental class  $[\mathcal{K}_{g,n}(\mathcal{X}, \beta)]^{vir}$  with cohomology classes of  $\mathcal{X}$  pulled back via the *evaluation maps*,

$$\bar{e}v : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \rightarrow \bar{I}(\mathcal{X})^{\times n}.$$

A detailed treatment can be found in [AGV08]. Note that the evaluation map  $\bar{e}v$  takes values in the *rigidified* inertia stack  $\bar{I}(\mathcal{X})$ .

When comparing Gromov-Witten theory of the product stack  $\mathcal{X}_1 \times \mathcal{X}_2$  with Gromov-Witten theory of the factors  $\mathcal{X}_1$  and  $\mathcal{X}_2$ , because of Lemma 4.1, it is more convenient to work in the framework where the evaluation maps take values in the inertia stack. This alternative framework is discussed in [AGV02, AGV08]. We adapt this into our setting, as follows.

Let  $\mathcal{X}$  be a smooth proper Deligne-Mumford stack with projective coarse moduli space  $X$ . Let  $\tilde{\tau}$  be a gerby  $\mathcal{X}$ -graph (see Definitions 2.14, 2.16). Over the stack  $\mathcal{K}(\mathcal{X}, \tilde{\tau})$  of  $\tilde{\tau}$ -marked twisted stable maps to  $\mathcal{X}$  there are universal marking gerbes  $\mathcal{G}_i, i \in S_{\tilde{\tau}}$ . The fiber product of all  $\mathcal{G}_i$  over  $\mathcal{K}(\mathcal{X}, \tilde{\tau})$ , which we denote by  $\mathcal{M}(\mathcal{X}, \tilde{\tau})$ , is the moduli stack of  $\tilde{\tau}$ -marked twisted stable maps to  $\mathcal{X}$  with *trivialized marked gerbes*. By construction the evaluation map of  $\mathcal{M}(\mathcal{X}, \tilde{\tau})$  fits into the following diagram

$$\begin{CD} \mathcal{M}(\mathcal{X}, \tilde{\tau}) @>{ev_{\tilde{\tau}}}>> I(\mathcal{X})^{\times \#S_{\tilde{\tau}}} \\ @V{\theta_{\tilde{\tau}}}VV @VVV \\ \mathcal{K}(\mathcal{X}, \tilde{\tau}) @>{\bar{ev}_{\tilde{\tau}}}>> \bar{I}(\mathcal{X})^{\times \#S_{\tilde{\tau}}} \end{CD}$$

As explained in [AGV02, AGV08], in order to define Gromov-Witten theory using  $\mathcal{M}(\mathcal{X}, \tilde{\tau})$  one has to work with the *weighted* virtual fundamental class, defined as follows.

**Definition 4.2.**

$$[\mathcal{M}(\mathcal{X}, \tilde{\tau})]^w := \left( \prod_{i \in S_{\tilde{\tau}}} \gamma(i) \right) \theta_{\tilde{\tau}}^* [\mathcal{K}(\mathcal{X}, \tilde{\tau})]^{vir}.$$

Theorem 3.13 has a counterpart for weighted virtual classes. Let  $\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{X}_1, \mathcal{X}_2, \tilde{\tau}, \tilde{\tau}_1,$  and  $\tilde{\tau}_2$  be as in Section 3.1. Consider the following diagram:

$$(20) \quad \begin{CD} \mathcal{M}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) @>{\tilde{h}}>> \mathcal{M}(\mathcal{X}_1, \tilde{\tau}_1) \times \mathcal{M}(\mathcal{X}_2, \tilde{\tau}_2) \\ @V{\theta'}VV @. @VV{\theta_{\tilde{\tau}_1} \times \theta_{\tilde{\tau}_2}}V \\ \mathcal{M} @>{h'}>> \mathcal{P} @>{q'}>> \mathcal{M}(\mathcal{X}_1, \tilde{\tau}_1) \times \mathcal{M}(\mathcal{X}_2, \tilde{\tau}_2) \\ @V{\theta_{\tilde{\tau}}}VV @VV{\rho}V @VV{\theta_{\tilde{\tau}_1} \times \theta_{\tilde{\tau}_2}}V \\ \mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) @>{h}>> P @>{q}>> \mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1) \times \mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2). \end{CD}$$

Here the bottom arrows are defined in (6).

**Theorem 4.3.**

$$(21) \quad \tilde{h}_* [\mathcal{M}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^w = \mathbf{c} \cdot \Delta^! ([\mathcal{M}(\mathcal{X}_1, \tilde{\tau}_1)]^w \times [\mathcal{M}(\mathcal{X}_2, \tilde{\tau}_2)]^w).$$

*Proof.* We calculate

$$\begin{aligned}
 \tilde{h}_*[\mathcal{M}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^w &= \left( \prod_{i \in S_{\tilde{\tau}}} \gamma(i) \right) \tilde{h}_* \theta_{\tilde{\tau}}^* [\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^{vir} \\
 &= \left( \prod_{i \in S_{\tilde{\tau}}} \gamma(i) \right) h'_* \theta'_* \theta'^* \theta_{12}^* [\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^{vir} \\
 &= (\deg \theta') \left( \prod_{i \in S_{\tilde{\tau}}} \gamma(i) \right) h'_* \theta_{12}^* [\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^{vir} \\
 &= (\deg \theta') \left( \prod_{i \in S_{\tilde{\tau}}} \gamma(i) \right) \rho^* h_* [\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^{vir}.
 \end{aligned}$$

By Theorem 3.13, we have

$$\begin{aligned}
 &\rho^* h_* [\mathcal{K}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^{vir} \\
 &= \mathbf{c} \cdot \rho^* \Delta^!([\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)]^{vir} \times [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)]^{vir}) \\
 &= \mathbf{c} \cdot \Delta^!(\theta_{\tilde{\tau}_1}^* [\mathcal{K}(\mathcal{X}_1, \tilde{\tau}_1)]^{vir} \times \theta_{\tilde{\tau}_2}^* [\mathcal{K}(\mathcal{X}_2, \tilde{\tau}_2)]^{vir}) \\
 &= \frac{\mathbf{c}}{\prod_{i \in S_{\tilde{\tau}_1}} \gamma(i) \prod_{i \in S_{\tilde{\tau}_2}} \gamma(i)} \Delta^!([\mathcal{M}(\mathcal{X}_1, \tilde{\tau}_1)]^w \times [\mathcal{M}(\mathcal{X}_2, \tilde{\tau}_2)]^w).
 \end{aligned}$$

We conclude by observing that

$$\deg \theta' = \frac{\deg \theta_{\tilde{\tau}}}{\deg \theta_{12}} = \frac{(\prod_{i \in S_{\tilde{\tau}}} \gamma(i))^{-1}}{\left( \prod_{i \in S_{\tilde{\tau}_1}} \gamma(i) \prod_{i \in S_{\tilde{\tau}_2}} \gamma(i) \right)^{-1}}. \quad \square$$

### 4.2. Gromov-Witten classes

Again let  $\mathcal{X}$  be a proper smooth Deligne-Mumford stack with projective coarse moduli space. Let  $\tilde{\tau}$  be a gerby  $\mathcal{X}$ -graph. Denote by  $\tau^s$  the absolute stabilization of the underlying modular graph  $\tau$ . There is a contraction morphism

$$st_{\tilde{\tau}} : \mathcal{M}(\mathcal{X}, \tilde{\tau}) \rightarrow \overline{M}(\tau^s).$$

**Definition 4.4.** Define a map

$$I_{\tilde{\tau}}^{\mathcal{X}} : H^*(I(\mathcal{X}), \mathbb{Q})^{\times \#S_{\tilde{\tau}}} \rightarrow H_*(\overline{M}(\tau^s), \mathbb{Q})$$

as follows. For  $\omega \in H^*(I(\mathcal{X}), \mathbb{Q})^{\times \#S_{\tilde{\tau}}}$ , define

$$I_{\tilde{\tau}}^{\mathcal{X}}(\omega) := \left( \prod_{i \in E_{\tilde{\tau}}} \gamma(i) \right) st_{\tilde{\tau}*}(ev_{\tilde{\tau}}^*(\omega) \cap [\mathcal{M}(\mathcal{X}, \tilde{\tau})]^w).$$

We call  $I_{\tilde{\tau}}^{\mathcal{X}}$  the *Gromov-Witten class* associated to  $\mathcal{X}$  and  $\tilde{\tau}$ .

**4.2.1. Gromov-Witten Cohomological Field Theory.** Let  $H$  be a  $\mathbb{Z}$ -graded  $\mathbb{Q}$ -vector space with a non-degenerate pairing  $h$ . Following [Man99], *Chapter 3, Section 4*, a structure of complete<sup>7</sup> Cohomological Field Theory (CohTh) on  $(H, h)$  is given by a collection of even linear maps (called *correlators*)

$$(22) \quad I_{g,n}^V : H^{\otimes n} \rightarrow H^*(\overline{M}_{g,n}, \mathbb{Q}),$$

defined for any  $g \geq 0$  and  $n + 2g - 3 \geq 0$ , satisfying the following axioms:

- 1)  $S_n$ -covariance with respect to the natural action of  $S_n$  on both sides of (22);
- 2) Splitting, i.e. compatibility with respect to boundary divisors. Let  $\sigma$  be a dual graph with two vertices of genus  $g_1$  and  $g_2$ , defining a partition  $S_1 \amalg S_2 = \{1, \dots, n\}$ . Let  $\phi_{\sigma} : \overline{M}_{g_1, n_1+1} \times \overline{M}_{g_2, n_2+1} \rightarrow \overline{M}_{g,n}$  denote the gluing morphism. Here  $n_1 = |S_1|, n_2 = |S_2|, g = g_1 + g_2$ . Then

$$\phi_{\sigma}^* I_{g,n}^V(\gamma_1 \otimes \dots \otimes \gamma_n) = \epsilon(\sigma) I_{g_1, n_1+1} \otimes I_{g_2, n_2+1} \left( \bigotimes_{p \in S_1} \gamma_p \otimes \Delta \otimes \bigotimes_{q \in S_2} \gamma_q \right),$$

where  $\Delta = \sum \Delta_a h^{ab} \otimes \Delta_b$  is the class of the diagonal and  $\epsilon(\sigma)$  is the sign of the permutation induced on the odd arguments  $\gamma_1, \dots, \gamma_n$ ;

- 3) Genus reduction. For  $g \geq 1$ , let us denote by  $\psi : \overline{M}_{g-1, n+2} \rightarrow \overline{M}_{g,n}$  the clutching morphism ([Knu83], *Section 3*). Then we require

$$\psi^* I_{g,n}^V(\gamma_1 \otimes \dots \otimes \gamma_n) = I_{g-1, n+2}(\gamma_1 \otimes \dots \otimes \gamma_n \otimes \Delta).$$

**Theorem 4.5.** *The classes  $I_{\tilde{\tau}}^{\mathcal{X}}$  define a cohomological field theory.*

*Proof.* This follows from compatibility relations of weighted virtual fundamental classes proven in [AGV02], *Proposition 5.3.1*. Consider the splitting

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<sup>7</sup>This means that it is defined for all genera.

axiom of a CohFT. Let  $\tilde{\tau}$  be a gerby  $\mathcal{X}$ -graph and  $\tilde{\tau}_1, \tilde{\tau}_2$  two gerby  $\mathcal{X}$ -graphs obtained from  $\tilde{\tau}$  by cutting an edge  $\{f, j_\tau f\} \in E_{\tilde{\tau}}$  that connects two vertices. Then we can form the following diagram

$$\begin{array}{ccc} \mathcal{M}(\mathcal{X}, \tilde{\tau}) \xleftarrow{\pi} \mathcal{M}(\mathcal{X}, \tilde{\tau}_1) \times_{I(\mathcal{X})} \mathcal{M}(\mathcal{X}, \tilde{\tau}_2) & \longrightarrow & \mathcal{M}(\mathcal{X}, \tilde{\tau}_1) \times \mathcal{M}(\mathcal{X}, \tilde{\tau}_2) \\ & & \downarrow \qquad \qquad \downarrow \\ & I(\mathcal{X}) \xrightarrow{\delta} & I(\mathcal{X}) \times I(\mathcal{X}). \end{array}$$

Here the fiber product  $\mathcal{M}(\mathcal{X}, \tilde{\tau}_1) \times_{I(\mathcal{X})} \mathcal{M}(\mathcal{X}, \tilde{\tau}_2)$  is taken using the evaluation maps at flags  $f, j_\tau f$ . The map  $\delta$  is the diagonal map. By definition of the stacks  $\mathcal{M}(\mathcal{X}, \tilde{\tau}), \mathcal{M}(\mathcal{X}, \tilde{\tau}_i)$ , the natural map  $\pi : \mathcal{M}(\mathcal{X}, \tilde{\tau}_1) \times_{I(\mathcal{X})} \mathcal{M}(\mathcal{X}, \tilde{\tau}_2) \rightarrow \mathcal{M}(\mathcal{X}, \tilde{\tau})$  is the universal gerbe at the node defined by the edge  $\{f, j_\tau f\}$ , and is of degree  $1/\gamma(f)$ . The splitting axiom follows from the following consideration:

$$\begin{aligned} & \left( \prod_{i \in E_{\tilde{\tau}}} \gamma(i) \right) [\mathcal{M}(\mathcal{X}, \tilde{\tau})]^w \\ &= \left( \prod_{i \in E_{\tilde{\tau}} \setminus \{f, j_\tau f\}} \gamma(i) \right) \gamma(f)^2 \pi_* \pi^* [\mathcal{M}(\mathcal{X}, \tilde{\tau})]^w \\ &= \left( \prod_{i \in E_{\tilde{\tau}} \setminus \{f, j_\tau f\}} \gamma(i) \right) \pi_* \delta^!([\mathcal{M}(\mathcal{X}, \tilde{\tau}_1)]^w \times [\mathcal{M}(\mathcal{X}, \tilde{\tau}_2)]^w) \\ & \qquad \qquad \qquad \text{by [AGV02], Proposition 5.3.1} \\ &= \pi_* \delta^! \left( \left( \prod_{i \in E_{\tilde{\tau}_1}} \gamma(i) \right) [\mathcal{M}(\mathcal{X}, \tilde{\tau}_1)]^w \times \left( \prod_{i \in E_{\tilde{\tau}_2}} \gamma(i) \right) [\mathcal{M}(\mathcal{X}, \tilde{\tau}_2)]^w \right). \end{aligned}$$

The genus-reduction axiom follows from a similar consideration. We omit the details. □

### 4.3. Product formula

Again let  $\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{X}_1, \mathcal{X}_2, \tilde{\tau}, \tilde{\tau}_1,$  and  $\tilde{\tau}_2$  be as in Section 3.1. As a corollary of the result Theorem 4.3 on virtual fundamental classes, we prove here a product formula for Gromov-Witten classes associated to  $\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{X}_1,$  and  $\mathcal{X}_2$ .

**Theorem 4.6.** *Let  $\omega_1 \otimes \omega_2 \in H^*(I(\mathcal{X}_1), \mathbb{Q})^{\times \#S_{\tilde{\tau}_1}} \otimes H^*(I(\mathcal{X}_2), \mathbb{Q})^{\times \#S_{\tilde{\tau}_2}}$ . Identify  $\omega_1 \otimes \omega_2$  with a class in  $H^*(I(\mathcal{X}_1 \times \mathcal{X}_2), \mathbb{Q})^{\times \#S_{\tilde{\tau}}}$  via the map*

$$H^*(I(\mathcal{X}_1), \mathbb{Q})^{\times \#S_{\tilde{\tau}_1}} \otimes H^*(I(\mathcal{X}_2), \mathbb{Q})^{\times \#S_{\tilde{\tau}_2}} \rightarrow H^*(I(\mathcal{X}_1 \times \mathcal{X}_2), \mathbb{Q})^{\times \#S_{\tilde{\tau}}}$$



induced by  $F_{\tilde{\tau}_i} \rightarrow F_{\tilde{\tau}}$ . Then we have in  $H_*(\overline{M}(\tau^s), \mathbb{Q})$

$$I_{\tilde{\tau}}^{\mathcal{X}_1 \times \mathcal{X}_2}(\omega_1 \otimes \omega_2) = I_{\tilde{\tau}_1}^{\mathcal{X}_1}(\omega_1) \cup I_{\tilde{\tau}_2}^{\mathcal{X}_2}(\omega_2).$$

*Proof.* We refer to diagrams (6) and (20). In diagram (6) denote the map  $P \rightarrow \mathfrak{B}^{tw} \rightarrow \overline{M}(\tau^s)$  by  $\epsilon$ . Then  $st_{\tilde{\tau}} : \mathcal{M}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau}) \rightarrow \overline{M}(\tau^s)$  factors as  $st_{\tilde{\tau}} = \epsilon \circ \rho \circ \tilde{h}$ . Also, the map  $st_{\tilde{\tau}_1} \times st_{\tilde{\tau}_2} : \mathcal{M}(\mathcal{X}_1, \tilde{\tau}_1) \times \mathcal{M}(\mathcal{X}_2, \tilde{\tau}_2) \rightarrow \overline{M}(\tau^s) \times \overline{M}(\tau^s)$  factors as  $st_{\tilde{\tau}_1} \times st_{\tilde{\tau}_2} = (s \times s) \circ a \circ (\theta_{\tilde{\tau}_1} \times \theta_{\tilde{\tau}_2})$ . By construction we have  $ev_{\tilde{\tau}}^*(\omega_1 \otimes \omega_2) = \tilde{h}^* q'^*(ev_{\tilde{\tau}_1}^*(\omega_1) \otimes ev_{\tilde{\tau}_2}^*(\omega_2))$ . Using these facts and Theorem 4.3 we now calculate

$$\begin{aligned} I_{\tilde{\tau}}^{\mathcal{X}_1 \times \mathcal{X}_2}(\omega_1 \otimes \omega_2) &= \left( \prod_{i \in E_{\tilde{\tau}}} \gamma(i) \right) st_{\tilde{\tau}*}(ev_{\tilde{\tau}}^*(\omega_1 \otimes \omega_2) \cap [\mathcal{M}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^w) \\ &= \left( \prod_{i \in E_{\tilde{\tau}}} \gamma(i) \right) \epsilon_* \rho_* \tilde{h}_*(\tilde{h}^* q'^*(ev_{\tilde{\tau}_1}^*(\omega_1) \otimes ev_{\tilde{\tau}_2}^*(\omega_2)) \cap [\mathcal{M}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^w) \\ &= \left( \prod_{i \in E_{\tilde{\tau}}} \gamma(i) \right) \epsilon_* \rho_*(q'^*(ev_{\tilde{\tau}_1}^*(\omega_1) \otimes ev_{\tilde{\tau}_2}^*(\omega_2)) \cap \tilde{h}_*[\mathcal{M}(\mathcal{X}_1 \times \mathcal{X}_2, \tilde{\tau})]^w) \\ &= \mathbf{c} \left( \prod_{i \in E_{\tilde{\tau}}} \gamma(i) \right) \epsilon_* \rho_*(q'^*(ev_{\tilde{\tau}_1}^*(\omega_1) \otimes ev_{\tilde{\tau}_2}^*(\omega_2)) \\ &\quad \cap \Delta^!([\mathcal{M}(\mathcal{X}_1, \tilde{\tau}_1)]^w \times [\mathcal{M}(\mathcal{X}_2, \tilde{\tau}_2)]^w)) \\ &= \mathbf{c} \left( \prod_{i \in E_{\tilde{\tau}}} \gamma(i) \right) \epsilon_* \rho_* \Delta^!((ev_{\tilde{\tau}_1}^*(\omega_1) \cap [\mathcal{M}(\mathcal{X}_1, \tilde{\tau}_1)]^w) \\ &\quad \times (ev_{\tilde{\tau}_2}^*(\omega_2) \cap [\mathcal{M}(\mathcal{X}_2, \tilde{\tau}_2)]^w)) \\ &= \mathbf{c} \left( \prod_{i \in E_{\tilde{\tau}}} \gamma(i) \right) \Delta^*(s \times s)_* a_*(\theta_{\tilde{\tau}_1} \times \theta_{\tilde{\tau}_2})_* ((ev_{\tilde{\tau}_1}^*(\omega_1) \cap [\mathcal{M}(\mathcal{X}_1, \tilde{\tau}_1)]^w) \\ &\quad \times (ev_{\tilde{\tau}_2}^*(\omega_2) \cap [\mathcal{M}(\mathcal{X}_2, \tilde{\tau}_2)]^w)) \\ &= \mathbf{c} \left( \prod_{i \in E_{\tilde{\tau}}} \gamma(i) \right) \Delta^*(st_{\tilde{\tau}_1*}(ev_{\tilde{\tau}_1}^*(\omega_1) \cap [\mathcal{M}(\mathcal{X}_1, \tilde{\tau}_1)]^w) \\ &\quad \otimes st_{\tilde{\tau}_2*}(ev_{\tilde{\tau}_2}^*(\omega_2) \cap [\mathcal{M}(\mathcal{X}_2, \tilde{\tau}_2)]^w)) \\ &= I_{\tilde{\tau}_1}^{\mathcal{X}_1}(\omega_1) \cup I_{\tilde{\tau}_2}^{\mathcal{X}_2}(\omega_2), \end{aligned}$$

where in the last line we use Proposition 3.10. The proof is completed.  $\square$

### 5. Application to trivial gerbes

In this Section we consider Gromov-Witten theory of trivial gerbes as an application of our main result Theorem 4.6. Let  $\mathcal{X}$  be a proper smooth Deligne-Mumford stack with projective coarse moduli space. Let  $G$  be a finite group and  $BG$  the classifying stack of  $G$ . The inertia stack of  $BG$  admits the following decomposition

$$I(BG) = \bigcup_{(g):\text{conjugacy class}} BC_G(g),$$

where  $C_G(g) \subset G$  is the centralizer subgroup of  $g$  in  $G$ . This yields a decomposition of the orbifold cohomology groups

$$H^*(I(BG), \mathbb{C}) = \bigoplus_{(g):\text{conjugacy class}} H^*(BC_G(g), \mathbb{C}) = \bigoplus_{(g):\text{conjugacy class}} \mathbb{C}\mathbf{1}_{(g)},$$

where  $\mathbf{1}_{(g)} \in H^0(BC_G(g), \mathbb{C})$  is the standard generator. Let

$$I(\mathcal{X}) = \bigcup_{i \in \mathcal{J}} \mathcal{X}_i$$

be the decomposition of  $I(\mathcal{X})$  into connected components. We know that

$$\begin{aligned} H^*(I(\mathcal{X} \times BG), \mathbb{C}) &\simeq H^*(I(\mathcal{X}), \mathbb{C}) \otimes H^*(I(BG), \mathbb{C}) \\ &\simeq \bigoplus_{(g):\text{conjugacy class}, i \in \mathcal{J}} H^*(\mathcal{X}_i, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}\mathbf{1}_{(g)}. \end{aligned}$$

Let  $\beta \in H_2(X, \mathbb{Z})$  be a curve class. Let  $\overline{M}_{g,n}(\mathcal{X} \times BG, \beta)$  be the moduli stack<sup>8</sup> of twisted stable maps with trivialized marked gerbes to  $\mathcal{X} \times BG$  of type  $(g, n, \beta)$ . Let  $(g_1), \dots, (g_n)$  be conjugacy classes of  $G$ . Consider integers  $k_1, \dots, k_n \geq 0$  and cohomology classes  $\gamma_j \in H^*(\mathcal{X}_{i_j}, \mathbb{C})$ ,  $1 \leq j \leq n$ , which we assume to be homogeneous. Consider the Gromov-Witten invariant

$$\left\langle \prod_{j=1}^n \tau_{k_j}(\gamma_j \otimes \mathbf{1}_{(g_j)}) \right\rangle_{g,n,\beta}^{\mathcal{X} \times BG} := \int_{[\overline{M}_{g,n}(\mathcal{X} \times BG, \beta)]^w} \prod_{j=1}^n ev_j^*(\gamma_j \otimes \mathbf{1}_{(g_j)}) \psi_j^{k_j}.$$

It follows from the definition that the descendant classes  $\psi_j$  on  $\overline{M}_{g,n}(\mathcal{X} \times BG, \beta)$  are obtained as pull-backs of descendant classes on  $\overline{M}_{g,n}(\mathcal{X}, \beta)$ . This

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<sup>8</sup>This was previously denoted by  $\mathcal{M}(\mathcal{X} \times BG, \beta)$ . In order to be consist with notations in [JK02], we change our notations from now on.

observation together with the property of virtual classes (i.e. Theorem 4.3) imply the following equality:

$$(23) \quad \left\langle \prod_{j=1}^n \tau_{k_j}(\gamma_j \otimes \mathbf{1}_{(g_j)}) \right\rangle_{g,n,\beta}^{\mathcal{X} \times BG} = \left\langle \prod_{j=1}^n \tau_{k_j}(\gamma_j) \right\rangle_{g,n,\beta}^{\mathcal{X}} \times \deg(\overline{M}_{g,n}(BG, (g_1), \dots, (g_n)) / \overline{M}_{g,n}).$$

Here  $\deg(\overline{M}_{g,n}(BG, (g_1), \dots, (g_n)) / \overline{M}_{g,n})$  is the degree of the natural map

$$\overline{M}_{g,n}(BG, (g_1), \dots, (g_n)) \rightarrow \overline{M}_{g,n}.$$

In [JK02] this number is denoted by  $\Omega_g^G((g_1), \dots, (g_n))$ . According to [JK02], *Theorem 3.6* the assignment

$$\Lambda_{g,n}^G(\mathbf{1}_{(g_1)} \otimes \dots \otimes \mathbf{1}_{(g_n)}) := \Omega_g^G((g_1), \dots, (g_n))$$

defines a CohFT.

Let  $\{V_\alpha\}_{\alpha=1}^r$  be the set of isomorphism classes of irreducible representations of  $G$ . Denote by  $\chi_\alpha$  the character of  $V_\alpha$ . For  $1 \leq \alpha \leq r$  define

$$f_\alpha := \frac{\dim V_\alpha}{|G|} \sum_{(g): \text{conjugacy class}} \chi_\alpha(g^{-1}) \mathbf{1}_{(g)}, \quad \nu_\alpha := \left( \frac{\dim V_\alpha}{|G|} \right)^2.$$

According to [JK02], *Proposition 4.2*, we have  $\Lambda_{g,n}^G(f_{\alpha_1} \otimes \dots \otimes f_{\alpha_n}) = 0$  unless  $\alpha_1 = \dots = \alpha_n =: \alpha$ , in which case  $\Lambda_{g,n}^G(f_\alpha \otimes \dots \otimes f_\alpha) = \nu_\alpha^{1-g}$ . It follows from multi-linearity that

$$\begin{aligned} & \left\langle \prod_{j=1}^n \tau_{k_j}(\gamma_j \otimes f_{\alpha_j}) \right\rangle_{g,n,\beta}^{\mathcal{X} \times BG} \\ &= \begin{cases} \nu_\alpha^{1-g} \left\langle \prod_{j=1}^n \tau_{k_j}(\gamma_j) \right\rangle_{g,n,\beta}^{\mathcal{X}} & \text{if } \alpha_1 = \dots = \alpha_n =: \alpha, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This may be formulated in terms of generating functions. Let

$$\{\gamma_j\}_{1 \leq j \leq \text{rank } H^*(I(\mathcal{X}), \mathbb{C})} \subset H^*(I(\mathcal{X}), \mathbb{C})$$

be an additive basis consisting of homogeneous elements. Let  $\{t_{j,k} | 1 \leq j \leq \text{rank } H^*(I(\mathcal{X}), \mathbb{C}), k \geq 0\}$  and  $\{t_{\alpha,j,k} | 1 \leq \alpha \leq r, 1 \leq j \leq \text{rank } H^*(I(\mathcal{X}), \mathbb{C})$ ,

$k \geq 0$  be two sets of variables. We may consider generating functions of genus  $g$  descendant Gromov-Witten invariants:

$$\begin{aligned} & \mathcal{F}_{\mathcal{X} \times BG}^g(\{t_{\alpha,j,k}\}_{1 \leq \alpha \leq r, 1 \leq j \leq \text{rank } H^*(I(\mathcal{X}, \mathbb{C}), k \geq 0); \mathcal{Q}) \\ := & \sum_{\substack{\alpha_1, \dots, \alpha_n; j_1, \dots, j_n; k_1, \dots, k_n \\ n, \beta}} \frac{Q^\beta}{n!} \prod_{l=1}^n t_{\alpha_l, j_l, k_l} \langle \prod_{l=1}^n \tau_{k_l}(\gamma_{j_l} \otimes f_{\alpha_l}) \rangle_{g, n, \beta}^{\mathcal{X} \times BG}; \\ & \mathcal{F}_{\mathcal{X}}^g(\{t_{j,k}\}_{1 \leq j \leq \text{rank } H^*(I(\mathcal{X}, \mathbb{C}), k \geq 0); \mathcal{Q}) \\ := & \sum_{\substack{j_1, \dots, j_n; k_1, \dots, k_n \\ n, \beta}} \frac{Q^\beta}{n!} \prod_{l=1}^n t_{j_l, k_l} \langle \prod_{l=1}^n \tau_{k_l}(\gamma_{j_l}) \rangle_{g, n, \beta}^{\mathcal{X}}. \end{aligned}$$

We thus have

$$(24) \quad \begin{aligned} & \mathcal{F}_{\mathcal{X} \times BG}^g(\{t_{\alpha,j,k}\}_{1 \leq \alpha \leq r, 1 \leq j \leq \text{rank } H^*(I(\mathcal{X}, \mathbb{C}), k \geq 0); \mathcal{Q}) \\ = & \sum_{\alpha=1}^r \nu_\alpha^{1-g} \mathcal{F}_{\mathcal{X}}^g(\{t_{\alpha,j,k}\}_{1 \leq j \leq \text{rank } H^*(I(\mathcal{X}, \mathbb{C}), k \geq 0); \mathcal{Q}). \end{aligned}$$

In terms of the *total descendant potential*  $\mathcal{D} := \exp(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}^g)$ , we have

$$(25) \quad \begin{aligned} & \mathcal{D}_{\mathcal{X} \times BG}^g(\{t_{\alpha,j,k}\}_{1 \leq \alpha \leq r, 1 \leq j \leq \text{rank } H^*(I(\mathcal{X}, \mathbb{C}), k \geq 0); \mathcal{Q}) \\ = & \prod_{\alpha=1}^r \mathcal{D}_{\mathcal{X}}^g(\{t_{\alpha,j,k}\}_{1 \leq j \leq \text{rank } H^*(I(\mathcal{X}, \mathbb{C}), k \geq 0); \mathcal{Q})_{\hbar \rightarrow \hbar/\nu_\alpha}. \end{aligned}$$

This verifies the decomposition conjecture for  $\mathcal{X} \times BG$ .

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