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A monotonicity formula for free boundary surfaces with respect to the unit ball

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We prove a monotonicity identity for compact surfaces with free boundaries inside the boundary of the unit ball in $\mathbb{R}^n$ that have square integrable mean curvature. As one consequence we obtain a Li-Yau type inequality in this setting, thereby generalizing results of Oliveira and Soret [19, Proposition 3], and Fraser and Schoen [11, Theorem 5.4].

In the final section of this paper we derive some sharp geometric inequalities for compact surfaces with free boundaries inside arbitrary orientable support surfaces of class C^2. Furthermore, we obtain a sharp lower bound for the L^1-tangent-point energy of closed curves in $\mathbb{R}^3$ thereby answering a question raised by Strzelecki, Szumańska, and von der Mosel [22].

1. Introduction

The main goal of this paper is to establish a monotonicity formula for compact free boundary surfaces (unless otherwise stated this means 2-dimensional, smooth, embedded) with respect to the unit ball in $\mathbb{R}^n$. The corresponding result for closed, i.e. compact and boundaryless, surfaces was proved by Simon [21]. (See also Kuwert and Schätzle [14] for a generalization to integer rectifiable 2-varifolds with square integrable generalized mean curvature.) For a closed surface Σ, and radii $0 < \sigma < \rho < \infty$ Simon's monotonicity identity reads as follows.

$$g_{x_0}(\rho) - g_{x_0}(\sigma) = \frac{1}{\pi} \int_{\Sigma \cap B_\rho(x_0) \setminus B_\sigma(x_0)} \left| \frac{1}{4}\vec{H} + \frac{(x-x_0)^\perp}{|x-x_0|^2} \right|^2 d\mathcal{H}^2,$$

where

$$g_{x_0}(r) := \frac{\mathcal{H}^2(\Sigma \cap B_r(x_0))}{\pi r^2} + \frac{1}{16\pi} \int_{\Sigma \cap B_r(x_0)} |\vec{H}|^2 \, d\mathcal{H}^2 + \frac{1}{2\pi r^2} \int_{\Sigma \cap B_r(x_0)} \vec{H} \cdot (x - x_0) \, d\mathcal{H}^2.$$

This monotonicity formula plays an important role in the existence proof of surfaces minimizing the Willmore functional [21]. It also yields an alternative proof of the so called Li-Yau inequality [17]. Very recently, Lamm and Schätzle [16] used it to establish a quantitative version of Codazzi's theorem, thereby extending results of De Lellis and Müller [7, 8] to arbitrary codimension.

In this paper we prove a monotonicity identity for compact free boundary surfaces with respect to the unit ball in $\mathbb{R}^n$, i.e. compact surfaces with non-empty boundary meeting the boundary of the unit ball orthogonally. In fact, our results hold in the varifold context (see Section 2 for the precise assumptions).

As a consequence we obtain area bounds, and the existence of the density at *every* point on the surface. As a limiting case of the monotonicity identity we obtain the Li-Yau type inequality

$$2\pi\theta_{max} \leq \frac{1}{4}\int_{\Sigma} |\vec{H}|^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} x \cdot \eta \, d\mathcal{H}^1, \tag{1}$$

where θ_{max} denotes the maximal multiplicity of the surface Σ (see Theorem 4.1).

A special case of (1) (for free boundary CMC surfaces inside the unit ball in $\mathbb{R}^3$) has appeared in a work of Ros and Vergasta [19, Proposition 3], attributing the result to Oliveira and Soret. The proof given in [19] seems to also work for any compact free boundary surface with respect to the unit ball in $\mathbb{R}^n$. Unaware of this result Fraser and Schoen independently established the inequality for free boundary minimal surfaces inside the unit ball in $\mathbb{R}^n$ (see [11, Theorem 5.4]). In this context we also mention the work of Brendle [6] in which the author generalizes the inequality [11, Theorem 5.4] to higher-dimensional free boundary minimal surfaces inside the unit ball in $\mathbb{R}^n$.

The paper is organized as follows. In Section 2 we introduce the notation and describe the setting we work in. In Section 3 we establish the monotonicity formula (Theorem 3.1) and prove the existence of the density (Theorem 3.4). In Section 4 we give some geometric applications that follow from the results of Section 3. Finally, in Section 5 we prove sharp geometric inequalities for compact free boundary surfaces with respect to arbitrary orientable support surfaces of class C^2. We also include a sharp lower bound for the L^1-tangent-point energy of closed curves in $\mathbb{R}^3$.

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2. The setting

We use essentially the same notation as in [14]. Unless stated otherwise we assume that μ is an integer rectifiable 2-varifold in $\mathbb{R}^n$ of compact support $\Sigma := \mathrm{spt}(\mu)$, $\Sigma \cap \partial B \neq \emptyset$, with generalized mean curvature $\vec{H} \in L^2(\mu; \mathbb{R}^n)$ such that

$$\int \mathrm{div}_\Sigma X \, d\mu = -\int \vec{H} \cdot X \, d\mu \tag{2}$$

for all vector fields $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ with $X \cdot \gamma = 0$ on ∂B, where $\gamma(x) = x$ denotes the outward unit normal to B (the open unit ball in $\mathbb{R}^n$). Furthermore, we assume that $\mu(\partial B) = 0$.

It follows from the work of Grüter and Jost [12] that μ has bounded first variation $\delta\mu$. Hence, by Lebesgue's decomposition theorem there exists a Radon measure $\sigma = |\delta\mu| \llcorner Z$ $(Z = \{x \in \mathbb{R}^n : D_\mu |\delta\mu|(x) = +\infty\})$ and a vector field $\eta \in L^1(\sigma; \mathbb{R}^n)$ with $|\eta| = 1$ σ-a.e. such that

$$\delta\mu(X) =^{def} \int \mathrm{div}_\Sigma X \, d\mu = -\int \vec{H} \cdot X \, d\mu + \int X \cdot \eta \, d\sigma \tag{3}$$

for all $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$. It easily follows from (2) that

$$\mathrm{spt}(\sigma) \subset \partial B \quad \text{and} \quad \eta \in \{\pm\gamma\} \ \ \sigma\text{-a.e.}.$$

We shall henceforth refer to such varifolds μ as compact *free boundary varifolds* (with respect to the unit ball).

In case μ is given by a smooth embedded surface Σ (i.e. $\mu = \mathcal{H}^2 \llcorner \Sigma$) η is the outward unit conormal to Σ and $\sigma = \mathcal{H}^1 \llcorner \partial\Sigma$, and we say that Σ is a compact free boundary surface (with respect to the unit ball).

Note that since Σ is compact we may use the position vector field as a test function to obtain

$$2\mu(\mathbb{R}^n) = -\int \vec{H} \cdot x \, d\mu + \int x \cdot \eta \, d\sigma. \tag{4}$$

3. The monotonicity formula

The following monotonicity identity is the free boundary analogue of the monotonicity identity [21, (1.2)], [14, (A.3)].

Theorem 3.1. (monotonicity identity) *For $x_0 \in \mathbb{R}^n$ consider the functions g_{x_0} and $\hat{g}_{x_0}$ given by*

$$g_{x_0}(r) := \frac{\mu(B_r(x_0))}{\pi r^2} + \frac{1}{16\pi}\int_{B_r(x_0)} |\vec{H}|^2 \, d\mu + \frac{1}{2\pi r^2}\int_{B_r(x_0)} \vec{H}\cdot(x-x_0)\, d\mu$$

and

$$\begin{aligned}\hat{g}_{x_0}(r) := {} & g_{\xi(x_0)}(r/|x_0|) \\ & - \frac{1}{\pi(|x_0|^{-1}r)^2}\int_{\hat{B}_r(x_0)} (|x-\xi(x_0)|^2 + P_x(x-\xi(x_0))\cdot x)\, d\mu \\ & - \frac{1}{2\pi(|x_0|^{-1}r)^2}\int_{\hat{B}_r(x_0)} \vec{H}\cdot(|x-\xi(x_0)|^2 x)\, d\mu \\ & + \frac{1}{2\pi}\int_{\hat{B}_r(x_0)} \vec{H}\cdot x\, d\mu + \frac{\mu(\hat{B}_r(x_0))}{\pi},\end{aligned}$$

for $x_0 \neq 0$, and

$$\hat{g}_0(r) = -\frac{\min(r^{-2},1)}{2\pi}\int x\cdot\eta\, d\sigma.$$

Here $\xi(x) := \frac{x}{|x|^2}$ and $\hat{B}_r(x_0) = B_{r/|x_0|}(\xi(x_0))$. Then for any $0 < \sigma < \rho < \infty$ we have

$$\begin{aligned}\frac{1}{\pi}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)} & \left|\frac{1}{4}\vec{H} + \frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu \\ + \frac{1}{\pi}\int_{\hat{B}_\rho(x_0)\setminus\hat{B}_\sigma(x_0)} & \left|\frac{1}{4}\vec{H} + \frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2}\right|^2 d\mu \\ = (g_{x_0}(\rho) + \hat{g}_{x_0}(\rho)) & - (g_{x_0}(\sigma) + \hat{g}_{x_0}(\sigma)),\end{aligned} \tag{5}$$

where the second integral in (5) is to be interpreted as 0 in case $x_0 = 0$. Here $(x-x_0)^\perp := (x-x_0) - P_x(x-x_0)$, where P_x denotes the orthogonal projection onto $T_x\mu$, the approximate tangent space of μ at x. In particular, $g + \hat{g}$ is non-decreasing.

Before we give a proof of the above theorem we note (cf. [9]) that the Neumann Green's function of the disk of radius R in $\mathbb{R}^2$ is, up to a multiplicative and additive constant, given by

$$G(x,y) = \log(|x-y|) + \log\left(\frac{|x|}{R}|\xi(x)-y|\right) + \frac{1}{2R^2}|y|^2,$$

where $\xi(x) := R^2 \frac{x}{|x|^2}$. We have, for $R = 1$,

$$(D_y G)(x, y) = -\frac{x - y}{|x - y|^2} - \frac{\xi(x) - y}{|\xi(x) - y|^2} - y.$$

Proof. (of the theorem) Let $x_0 \in \mathbb{R}^n$. We define

$$Y(x) := \begin{cases} \frac{x - x_0}{|x - x_0|^2} + \frac{x - \xi(x_0)}{|x - \xi(x_0)|^2} - x, & x_0 \neq 0 \\ \frac{x}{|x|^2} - x, & x_0 = 0. \end{cases}$$

For $0 < \sigma < \rho < \infty$ we define the vector field X by

$$X(x) := X_1(x) + X_2(x), \tag{6}$$

where we set

$$X_1(x) := (|x - x_0|_\sigma^{-2} - \rho^{-2})^+ (x - x_0)$$

and

$$X_2(x) := \begin{cases} (|x - \xi(x_0)|_{\sigma |x_0|^{-1}}^{-2} - |x_0|^2 \rho^{-2})^+ (x - \xi(x_0)) \\ \quad -\sigma^{-2} \min(|x_0||x - \xi(x_0)|, \sigma)^2 x \\ \quad +\rho^{-2} \min(|x_0||x - \xi(x_0)|, \rho)^2 x, & x_0 \neq 0 \\ -\sigma^{-2} \min(1, \sigma)^2 x + \rho^{-2} \min(1, \rho)^2 x, & x_0 = 0, \end{cases}$$

and where $|v|_\sigma := \max(|v|, \sigma)$.

First, assume that $x_0 \neq 0$. Then, we set for $r > 0$

$$\hat{B}_r(x_0) = B_{r/|x_0|}(\xi(x_0)).$$

To simplify notation, we shall write B_r and $\hat{B}_r$ instead of $B_r(x_0)$ and $\hat{B}_r(x_0)$, respectively. We may decompose $\mathbb{R}^n$ into a disjoint union over the elements of the family of sets $\mathcal{F}_1$ or $\mathcal{F}_2$ given by

$$\mathcal{F}_1 := \{B_\sigma,\, B_\rho \setminus B_\sigma,\, \mathbb{R}^n \setminus B_\rho\} \quad \text{and} \quad \mathcal{F}_2 := \{\hat{B}_\sigma,\, \hat{B}_\rho \setminus \hat{B}_\sigma,\, \mathbb{R}^n \setminus \hat{B}_\rho\},$$

respectively. For $x \in \partial B$ we have $|x - x_0| = |x_0||x - \xi(x_0)|$. Therefore, ∂B can be decomposed into a disjoint union over the elements of the family of sets $\mathcal{F}_{\partial B}$ given by

$$\mathcal{F}_{\partial B} := \{\partial B \cap (B_\sigma \cap \hat{B}_\sigma),\, \partial B \cap [(B_\rho \setminus B_\sigma) \cap (\hat{B}_\rho \setminus \hat{B}_\sigma)],\, \partial B \setminus (B_\rho \cup \hat{B}_\rho)\},$$

and so we have for $x \in \partial B$

$$X(x) = \begin{cases} (\sigma^{-2} - \rho^{-2})|x - x_0|^2 Y(x), & 0 \le |x - x_0| \le \sigma \\ Y(x) - \rho^{-2}|x - x_0|^2 Y(x), & \sigma < |x - x_0| < \rho \\ 0, & \rho \le |x - x_0|. \end{cases} \tag{7}$$

This implies that X is a valid test vector field in (2) in case $\partial B_\sigma, \partial \hat{B}_\sigma, \partial B_\rho$ and $\partial \hat{B}_\rho$ have μ measure zero, i.e. for a.e. σ and ρ. We compute

$$\int_A \mathrm{div}_\Sigma X_i \, d\mu \quad \text{and} \quad \int_A \vec{H} \cdot X_i \, d\mu$$

for all sets $A \in \mathcal{F}_i$, $i = 1, 2$, separately. We have

$$\begin{aligned} \int \mathrm{div}_\Sigma X_2 \, d\mu &= \sum_{A \in \mathcal{F}_2} \int_A \mathrm{div}_\Sigma X_2 \, d\mu \\ &= 2|x_0|^2 \sigma^{-2} \mu(\hat{B}_\sigma) - 2|x_0|^2 \rho^{-2} \mu(\hat{B}_\rho) \\ &\quad - 2|x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} |x - \xi(x_0)|^2 \, d\mu + 2|x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} |x - \xi(x_0)|^2 \, d\mu \\ &\quad - 2|x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} P_x(x - \xi(x_0)) \cdot x \, d\mu \\ &\quad + 2|x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} P_x(x - \xi(x_0)) \cdot x \, d\mu \\ &\quad + 2 \int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \frac{|(x - \xi(x_0))^\perp|^2}{|x - \xi(x_0)|^4} \, d\mu \\ &\quad - 2\mu(\hat{B}_\rho \setminus \hat{B}_\sigma), \end{aligned}$$

and

$$\begin{aligned} &\int \vec{H} \cdot X_2 \, d\mu = \sum_{A \in \mathcal{F}_2} \int_A \vec{H} \cdot X_2 \, d\mu \\ &= |x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} \vec{H} \cdot (x - \xi(x_0)) \, d\mu - |x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} \vec{H} \cdot (x - \xi(x_0)) \, d\mu \\ &\quad - |x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} \vec{H} \cdot (|x - \xi(x_0)|^2 x) \, d\mu + |x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} \vec{H} \cdot (|x - \xi(x_0)|^2 x) \, d\mu \\ &\quad + \int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \vec{H} \cdot \frac{x - \xi(x_0)}{|x - \xi(x_0)|^2} \, d\mu - \int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \vec{H} \cdot x \, d\mu. \end{aligned}$$

Using the fact that for any vector $v \in \mathbb{R}^n$

$$2\left|\frac{1}{4}\vec{H} + v^{\perp}\right|^2 = \frac{1}{8}|\vec{H}|^2 + 2|v^{\perp}|^2 + \vec{H}\cdot v, \tag{8}$$

where we used Brakke's orthogonality theorem (cf. [5, Chapter 5]), we get that

$$\begin{aligned}
&\int \mathrm{div}_{\Sigma} X_2 \, d\mu + \int \vec{H}\cdot X_2 \, d\mu \\
&= 2|x_0|^2\sigma^{-2}\mu(\hat{B}_\sigma) - 2|x_0|^2\rho^{-2}\mu(\hat{B}_\rho) - \frac{1}{8}\int_{\hat{B}_\rho\setminus\hat{B}_\sigma} |\vec{H}|^2 \, d\mu \\
&\quad + |x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma} \vec{H}\cdot(x-\xi(x_0))\, d\mu - |x_0|^2\rho^{-2}\int_{\hat{B}_\rho} \vec{H}\cdot(x-\xi(x_0))\, d\mu \\
&\quad - 2|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma} |x-\xi(x_0)|^2\, d\mu + 2|x_0|^2\rho^{-2}\int_{\hat{B}_\rho} |x-\xi(x_0)|^2\, d\mu \\
&\quad - 2|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma} P_x(x-\xi(x_0))\cdot x\, d\mu + 2|x_0|^2\rho^{-2}\int_{\hat{B}_\rho} P_x(x-\xi(x_0))\cdot x\, d\mu \\
&\quad - |x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma} \vec{H}\cdot(|x-\xi(x_0)|^2 x)\, d\mu + |x_0|^2\rho^{-2}\int_{\hat{B}_\rho} \vec{H}\cdot(|x-\xi(x_0)|^2 x)\, d\mu \\
&\quad - 2\mu(\hat{B}_\rho\setminus\hat{B}_\sigma) - \int_{\hat{B}_\rho\setminus\hat{B}_\sigma} \vec{H}\cdot x\, d\mu + 2\int_{\hat{B}_\rho\setminus\hat{B}_\sigma} \left|\frac{1}{4}\vec{H} + \frac{(x-\xi(x_0))^{\perp}}{|x-\xi(x_0)|^2}\right|^2 d\mu.
\end{aligned}$$

Similarly, (in fact exactly as in [14]) we get that

$$\begin{aligned}
&\int \mathrm{div}_{\Sigma} X_1 \, d\mu + \int \vec{H}\cdot X_1 \, d\mu \\
&= 2\sigma^{-2}\mu(B_\sigma) - 2\rho^{-2}\mu(B_\rho) - \frac{1}{8}\int_{B_\rho\setminus B_\sigma} |\vec{H}|^2\, d\mu \\
&\quad + \sigma^{-2}\int_{B_\sigma} \vec{H}\cdot(x-x_0)\, d\mu - \rho^{-2}\int_{B_\rho} \vec{H}\cdot(x-x_0)\, d\mu \\
&\quad + 2\int_{B_\rho\setminus B_\sigma} \left|\frac{1}{4}\vec{H} + \frac{(x-x_0)^{\perp}}{|x-x_0|^2}\right|^2 d\mu.
\end{aligned}$$

Since, as mentioned above $X = X_1 + X_2$ is an admissible vector field for (2), we get after rearranging that

$$
\begin{aligned}
&2\int_{B_\rho\setminus B_\sigma}\left|\frac{1}{4}\vec{H}+\frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu+2\int_{\hat{B}_\rho\setminus\hat{B}_\sigma}\left|\frac{1}{4}\vec{H}+\frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2}\right|^2 d\mu\\
&=2\rho^{-2}\mu(B_\rho)-2\sigma^{-2}\mu(B_\sigma)+2|x_0|^2\rho^{-2}\mu(\hat{B}_\rho)-2|x_0|^2\sigma^{-2}\mu(\hat{B}_\sigma)\\
&\quad+\frac{1}{8}\int_{B_\rho\setminus B_\sigma}|\vec{H}|^2\,d\mu+\frac{1}{8}\int_{\hat{B}_\rho\setminus\hat{B}_\sigma}|\vec{H}|^2\,d\mu+2\mu(\hat{B}_\rho\setminus\hat{B}_\sigma)\\
&\quad+\rho^{-2}\int_{B_\rho}\vec{H}\cdot(x-x_0)\,d\mu-\sigma^{-2}\int_{B_\sigma}\vec{H}\cdot(x-x_0)\,d\mu\\
&\quad+|x_0|^2\rho^{-2}\int_{\hat{B}_\rho}\vec{H}\cdot(x-\xi(x_0))\,d\mu-|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma}\vec{H}\cdot(x-\xi(x_0))\,d\mu\\
&\quad-|x_0|^2\rho^{-2}\int_{\hat{B}_\rho}\vec{H}\cdot(|x-\xi(x_0)|^2x)\,d\mu+|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma}\vec{H}\cdot(|x-\xi(x_0)|^2x)\,d\mu\\
&\quad-2|x_0|^2\rho^{-2}\int_{\hat{B}_\rho}P_x(x-\xi(x_0))\cdot x\,d\mu+2|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma}P_x(x-\xi(x_0))\cdot x\,d\mu\\
&\quad-2|x_0|^2\rho^{-2}\int_{\hat{B}_\rho}|x-\xi(x_0)|^2\,d\mu+2|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma}|x-\xi(x_0)|^2\,d\mu\\
&\quad+\int_{\hat{B}_\rho\setminus\hat{B}_\sigma}\vec{H}\cdot x\,d\mu.
\end{aligned}
$$

In view of the definition of g and $\hat{g}$ we may rewrite this as

$$
\begin{aligned}
&\frac{1}{\pi}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\left|\frac{1}{4}\vec{H}+\frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu\\
&\quad+\frac{1}{\pi}\int_{\hat{B}_\rho(x_0)\setminus\hat{B}_\sigma(x_0)}\left|\frac{1}{4}\vec{H}+\frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2}\right|^2 d\mu\\
&=(g_{x_0}(\rho)+\hat{g}_{x_0}(\rho))-(g_{x_0}(\sigma)+\hat{g}_{x_0}(\sigma)).
\end{aligned}
$$

Now, assume that $x_0=0$. Then (7) still holds, and we may again test (2) with X. (Again first for a.e. σ and ρ.) We write B_r instead of $B_r(0)$, and may decompose $\mathbb{R}^n$ into a disjoint union over the elements of the family of sets $\mathcal{F}$ given by

$$\mathcal{F}:=\{B_\sigma,\,B_\rho\setminus B_\sigma,\,\mathbb{R}^n\setminus B_\rho\}.$$

Recalling that

$$X_1(x):=(|x|_\sigma^{-2}-\rho^{-2})^+x$$

and

$$X_2(x):=(\min(\rho^{-2},1)-\min(\sigma^{-2},1))x,$$

we compute

$$\int_A \mathrm{div}_\Sigma X_1\, d\mu \quad \text{and} \quad \int_A \vec{H}\cdot X_1\, d\mu$$

for all sets $A \in \mathcal{F}$. We have

$$\begin{aligned}\int \mathrm{div}_\Sigma X\, d\mu &= \int \mathrm{div}_\Sigma X_1\, d\mu + \int \mathrm{div}_\Sigma X_2\, d\mu\\ &= 2\sigma^{-2}\mu(B_\sigma) - 2\rho^{-2}\mu(B_\rho)\\ &\quad + 2\int_{B_\rho\setminus B_\sigma} \frac{|x^\perp|^2}{|x|^4}\, d\mu\\ &\quad + 2(\min(\rho^{-2},1) - \min(\sigma^{-2},1))\mu(\mathbb{R}^n)\end{aligned}$$

and

$$\begin{aligned}-\int \vec{H}\cdot X\, d\mu &= -\int \vec{H}\cdot X_1\, d\mu - \int \vec{H}\cdot X_2\, d\mu\\ &= -\sigma^{-2}\int_{B_\sigma}\vec{H}\cdot x\, d\mu + \rho^{-2}\int_{B_\rho}\vec{H}\cdot x\, d\mu\\ &\quad - \int_{B_\rho\setminus B_\sigma}\vec{H}\cdot(|x|^{-2}x)\, d\mu\\ &\quad - (\min(\rho^{-2},1) - \min(\sigma^{-2},1))\int \vec{H}\cdot x\, d\mu.\end{aligned}$$

Using again (8) we get

$$\begin{aligned}2\int_{B_\rho\setminus B_\sigma}\left|\frac{1}{4}\vec{H} + \frac{x^\perp}{|x|^2}\right|^2 d\mu &= 2\rho^{-2}\mu(B_\rho) - 2\sigma^{-2}\mu(B_\sigma) + \frac{1}{8}\int_{B_\rho\setminus B_\sigma}|\vec{H}|^2\, d\mu\\ &\quad - 2(\min(\rho^{-2},1) - \min(\sigma^{-2},1))\mu(\mathbb{R}^n)\\ &\quad + \rho^{-2}\int_{B_\rho}\vec{H}\cdot x\, d\mu - \sigma^{-2}\int_{B_\sigma}\vec{H}\cdot x\, d\mu\\ &\quad - (\min(\rho^{-2},1) - \min(\sigma^{-2},1))\int \vec{H}\cdot x\, d\mu.\end{aligned}$$

In view of the definition of g_0 and $\hat{g}_0$, and equation (4) we may rewrite this as

$$\frac{1}{\pi}\int_{B_\rho(0)\setminus B_\sigma(0)}\left|\frac{1}{4}\vec{H} + \frac{x^\perp}{|x|^2}\right|^2 d\mu = (g_0(\rho) + \hat{g}_0(\rho)) - (g_0(\sigma) + \hat{g}_0(\sigma)).$$

This equality which was proved for a.e. σ and ρ is obviously also true for every σ and ρ by an approximation argument. □

Proposition 3.2. *For every $x_0 \in \mathbb{R}^n$ the tilde-density*

$$\widetilde{\theta}^2(\mu, x_0) := \begin{cases} \lim_{r\downarrow 0} \left(\frac{\mu(B_r(x_0))}{\pi r^2} + \frac{\mu(\hat{B}_r(x_0))}{\pi(|x_0|^{-1}r)^2} \right), & x_0 \neq 0, \\ \lim_{r\downarrow 0} \frac{\mu(B_r(0))}{\pi r^2} \end{cases}$$

exists. Moreover, the function $x \mapsto \widetilde{\theta}^2(\mu, x)$ is upper semicontinuous in $\mathbb{R}^n$.

Remark 3.3. Since $\hat{B}_r(x_0) = B_r(x_0)$ for $x_0 \in \partial B$ we have that $\widetilde{\theta}^2(\mu, \cdot) = 2\theta^2(\mu, \cdot)$ on ∂B.

Proof. Set, in case $x_0 \neq 0$,

$$\begin{aligned} R(r) := &\frac{1}{2\pi r^2} \int_{B_r} \vec{H} \cdot (x - x_0)\, d\mu + \frac{1}{2\pi(|x_0|^{-1}r)^2} \int_{\hat{B}_r} \vec{H} \cdot (x - \xi(x_0))\, d\mu \\ &- \frac{1}{\pi(|x_0|^{-1}r)^2} \int_{\hat{B}_r} (|x - \xi(x_0)|^2 + P_x(x - \xi(x_0)) \cdot x)\, d\mu \\ &- \frac{1}{2\pi(|x_0|^{-1}r)^2} \int_{\hat{B}_r} \vec{H} \cdot (|x - \xi(x_0)|^2 x)\, d\mu. \end{aligned}$$

We estimate with Hölder's inequality

$$\begin{aligned} |R(r)| \leq &\left(\frac{\mu(B_r)}{\pi r^2}\right)^{\frac{1}{2}} \left(\frac{1}{4\pi} \int_{B_r} |\vec{H}|^2\, d\mu\right)^{\frac{1}{2}} \\ &+ \left(\frac{\mu(\hat{B}_r)}{\pi(|x_0|^{-1}r)^2}\right)^{\frac{1}{2}} \left(\frac{1}{4\pi} \int_{\hat{B}_r} |\vec{H}|^2\, d\mu\right)^{\frac{1}{2}} \\ &+ \frac{\mu(\hat{B}_r)}{\pi} + d\left(\frac{\mu(\hat{B}_r)}{\pi(|x_0|^{-1}r)^2}\right)^{\frac{1}{2}} \left(\frac{\mu(\hat{B}_r)}{\pi}\right)^{\frac{1}{2}} \\ &+ d\left(\frac{\mu(\hat{B}_r)}{\pi}\right)^{\frac{1}{2}} \left(\frac{1}{4\pi} \int_{\hat{B}_r} |\vec{H}|^2\, d\mu\right)^{\frac{1}{2}}, \end{aligned} \tag{9}$$

where $d := \sup\{|x| : x \in \Sigma\}$. Moreover, for $\varepsilon > 0$

$$\begin{aligned} |R(r)| \leq &\,\varepsilon\frac{\mu(B_r)}{\pi r^2} + \frac{1}{16\pi\varepsilon} \int_{B_r} |\vec{H}|^2\, d\mu + \varepsilon\frac{\mu(\hat{B}_r)}{\pi(|x_0|^{-1}r)^2} + \frac{1}{16\pi\varepsilon} \int_{\hat{B}_r} |\vec{H}|^2\, d\mu \\ &+ \frac{\mu(\hat{B}_r)}{\pi} + \varepsilon\frac{\mu(\hat{B}_r)}{\pi(|x_0|^{-1}r)^2} + \frac{1}{4\varepsilon} d^2 \frac{\mu(\hat{B}_r)}{\pi} + \frac{1}{4\pi} \int_{\hat{B}_r} |\vec{H}|^2\, d\mu + d^2 \frac{\mu(\hat{B}_r)}{4\pi}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\frac{\mu(B_\sigma)}{\pi\sigma^2} + \frac{\mu(\hat{B}_\sigma)}{\pi(|x_0|^{-1}\sigma)^2} \\
&\le \frac{\mu(B_\rho)}{\pi\rho^2} + \frac{\mu(\hat{B}_\rho)}{\pi(|x_0|^{-1}\rho)^2} + \frac{1}{16\pi}\int_{(B_\rho\cup\hat{B}_\rho)\setminus(B_\sigma\cup\hat{B}_\sigma)} |\vec{H}|^2\,d\mu \\
&\quad + \frac{1}{2\pi}\int_{\hat{B}_\rho\setminus\hat{B}_\sigma} \vec{H}\cdot x\,d\mu + \frac{\mu(\hat{B}_\rho\setminus\hat{B}_\sigma)}{\pi} + R(\rho) - R(\sigma).
\end{aligned}$$

Using (9) and

$$\int_{\hat{B}_\rho\setminus\hat{B}_\sigma} \vec{H}\cdot x\,d\mu \le \frac{1}{4}\int_{\hat{B}_\rho} |\vec{H}|^2\,d\mu + d^2\mu(\hat{B}_\rho),$$

we infer, upon redefining $0<\varepsilon<1$, that

$$\begin{aligned}
\frac{\mu(B_\sigma)}{\pi\sigma^2} + \frac{\mu(\hat{B}_\sigma)}{\pi(|x_0|^{-1}\sigma)^2} &\le (1+\varepsilon)\left(\frac{\mu(B_\rho)}{\pi\rho^2} + \frac{\mu(\hat{B}_\rho)}{\pi(|x_0|^{-1}\rho)^2}\right) \\
&\quad + C(\varepsilon)\int_{B_\rho}|\vec{H}|^2\,d\mu + C(\varepsilon)\int_{\hat{B}_\rho}|\vec{H}|^2\,d\mu \\
&\quad + C(\varepsilon)\left(1+d^2\right)\mu(\hat{B}_\rho).
\end{aligned} \tag{10}$$

We infer that

$$\limsup_{\sigma\downarrow 0}\left(\frac{\mu(B_\sigma)}{\pi\sigma^2} + \frac{\mu(\hat{B}_\sigma)}{\pi(|x_0|^{-1}\sigma)^2}\right) < \infty,$$

and in view of (9) that

$$\lim_{r\downarrow 0}|R(r)| = 0.$$

Theorem 3.1 implies that the tilde-density $\widetilde{\theta}^2(\mu,x_0)$ exists, and that

$$\widetilde{\theta}^2(\mu,x_0) = \lim_{\sigma\downarrow 0}(g_{x_0}(\sigma) + \hat{g}_{x_0}(\sigma)).$$

Hence also

$$\begin{aligned}
\widetilde{\theta}^2(\mu,x_0) &\le (1+\varepsilon)\left(\frac{\mu(B_\rho)}{\pi\rho^2} + \frac{\mu(\hat{B}_\rho)}{\pi(|x_0|^{-1}\rho)^2}\right) \\
&\quad + C(\varepsilon)\int_{B_\rho}|\vec{H}|^2\,d\mu + C(\varepsilon)\int_{\hat{B}_\rho}|\vec{H}|^2\,d\mu \\
&\quad + C(\varepsilon)\left(1+d^2\right)\mu(\hat{B}_\rho).
\end{aligned} \tag{11}$$

Now, assume $x_0 = 0$, then set

$$R(r) := \frac{1}{2\pi r^2} \int_{B_r} \vec{H} \cdot x \, d\mu,$$

and we have that

$$|R(r)| \leq \left(\frac{\mu(B_r)}{\pi r^2} \right)^{\frac{1}{2}} \left(\frac{1}{4\pi} \int_{B_r} |\vec{H}|^2 \, d\mu \right)^{\frac{1}{2}} \tag{12}$$

and for $\varepsilon > 0$

$$|R(r)| \leq \varepsilon \frac{\mu(B_r)}{\pi r^2} + \frac{1}{16\pi\varepsilon} \int_{B_r} |\vec{H}|^2 \, d\mu.$$

Hence,

$$\frac{\mu(B_\sigma)}{\pi\sigma^2} \leq (1+\varepsilon) \frac{\mu(B_\rho)}{\pi\rho^2} + C(\varepsilon) \int_{B_\rho} |\vec{H}|^2 \, d\mu + C(\varepsilon)(1 - \min(\rho^{-2}, 1)) \, \sigma(\partial B),$$

where we used that $\mathrm{spt}(\sigma) \subset \partial B$. We infer that

$$\limsup_{\sigma \downarrow 0} \frac{\mu(B_\sigma)}{\pi\sigma^2} < \infty,$$

and in view of (12) that

$$\lim_{r \downarrow 0} |R(r)| = 0.$$

Theorem 3.1 implies that the density $\theta^2(\mu, 0)$ exists, and that

$$\theta^2(\mu, 0) = \lim_{\sigma \downarrow 0} g_0(\sigma),$$

where we used that $\hat{g}_0(r) \equiv -\frac{1}{2\pi} \int x \cdot \eta \, d\sigma$ for all $0 < r \leq 1$. Hence also

$$\begin{aligned} \widetilde{\theta}^2(\mu, 0) = \theta^2(\mu, 0) \leq (1+\varepsilon) \frac{\mu(B_\rho)}{\pi\rho^2} &+ C(\varepsilon) \int_{B_\rho} |\vec{H}|^2 \, d\mu \\ &+ C(\varepsilon)(1 - \min(\rho^{-2}, 1)) \, \sigma(\partial B). \end{aligned} \tag{13}$$

Now, let x_j be a sequence in $\mathbb{R}^n$ such that $x_j \to x_0$. Then (11) and (13) with x_0 replaced by x_j implies

$$
\begin{aligned}
&\frac{\mu(\overline{B}_\rho)}{\pi\rho^2} + \frac{\mu(\overline{\hat{B}}_\rho)}{\pi(|x_0|^{-1}\rho)^2} \geq \limsup_{j\to\infty}\left(\frac{\mu(B_\rho(x_j))}{\pi\rho^2} + \frac{\mu(\hat{B}_\rho(x_j))}{\pi(|x_j|^{-1}\rho)^2}\right)\\
&\geq \frac{1}{1+\varepsilon}\limsup_{j\to\infty}\Big(\widetilde{\theta}^2(\mu,x_j) - C(\varepsilon)\int_{B_\rho(x_j)\cup\hat{B}_\rho(x_j)} |\vec{H}|^2\,d\mu\\
&\qquad - C(\varepsilon)(1+d^2)\mu(\hat{B}_\rho(x_j)) - C(\varepsilon)(1-\min(\rho^{-2},1))\,\sigma(\partial B).\Big)\\
&\geq \frac{1}{1+\varepsilon}\Big(\limsup_{j\to\infty}\widetilde{\theta}^2(\mu,x_j) - C(\varepsilon)\int_{B_{2\rho}(x_0)\cup\hat{B}_{2\rho}(x_0)} |\vec{H}|^2\,d\mu\\
&\qquad - C(\varepsilon)\left(1+d^2\right)\mu(\hat{B}_{2\rho}(x_0)) - C(\varepsilon)(1-\min(\rho^{-2},1))\,\sigma(\partial B).\Big),
\end{aligned}
$$

where we interpret $\hat{B}_r(0) = \emptyset$ and $\frac{\mu(\overline{\hat{B}}_\rho(0))}{\pi(|0|^{-1}\rho)^2} = 0$. Letting $\rho \downarrow 0$ and then $\varepsilon \downarrow 0$ implies the upper semicontinuity. □

Since Σ is compact we may estimate

$$
\begin{aligned}
|R(r)| \leq{}& \frac{1}{2\pi r}\mu(B_r)^{\frac{1}{2}}\left(\int_{B_r}|\vec{H}|^2\,d\mu\right)^{\frac{1}{2}} + \frac{C(d,|x_0|)}{r^2}\mu(\hat{B}_r)\\
&+ \frac{C(d,|x_0|)}{r^2}\mu(\hat{B}_r)^{\frac{1}{2}}\left(\int_{\hat{B}_r}|\vec{H}|^2\,d\mu\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence,

$$\lim_{r\to\infty}|R(r)| = 0.$$

Also, by (3) and (4),

$$
\begin{aligned}
\lim_{r\to\infty}(g_{x_0}(r) + \hat{g}_{x_0}(r)) &= \frac{1}{8\pi}\int|\vec{H}|^2\,d\mu + \frac{1}{2\pi}\int\vec{H}\cdot x\,d\mu + \frac{\mu(\mathbb{R}^n)}{\pi}\\
&= \frac{1}{8\pi}\int|\vec{H}|^2\,d\mu + \frac{1}{2\pi}\int x\cdot\eta\,d\sigma
\end{aligned}
$$

for $x_0 \neq 0$, and

$$\lim_{r\to\infty}(g_0(r) + \hat{g}_0(r)) = \frac{1}{16\pi}\int|\vec{H}|^2\,d\mu.$$

Summarizing, we have proved the following theorem:

Theorem 3.4. *For every $x_0 \in \mathbb{R}^n$ the tilde-density*

$$\widetilde{\theta}^2(\mu, x_0) := \begin{cases} \lim_{r\downarrow 0} \left(\frac{\mu(B_r(x_0))}{\pi r^2} + \frac{\mu(\hat{B}_r(x_0))}{\pi(|x_0|^{-1}r)^2} \right), & x_0 \neq 0, \\ \lim_{r\downarrow 0} \frac{\mu(B_r(0))}{\pi r^2} \end{cases}$$

exists. The function $x \mapsto \widetilde{\theta}^2(\mu, x)$ is upper semicontinuous. Moreover, we have for all $0 < \sigma < \rho < \infty$

1) **(area bound)**

$$\begin{cases} \sigma^{-2}\mu(B_\sigma(x_0)) + (\sigma/|x_0|)^{-2}\mu(\hat{B}_\sigma(x_0)) \leq C, & x_0 \neq 0, \\ \sigma^{-2}\mu(B_\sigma(0)) \leq C, \end{cases}$$

for $C = C(d, \mu(\mathbb{R}^n), \|\vec{H}\|_{L^2})$,

2) **(density bound)**

$$\begin{aligned} \widetilde{\theta}^2(\mu, x_0) \leq (1+\varepsilon)\frac{\mu(B_\rho(x_0))}{\pi\rho^2} &+ (1+\varepsilon)\frac{\mu(\hat{B}_\rho(x_0))}{\pi(|x_0|^{-1}\rho)^2} \\ &+ C(\varepsilon)\int_{B_\rho(x_0)} |\vec{H}|^2 \, d\mu + C(\varepsilon)\int_{\hat{B}_\rho(x_0)} |\vec{H}|^2 \, d\mu \\ &+ C(\varepsilon)\left(1 + d^2\right)\mu(\hat{B}_\rho(x_0)) \end{aligned}$$

and

$$\theta^2(\mu, 0) \leq (1+\varepsilon)\frac{\mu(B_\rho)}{\pi\rho^2} + C(\varepsilon)\int_{B_\rho} |\vec{H}|^2 \, d\mu + C(\varepsilon)(1 - \min(\rho^{-2}, 1))\,\sigma(\partial B),$$

and

3) **(integral identity)**

$$\begin{aligned} &\frac{1}{\pi}\int \left| \frac{1}{4}\vec{H} + \frac{(x-x_0)^\perp}{|x-x_0|^2} \right|^2 d\mu + \frac{1}{\pi}\int \left| \frac{1}{4}\vec{H} + \frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2} \right|^2 d\mu \\ &= \frac{1}{8\pi}\int |\vec{H}|^2 \, d\mu + \frac{1}{2\pi}\int x \cdot \eta \, d\sigma - \widetilde{\theta}^2(\mu, x_0) \qquad \textit{for } x_0 \neq 0, \end{aligned}$$

and

$$\frac{1}{\pi}\int \left| \frac{1}{4}\vec{H} + \frac{x^\perp}{|x|^2} \right|^2 d\mu = \frac{1}{16\pi}\int |\vec{H}|^2 \, d\mu + \frac{1}{2\pi}\int x \cdot \eta \, d\sigma - \theta^2(\mu, 0).$$

4. Applications

The Willmore energy $\mathcal{W}(F)$ of a smooth immersed compact orientable surface $F : \Sigma \to \mathbb{R}^n$ with boundary $\partial\Sigma$ is given by

$$\mathcal{W}(F) := \frac{1}{4}\int_\Sigma H^2 \, d\mathcal{H}^2_{F^*\delta} + \int_{\partial\Sigma} \kappa_g \, d\mathcal{H}^1_{F^*\delta},$$

where κ_g denotes the geodesic curvature of $\partial\Sigma$ as a submanifold of Σ (cf. [20]). By the Gauss equations and the Gauss-Bonnet theorem we have that

$$\mathcal{W}(F) = \frac{1}{2}\int_\Sigma |A^\circ|^2 \, d\mathcal{H}^2_{F^*\delta} + 2\pi\chi(\Sigma),$$

where A° denotes the tracefree part of the second fundamental form, and $\chi(\Sigma)$ denotes the Euler characteristic of Σ. Since $\chi(\Sigma) = 2 - 2g(\Sigma) - r(\Sigma)$, $g(\Sigma) =$ genus of Σ, $r(\Sigma) =$ number of boundary components of Σ, we have that

$$\mathcal{W}(F) \geq 2\pi$$

for topological disks. For free boundary surfaces with respect to the unit ball we have that

$$\kappa_g = D_\tau \eta \cdot \tau = D_\tau(\eta \cdot x \, x) \cdot \tau = x \cdot \eta, \quad (\tau \in T(\partial\Sigma), |\tau| = 1)$$

hence the Willmore energy may be rewritten as

$$\mathcal{W}(F) = \frac{1}{4}\int_\Sigma |\vec{H}|^2 \, d\mathcal{H}^2_{F^*\delta} + \int_{\partial\Sigma} x \cdot \eta \, d\mathcal{H}^1_{F^*\delta}.$$

Motivated by the smooth case we may define the Willmore energy $\mathcal{W}(\mu)$ of a free boundary varifold μ with respect to the unit ball by

$$\mathcal{W}(\mu) = \frac{1}{4}\int |\vec{H}|^2 \, d\mu + \int x \cdot \eta \, d\sigma.$$

Theorem 4.1. *For any immersion $F : \Sigma \to \mathbb{R}^n$ of a compact free boundary surface with respect to the unit ball in $\mathbb{R}^n$ and the image varifold $\mu =$*

$\theta\mathcal{H}^2 \llcorner F(\Sigma)$, where $\theta(x) = \mathcal{H}^0(F^{-1}(\{x\}))$, we have

$$\mathcal{H}^0(F^{-1}(\{x, \xi(x)\})) = \widetilde{\theta}^2(\mu, x) \leq \frac{1}{2\pi}\mathcal{W}(F),$$

in particular

$$W(F) \geq 2\pi, \tag{14}$$

and if

$$W(F) < 4\pi,$$

then F is an embedding. Moreover, equality in (14) *implies that F parametrizes a round spherical cap or a flat unit disk.*

Proof. The inequalities follow from Theorem 3.4. Assume now equality in (14) holds. In particular, we have that F is an embedding, and we may identify Σ with $F(\Sigma)$. The proof now follows from Proposition 4.3 below. □

Remark 4.2. The estimate is sharp, as can be seen by taking the union of two distinct free boundary flat disks.
It is also interesting to note that in case $0 \in \Sigma$ we have the stronger inequality

$$2\pi\theta^2(\mu, 0) + \frac{1}{8}\int |\vec{H}|^2 \, d\mu \leq \mathcal{W}(\mu).$$

Proposition 4.3. *Let $\mu \neq 0$ be a compact integer rectifiable free boundary 2-varifold with respect to ∂B such that*

$$\mathcal{W}(\mu) = 2\pi.$$

Then $\mu = \mathcal{H}^2 \llcorner \Sigma$, where Σ is a round spherical cap or a flat unit disk.

Proof. It follows from Theorem 3.4 that the tilde-density $\widetilde{\theta}^2(\mu, x)$ exists and is ≥ 1 for every $x \in \Sigma$. The assumption together with Theorem 3.4 then yield that $\widetilde{\theta}^2(\mu, x) = 1$ for every $x \in \Sigma$. In particular, we conclude that $\theta^2(\mu, x) = 1$ for every $x \in \Sigma \setminus \partial B$ and $\theta^2(\mu, x) = 1/2$ for every $x \in \Sigma \cap \partial B$. Since $\mu \neq 0$ and Σ is compact the area estimate in Theorem 3.4 implies that there exists a radius $R > 0$ such that $\Sigma \setminus B_R(x) \neq \emptyset$ for all $x \in \Sigma$. Pick any point $x_0 \in \Sigma$, then

$$
\begin{aligned}
&1+\frac{1}{\pi}\int\left|\frac{1}{4}\vec{H}+\frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu+\frac{1}{\pi}\int\left|\frac{1}{4}\vec{H}+\frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2}\right|^2 d\mu\\
&=\frac{1}{2\pi}\mathcal{W}(\mu)=1.
\end{aligned}
$$

We conclude that

$$
\frac{1}{4}\vec{H}(x)+\frac{(x-x_0)^\perp}{|x-x_0|^2}=0 \quad \text{for } \mu\text{-a.e. } x\in\Sigma. \tag{15}
$$

In particular,

$$
|\vec{H}(x)|=4\left|\frac{(x-x_0)^\perp}{|x-x_0|^2}\right|\leq\frac{8}{R} \quad \text{for } \mu\text{-a.e. } x\in\Sigma\setminus B_{\frac{R}{2}}(x_0).
$$

And similarly, picking a second point $x_1\in\Sigma\setminus B_R(x_0)$ we conclude that $|\vec{H}(x)|\leq\frac{8}{R}$ for μ-a.e. $x\in\Sigma\setminus B_{\frac{R}{2}}(x_1)$. Since $B_{\frac{R}{2}}(x_0)\cap B_{\frac{R}{2}}(x_1)=\emptyset$ we have that $|\vec{H}(x)|\leq\frac{8}{R}$ for μ-a.e. $x\in\Sigma$. In particular, $|\vec{H}|\in L^\infty(\mu)$. By Allard's regularity theorem [1], Grüter-Jost's free boundary version [12], and Theorem 3.4 we conclude that Σ is a $C^{1,\alpha}$ manifold with boundary. We consider two cases:

First suppose that Σ is a free boundary minimal surface (cf. [6]). Then writing Σ locally as the graph of a $C^{1,\alpha}$ function elliptic regularity theory (see for example [15]) implies that Σ is smooth. For any given point $y\in\Sigma$ we have that

$$
\frac{(x-y)^{\perp_x}}{|x-y|^2}=0 \quad \text{for } x\in\Sigma\setminus\{y\},
$$

where ${}^{\perp_x}$ stands for the orthogonal projection onto the normal space of Σ at x. In particular, $y-x\in T_x\Sigma$ for all $y\in\partial\Sigma$ and all points $x\in\Sigma\setminus\partial\Sigma$. Hence, $\partial\Sigma$ is contained in a 2-dimensional plane. The maximum principle implies that Σ is itself contained in this plane. Since Σ is compact and $\partial\Sigma\subset\partial B$, Σ must be equal to a flat unit disk.

Now assume that Σ is not minimal. Then there exists a point $x_0\in\Sigma\setminus\partial\Sigma$ such that $\vec{H}(x_0)\neq 0$ and equality holds in (15). After possibly rotating Σ we may assume that $T_{x_0}\Sigma=\operatorname{span}\{e_1,e_2\}$ and that $\vec{H}(x_0)=\frac{2}{r}\,e_3$ for some $r\neq 0$. This implies that for $j=4,\dots,n$

$$
0=\vec{H}(x_0)\cdot e_j=4\frac{(x-x_0)^{\perp_{x_0}}}{|x-x_0|^2}\cdot e_j=4\frac{(x-x_0)_j}{|x-x_0|^2} \tag{16}
$$

for all $x \in \Sigma \setminus \{x_0\}$. (First for μ-almost all points, and by continuity in x of the right hand side of equation (16) for all points.) This implies that $\Sigma \subset x_0 + \mathbb{R}^3 \times \{0\}$. On the other hand,

$$\frac{2}{r} = \vec{H}(x_0) \cdot e_3 = 4\frac{(x - x_0)_3}{|x - x_0|^2},$$

i.e. $\frac{1}{r}|x - x_0|^2 = 2(x - x_0)_3$, or equivalently

$$r^2 = (x - x_0)_1^2 + (x - x_0)_2^2 + ((x - x_0)_3 - r)^2 = |x - (x_0 + re_3)|^2$$

for all $x \in \Sigma \setminus \{x_0\}$, and $\Sigma \subset \partial B_r(x_0 + re_3) \cap \mathbb{R}^3 \times \{0\}$. Since $\partial\Sigma \subset \partial B$ we must have that either $\Sigma = (\partial B_r(x_0 + re_3) \cap \mathbb{R}^3 \times \{0\}) \cap \overline{B}$ or $\Sigma = (\partial B_r(x_0 + re_3) \cap \mathbb{R}^3 \times \{0\}) \setminus B$. □

An immediate corollary of Theorem 4.1 is the following very special case of a Theorem due to Ekholm, White, and Wienholtz [10].

Corollary 4.4. *Any immersed compact free boundary minimal surface with respect to the unit ball of boundary length strictly less that 4π (or equivalently of area strictly less that 2π) must be embedded.*

Remark 4.5. Bourni and Tinaglia [4] have extended the result of Ekholm, White, and Wienholtz to surfaces with small L^p-norm of the mean curvature with $p \geq 2$.

5. Geometric inequalites for free boundary surfaces

In this section we consider free boundary surfaces with respect to an orientable C^2-hypersurface S with outward unit normal γ that meet S from the inside. More precisely, we make the following assumptions.

We assume that μ is an integer rectifiable 2-varifold in $\mathbb{R}^n$ of compact support $\Sigma := \operatorname{spt}(\mu)$, $\Sigma \cap S \neq \emptyset$, with generalized mean curvature $\vec{H} \in L^p(\mu; \mathbb{R}^n)$, $p > 2$, such that

$$\int \operatorname{div}_\Sigma X \, d\mu = -\int \vec{H} \cdot X \, d\mu + \int X \cdot \gamma \, d\sigma \tag{17}$$

for all $X \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, and where $\sigma = |\delta\mu| \llcorner Z$ $(Z = \{x \in \mathbb{R}^n : D_\mu|\delta\mu|(x) = +\infty\})$. By [12, Corollary 3.2] we have that the density

$$\theta^2(\mu, x_0) = \lim_{r \downarrow 0} \frac{\mu(B_r(x_0))}{\pi r^2}$$

exists at every point $x_0 \in \operatorname{spt}(\mu)$, and that $\theta^2(\mu, x_0) \geq 1/2$ for every point $x_0 \in \operatorname{spt}(\sigma)$.

Lemma 5.1. *For every $x_0 \in \mathbb{R}^n$ we have*

$$\lim_{r \downarrow 0} \sigma(B_r(x_0)) = 0.$$

Proof. Let $x_0 \in \operatorname{spt}(\sigma) \subset S$. For $r > 0$ small enough so that the oriented distance function d_S of S is of class C^2. Let $\varphi \in C^1_c(\mathbb{R}^n)$, $0 \leq \varphi \leq 1$, be such that $\varphi = 1$ on $B_r(x_0)$, $\varphi = 0$ outside $B_{2r}(x_0)$, and $|D\varphi| \leq c$ for some constant c independent of r. Testing (17) with $X = -\varphi \, D d_S$ we obtain

$$\begin{aligned}\sigma(B_r(x_0)) &\leq \int \varphi \, d\sigma \leq \int \varphi |D^2 d_S| + |D\varphi| \, d\mu + \int_{B_{2r}(x_0)} |\vec{H}| \, d\mu \\ &\leq \left(C(S) + \frac{c}{r} \right) \mu(B_{2r}(x_0)) + \int_{B_{2r}(x_0)} |\vec{H}| \, d\mu,\end{aligned}$$

which by [12, Theorem 3.4] goes to zero as $r \downarrow 0$. □

We need the following definition.

Definition 5.2 (cf. [2]). (interior and exterior ball curvatures) *The* interior (exterior) ball curvature $\overline{\kappa}(x)$ $(\underline{\kappa}(x))$ *of (S, γ) at $x \in S$ is defined by*

$$\overline{\kappa}(x) := \sup_{y \in S \setminus \{x\}} Z(x, y) \quad \left(\underline{\kappa}(x) := \inf_{y \in S \setminus \{x\}} Z(x, y) \right),$$

where

$$Z(x, y) := \frac{2(x - y) \cdot \gamma(x)}{|x - y|^2}.$$

The ball curvature $\kappa(x)$ *of S at $x \in S$ is defined by* $\kappa(x) := \max\{\overline{\kappa}(x), -\underline{\kappa}(x)\} \geq 0$. *For a subset A of S we set*

$$\overline{\kappa}_A(x) := \sup_{y \in A \setminus \{x\}} Z(x, y) \quad \left(\underline{\kappa}_A(x) := \inf_{y \in A \setminus \{x\}} Z(x, y) \right),$$

and $\kappa(x) := \max\{\overline{\kappa}_A(x), -\underline{\kappa}_A(x)\} \geq 0$.

Remark 5.3. In case $S = \partial\Omega$ for a bounded and convex set Ω the interior (exterior) ball curvature is the curvature of the largest (smallest) ball enclosed by (enclosing) Ω and touching $\partial\Omega$ at x.

Writing S locally as a graph over its tangent plane one easily verifies the following lemma.

Lemma 5.4. *For any compact sets $K_1, K_2 \subset S$ we have*

$$\sup_{K_2} \kappa_{K_1} < \infty.$$

We test equation (17) with $X = \varphi|x - x_0|^{-2}(x - x_0)$, where $\varphi(x) = (|x - x_0|_\sigma^{-2} - \rho^{-2})^+|x - x_0|^2 \geq 0$, and where $x_0 \in S$. We have

$$\int X \cdot \eta \, d\sigma = \sigma^{-2} \int_{B_\sigma} (x - x_0) \cdot \gamma \, d\sigma - \rho^{-2} \int_{B_\rho} (x - x_0) \cdot \gamma \, d\sigma + \int_{B_\rho \setminus B_\sigma} \frac{x - x_0}{|x - x_0|^2} \cdot \gamma \, d\sigma,$$

where the double usage of the symbol σ should not lead to confusion. Then for a.e. $0 < \sigma < \rho < \infty$ we have

$$\begin{aligned} &\frac{1}{\pi} \int_{B_\rho(x_0) \setminus B_\sigma(x_0)} \left| \frac{1}{4}\vec{H} + \frac{(x - x_0)^\perp}{|x - x_0|^2} \right|^2 d\mu \\ &- \frac{1}{4\pi} \int_{B_\rho(x_0) \setminus B_\sigma(x_0)} \frac{2(x - x_0)}{|x - x_0|^2} \cdot \gamma \, d\sigma \\ &= (g_{x_0}(\rho) + b_{x_0}(\rho)) - (g_{x_0}(\sigma) + b_{x_0}(\sigma)), \end{aligned}$$

where

$$b_{x_0}(r) = -\frac{1}{2\pi r^2} \int_{B_r} (x - x_0) \cdot \gamma \, d\sigma.$$

We note that this identity was originally derived in [21] for smooth surfaces. Using Lemma 5.4 and the fact that (by Lemma 5.1)

$$|b_{x_0}(r)| \leq \frac{\sigma(B_r)}{4\pi} \sup_{B_r} \kappa_{\mathrm{spt}(\sigma)} \to 0 \quad \text{as } r \to 0$$

one easily concludes that one can let $\rho \to \infty$ and $\sigma \to 0$ to obtain

$$\begin{aligned} &2\theta^2(\mu, x_0) + \frac{2}{\pi} \int \left| \frac{1}{4}\vec{H} + \frac{(x - x_0)^\perp}{|x - x_0|^2} \right|^2 d\mu \\ &= \frac{1}{8\pi} \int |\vec{H}|^2 \, d\mu + \frac{1}{2\pi} \int \frac{2(x - x_0) \cdot \gamma}{|x - x_0|^2} \, d\sigma. \end{aligned} \tag{18}$$

Even though the identity (18) is well known [21], the geometric interpretation of the boundary term does not seem to have been exploited thus far. The quantity

$$Z(x, x_0) = \frac{2(x - x_0) \cdot \gamma(x)}{|x - x_0|^2}$$

is the curvature of the tangent ball, plane, or ball complement of S at x passing through x_0.

Proposition 5.5. *We have*

$$2\pi \leq \frac{1}{4} \int |\vec{H}|^2 \, d\mu + \int \overline{\kappa}_{\mathrm{spt}(\sigma)} \, d\sigma.$$

Moreover, equality holds if and only if Σ is a round spherical cap or a flat unit disk.

Proof. The inequality follows immediately from (18), the definition of $\overline{\kappa}_{\mathrm{spt}(\sigma)}$, and the fact that the density at a boundary point is at least 1/2. Now assume that equality holds. Then for σ-a.e. $x \in \mathrm{spt}(\sigma)$ we have that

$$\overline{\kappa}_{\mathrm{spt}(\sigma)}(x) = Z(x, y) \quad \text{for all } y \in \mathrm{spt}(\sigma) \setminus \{x\}. \tag{19}$$

Moreover, by (19) we see that $\mathrm{spt}(\sigma)$ must lie on the tangent sphere of S at x. Since this is true for σ-a.e. point $x \in \mathrm{spt}(\sigma)$ there exists a *single* sphere that is the tangent sphere of S at every point $x \in \mathrm{spt}(\sigma)$. After rescaling and translating we are in the situation of Proposition 4.3, which completes the proof. □

Remark 5.6. A weaker, but also sharp, inequality that can be obtained from (18) was observed by Rivière [18, Lemma 1.2].

Lemma 5.7. *Let Ω be a convex domain of class C^2. Then*

$$\sup_{x \in \partial\Omega} \overline{\kappa} = \sup_{v \in T(\partial\Omega), |v|=1} A^{\partial\Omega}(v, v) \quad \text{and} \quad \inf_{x \in \partial\Omega} \underline{\kappa} = \inf_{v \in T(\partial\Omega), |v|=1} A^{\partial\Omega}(v, v),$$

where $A^{\partial\Omega} = \{h_{ij}^{\partial\Omega}\}$ denotes the second fundamental form of $\partial\Omega$ with outward unit normal γ.

Proof. We have

$$\overline{\kappa}(x) \geq \limsup_{y \to x} \frac{2(x-y)\cdot\gamma(x)}{|x-y|^2} = \sup_{v \in T_x\partial\Omega, |v|=1} A^{\partial\Omega}(x)(v,v),$$

which establishes one inequality. Now assume by contradiction that the inequality is strict, i.e.

$$\sup_{\partial\Omega} \overline{\kappa} > \sup_{v \in T_x\partial\Omega, |v|=1} A^{\partial\Omega}(v,v). \tag{20}$$

By (20) we can find two distinct points $\overline{x}, \overline{y} \in \partial\Omega$ such that

$$Z(\overline{x}, \overline{y}) = \sup_{\partial\Omega} \overline{\kappa} =: R^{-1}.$$

By definition of $\overline{\kappa}$ we have that for every $x \in \partial\Omega$

$$B_R(x - R\gamma(x)) \subset \Omega,$$

and since $Z(\overline{x}, \overline{y}) = R^{-1}$ we also have that

$$\overline{y} \in \partial B_R(\overline{x} - R\gamma(\overline{x})). \tag{21}$$

W.l.o.g. we assume that $\overline{x} - R\gamma(\overline{x}) = 0$. Since Ω is convex we have that

$$\Omega \subset \{\overline{x} + x : x \cdot \overline{x} < 0\} \cap \{\overline{y} + x : x \cdot \overline{y} < 0\} =: W.$$

That is, Ω is contained inside the slab or the wedge bounded by its affine tangent spaces at $\overline{x}$ and $\overline{y}$. We consider two cases. First assume that W is a wedge, i.e.

$$P := \operatorname{span}\{\overline{x}, \overline{y}\}$$

is a 2-dimensional subspace of $\mathbb{R}^n$. Then $\Omega \cap P$ is contained inside the cone $W \cap P$. By convexity and by definition of $\sup_{\partial\Omega} \overline{\kappa} = R^{-1}$ we must have that the segment

$$\partial B_R(0) \cap \{x : x \cdot (\gamma(\overline{x}) + \gamma(\overline{y})) \geq 0\} \cap P$$

is completely contained inside $\partial\Omega$, which however contradicts (20). Now, assume that W is a slab, i.e. $\overline{x}$ and $\overline{y}$ are co-linear. Choose a point $z \in \partial\Omega \cap W$.

(If no such point existed, we would have $\Omega = W$, contradicting (20).) Now let

$$P := \operatorname{span}\{\overline{x}, z\}.$$

Arguing similarly to the first case we see that $\partial\Omega$ must contain a circular segment of radius R inside P connecting $\overline{x}$ and z, which again contradicts (20). This establishes the first claim. The proof of the second claim is similar. $\square$

Corollary 5.8. *Suppose $S = \partial\Omega$ for a convex set $\Omega \subset \mathbb{R}^n$ such that $h_{ij}^{\partial\Omega} \leq k\,\delta_{ij}$. Then*

$$2\pi \leq \frac{1}{4}\int |\vec{H}|^2\,d\mu + k\,\sigma(\mathbb{R}^n).$$

Suppose $S = \partial(\mathbb{R}^n \setminus \Omega)$ for a convex set $\Omega \subset \mathbb{R}^n$ such that $h_{ij}^{\partial\Omega} \geq k\,\delta_{ij}$. Then

$$2\pi \leq \frac{1}{4}\int |\vec{H}|^2\,d\mu - k\,\sigma(\mathbb{R}^n).$$

Moreover, equality holds if and only if Σ is a round spherical cap or a flat unit disk.

Remark 5.9. The assumption that $\vec{H} \in L^p(\mu;\mathbb{R}^n)$ with $p > 2$ was only needed to ensure that the singular part σ of the total variation measure $|\delta\mu|$ has no point masses which ensures that the integral

$$\int \frac{2(x-x_0)\cdot\gamma}{|x-x_0|^2}\,d\sigma$$

exists, and to ensure that the density at every boundary point is at least $1/2$. Alternatively, we could have supposed that $p = 2$ and that μ is the image varifold of a C^1-immersion.

Some observations concerning the L^1-tangent-point energy

Integration of (18) yields

$$2\pi \leq \frac{1}{4}\int |\vec{H}|^2\,d\mu + \fint\int \frac{2\operatorname{dist}(x-y, T_x\partial\Omega)}{|x-y|^2}\,d\sigma(x)\,d\sigma(y).$$

We note that in case σ is 1-rectifiable the double integral can be estimated in terms of the so called (cf. [23]) L^1-*tangent-point energy* $\mathcal{E}_1(\sigma)$. By definition we have

$$\mathcal{E}_p(\sigma) := \int\int \frac{1}{R_{tp}(x,y)^p}\, d\sigma(x)\, d\sigma(y),$$

where $R_{tp}(x,y)$ denotes the so called (cf. [23]) *tangent-point radius* of σ at (x,y) given by

$$R_{tp}(x,y) = \frac{|x-y|^2}{2\mathrm{dist}(x-y, T_x\sigma)}.$$

This leads to the following.

Proposition 5.10. *Let Γ be a closed curve in $\mathbb{R}^3$ of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Then*

$$2\pi\mathcal{H}^1(\Gamma) \leq \mathcal{E}_1(\Gamma), \tag{22}$$

with equality only if Γ is a planar, convex curve.

Proof. Let Σ be a compact orientable minimal surface with boundary $\partial\Sigma = \Gamma$. Such a surface may be obtained by solving the Plateau problem. See for example [13] and the references therein. The identity (18) in this context still holds with γ replaced by η, the outward unit conormal of Σ. Integrating the identity (18) over $\partial\Sigma = \Gamma$ yields

$$\begin{aligned} &2\pi\mathcal{H}^1(\Gamma) + 4\int_\Gamma\int_\Sigma \frac{\left|(x-y)^{\perp_x}\right|^2}{|x-y|^4}\, d\mathcal{H}^2(x)\mathcal{H}^1(y) \\ &= \int_\Gamma\int_\Gamma \frac{2(x-y)\cdot\eta(x)}{|x-y|^2}\, d\mathcal{H}^1(x)\, d\mathcal{H}^1(y), \end{aligned}$$

which is no greater than

$$\int_\Gamma\int_\Gamma \frac{2\mathrm{dist}(x-y, T_x\Gamma)}{|x-y|^2}\, d\mathcal{H}^1(x)\, d\mathcal{H}^1(y) = \mathcal{E}_1(\Gamma).$$

This establishes the inequality (22). Now assume that equality holds in (22). Then for any given point $y \in \Gamma$

$$\frac{(x-y)^{\perp_x}}{|x-y|^2} = 0 \quad \text{for } x \in \Sigma \setminus \{y\}.$$

Arguing as in the proof of Proposition 4.3 we see that Σ is contained in a 2-dimensional plane. Since in the equality case we have equalities everywhere in our estimates we also conclude that

$$(x - y) \cdot \eta(x) = \operatorname{dist}(x - y, T_x\Gamma) \geq 0 \quad \text{for all } x, y \in \Gamma.$$

That is, Γ is convex. In particular, Γ must be connected. □

Remark 5.11. After informing Simon Blatt about our inequality (22) he communicated to us the following alternative proof of Proposition 5.10 that works for closed C^1-curves in $\mathbb{R}^n$.

Proof. ([3]) Let $y \in \Gamma$. Choose an arc length parametrization starting at y, i.e. let $c : [0, L] \to \mathbb{R}^n$ be a curve with $c(0) = c(L) = y$, $|c'(s)| \equiv 1$, and $\operatorname{trace}(c) = \Gamma$. We define the curve w by

$$w(s) := \frac{c(s) - c(0)}{|c(s) - c(0)|}.$$

The curve w is of class C^1 on the open interval $(0, L)$, has limits $\lim_{s\downarrow 0} w(s) = c'(0)$ and $\lim_{s\uparrow L} w(s) = -c'(0)$, and maps into the unit sphere $\mathbb{S}^{n-1}$. Thus we have

$$\pi = \lim_{\varepsilon\downarrow 0} \operatorname{dist}(w(\varepsilon), w(L-\varepsilon)) \leq \liminf_{\varepsilon\downarrow 0} \int_\varepsilon^{L-\varepsilon} |w'(s)|\, ds = \int_0^L |w'(s)|\, ds.$$

A straightforward calculation shows that

$$|w'(s)| = \frac{1}{2} \frac{1}{R_{tp}(c(s), c(0))},$$

and therefore

$$2\pi \leq \int_\Gamma \frac{1}{R_{tp}(x, y)}\, d\mathcal{H}^1(x).$$

Integrating over y yields the desired inequality. Note that we have equality if and only if the curve w is a geodesic in $\mathbb{S}^{n-1}$, that is if *and only* if c is planar and convex. □

Applying Hölder's inequality twice we immediately obtain the following.

Corollary 5.12. *Let Γ be a closed curve in $\mathbb{R}^n$ of class C^1. Then for any $p > 1$ we have*

$$2\pi \leq \mathcal{E}_p(\Gamma)^{\frac{1}{p}} \mathcal{H}^1(\Gamma)^{1-\frac{2}{p}}$$

with equality if and only if Γ is a round circle.

Remark 5.13. Corollary 5.12 answers a question raised by Strzelecki, Szumańska, and von der Mosel [22].

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