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Existence of isoperimetric regions in non-compact Riemannian manifolds under Ricci or scalar curvature conditions

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We prove existence of isoperimetric regions for every volume in non-compact Riemannian n-manifolds (M, g), $n \geq 2$, having Ricci curvature $\mathrm{Ric}_g \geq (n-1)k_0 g$ and being C^0-locally asymptotic to the simply connected space form of constant sectional curvature $k_0 \leq 0$; moreover in case $k_0 = 0$ we show that the isoperimetric regions are indecomposable. Our results apply to a large class of physically and geometrically relevant examples: Eguchi-Hanson metric and more generally ALE gravitational instantons, asymptotically hyperbolic Einstein manifolds, Bryant type solitons, etc. Finally, under assumptions on the scalar curvature, we prove existence of isoperimetric regions of small volume.

1. Introduction

If (M, g) is a *compact* Riemannian n-manifold, then standard techniques of geometric measure theory ensure existence of isoperimetric regions (roughly speaking $\Omega \subset M$ is an isoperimetric region if its boundary has least area among the boundaries of regions having the same volume of Ω; for the precise notions see Section 2).

In case M is *non-compact* the question of existence of isoperimetric regions is completely non-trivial and the few known existence results are quite specific. A simple example where existence fails is the right hyperbolic paraboloid M_λ defined by the equation $z = \lambda xy$: here there is no isoperimetric region for any value of the area (see [52] by M. Ritoré). More dramatically, it can happen that isoperimetric regions exist just for *some* value of the area (see [17] where A. Cañete and M. Ritoré perform a complete study of isoperimetry in the case of quadrics of revolution). Nevertheless there are

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some cases when the existence of isoperimetric regions for every volume is known:

1) (M, g) is complete non-compact but its *isometry group acts cocompactly* (see [43] by F. Morgan and D. L. Johnson, [40] by F. Morgan, or [28] by M. Galli and M. Ritoré in the context of sub-Riemannian contact manifolds).

2) (M, g) is connected complete non-compact but with *finite volume* (this is an easy consequence of Theorem 2.1 in [53] by M. Ritoré and C. Rosales).

3) The non-compact non simply connected surfaces constructed in [31] by R. Grimaldi and P. Pansu.

4) In several cases when (M, g) is a *cone*, the isoperimetric regions exist for every volume and are characterized (see [44] by F. Morgan and M. Ritoré, and [53] by M. Ritoré and C. Rosales); for *warped products* see [11] by H. Bray and F. Morgan.

5) If (M, g) is a complete plane with non-negative curvature (see [51] by M. Ritoré).

The lack of compactness in the variational problem is due to the fact that the minimizing sequences might split into a part converging nicely to an isoperimetric region, and in another part of positive volume going to infinity. More precisely, let us recall a fundamental result due to Ritoré and Rosales ([53, Theorem 2.1]) making this statement clear. To this aim, given a Riemannian manifold (M, g) let us say that a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of finite perimeter subsets of M *diverges* if for every compact subset $K \subset\subset M$, there exists $N \in \mathbb{N}$ such that

$$\Omega_k \cap K = \emptyset \quad \forall k \geq N.$$

Theorem 1.1 (Ritoré–Rosales '04). *Let (M^n, g) be a complete connected Riemannian n-manifold. For every minimizing sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of sets of volume v, there exists a finite perimeter set $\Omega \subset M$ and sequences of sets of finite perimeter $\{\Omega_k^c\}_{k\in\mathbb{N}}$, $\{\Omega_k^d\}_{k\in\mathbb{N}}$, with $\Omega_k = \Omega_k^c \cup \Omega_k^d$ and $\Omega_k^c \cap \Omega_k^d = \emptyset$, such that the following hold:*

1) $V(\Omega) \leq v$, $\mathcal{P}(\Omega) \leq I_M(v)$,

2) $V(\Omega_k^c) + V(\Omega_k^d) = v$, $\lim_{k\to\infty}[\mathcal{P}(\Omega_k^c) + \mathcal{P}(\Omega_k^d)] = I_M(v)$,

3) the sequence $\{\Omega_k^d\}_{k\in\mathbb{N}}$ diverges,

4) *there exists a finite perimeter set $\Omega \subset M$ such that, passing to a subsequence $\{k_j\}_{j\in\mathbb{N}}$, $\{\Omega^c_{k_j}\}_{j\in\mathbb{N}}$ converges to Ω in the sense of finite perimeter sets. In particular, $\lim_{j\to\infty}\mathcal{P}(\Omega^c_{k_j}) = \mathcal{P}(\Omega)$ and $\lim_{k_j\to\infty} V(\Omega^c_{k'}) = V(\Omega)$,*

5) *Ω is an isoperimetric region (possibly empty) for the volume it encloses.*

The second author [47] refined this result by analyzing the diverging part of the minimizing sequences using the theory of $C^{m,\alpha}$-pointed convergence of manifolds (see Section 2, for a more comprehensive treatment of this notion of convergence of manifolds see [50]).

The main goal of the present work is to add, to the previous list, a class of manifolds admitting isoperimetric regions *for all volumes*. This is the content of the next theorem, whose proof is achieved by combining the generalized existence of isoperimetric regions developed by the second author in [47] (see Theorem 2.7 and Remark 2.8) and results of F. Morgan and D. L. Johnson in [43] (actually we need a slight generalization to the complete non compact setting, see Proposition 3.2).

Theorem 1.2. *Let $k_0 \in (-\infty, 0]$ and let (M^n, g) be an $n \geq 2$ dimensional complete Riemannian manifold satisfying the following assumptions:*

1) *(M^n, g) is C^0-locally asymptotic to the simply connected n-dimensional space form of constant sectional curvature k_0, i.e., for every diverging sequence of points p_j the sequence of pointed manifolds (M, g, p_j) converges in C^0-pointed topology to $(\mathbb{M}^n_{k_0}, x_0)$ (x_0 is any point in $\mathbb{M}^n_{k_0}$),*

2) *$Ric_g \geq (n-1)k_0 g$.*

Then for every $v > 0$ there exists an isoperimetric region Ω_v of volume v such that

$$\mathcal{P}(\Omega_v) = I_M(v). \tag{1}$$

Moreover if $k_0 = 0$ (i.e. $\mathrm{Ric}_g \geq 0$ and (M, g) is C^0-locally asymptotically euclidean) then the isoperimetric regions are indecomposable.

Roughly speaking, the last sentence says that the isoperimetric regions are connected if $k_0 = 0$. For the precise notion of indecomposability see Section 2; see Definition 2.2 for the concept of C^0-pointed convergence of manifolds. Instead, Equation (1) states that the perimeter of Ω_v (By regularity theory this corresponds to the $(n-1)$-Hausdorff measure of the topological boundary $\partial\Omega$ of the suitable L^1-representative, see Proposition 2.1) achieves the infimum of the perimeters of the finite perimeter subsets of M having fixed enclosed volume v (see Section 2 for the precise notions).

To our knowledge, this is the first existence result valid *for all volumes* and *all dimensions* in the *non-compact* case under just geometric curvature assumptions and asymptotic conditions on the ambient manifold.

Remark 1.1 (Examples). *The class of manifolds satisfying the assumptions of Theorem 1.2 contains many geometrically and physically relevant examples.*

- Eguchi-Hanson and more generally ALE gravitational instantons. *The first example of such manifolds was discovered by Eguchi and Hanson in [23].The Eguchi-Hanson example was then generalized by Gibbons and Hawking [29], see also the work by Hitchin [35]. These metrics constitute the building blocks of the Euclidean quantum gravity theory of Hawking (see [32], [33]). The ALE Gravitational Instantons were classified in 1989 by Kronheimer (see [37], [38]).*

- Asymptotically hyperbolic Einstein manifolds. *Such metrics were studied by Penrose [49]. More recently, asymptotically hyperbolic Einstein metrics have begun to play a central role in the "AdS/CFT correspondence" of quantum field theory (see for example [2]). Regarding the existence of such metrics see for instance the work by Graham and Lee [30], later extended by Biquard [9], Lee [39] and Anderson [3].*

- Bryant type solitons. *The Bryant soliton, discovered by R. Bryant [15], are special but fundamental solutions to the Ricci flow (see for instance the work of Brendle [12] and [13] for higher dimension). Such metrics are complete, have non-negative Ricci curvature (they actually satisfy the stronger condition of having nonnegative curvature operator) and are locally C^0-asymptotically Euclidean. Other soliton examples fitting our assumptions are given by Catino-Mazzieri in [16].*

Remark 1.2 (Isoperimetry and General Relativity). *The existence and the description of isoperimetric regions is an important issue in general relativity. To name a few examples, D. Christodoulou and S.T. Yau proved in [18] that the Hawking mass of isoperimetric spheres is non-negative (provided the scalar curvature of the ambient manifold is non-negative); H. Bray in [10] gave a proof of a special case of the Riemannian Penrose inequality using isoperimetric techniques; G. Huisken in [36] proposed a definition* *of mass using just isoperimetric concepts; H. Bray and F. Morgan in [11] characterized isoperimetric regions in certain spherically symmetric manifolds, in particular in Schwarzshild; M. Eichmair and J. Metzger in [24], [25] and [26] described the isoperimetric regions of large volume in initial data sets for the Einstein's equations; J. Corvino, A. Gerek, M. Greenberg, B. Krummel in [21] characterized the isoperimetric regions in the spatial Reissner-Nordstrom and Schwarzschild anti-de Sitter manifolds; S. Brendle and M. Eichmair in [14] characterized isoperimetric regions in the "doubled" Schwarzschild manifold.*

In order to prove Theorem 1.2 in Section 4, in Section 3 we prove some general properties of the isoperimetric regions and of the isoperimetric profile function of a non-compact Riemannian manifold: a Bishop-type rigidity result (see Proposition 3.2), a second order differential inequality satisfied by the isoperimetric profile function (see Theorem 3.3) which implies that in case the manifold has non negative Ricci curvature then the isoperimetric profile is striclty concave and the isoperimetric regions are indecomposable (see Corollary 3.4).
These properties are classical for compact ambient manifolds (see for instance the work of V. Bayle [6] and the classical paper by F. Morgan and D. L. Johnson [43]), the novelty here is that the ambient manifold is complete non compact.

Using the results of Section 3 we are also able to perform a finer analysis of the minimizing sequences for the perimeter under the volume constraint in case the manifold has non-negative Ricci tensor, $\mathrm{Ric} \geq 0$: roughly speaking either they converge to an isoperimetric region or they diverge, but they cannot split into a converging and a diverging part. For the precise statement see Corollary 3.5.

The previous existence theorem is based on assumptions on the *Ricci* curvature; actually, as the following theorem points out, if one is interested in the existence of isoperimetric regions of *small volume* it is enough to ask conditions on the *scalar* curvature. The central role that the scalar curvature plays in the study of small isoperimetric regions had already been observed by O. Druet in [22] and by the second author in [45] in case of *compact* ambient manifolds (for the extension to the non compact case see [46]); here, using results from [46] by the second author and the expansion of the area of small geodesics spheres, we manage to prove the following theorem in the *non compact* framework.

Theorem 1.3. *Let (M,g) be an $n \geq 2$ dimensional Riemannian manifold of $C^{2,\alpha}$-bounded geometry (in the sense of Definition 4.1) and let $S \in \mathbb{R}$. Suppose that (M,g) satisfies the following assumptions:*

1) *for every $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset\subset M$ such that the scalar curvature*

$$\mathrm{Scal}_g(p) \leq S + \epsilon \quad \forall p \in M \backslash K_\epsilon,$$

2) *there exists a point $\bar{p} \in M$ such that $\mathrm{Scal}_g(\bar{p}) > S$.*

Then there exists a small $v_0 > 0$ such that for any $0 < v \leq v_0$ there exists an isoperimetric region of volume v_0. Moreover such an isoperimetric region is a pseudo-bubble having center of mass in a point $\bar{p}_v$ which is converging, as $v \to 0$, to the set $\mathcal{M}$ of points of global maximum of the scalar curvature Scal_g in the following sense: for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $v \in (0,\delta)$ then $\bar{p}_v$ is contained in the ε-neighbourhood of $\mathcal{M}$, i.e.

$$\inf_{x \in \mathcal{M}} d(\bar{p}_v, x) \leq \varepsilon, \quad \forall v \in (0,\delta),$$

where, of course, d denotes the Riemannian distance.

For the concept of $C^{2,\alpha}$-bounded geometry see Definition 2.4, for the precise notions of pseudo-bubble and center of mass see Definitions 2.10 and 2.11.

Remark 1.3. *Theorem 1.3 is also interesting in connection with Theorem 1.2. Indeed, if the Riemannian manifold (M,g) satisfies the assumptions of Theorem 1.2 and moreover there exists a point $\bar{p} \in M$ where $\mathrm{Scal}_g(\bar{p}) > n(n-1)k_0$ then Theorem 1.2 ensures existence of isoperimetric regions for every volume, and Theorem 1.3 says that these isoperimetric regions, for small volumes, are pseudo-bubbles centered near the points of maximal scalar curvature.*

The article is organized in the following way: in Section 2 we recall the notions and the known results used throughout the paper, in Section 3 we prove some general properties of the isoperimetric profile function of a non-compact Riemannian manifold, finally in Section 4 we prove the main theorems (we also give alternative proofs of Theorem 1.2 using the second variation or using differential inequalities).

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2. Notation and preliminaries

Let (M^n, g) be a smooth complete Riemannian n-manifold. The n-dimensional and k-dimensional Hausdorff measures of a set $\Omega \subset M$ will be denoted by $V(\Omega)$ and $\mathcal{H}^k(\Omega)$, respectively. For any measurable set $\Omega \subset M$ we denote with $\mathcal{P}(\Omega)$ the *perimeter* of Ω defined by

$$\mathcal{P}(\Omega) := \sup\left\{\int_\Omega \mathrm{div}X\, d\mathcal{H}^{n+1} : |X|_\infty \leq 1\right\},$$

where X is a smooth vector field with compact support in M, $|X|_\infty$ is the sup-norm, and $\mathrm{div}X$ is the divergence of X.

A measurable subset $\Omega \subset M$ is of *finite perimeter* if $\mathcal{P}(\Omega) < \infty$ and we denote with τ_M the family of all finite perimeter subsets of M. A finite perimeter set Ω is said *indecomposable* if there do not exist disjoint non-empty finite perimeter sets Ω_1, Ω_2 of positive volume such that $\Omega = \Omega_1 \cup \Omega_2$, and $\mathcal{P}(\Omega) = \mathcal{P}(\Omega_1) + \mathcal{P}(\Omega_2)$ (for more details see [1]).

The *isoperimetric profile* of M is the function $I_M : (0, V(M)) \to [0, +\infty)$ given by

$$I_M(v) := \inf\{\mathcal{P}(\Omega) : \Omega \in \tau_M, V(\Omega) = v\}.$$

If there exists a finite perimeter set $\Omega \in \tau_M$ satisfying $V(\Omega) = v$ and $I_M(v) = \mathcal{P}(\Omega)$, such an Ω will be called an *isoperimetric region*, and we say that $I_M(v)$ is *achieved*. A *minimizing sequence* of sets of volume v is a sequence of finite perimeter sets $\{\Omega_k\}_{k\in\mathbb{N}}$ such that $V(\Omega) = v$ for all $k \in \mathbb{N}$ and

$\lim_{k\to\infty}\mathcal{P}(\Omega_k)=I_M(v)$. Recall that a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ *converges in the finite perimeter sense* to a set Ω if $\chi_{\Omega_k}\to\chi_\Omega$ in $L^1_{loc}(M)$ and $\lim_{k\to\infty}\mathcal{P}(\Omega_k)=\mathcal{P}(\Omega)$, where χ_{Ω_k} and χ_Ω denote the characteristic functions of Ω_k and Ω, respectively.

Of course the existence of isoperimetric regions does not always occur in general, but if an isoperimetric region does exist, then the following classical regularity theorem holds (for the proof see [41]).

Proposition 2.1 (Regularity). *Let (M^n,g) be a smooth Riemannian n-manifold and $v\in]0,Vol(M)[$. Assume that the isoperimetric profile is achieved at v by a finite perimeter set $\Omega\subset M$: $\mathcal{P}(\Omega)=I_M(v)$. Then, up to replacing Ω by an L^1-equivalent finite perimeter set, the following hold:*

1) *the topological boundary $\partial\Omega$ is the disjoint union of a regular part $\partial\Omega_r$ and a singular one $\partial\Omega_s$. For each point $p\in\partial\Omega_r$ there exists a neighborhood $U_p\subset M$ such that $\partial\Omega\cap U_p$ is a smooth hypersurface of constant mean curvature. Moreover the Hausdorff dimension of $\partial\Omega_s$ is less than or equal to $n-8$. In particular it holds $\mathcal{P}(\Omega)=\mathcal{H}^{n-1}(\partial\Omega_r)$ and, if $n<8$, then $\partial\Omega_s=\emptyset$.*
2) *$\partial\Omega_r$ is locally equipped with a smooth outward pointing unit normal vector field ν.*

Now, in order to state the generalized existence theorem of the second author (a tool used throughout the paper), we recall the basics of the theory of $C^{m,\alpha}$-pointed convergence of manifolds (for more details see [50]).

Definition 2.2. *Let $m\in\mathbb{N}$, $\alpha\in[0,1]$, (M,g) be a smooth manifold with a $C^{m,\alpha}$-metric g and let $p\in M$. A sequence of pointed smooth complete Riemannian n-manifolds is said to converge in the pointed $C^{m,\alpha}$-topology to the manifold (M,g,p), and we write $(M_i,g_i,p_i)\to(M,g,p)$, if for every $R>0$ we can find a domain Ω_R with $B(p,R)\subseteq\Omega_R\subseteq M$, a natural number $\nu_R\in\mathbb{N}$, and $C^{m+1,\alpha}$-embeddings $F_{i,R}:\Omega_R\to M_i$ for large $i\geq\nu_R$ such that $B(p_i,R)\subseteq F_{i,R}(\Omega_R)$ and $F^*_{i,R}(g_i)\to g$ on Ω_R in the $C^{m,\alpha}$-topology.*

Remark 2.3. *Whitney proved (see for instance Theorem 2.9 in [34]) that if $\mathcal{A}$ is a C^r differentiable structure on a topological manifold M, $r\geq 1$, then for every $r<s\leq\infty$ there exists a compatible C^s differentiable structure $\mathcal{B}\subset\mathcal{A}$, and $\mathcal{B}$ is unique up to C^s diffeomorphism. Therefore the assumption that the limit manifold M is smooth is somehow unnecessary, but we keep it for simplicity.*

Definition 2.4. *Let $m \in \mathbb{N}$ and $\alpha \in [0,1]$ be given. We say that a complete Riemannian n-manifold (M,g) has $C^{m,\alpha}$-**locally asymptotic bounded geometry** if the following holds:*

1) *There exists a constant $k \in \mathbb{R}$ such that $\mathrm{Ric}_g \geq k(n-1)g$,*
2) *The volume of unit balls is uniformly bounded below: $\inf_{p\in M} V(B(p,1)) \geq v_0 > 0$.*
3) *For every diverging sequence of points $(p_j)_{j\in\mathbb{N}}$ there exists a subsequence $(p_{j_l})_{l\in\mathbb{N}}$ and a pointed smooth n-dimensional Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ such that the sequence of pointed manifolds $(M, g, p_{j_l}) \to (M_\infty, g_\infty, p_\infty)$ in the pointed $C^{m,\alpha}$-topology.*

Remark 2.5. *Notice that if we assume all the C^0-pointed limit manifolds to be isometric to a fixed space form $\mathbb{M}^n_{k_0}$, then clearly the second condition is fulfilled. More generally such condition is satisfied if (M,g) has positive injectivity radius, $Inj_M > 0$; indeed Croke proved (see Proposition 14 in [19] and the discussion at page 2 in [20]; see also [8]) that there exists a constant C_n (depending only on $n = dimM$) such that if $r \leq \frac{Inj_M}{2}$ then $Vol(B(p,r)) \geq C_n r^n$ for every $p \in M$.*

Remark 2.6. *The lower bound on the Ricci tensor $\mathrm{Ric}_g \geq k_0(n-1)g$ is stable under pointed C^0-convergence in the following sense. Let (M,g) be a Riemannian n-manifold with C^0-locally asymptotic bounded geometry in above sense with $\mathrm{Ric}_g \geq k_0(n-1)g$, for some $k_0 \leq 0$. Then any limit manifold (M_∞, g_∞) still satisfies the same lower Ricci curvature bound $\mathrm{Ric}_{g_\infty} \geq k_0(n-1)g_\infty$.*

Such stability property follows directly by the stability of Ricci curvature lower bounds under pointed measured Gromov-Hausdorff convergence (which is clearly implied by the convergence in pointed C^0-topology), see for instance [54, Theorem 29.9 and Theorem 29.25].

Now we recall the generalized existence theorem of the second author (Theorems 1 and 2 in [47]).

Theorem 2.7. *Let (M,g) be a Riemannian n-manifold with C^0-locally asymptotic bounded geometry in the sense of Definition 2.4. Given a positive volume $0 < v < V(M)$, there are a finite number of limit manifolds at infinity such that their disjoint union with M contains an isoperimetric region of volume v and perimeter $I_M(v)$. Moreover, the number of limit manifolds is at worst linear in v.*

Remark 2.8. *The Theorem above was stated in [47] under slightly different assumptions used in the proof just to ensure that the manifolds at infinity are at least $C^{2,\alpha}$ with $C^{1,\alpha}$ metric and that volume and perimeter pass to the limit. These technical requirements are of course satisfied under the assumptions of Theorem 2.7; indeed the C^0-convergence of the metric tensors ensures the convergence of the volume and of the perimeter (this is clear on smooth sets, so by approximation it holds on all finite perimeter sets).*

Recall also the following useful result (see Theorem 3 in [47]).

Theorem 2.9. *Let (M, g) be a complete Riemannian manifold with C^0-locally asymptotic bounded geometry in the sense of Definition 2.4. Then isoperimetric regions are bounded.*

Now we recall the notion of pseudo-bubble which will be useful to study the existence of isoperimetric regions of small volume. Call U_pM the fiber over p of the unit tangent bundle (also called the sphere bundle) of the Riemannian manifold (M, g).

Definition 2.10. *A* ***pseudo-bubble*** *is a hypersurface ΨB embedded in M such that there exists a point $p \in M$ and a function w belonging to $C^{2,\alpha}(U_pM \backsimeq \mathbb{S}^{n-1}, \mathbb{R})$, such that ΨB is the graph of w in normal polar coordinates centered at p, i.e.*

$$\Psi B = \{exp_p(w(\theta)\theta), \quad \theta \in U_pM\}$$

and the mean curvature $H(w) = H_0 + \phi$ of the normal graph is a real constant H_0 plus a function ϕ, where ϕ is a first spherical harmonic function on $U_pM \backsimeq \mathbb{S}^{n-1}$.

Recall also the notion of *Riemannian center of mass.*

Definition 2.11. *Let $\Sigma \subset M$ be an embedded compact hypersurface in the n-dimensional Riemannian manifold (M, g) and let μ the induced volume measure on Σ. Consider the function $\mathcal{E}_\Sigma : M \to [0, +\infty[$*

$$\mathcal{E}_\Sigma(x) := \int_\Sigma d^2(x, y) d\mu(y),$$

where d is the Riemannian distance on M. The ***center of mass*** *of Σ is the minimum point of $\mathcal{E}_\Sigma$ in M.*

Notice that, since Σ is compact, by the Dominated Convergence Theorem, the function $\mathcal{E}_\Sigma$ is continuous and coercive, hence the existence of a minimum is guaranteed. Notice also that although uniqueness of this minimum point does not hold in general, it does in the cases we are interested, namely pseudo-bubbles of small diameter.

3. Some general properties of the isoperimetric profile valid for (possibly non-compact) manifolds of bounded geometry

Some classical properties of the isoperimetric profile for compact manifolds are also valid for non-compact manifolds (sometimes assuming bounded geometry). This section is devoted to prove some of them.

Proposition 3.1. *Let (M,g) be a Riemannian n-manifold with C^0-locally asymptotic bounded geometry in the sense of Definition 2.2. Then the isoperimetric profile $I_M :]0,V(M)[\to [0,+\infty[$ is absolutely continuous and twice differentiable almost everywhere.*

Proof. The continuity of the isoperimetric profile for general complete non compact manifolds is not a trivial issue, and indeed in general it may fail [48]; but under the above assumptions it was proved together with the aforementioned stronger regularity properties in [27, Theorem 1 and Corollary 1]. □

The following theorem is stated and proved in [43, Theorem 3.4-3.5] in case of a compact ambient manifold but, as was pointed out to the authors by C. Rosales, the same proof holds for manifolds which are merely complete.

Proposition 3.2. *Let (M,g) be a smooth, complete, connected n-dimensional Riemannian manifold and assume the following lower bound on the Ricci curvature:*

$$\mathrm{Ric}_{(M,g)}(.,.) \geq (n-1)k_0 g(.,.) \quad k_0 \in \mathbb{R}.$$

For a given volume $v \in]0,V(M)[$, let $\Omega \subset M$ be an isoperimetric region of volume v and perimeter $\mathcal{P}_M(\Omega)$. Then

$$\mathcal{P}_M(\Omega) \leq \mathcal{P}_0(B_v)$$

where $\mathcal{P}_0(B_v)$ is the perimeter of a ball B_v of volume v in the simply connected space form M_0 of constant curvature k_0. Suppose further that for *some volume v_0, $I_M(v_0) = I_{M_0}(v_0)$. Then M has constant sectional curvature k_0 and all metric balls of volume v_0 contained in M are isometric to $B_{M_0}(v_0)$, the metric ball of volume v_0 contained in M_0.*

Proof. Fix a volume $v_0 \in]0, V(M)[$ and take any metric ball $B_M(v_0) \subset M$ of volume v_0. Since the proof of Theorem 3.5 in [43] (stated for compact M) relies only on the part of M *inside* the metric ball, the same argument holds for complete (possibly non-compact) M. It follows that

$$\mathcal{P}_M(B_M(v_0)) \leq \mathcal{P}_0(B_{M_0}(v_0)) \tag{2}$$

where $B_{M_0}(v_0)$ is the ball of volume v_0 in the space form of constant sectional curvature k_0, moreover if equality holds then $B_M(v_0)$ is isometric to $B_{M_0}(v_0)$.

Since $B_M(v_0)$ is a competitor in M between regions of volume v_0 while $B_{M_0}(v_0)$ is minimizer in M_0, we have

$$I_M(v_0) \leq \mathcal{P}_M(B_M(v_0)) \leq \mathcal{P}_0(B_{M_0}(v_0)) = I_{M_0}(v_0).$$

If $I_M(v_0) = I_{M_0}(v_0)$ then all metric balls of volume v_0 contained in M are isometric to $B_{M_0}(v_0)$. Covering of M with metric balls of volume v_0, we conclude that M has constant sectional curvature k_0. $\square$

Now, using geometric differential inequalities, we are going to prove two useful properties of the isoperimetric profile of a manifold with C^0-locally asymptotic bounded geometry. First recall that given a function $f : \mathbb{R} \to \mathbb{R}$, one denotes

$$\overline{D^2 f}(x_0) := \limsup_{h \to 0^+} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2}, \quad \forall x_0 \in \mathbb{R}. \tag{3}$$

The following theorem for compact manifolds is due to Bayle (see [6, Theorem 2.1] or [7, Theorem 2.2.1]).

Theorem 3.3. *Let (M, g) be a Riemannian n-manifold, $n \geq 2$, with C^0-locally asymptotic bounded geometry in the sense of Definition 2.2. Let us assume that*

$$\mathrm{Ric}_g \geq (n-1) k_0\, g, \quad k_0 \in \mathbb{R}.$$

Then the normalized isoperimetric profile $Y_{(M,g)} := I_M^{\frac{n}{n-1}}$ satisfies the following second order differential inequality

$$\forall v > 0 \quad \overline{D^2 Y_{(M,g)}}(v) \leq -n k_0\, Y_{(M,g)}(v)^{\frac{2-n}{n}}, \tag{4}$$

with equality in the case of the simply connected space form of constant sectional curvature k_0; moreover if equality holds for a certain v_0, then all the isoperimetric regions with volume v_0 have totally umbilic boundary along which the Ricci curvature, evaluated on unit normal directions, equals $(n-1)k_0$.

Proof. The proof of Bayle relies on the Ricci curvature lower bound and on the fact that for every $v \in]0, V(M)[$ the isoperimetric profile $I_M(v)$ is achieved by a region with the regularity stated in Proposition 2.1. Using the Generalized Existence Theorem 2.7, for every $v > 0$ there exists an isoperimetric region $D = D_1 \cup_{i=2}^{N} D_{\infty,i}$ where $D_1 \subset M, D_{\infty,i} \subset M_{\infty,i}$ are isoperimetric regions in their own manifolds; moreover D is an isoperimetric region in the manifold given by the disjoint union $M \mathring{\cup}_{i=2}^{\infty} M_{\infty,i}$ for its own volume. Therefore, recalling Remark 2.5 and Remark 2.6, the argument of Bayle (see [7] Theorem 2.2.1 and the computations on pages 45-51; see also [43] Proposition 3.3) can be repeated, bringing the desired conclusion. □

Corollary 3.4. *Let (M, g) be a Riemannian n-manifold, $n \geq 2$, with non negative Ricci curvature:* $\mathrm{Ric}_g \geq 0$. *Assume also that (M, g) has C^0-locally asymptotic bounded geometry in the sense of Definition 2.2. Then the isoperimetric profile function $I_M : [0, V(M)[\to \mathbb{R}$ is strictly concave. In particular, since $I_M(0) = 0$, I_M is strictly subadditive, and this implies that every isoperimetric region is indecomposable.*

Proof. Since in this case $k_0 = 0$, by the differential inequality (4) we get

$$\forall v > 0 \quad \overline{D^2 Y_{(M,g)}}(v) \leq 0,$$

so the function Y_M is concave (see [7] Proposition B.2.1 pag.181). Now observe that $I_M = Y_{(M,g)}^{\frac{n-1}{n}}$; since the exponent is $\frac{n-1}{n} < 1$, it follows that I_M is strictly concave. Of course a continuous strictly concave function on $]0, \infty[$ which is null at 0 is strictly subadditive (for the simple proof see for example [7], Lemma B.1.4). Now let Ω_v be an isoperimetric region in volume $v > 0$. If by contradiction $\Omega_v = \Omega^1 \cup \Omega^2$ is a decomposition of Ω, say $0 < v_1 = V(\Omega_1)$ and $0 < v_2 = V(\Omega_2)$, then by the assumed subadditivity of the isoperimetric profile we reach the contradiction

$$I_M(v) = \mathcal{P}(\Omega_v) = \mathcal{P}(\Omega_1) + \mathcal{P}(\Omega_2) \geq I_M(v_1) + I_M(v_2) > I_M(v). \qquad \square$$

As a second corollary we can prove the following refinement of Theorem 1.1 due to Ritoré-Rosales.

Corollary 3.5. *Let (M^n, g) be a complete connected Riemannian n-manifold with $\mathrm{Ric}_g \geq 0$. Assume also that (M, g) has C^0-locally asymptotic bounded geometry in the sense of Definition 2.2. Fix $v \in]0, V(M)[$ and consider any minimizing sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ for the volume v. Then there exist sequences of sets of finite perimeter $\{\Omega_k^1\}_{k\in\mathbb{N}}$, $\{\Omega_k^2\}_{k\in\mathbb{N}}$ and a subsequence $\{k_j\}_{j\in\mathbb{N}}$ such that*

$$\begin{aligned}&\Omega_k = \Omega_k^1 \cup \Omega_k^2, \quad \Omega_k^1 \cap \Omega_k^2 = \emptyset \textit{ for all } k \in \mathbb{N}, \quad \textit{and}\\ &\lim_{j\to\infty} V(\Omega_{k_j}^1) = v, \ \lim_{j\to\infty} V(\Omega_{k_j}^2) = 0.\end{aligned}$$

Moreover, either $\{\Omega_{k_j}^1\}_{j\in\mathbb{N}}$ diverges or there exists an isoperimetric region $\Omega \subset M$ for the volume v such that $\{\Omega_{k_j}^1\}_{j\in\mathbb{N}}$ converges to Ω in the sense of finite perimeter.

Proof. Applying Theorem 1.1 to the minimizing sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ we obtain the sequences of sets of finite perimeter $\{\Omega_k^c\}_{k\in\mathbb{N}}$, $\{\Omega_k^d\}_{k\in\mathbb{N}}$ with the stated properties. Let

$$v_1 = V(\Omega) = \lim_{j\to\infty} V(\Omega_{k_j}^c) \quad \text{and} \quad v_\infty = v - v_1 = \lim_{j\to\infty} V(\Omega_{k_j}^d).$$

The conclusion follows if we prove that either $v_1 = v$ and $v_\infty = 0$ or that $v_1 = 0$ and $v_\infty = v$.

Assume by contradiction that both v_1 and v_∞ are strictly positive. Combining items 2, 4 and 5 of Theorem 1.1, we infer

$$I_M(v) = \lim_{k\to\infty} [\mathcal{P}(\Omega_k^c) + \mathcal{P}(\Omega_k^d)] = I_M(v_1) + \lim_{k\to\infty} \mathcal{P}(\Omega_k^d).$$

Using the trivial inequality $\mathcal{P}(\Omega_k^d) \geq I_M(V(\Omega_k^d))$ we can continue the above chain of inequalities, obtaining

$$\begin{aligned} I_M(v) &\geq I_M(v_1) + \limsup_{k\to\infty} I_M(V(\Omega_k^d))\\ &\geq I_M(v_1) + \lim_{j\to\infty} I_M(V(\Omega_{k_j}^d))\\ &= I_M(v_1) + I_M(v_\infty), \end{aligned}$$

where, in the last equality, we used that $\lim_{j\to\infty} V(\Omega_{k_j}^d) = v_\infty$ together with the continuity of the isoperimetric profile ensured by Proposition 3.1. Therefore $I_M(v) \geq I_M(v_1) + I_M(v_\infty)$, and if both v_1 and v_∞ are strictly positive this contradicts the strict subadditivity of I_M stated in Corollary 3.4. $\square$

4. Existence and properties of isoperimetric regions

4.1. Proof of Theorem 1.2

Recall that the isoperimetric regions in the n-dimensional simply connected space form $\mathbb{M}^n_{k_0}$ of constant sectional curvature k_0 are metric balls (no matter where the center is). Therefore it is clear that, applying the generalized existence Theorem 2.7 to a manifold (M^n, g) satisfying the assumptions of Theorem 1.2, there is at most one component of the generalized isoperimetric region D placed in the manifold $\mathbb{M}^n_{k_0}$ at infinity. More precisely we have that

$$D = D_1 \cup D_\infty \quad \text{where } D_1 \subset M \text{ and } D_\infty \subset \mathbb{M}^n_{k_0}, \tag{5}$$

with $v = v_1 + v_\infty$ where $v_1 = V_M(D_1)$ and $v_\infty := V_{\mathbb{M}^n_{k_0}}(D_\infty)$. Since both $D_\infty \subset \mathbb{M}^n_{k_0}$ and $D_1 \subset M$ are isoperimetric regions for their own volume, we have that

$$D_\infty \subset \mathbb{M}^n_{k_0} \quad \text{is a metric ball: } D_\infty = B_{\mathbb{M}^n_{k_0}}(v_\infty), \tag{6}$$

$$D_1 \subset M \quad \text{is bounded,} \tag{7}$$

where $B_{\mathbb{M}^n_{k_0}}(v_\infty)$ is a metric ball in $\mathbb{M}^n_{k_0}$ of volume v_∞ and the second statement is ensured by Theorem 2.9.

If $D_\infty = \emptyset$ the conclusion follows, so we can assume that $D_\infty \neq \emptyset$ and $v_\infty := V_{\mathbb{M}^n_{k_0}}(D_\infty) > 0$. Let us consider a metric ball $B_M(v_\infty) \subset M$ of volume v_∞ placed at positive distance from D_1 (this is possible thanks to (7) and the assumed asymptotic behaviour of (M, g)). By formula (2) in the proof of Proposition 3.2, we have that

$$\mathcal{P}_M(B_M(v_\infty)) \leq \mathcal{P}_{\mathbb{M}^n_{k_0}}(D_\infty). \tag{8}$$

Therefore if we move all the volume v_∞ which stays in the manifold at infinity $\mathbb{M}^n_{k_0}$ in any metric ball contained in the original manifold M, we do not increase the perimeter and

$$\begin{aligned} I_M(v) &= \mathcal{P}_M(D_1) + \mathcal{P}_{\mathbb{M}^n_{k_0}}(D_\infty) \\ &\geq \mathcal{P}_M(D_1) + \mathcal{P}_M(B_M(v_\infty)) = \mathcal{P}_M(D_1 \cup B_M(v_\infty)), \end{aligned}$$

where we used Theorem 2.7 for the first equality and the fact that D_1 and $B_M(v_\infty)$ are at positive distance for the final equality.

Since D_1 and $B_M(v_\infty)$ are disjoint, then $V(D_1 \cup B_M(v_\infty)) = v_1 + v_\infty = v$ and we conclude that $D_1 \cup B_M(v_\infty)$ is an isoperimetric region in M for the volume v. Since $v > 0$ was arbitrary the theorem is proved.

The indecomposability of the isoperimetric regions in case $\mathrm{Ric}_g \geq 0$ is ensured by Corollary 3.4. $\square$

4.1.1. An alternative proof of Theorem 1.2 via second variation. Let (M^n, g) satisfy the hypothesis of Theorem 1.2 for some $k_0 \leq 0$, so that $\mathrm{Ric}_g \geq k_0(n-1)g$ and the manifold is C^0-locally asymptotic to the simply connected n-dimensional space form $\mathbb{M}^n_{k_0}$ of constant sectional curvature k_0. For simplicity let us also assume here that M is orientable. For a fixed $v > 0$, we want to show that there exists an isoperimetric region in M of volume v. Theorem 2.7 ensures the existence of a generalized isoperimetric region

$$D = D_1 \cup D_\infty \quad \text{where } D_1 \subset M \text{ and } D_\infty \subset \mathbb{M}^n_{k_0}$$

are such that $V_M(D_1) = v_1$, $V_{\mathbb{M}^n_{k_0}}(D_\infty) = v_\infty$ with $v_1 + v_\infty = v$ and D_1 (resp. D_∞) is an isoperimetric region in M (resp. in $\mathbb{M}^n_{k_0}$) for its own volume v_1 (resp. v_∞).

The structure of the proof is the following: first we show that D is connected, so either $D = D_1$ or $D = D_\infty$, then we prove that it must be $D = D_1$.

STEP 1: $D = D_1 \subset M$ or $D = D_\infty \subset \mathbb{M}^n_{k_0}$.

Let us start assuming $dim(M) = n < 8$, since in this case the proof is very short (later we will explain how to handle the general case). As D is an isoperimetric domain in $M \cup \mathbb{M}^n_{k_0}$, its boundary is a smooth stable CMC hypersurface of finite area, in particular it follows that the mean curvature of ∂D_1 and of ∂D_2 are equal to the same constant H. If by contradiction $D_1 \neq \emptyset$ and $D_\infty \neq \emptyset$, then $0 < \mathcal{P}_M(D_1), \mathcal{P}_{\mathbb{M}^n_{k_0}}(D_\infty) < \infty$ and there exist $c_1, c_\infty \in \mathbb{R}\backslash\{0\}$ such that

$$c_1\, \mathcal{P}_M(D_1) = c_\infty\, \mathcal{P}_{\mathbb{M}^n_{k_0}}(D_\infty). \tag{9}$$

Denote by ν_1 and ν_∞ the outward pointing unit normal vectors to ∂D_1 and ∂D_∞, and consider the variation of D composed by varying D_1 in the direction $c_1\nu_1$ and varying D_∞ in the direction of $-c_\infty\nu_\infty$. Observe that (9) implies that this is an admissible variation (it has null mean value so it is volume preserving to first order). Since the first variation of the perimeter $\mathcal{P}$ of D with respect to null mean value deformations is null (recall that ∂D is union of smooth hypersurfaces of constant mean curvature), it is interesting to compute the second variation of $\mathcal{P}$ in the specified direction. The standard expression of the second variation of the area (see for example [4, Proposition 2.5]) gives

$$\begin{aligned}(10)\quad \mathcal{P}''(D) &= \mathcal{P}''_M(D_1) + \mathcal{P}''_{\mathbb{R}^n}(D_\infty)\\ &= -c_1^2 \int_{\partial D_1} \left(\sigma_1^2 + \mathrm{Ric}_g(\nu_1,\nu_1)\right) - c_\infty^2 \int_{\partial D_\infty} \left(\sigma_\infty^2 + (n-1)k_0\right),\end{aligned}$$

where σ_1 (resp. σ_∞) is the norm of the second fundamental form of ∂D_1 (resp. ∂D_∞). Since by Theorem 2.7 we know that D_∞ is an isoperimetric region in $\mathbb{M}^n_{k_0}$, then D_∞ must be a geodesic sphere of some radius $r > 0$ (see for instance again [4]); but such surface in $\mathbb{M}^n_{k_0}$ is totally umbilic and its mean curvature is given by the expression

$$(11)\qquad H = H_{k_0}(r) := (n-1)\sqrt{|k_0|}\coth\left(\sqrt{|k_0|}r\right) > (n-1)\sqrt{|k_0|}.$$

Therefore $\sigma_\infty^2 = H^2/(n-1) > -k_0(n-1)$ and we have

$$(12)\qquad \int_{\partial D_\infty} \left(\sigma_\infty^2 + (n-1)k_0\right) > 0.$$

On the other hand, since the mean curvature of D_1 equals the mean curvature of D_∞, using again (11) we also have $\sigma_1^2 \geq H^2/n > -k_0(n-1)$; recalling that $\mathrm{Ric}_g(\nu_1,\nu_1) \geq k_0(n-1)$ by assumption, we get

$$(13)\qquad \int_{\partial D_1} \left(\sigma_1^2 + \mathrm{Ric}_g(\nu_1,\nu_1)\right) > 0.$$

Combining (12) and (13) we can conclude that $\mathcal{P}''(D) < 0$, which contradicts the stability of ∂D.

In the case of general dimension n for the ambient manifold M, we can use a classical argument employing cutoff functions. This trick was attributed in [5] (Section 7) to P. Berard, G. Besson, S. Gallot, proved in detail in [7] (Proposition A.0.5) and in [44] (Lemma 3.1), and used for example in [43] in Section 2. The argument is as follows: If $\Omega \subset M$ is an isoperimetric region and $\partial\Omega_r, \partial\Omega_s$ are the regular and the singular part respectively, then by the aforementioned Proposition A.0.5 in [7] there exist a function $\phi_\epsilon : \partial\Omega \to [0,1]$ for each $\epsilon > 0$ with the following properties:

- $\phi_{\epsilon|\partial\Omega_r} \in C^\infty(\partial\Omega_r, [0,1])$, with $\mathrm{spt}(\phi_\epsilon) \subset\subset \partial\Omega_r$;
- $\mathcal{P}(\Omega) - \epsilon \leq \int_{\partial\Omega} \phi_\epsilon d\mathcal{H}_{n-1} \leq \mathcal{P}(\Omega)$
  and $\mathcal{P}(\Omega) - \epsilon \leq \int_{\partial\Omega} \phi_\epsilon^2 d\mathcal{H}_{n-1} \leq \mathcal{P}(\Omega)$,

- $\frac{\int_{\partial\Omega} ||\nabla\phi_\epsilon||^2 d\mathcal{H}_{n-1}}{\int_{\partial\Omega} \phi_\epsilon^2 d\mathcal{H}_{n-1}} \leq \epsilon$.

For small $\epsilon > 0$, consider the standard expression of $\mathcal{P}''(D)$ with variation field $c_{1,\epsilon}\phi_\epsilon\nu_1 - c_{\infty,\epsilon}\nu_\infty$, where ϕ_ϵ is as before with $\Omega = D_1$ and $c_{1,\epsilon}, c_{\infty,\epsilon}$ chosen in such a way that the variation has null mean value. Since (12) was true for any dimension (the isoperimetric problem in space forms is fully solved in any dimension by geodesic spheres [4]), letting $\epsilon \to 0$ in the second variation formula for D_1 gives again (13). As before we conclude that $\mathcal{P}''(D) < 0$, contradicting the stability of ∂D.

An alternative proof of STEP 1 in the case $k_0 = 0$:

Since the enlarged manifold $M \cup \mathbb{R}^n$ has non-negative Ricci curvature, C^0-locally asymptotic bounded geometry and of course the limit manifolds are all isometric to $\mathbb{R}^n$, by Corollary 3.4 the isoperimetric regions are indecomposable, so either $D = D_1 \subset M$ or $D = D_\infty \subset \mathbb{R}^n$.

STEP 2: $D = D_1 \subset M$.

By Step 1, either $D = D_1 \subset M$ or $D = D_\infty \subset \mathbb{M}^n_{k_0}$. If $D = D_1$ we have finished, so we can assume $D = D_\infty$. Recalling that $V(D) = v$, Theorem 2.7 yield that

$$I_M(v) = \mathcal{P}_{M\cup\mathbb{M}^n_{k_0}}(D) = \mathcal{P}_{\mathbb{M}^n_{k_0}}(D_\infty) = I_{\mathbb{M}^n_{k_0}}(v). \tag{14}$$

Now, by Proposition 3.2 all the metric balls $B_M(p_0, v) \subset M$ of volume v are isometric to the geodesic ball $B_{\mathbb{M}^n_{k_0}}(v) \subset \mathbb{M}^n_{k_0}$ of volume v, and in particular the area of the boundaries are equal. We conclude

$$I_M(v) \leq P_M(B_M(p_0, v)) = \mathcal{P}_{\mathbb{M}^n_{k_0}}(B_{\mathbb{M}^n_{k_0}}(v)) = I_{\mathbb{M}^n_{k_0}}(v) = I_M(v)$$

where we used (14) in the last equality. Therefore $I_M(v) = \mathcal{P}_M(B_M(p_0, v))$, i.e. $B_M(p_0, v)$ is an isoperimetric region in volume v for every $p_0 \in M$, and the theorem follows since $v > 0$ was arbitrary. We remark that in the latter case M is locally isometric to $\mathbb{M}^n_{k_0}$. □

4.2. Existence of isoperimetric regions of small volume under assumptions on the scalar curvature

In this section we prove Theorem 1.3, i.e. the existence of isoperimetric regions of small volumes in non-compact manifolds of any dimension under assumptions on the scalar curvature alone. In order to quote results already in the literature (in particular [46]) we need to assume the following stronger (compared to the notion of $C^{m,\alpha}$-locally asymptotic bounded geometry of Definition 2.4 that we used throughout the paper) notion of $C^{m,\alpha}$-bounded geometry (for more details see [50]).

Definition 4.1. *Let (M, g) be a complete Riemannian n-manifold. We say that (M, g) has $C^{m,\alpha}$-bounded geometry if there exist $r, Q > 0$ with the following property: there exist charts $\psi_s : \mathbb{R}^n \supseteq B(0, r) \to U_s \subseteq M$ such that*

(i): For all $p \in M$ there exists U_s such that $B(p, \frac{1}{10}e^{-Q}r) \subseteq U_s$.

(ii): $|D\psi_s| \leq e^Q$ on $B(0, r)$ and $|D\psi_s^{-1}| \leq e^Q$ on U_s.

(iii): $r^{|j|+\alpha}||D^j g_s||_\alpha \leq Q$ for all multi indices j with $0 \leq |j| \leq m$, where g_s is the matrix of functions of metric coefficients in the ψ_s coordinates regarded as a matrix on $B(0, r)$.

PROOF OF THEOREM 1.3: From Lemma 3.6 in [46], there exists a small $v_0 > 0$ such that for any $0 < v < v_0$, the isoperimetric profile $I_M(v)$ is achieved in the enlarged manifold $M \cup M_\infty$, where M_∞ is given by a compactness argument in the theory of pointed convergence of manifolds (see [46], and note that we have changed the notation a bit from that in the cited paper, where M_∞ may coincide with M, while here M denotes the original manifold and M_∞ denotes the manifold we are attaching at infinity in case a minimizing sequence is diverging). From Lemma 3.7, the minimizer is a pseudo-bubble (for the precise notion see Definition 2.10) ΨB_v contained either in M or in M_∞.

We now show that ΨB_v must be contained in M, from which the theorem follows. Suppose by contradiction that $\Psi B_v \subset M_\infty$. Then the expansion of the isoperimetric profile I_{M_∞} for small volume (see formula (2) in Theorem 2 in [46]) is

$$I_M(v) = I_{M_\infty}(v) = c_n v^{\frac{n-1}{n}} \left(1 - \frac{S_\infty}{2n(n+2)} \left(\frac{v}{\omega_n}\right)^{\frac{2}{n}} + o\left(v^{\frac{2}{n}}\right)\right), \tag{15}$$

where c_n is the Euclidean isoperimetric constant, $S_\infty := \sup_{M_\infty} \mathrm{Scal}_{g_\infty}$, and ω_n is the volume of the n dimensional ball of radius 1. Notice that since (M, g) has $C^{2,\alpha}$-bounded geometry, the asymptotic bounds on the curvature of M are transferred to the $C^{2,\alpha}$-limit manifold M_∞, so under our assumptions we have that

$$S_\infty \leq S.$$

On the other hand, taking a point $\bar{p} \in M$ where $\mathrm{Scal}_g(\bar{p}) > S$, the same computations show that on small geodesic balls $B_{\bar{p},v}$ of volume v centered at $\bar{p}$ we have

$$\mathcal{P}_M(B_{\bar{p},v}) = c_n v^{\frac{n-1}{n}} \left(1 - \frac{\mathrm{Scal}_g(\bar{p})}{2n(n+2)} \left(\frac{v}{\omega_n}\right)^{\frac{2}{n}} + o\left(v^{\frac{2}{n}}\right)\right). \tag{16}$$

Since $\mathrm{Scal}_g(\bar{p}) > S \geq S_\infty$, the combination of (15) and (16) gives the contradiction

$$I_M(v) \leq A_M(S_{\bar{p},v}) < I_{M_\infty}(v) = I_M(v), \quad \text{for small } v > 0.$$

Finally, from Theorem 1 in [46], the isoperimetric regions of small fixed volume v are pseudo-bubbles with center of mass $\bar{p}_v$ converging to the set $\mathcal{M}$ of points of global maximum of the scalar curvature Scal_g as $v \to 0$ in the following sense: for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $v \in (0, \delta)$ then $\bar{p}_v$ is contained in the ε-neighbourhood of $\mathcal{M}$, i.e.

$$\inf_{x \in \mathcal{M}} d(\bar{p}_v, x) \leq \varepsilon, \quad \forall v \in (0, \delta),$$

where, of course, d denotes the Riemannian distance.

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