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Rigidity theorems of λ-hypersurfaces

Qing-Ming Cheng, Shiho Ogata, and Guoxin Wei

Since n-dimensional λ-hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ are critical points of the weighted area functional for the weighted volume-preserving variations, in this paper, we study the rigidity properties of complete λ-hypersurfaces. We give some gap theorems of complete λ-hypersurfaces with polynomial area growth. By making use of the generalized maximum principle for $\mathcal{L}$ of λ-hypersurfaces, we prove a rigidity theorem of complete λ-hypersurfaces.

1. Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth n-dimensional immersed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. In [4], Cheng and Wei have introduced notation of the weighted volume-preserving mean curvature flow, which is defined as the following: a family $X(\cdot, t)$ of smooth immersions

$$X(\cdot, t) : M \to \mathbb{R}^{n+1}$$

with $X(\cdot, 0) = X(\cdot)$ is called *a weighted volume-preserving mean curvature flow* if

$$\frac{\partial X(t)}{\partial t} = -\alpha(t)N(t) + \mathbf{H}(t) \tag{1.1}$$

holds, where

$$\alpha(t) = \frac{\int_M H(t)\langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu},$$

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$\mathbf{H}(t) = \mathbf{H}(\cdot, t)$ and $N(t)$ denote the mean curvature vector and the normal vector of hypersurface $M_t = X(M^n, t)$ at point $X(\cdot, t)$, respectively and N is the unit normal vector of $X : M \to \mathbb{R}^{n+1}$. One can prove that the flow (1.1) preserves the weighted volume $V(t)$ defined by

$$V(t) = \int_M \langle X(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu.$$

The weighted area functional $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is defined by

$$A(t) = \int_M e^{-\frac{|X(t)|^2}{2}} d\mu_t,$$

where $d\mu_t$ is the area element of M in the metric induced by $X(t)$. Let $X(t) : M \to \mathbb{R}^{n+1}$ with $X(0) = X$ be a variation of X. If $V(t)$ is constant for any t, we call $X(t) : M \to \mathbb{R}^{n+1}$ *a weighted volume-preserving variation of* X. Cheng and Wei [4] have proved that $X : M \to \mathbb{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for all weighted volume-preserving variations if and only if there exists constant λ such that

$$\langle X, N \rangle + H = \lambda. \tag{1.2}$$

An immersed hypersurface $X(t) : M \to \mathbb{R}^{n+1}$ is called *a λ-hypersurface* if the equation (1.2) is satisfied.

Remark 1.1. *If $\lambda = 0$, then the λ-hypersurface is a self-shrinker of the mean curvature flow. Hence, the λ-hypersurface is a generalization of the self-shrinker.*

Example 1.1. The n-dimensional sphere $S^n(r)$ with radius $r > 0$ is a compact λ-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{n}{r} - r$.

Example 1.2. For $1 \leq k \leq n-1$, the n-dimensional cylinder $S^k(r) \times \mathbb{R}^{n-k}$ with radius $r > 0$ is a complete and non-compact λ-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{k}{r} - r$.

Example 1.3. The n-dimensional Euclidean space $\mathbb{R}^n$ is a complete and non-compact λ-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = 0$.

Definition 1.1. If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional hypersurface in $\mathbb{R}^{n+1}$, we say that M has polynomial area growth if there exist constant C and d

such that for all $r \geq 1$,

$$\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0)\cap X(M)} d\mu \leq Cr^d, \tag{1.3}$$

where $B_r(0)$ is a standard ball in $\mathbb{R}^{n+1}$ with radius r and centered at the origin.

In [4], Cheng and Wei have studied properties of complete λ-hypersurfaces with polynomial area growth. They have proved that a complete and non-compact λ-hypersurface $X : M \to \mathbb{R}^{n+1}$ in the Euclidean space $\mathbb{R}^{n+1}$ has polynomial area growth if and only if $X : M \to \mathbb{R}^{n+1}$ is a complete proper hypersurface. Furthermore, there is a positive constant C such that for $r \geq 1$,

$$\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0)\cap X(M)} d\mu \leq Cr^{n+\frac{\lambda^2}{2}-2\beta-\frac{\inf H^2}{2}}, \tag{1.4}$$

where $\beta = \frac{1}{4}\inf(\lambda - H)^2$.

In this paper, we study the rigidity theorems of complete λ-hypersurfaces. We will prove the following:

Theorem 1.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete λ-hypersurface with polynomial area growth in the Euclidean space $\mathbb{R}^{n+1}$. Then either*

1) *$X : M \to \mathbb{R}^{n+1}$ is isometric to the sphere $S^n(r)$ with radius $r > 0$ or*
2) *$X : M \to \mathbb{R}^{n+1}$ is isometric to the Euclidean space $\mathbb{R}^n$ or*
3) *$X : M \to \mathbb{R}^{n+1}$ is isometric to the cylinder $S^1(r) \times \mathbb{R}^{n-1}$ or*
4) *$X : M \to \mathbb{R}^{n+1}$ is isometric to the cylinder $S^{n-1}(r) \times \mathbb{R}$ or*
5) *$X : M \to \mathbb{R}^{n+1}$ is isometric to the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ for $2 \leq k \leq n-2$ or*
6) *there exists $p \in M$ such that the squared norm S of the second fundamental form and the mean curvature H of $X : M \to \mathbb{R}^{n+1}$ satisfy*

$$\left(\sqrt{S(p) - \frac{H^2(p)}{n}} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H(p)-\lambda)^2 > 1 + \frac{n\lambda^2}{4(n-1)}. \tag{1.5}$$

Corollary 1.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete λ-hypersurface with polynomial area growth in the Euclidean space $\mathbb{R}^{n+1}$. If* *the squared norm S of the second fundamental form and the mean curvature H of $X : M \to \mathbb{R}^{n+1}$ satisfies*

$$\left(\sqrt{S-\frac{H^2}{n}}+|\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2+\frac{1}{n}(H-\lambda)^2\leq 1+\frac{n\lambda^2}{4(n-1)}, \tag{1.6}$$

then $X : M \to \mathbb{R}^{n+1}$ is isometric to one of the following:

1) *the sphere $S^n(r)$ with radius $r \leq \sqrt{n}$,*
2) *the Euclidean space $\mathbb{R}^n$,*
3) *the cylinder $S^1(r) \times \mathbb{R}^{n-1}$ with radius $r > 0$ and $n = 2$ or with radius $r \geq 1$ and $n > 2$,*
4) *the cylinder $S^{n-1}(r) \times \mathbb{R}$ with radius $r > 0$ and $n = 2$ or with radius $r \leq \sqrt{n-1}$ and $n > 2$,*
5) *the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ for $2 \leq k \leq n-2$.*

Remark 1.2. *If $\lambda = 0$, that is, $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete self-shrinker, the condition* (1.6) *becomes $S \leq 1$. Hence, the above theorem is a generalization of Cao and Li [1] and Le and Sesum [11] to λ-hypersurfaces. On study of complete self-shrinkers, see [2], [3], [5], [6], [7, 8], [9, 10].*

Theorem 1.2. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete λ-hypersurface with polynomial area growth in the Euclidean space $\mathbb{R}^{n+1}$. If*

$$\left(H-\frac{\lambda}{2}\right)^2\geq n+\frac{\lambda^2}{4}, \tag{1.7}$$

then $(H - \frac{\lambda}{2})^2 \equiv n + \frac{\lambda^2}{4}$ and M is isometric to the sphere $S^n(r)$ with radius $r > 0$.

For compact case, we give the following:

Proposition 1.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional compact λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If*

$$\left(H-\frac{\lambda}{2}\right)^2\leq n+\frac{\lambda^2}{4}, \tag{1.8}$$

then $(H - \frac{\lambda}{2})^2 \equiv n + \frac{\lambda^2}{4}$ and M is isometric to the sphere $S^n(r)$ with radius $r > 0$.

If we do not assume that $X : M \to \mathbb{R}^{n+1}$ has polynomial area growth, we can prove the following:

Theorem 1.3. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If the squared norm S of the second fundamental form and the mean curvature H of $X : M \to \mathbb{R}^{n+1}$ satisfy*

$$\sup\Big\{\Big(\sqrt{S - \frac{H^2}{n}} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\Big)^2 + \frac{1}{n}(H-\lambda)^2\Big\} < 1 + \frac{n\lambda^2}{4(n-1)}, \tag{1.9}$$

then $X : M \to \mathbb{R}^{n+1}$ is isometric to one of the following:

1) *the sphere $S^n(r)$ with radius $r < \sqrt{n}$,*
2) *the Euclidean space $\mathbb{R}^n$.*

Remark 1.3. *If $\lambda = 0$, that is, $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete self-shrinker, the condition* (1.9) *becomes* $\sup S < 1$. *The above theorem is a generalization of Cheng and Peng [2] to λ-hypersurfaces.*

2. Proofs of theorems for λ-hypersurfaces

In order to prove our theorems, we prepare several fundamental formulas. Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional connected hypersurface of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We choose a local orthonormal frame field $\{e_A\}_{A=1}^{n+1}$ in $\mathbb{R}^{n+1}$ with dual coframe field $\{\omega_A\}_{A=1}^{n+1}$, such that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n. Then we have

$$dX = \sum_i \omega_i e_i, \quad de_i = \sum_j \omega_{ij} e_j + \omega_{in+1} e_{n+1}$$

and

$$de_{n+1} = \sum_i \omega_{n+1i} e_i.$$

We restrict these forms to M^n, then

$$\omega_{n+1} = 0, \quad \omega_{n+1i} = -\sum_{j=1}^n h_{ij}\omega_j, \quad h_{ij} = h_{ji},$$

where h_{ij} denotes component of the second fundamental form of $X : M^n \to \mathbb{R}^{n+1}$. $H = \sum_{j=1}^n h_{jj}$ is the mean curvature and $II = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j N$ is the second fundamental form of $X : M^n \to \mathbb{R}^{n+1}$ with $N = e_{n+1}$. Let

$$h_{ijk} = \nabla_k h_{ij} \quad \text{and} \quad h_{ijkl} = \nabla_l \nabla_k h_{ij},$$

where ∇_j is the covariant differentiation operator. Gauss equations, Codazzi equations and Ricci formulas are given by

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.1}$$

$$h_{ijk} = h_{ikj}, \tag{2.2}$$

$$h_{ijkl} - h_{ijlk} = \sum_{m=1}^{n} h_{im}R_{mjkl} + \sum_{m=1}^{n} h_{mj}R_{mikl}, \tag{2.3}$$

where R_{ijkl} is component of the curvature tensor. For a function F, we denote covariant derivatives of F by $F_{,i} = \nabla_i F$, $F_{,ij} = \nabla_j \nabla_i F$. For λ-hypersurfaces, an elliptic operator $\mathcal{L}$ is given by

$$\mathcal{L}f = \Delta f - \langle X, \nabla f\rangle, \tag{2.4}$$

where Δ and ∇ denote the Laplacian and the gradient operator of the λ-hypersurface, respectively. The $\mathcal{L}$ operator is introduced by Colding and Minicozzi in [6] for self-shrinkers and by Cheng and Wei [4] for λ-hypersurfaces.

The following lemma due to Colding and Minicozzi [6] is needed in order to prove our results.

Lemma 2.1. *Let $X : M \to \mathbb{R}^{n+1}$ be a complete hypersurface. If u, v are C^2 functions satisfying*

$$\int_M (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|)e^{-\frac{|X|^2}{2}} d\mu < +\infty, \tag{2.5}$$

then we have

$$\int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}} d\mu = -\int_M \langle \nabla u, \nabla v\rangle e^{-\frac{|X|^2}{2}} d\mu. \tag{2.6}$$

Proof of Theorem 1.1. Since $\langle X, N\rangle + H = \lambda$, one has

$$H_{,i} = \sum_j h_{ij}\langle X, e_j\rangle, \tag{2.7}$$

$$H_{,ik} = \sum_j h_{ijk}\langle X, e_j\rangle + h_{ik} + \sum_j h_{ij}h_{jk}(\lambda - H).$$

From the Codazzi equation (2.2), we infer

$$\Delta H = \sum_i H_{,ii} = \sum_i H_{,i}\langle X, e_i\rangle + H + S(\lambda - H).$$

Hence, we get

$$\mathcal{L}H = \Delta H - \sum_i \langle X, e_i\rangle H_{,i} = H + S(\lambda - H), \tag{2.8}$$

$$\frac{1}{2}\mathcal{L}H^2 = |\nabla H|^2 + H^2 + S(\lambda - H)H. \tag{2.9}$$

By making use of the Ricci formulas, the Gauss equations and the Codazzi equations, we have

$$\begin{aligned}
\mathcal{L}h_{ij} &= \Delta h_{ij} - \sum_k \langle X, e_k\rangle h_{ijk} \\
&= \sum_k h_{ijkk} - \sum_k \langle X, e_k\rangle h_{ijk} \\
&= (1 - S)h_{ij} + \lambda \sum_k h_{ik}h_{kj}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\frac{1}{2}\mathcal{L}S &= \frac{1}{2}\Big\{\Delta \sum_{i,j}(h_{ij})^2 - \sum_k \langle X, e_k\rangle \big(\sum_{i,j}(h_{ij}^2)\big)_{,k}\Big\} \\
&= \sum_{i,j,k} h_{ijk}^2 + (1 - S)\sum_{i,j} h_{ij}^2 + \lambda \sum_{i,j,k} h_{ik}h_{kj}h_{ji} \\
&= \sum_{i,j,k} h_{ijk}^2 + (1 - S)S + \lambda f_3,
\end{aligned}$$

where $f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}$.
Taking $\{e_1, e_2, \cdots, e_n\}$ such that

$$h_{ij} = \lambda_i \delta_{ij}$$

at a point p and putting $\mu_i = \lambda_i - \frac{H}{n}$, we have

$$\begin{aligned}
f_3 = \sum_i \lambda_i^3 &= \sum_i \left(\mu_i + \frac{H}{n}\right)^3 \\
&= B_3 + \frac{3}{n}HB + \frac{1}{n^2}H^3
\end{aligned}$$

with $B=\sum_i \mu_i^2 = S-\frac{H^2}{n}$ and $B_3=\sum_i \mu_i^3$. Thus, we have

$$\begin{aligned}
\frac{1}{2}\mathcal{L}B &= \frac{1}{2}\mathcal{L}S - \frac{1}{2}\mathcal{L}\frac{H^2}{n} \\
&= \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + (1-S)S + \lambda f_3 - \frac{H^2}{n} - S(\lambda - H)\frac{H}{n} \\
&= \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + (1-B)B - \frac{1}{n}H^2 B + \lambda B_3 + \frac{2}{n}\lambda H B.
\end{aligned}$$

Since

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = B,$$

it is not hard to prove

$$|B_3| \le \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}} \tag{2.10}$$

and the equality holds if and only if at least, $n-1$ of μ_i are equal.

Thus, we have

$$\begin{aligned}
\frac{1}{2}\mathcal{L}B \ge & \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \\
& + (1-B)B - \frac{1}{n}H^2 B - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}B^{\frac{3}{2}} + \frac{2}{n}\lambda H B \\
= & \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \\
& + B\left(1 - B - \frac{1}{n}H^2 - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}B^{\frac{1}{2}} + \frac{2}{n}\lambda H\right) \\
= & \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \\
& + B\left(1 + \frac{n\lambda^2}{4(n-1)} - \frac{1}{n}(H-\lambda)^2 - (\sqrt{B} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2\right).
\end{aligned}$$

Since $X: M \to \mathbb{R}^{n+1}$ has polynomial area growth, according to the results of Cheng and Wei in [4], we can apply the lemma 2.1 to functions 1 and

$B = S - \frac{H^2}{n}$. Hence, we have

$$0 \geq \int_M \Big\{\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2\Big\} e^{-\frac{|X|^2}{2}} d\mu$$
$$+ \int_M B\left(1 + \frac{n\lambda^2}{4(n-1)} - \frac{1}{n}(H-\lambda)^2 - (\sqrt{B} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2\right) e^{-\frac{|X|^2}{2}} d\mu.$$

From the Codazzi equations and the Schwarz inequality, we have

$$\sum_{i,j,k} h_{ijk}^2 = 3\sum_{i\neq k} h_{iik}^2 + \sum_i h_{iii}^2 + \sum_{i\neq j\neq k\neq i} h_{ijk}^2, \quad \frac{1}{n}|\nabla H|^2 \leq \sum_{i,k} h_{iik}^2,$$
$$\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \geq 2\sum_{i\neq k} h_{iik}^2 + \sum_{i\neq j\neq k\neq i} h_{ijk}^2 \geq 0$$

and the equality holds if and only if $h_{ijk} = 0$ for any i, j, k. Therefore, we get either $B \equiv 0$ and $X : M \to \mathbb{R}^{n+1}$ is totally umbilical; or there exists $p \in M$ such that

$$\left(\sqrt{S(p) - \frac{H^2(p)}{n}} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H(p)-\lambda)^2 > 1 + \frac{n\lambda^2}{4(n-1)}; \tag{2.11}$$

or for any point of M

$$\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 = 0,$$
$$\left(\sqrt{S - \frac{H^2}{n}} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2 = 1 + \frac{n\lambda^2}{4(n-1)}.$$

Hence, we know that the second fundamental form is parallel, $X : M \to \mathbb{R}^{n+1}$ is an isoparametric complete hypersurface. If $\lambda = 0$, then $X : M \to \mathbb{R}^{n+1}$ is isometric to the sphere $S^n(\sqrt{n})$, the Euclidean space $\mathbb{R}^n$, the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$. If $\lambda \neq 0$, then we have from (2.10) that the number of the distinct principal curvatures is two and one of them is simple, $X : M \to \mathbb{R}^{n+1}$ is isometric to the sphere $S^n(r)$, the Euclidean space $\mathbb{R}^n$, the cylinder $S^1(r) \times \mathbb{R}^{n-1}$, the cylinder $S^{n-1}(r) \times \mathbb{R}$. The proof of the theorem 1.1 is completed. $\square$

Proof of Corollary 1.1. For $S^n(r)$, we have

$$H = \frac{n}{r}, \quad S = \frac{n}{r^2}, \quad \lambda = H - r = \frac{n - r^2}{r},$$

then

$$\begin{aligned}
(2.12) \qquad & \left(\sqrt{S - \frac{H^2}{n}} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2 - 1 - \frac{n\lambda^2}{4(n-1)} \\
&= -\frac{(n-r^2)^2}{nr^2} + \frac{r^2}{n} - 1 \\
&= \frac{1}{r^2}(r^2 - n).
\end{aligned}$$

For $S^1(r) \times \mathbb{R}^{n-1}$, we have

$$H = \frac{1}{r}, \quad S = \frac{1}{r^2}, \quad \lambda = H - r = \frac{1 - r^2}{r},$$

then

$$\begin{aligned}
(2.13) \qquad & \left(\sqrt{S - \frac{H^2}{n}} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2 - 1 - \frac{n\lambda^2}{4(n-1)} \\
&= \frac{n-2}{nr^2}(1 - r^2 + |1 - r^2|).
\end{aligned}$$

For $S^{n-1}(r) \times \mathbb{R}$, we have

$$H = \frac{n-1}{r}, \quad S = \frac{n-1}{r^2}, \quad \lambda = H - r = \frac{n - 1 - r^2}{r},$$

then

$$\begin{aligned}
(2.14) \qquad & \left(\sqrt{S - \frac{H^2}{n}} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2 - 1 - \frac{n\lambda^2}{4(n-1)} \\
&= \frac{n-2}{nr^2}(r^2 - (n-1) + |r^2 - (n-1)|).
\end{aligned}$$

Combining the theorem 1.1, (2.12), (2.13) and (2.14), we can finish the proof of the Corollary 1.1. □

Proof of Theorem 1.2. By a direct calculation, one obtains

$$\begin{aligned}\frac{1}{2}\Delta|X|^2 &=< \Delta X, X > + \sum_i < X_{,i}, X_{,i} > \\ &= H < N, X > + n \\ &= n + \frac{\lambda^2}{4} - \left(H - \frac{\lambda}{2}\right)^2 .\end{aligned} \tag{2.15}$$

Since the assumption of polynomial area growth, we have

$$\int_M (\Delta|X|^2) e^{-\frac{|X|^2}{2}} d\mu < +\infty, \quad \int_M |\nabla|X|^2|^2 e^{-\frac{|X|^2}{2}} d\mu < +\infty,$$

then we can apply the lemma 2.1 to function 1 and $|X|^2$ and obtain

$$\begin{aligned}\frac{1}{4}\int_M |\nabla|X|^2|^2 e^{-\frac{|X|^2}{2}} d\mu &= \frac{1}{2}\int_M (\Delta|X|^2) e^{-\frac{|X|^2}{2}} d\mu \\ &= \int_M \left(n + \frac{\lambda^2}{4} - \left(H - \frac{\lambda}{2}\right)^2\right) e^{-\frac{|X|^2}{2}} d\mu.\end{aligned}$$

From $(H - \frac{\lambda}{2})^2 \geq n + \frac{\lambda^2}{4}$, we get

$$\left(H - \frac{\lambda}{2}\right)^2 = n + \frac{\lambda^2}{4}, \quad < X, X >= r^2, \tag{2.16}$$

namely, M is isometric to the sphere $S^n(r)$ with radius $r > 0$. It completes the proof of the theorem 1.2. □

Proof of Proposition 1.1. Integrating (2.15) over M and using the Stokes formula, one concludes

$$\int_M \left(n + \frac{\lambda^2}{4} - \left(H - \frac{\lambda}{2}\right)^2\right) d\mu = 0, \tag{2.17}$$

then it follows from $(H - \frac{\lambda}{2})^2 \leq n + \frac{\lambda^2}{4}$ that

$$\left(H - \frac{\lambda}{2}\right)^2 = n + \frac{\lambda^2}{4} \tag{2.18}$$

and M is isometric to the sphere $S^n(r)$ with radius $r > 0$. It completes the proof of the proposition 1.1. □

By making use of the same assertions as in Cheng and Peng [2], we know that the following generalized maximum principle holds.

Theorem 2.1. (*Generalized maximum principle for $\mathcal{L}$-operator*) *Let $X : M^n \to \mathbb{R}^{n+1}$ be a complete λ-hypersurface with Ricci curvature bounded from below. Let f be any C^2-function bounded from above on this λ-hypersurface. Then, there exists a sequence of points $\{p_k\} \subset M$, such that*

$$\begin{aligned} &\lim_{k\to\infty} f(X(p_k)) = supf, \\ &\lim_{k\to\infty} |\nabla f|(X(p_k)) = 0, \\ &\limsup_{k\to\infty} \mathcal{L}f(X(p_k)) \leq 0. \end{aligned} \tag{2.19}$$

Proof of Theorem 1.3. From the proof in the theorem 1.1, we have

$$\begin{aligned} \frac{1}{2}\mathcal{L}B \geq & \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \\ & + B\left(1 + \frac{n\lambda^2}{4(n-1)} - \frac{1}{n}(H-\lambda)^2 - (\sqrt{B} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2\right) \end{aligned}$$

and

$$\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \geq 2\sum_{i\neq k} h_{iik}^2 + \sum_{i\neq j\neq k\neq i} h_{ijk}^2 \geq 0.$$

Hence, we obtain

$$\frac{1}{2}\mathcal{L}B \geq B\left(1 + \frac{n\lambda^2}{4(n-1)} - \frac{1}{n}(H-\lambda)^2 - (\sqrt{B} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2\right).$$

Since

$$\sup\left\{(\sqrt{S - \frac{H^2}{n}} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2 + \frac{1}{n}(H-\lambda)^2\right\} < 1 + \frac{n\lambda^2}{4(n-1)},$$

we know H^2 and S are bounded. Hence, from the Gauss equations, we infer that the Ricci curvature is bounded from below. Applying the generalized maximum principle for $\mathcal{L}$ of λ-hypersurfaces to function B, there exists a sequence of points $\{p_k\} \subset M$ such that

$$0 \geq \sup B \left(1 + \frac{n\lambda^2}{4(n-1)} - \sup\left\{\frac{1}{n}(H-\lambda)^2 + (\sqrt{B} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2\right\}\right).$$

Hence, $\sup B = 0$, that is, $S \equiv \frac{H^2}{n}$. It follows from (2.12) that $X : M \to \mathbb{R}^{n+1}$ is isometric to

1) the sphere $S^n(r)$ with radius $r < \sqrt{n}$ or
2) the Euclidean space $\mathbb{R}^n$.

It completes the proof of the theorem 1.3. □

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Department of Applied Mathematics
Faculty of Sciences, Fukuoka University
814-0180, Fukuoka, Japan
E-mail address: cheng@fukuoka-u.ac.jp

Department of Applied Mathematics
Graduate School of Sciences, Fukuoka University
814-0180, Fukuoka, Japan

School of Mathematical Sciences
South China Normal University, 510631, Guangzhou, China
E-mail address: weiguoxin@tsinghua.org.cn

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