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A new open form of the weak maximum principle and geometric applications

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The aim of this paper is to give various height estimates and to establish some geometrical constraints for non-compact hypersurfaces with constant mean curvature or, more generally, constant higher order mean curvature into warped product manifolds. Results are sharp and agree with those in the compact case already considered in the literature. The main technical tool of the paper is a new form of the weak maximum principle for a very large class of differential operators that, despite of its simplicity, reveals very interesting for further applications.

1. Introduction

Let $(M, \langle , \rangle)$ be a, non necessarily complete, Riemannian manifold and consider its Laplace-Beltrami operator Δ. We say that M satisfies the weak maximum principle (WMP for short) for Δ if for each function u of class (say) $\mathcal{C}^2$ on M such that $u^* = \sup_M u < +\infty$ and each $\gamma < u^*$ we have

$$\inf_{\Omega_\gamma} \Delta u \leq 0 \tag{1.1}$$

with Ω_γ the super level set

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\}.$$

It is well known that the WMP is equivalent to the stochastic completeness of the Brownian motion associated to Δ (see [22]) and to the validity of the Khas'minskii test (see [23] and [20]). In this paper we introduce still another form of the WMP that is somehow localized on the open sets with non-empty boundary of the manifold M. Specifically we prove a further

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equivalence with the following property that, from now on, we call the open form of the WMP: *for each open set* $\Omega \subset M$ *with* $\partial\Omega \neq \emptyset$, *for each* $f \in \mathcal{C}^0(\mathbb{R})$ *and for each* $v \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ *satisfying*

$$\begin{cases} \Delta v \geq f(v) \ \text{ on } \Omega; \\ \sup_\Omega v < +\infty, \end{cases}$$

we have that

$$\text{either} \qquad \sup_\Omega v = \sup_{\partial\Omega} v \qquad \text{or} \qquad f(\sup_\Omega v) \leq 0.$$

As a matter of fact, we extend our investigation to a general class of differential operators $L = L_{\varphi,T,X}$ (see the precise definition in (2.3) below) that, besides the Laplacian, naturally arise in geometric contexts: for instance, this class includes the mean curvature operator and some of its variants; the operators L_k associated to the Newton tensors of a two-sided hypersurface in a Riemannian manifold, and so on. As it will become apparent from the statement of Theorem 2.8, this new open form of the WMP reminds of the classical formulation of parabolicity given by Alfhors for Riemann surfaces. In fact, introducing and appropriate notion, that we call strong parabolicity, for the operator L at hand, we are able to formulate strong parabolicity in a way that parallels Alfhors formulation. We note however that strong parabolicity implies the more familar parabolicity on the constancy of L-subharmonic functions bounded above, but the contrary may fail. Nevertheless, we show that under the usual property: " the sup of a subsolution with a constant is still a subsolution", the two notions are in fact equivalent in an appropriate functional class. Note that the above property is shared by a large class of operators as mentioned in Section 2. For details see for instance [8] and [9].

Although the above equivalence is not hard to prove, this open form of the WMP enables us to draw a number of new interesting geometric conclusions. For instance, in Section 3 we give some applications to hypersurfaces in warped product spaces as the following examples (for notations and details see Section 3).

Corollary 3.5. *Let* $(\mathbb{P}^n, \langle , \rangle_{\mathbb{P}})$ *be a complete* n*-dimensional Riemannian manifold. Fix an origin* $o \in \mathbb{P}$ *and suppose that*

$$\liminf_{r\to+\infty} \frac{\log \mathrm{vol} B_r}{r^2} < +\infty, \tag{1.2}$$

where B_r is the geodesic ball in $\mathbb{P}$ centered at o and with radius r. Let $\overline{M} = \mathbb{R} \times_\varrho \mathbb{P}^n$ and for $u \in \mathcal{C}^\infty(\mathbb{P})$ let $\Sigma(u)$ be a graph in $\overline{M}$ with $H^* = \sup_{\mathbb{P}} H \leq 0$. Assume that u and $|Du|_{\mathbb{P}}$ are bounded above. Then either $\Sigma(u)$ is a slice $\{u_0\} \times \mathbb{P}$ (with $\mathcal{H}(u_0) = H^* \equiv H$) or $\mathcal{H}(u^*) \leq H^*$, with $u^* = \sup_{\mathbb{P}} u$.

As an immediate consequence we deduce

Corollary 1.1. *Let $\mathbb{P}^n$ be a complete n-dimensional Riemannian manifold satisfying condition (1.2) and consider $\overline{M} = \mathbb{R} \times_\varrho \mathbb{P}^n$ with $\varrho' > 0$. There exists no entire minimal graph $\Sigma(u)$ in $\overline{M}$ with u and $|Du|_{\mathbb{P}}$ bounded above.*

In the next corollary we consider the case of a graph over a horosphere in hyperbolic space $\mathbb{H}^{n+1}$. In fact the result extends to a large class of manifolds called pseudo-hyperbolic manifolds, according to the terminology introduced by Tashiro [25]. These are obtained as warped product spaces of the form $\mathbb{R} \times_\varrho \mathbb{P}^n$, where the warping function is a positive solution, for some $c < 0$, of the ordinary differential equation $\varrho'' + c\varrho = 0$ on $\mathbb{R}$. Thus, either $\varrho(t) = \cosh(\sqrt{-c}t)$ or $\varrho(t) = e^{\sqrt{-c}t}$. Note that if $\mathbb{P}^n$ is Ricci flat then $\mathbb{R} \times_\varrho \mathbb{P}^n$ is Einstein with negative Ricci curvature, and if $\mathbb{P}^n$ is flat then $\mathbb{R} \times_\varrho \mathbb{P}^n$ is a negatively curved space form. Tashiro terminology is due to the fact that with suitable choices of the fiber, we obtain representations of the hyperbolic space. To realize this (and for more details we refer to Montiel [18]), we look at the hyperbolic space $\mathbb{H}^{n+1}$ of constant sectional curvature -1, as a hypersphere in the Lorentz-Minkowski space, precisely as a connected component of the hyperquadric

$$\{x \in \mathbb{R}^{n+2}_1, \langle x, x\rangle_L = -1\}$$

where $\langle , \rangle_L$ is the standard flat Lorentzian product in $\mathbb{R}^{n+2}$. Fix $a \in \mathbb{R}^{n+2}$ and consider the (closed, see [18] for this notion) conformal vector field

$$T_x = a + \langle a, x\rangle_L x, \;\; x \in \mathbb{H}^{n+1}.$$

Depending on the casual character of a we have different foliations of $\mathbb{H}^{n+1}$ and hence different descriptions of it as a warped product space. Namely, if a is timelike, $\mathbb{H}^{n+1}$ is foliated by spheres and can be described as the warped product $\mathbb{R}^+ \times_{\sinh t} \mathbb{S}^n$; if a is lightlike the hyperbolic space is foliated by horospheres and it can be viewed as $\mathbb{R} \times_{e^t} \mathbb{R}^n$ and finally if a is spacelike the vector field T generates a foliation of $\mathbb{H}^{n+1}$ by means of totally geodesic hyperplanes and it can be represented as the warped product $\mathbb{R} \times_{\cosh t} \mathbb{H}^n$. Therefore, choosing $\mathbb{P}^n = \mathbb{R}^n$ the Euclidean space and $\varrho(t) = e^t$, Corollary 1.1 gives the following

Corollary 1.2. *Let $\mathbb{H}^{n+1}$ be the hyperbolic space. There exists no entire minimal graph $\Sigma(u)$ over a horosphere with u and $|Du|_{\mathbb{P}}$ bounded above.*

We should remark that, in many instances, boundedness of u implies that of $|Du|_{\mathbb{P}}$. This is certainly the case if $\varrho = 1$ and for Killing graphs. Even if we require here only boundedness of u from above, in some of the proofs below u will be automatically bounded on the set where boundedness of $|Du|_{\mathbb{P}}$ is needed. Thus this latter assumption can be dropped in many cases. We have chosen the above version of the statements of the corollaries for the sake of simplicity.

These results compare with some recent results on the half-space theorem given by Rosenberg, Schulze and Spruck [24] and Mazet [21]. At this respect, the foundational result was given by Hoffman and Meeks [16], who proved that a proper minimal surface in $\mathbb{R}^3$ which lies on one side of a plane must be a parallel plane; in other words, a proper minimal surface Σ in $(-\infty, 0] \times \mathbb{R}^2$ must be a slice $\Sigma = \{c\} \times \mathbb{R}^2$, $c \leq 0$. They called this the half-space theorem. Motivated by this fact, Rosenberg, Schulze and Spruck proved in [24] that, if $\mathbb{P}^n$ is a complete recurrent Riemannian manifold with bounded sectional curvature, then every proper minimal hypersurface Σ in $(-\infty, 0] \times \mathbb{P}^n$ must be a slice $\Sigma = \{c\} \times \mathbb{P}^n$, $c \leq 0$. They also proved that the same happens for positive entire minimal graphs when $\mathbb{P}^n$ is a complete Riemannian manifold with nonnegative Ricci curvature and sectional curvatures bounded from below. In a more general setting and following [21], if Σ_0 is a given hypersurface with constant mean curvature H_0 in an $(n+1)$-dimensional ambient space $\overline{M}^{n+1}$, a half-space theorem with respect to Σ_0 says that hypersurfaces Σ with the same constant mean curvature H_0 which lie on one side of Σ_0 are classified. In this sense, our results can be seen as kind of half-space theorems. For instance, it follows from Corollary 3.5 that if $\mathbb{P}^n$ is a complete n-dimensional Riemannian manifold satisfying condition (1.2) and $\Sigma(u)$ is an entire graph with constant mean curvature $H \leq 0$ contained in a lower half-space $(-\infty, a] \times_\varrho \mathbb{P}^n$ and having $\sup_{\mathbb{P}} |Du|_{\mathbb{P}} < +\infty$, then either $\Sigma(u)$ is a slice $\{u_0\} \times \mathbb{P}$ (with $\mathcal{H}(u_0) = H$) or $\mathcal{H}(u^*) \leq H$, with $u^* = \sup_{\mathbb{P}} u$. In particular, if $\varrho' > 0$ on $(-\infty, a]$, there exists no entire graph with constant mean curvature $H \leq 0$ contained in the lower half-space $(-\infty, a] \times_\varrho \mathbb{P}^n$ and having $\sup_{\mathbb{P}} |Du|_{\mathbb{P}} < +\infty$.

As another example of the type of applications that we can derive, we have the following simple consequence of Theorem 3.6

Corollary 1.3. *Let $F : \Sigma \to \mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}^n$ be a stochastically complete oriented hypersurface with constant mean curvature such that, for a correct* *orientation of the normal N, $H \geq 0$. Suppose that the height function $h = \pi_{\mathbb{R}} \circ F$ is bounded above on Σ,*

$$h^* = \sup_\Sigma h(x) < +\infty.$$

Then $H \geq 1$. In particular if $H \in [0,1)$ the hypersurface cannot be contained in a lower half-space $(-\infty, a] \times_{e^t} \mathbb{R}^n$ so that it must have at least one top end.

Using Theorem 3.7 we can extend this result to the case of higher order mean curvatures as follows.

Corollary 1.4. *Let $F : \Sigma \to \mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}^n$ be a complete oriented hypersurface with sectional curvatures bounded from below and constant k-mean curvature $H_k \neq 0$ for some $2 \leq k \leq n$. Assume*

$$\sup_\Sigma |H_1| < +\infty$$

and the existence of an elliptic point on Σ so that, for a correct orientation of the normal N, $H_k > 0$. Suppose that for this orientation the angle $\Theta = \langle \partial_t, N \rangle \leq 0$ on Σ. Let $h = \pi_{\mathbb{R}} \circ F$ be the height function and assume

$$h^* = \sup_\Sigma h(x) < +\infty.$$

Then $H_k \geq 1$. In particular if $H_k \in (0,1)$ the hypersurface cannot be contained in a lower half-space $(-\infty, a] \times_{e^t} \mathbb{R}^n$ so that it must have at least one top end.

On the other hand, in Section 4 we give some applications to hypersurfaces in product spaces. For instance, the next result is a specific version of Theorem 4.1 and it extends Theorem 3.5 of Alías and Dajczer [2] from compact to complete hypersurfaces.

Theorem 1.5. *Let $F : \Sigma^n \to \mathbb{R} \times \mathbb{P}^n$ be a complete hypersurface with constant mean curvature $H > 0$. Assume that*

$$\beta = \sup_\Sigma \Theta < 0$$

and suppose that $K_{\mathbb{P}} \geq -\alpha$ and $H^2 > \alpha$, for some $\alpha > 0$. Furthermore, assume that the Weingarten operator A of Σ satisfies

$$|A(x)| \leq G(r(x))$$

for some $G \in \mathcal{C}^1([0, +\infty))$ *satisfying*

$$\text{(i) } G(0) > 0, \quad \text{(ii) } G'(t) \geq 0 \quad \textit{and} \quad \text{(iii) } 1/G(t) \notin L^1(+\infty),$$

where $r(x)$ *denotes the distance in* Σ *from some fixed origin* o*.*

If $\Omega \subset \Sigma$ *is an open set with* $\partial\Omega \neq \emptyset$ *for which* $F(\Omega)$ *is contained in a slab and* $F(\partial\Omega) \subset \mathbb{P}_0 = \{0\} \times \mathbb{P}^n$*, then*

$$F(\Omega) \subset \left[0, \frac{(1+\beta)H}{H^2 - \alpha}\right] \times \mathbb{P}^n.$$

In Corollary 4.2, assuming that Σ is parabolic and that $\mathrm{Ric}_{\mathbb{P}} \geq 0$, we obtain a version of Theorem 4.1 that compares directly with some work of Cheng and Rosenberg [10] obtained in the compact case. This is somehow expected since parabolicity often reveals the right generalization of the topological assumption of compactness. This result and the above observation are generalized to higher order mean curvatures in Theorem 4.3 and in Corollary 4.4 of Subsection 4.3.

We conclude this presentation of geometric results by recalling Theorem 5.1 in Section 5, where we show, under very mild and, in a sense, local assumptions, that a Killing graph with constant mean curvature has to be indeed minimal. This generalizes some recent works of ours [6] where the main assumption was of a global nature.

2. An equivalent form of the weak maximum principle

The aim of this section is to present another form of the weak maximum principle which turns out to be very useful in geometric applications. We focus our attention on the next general class of operators that we consider, for instance, in [1] and [6]. We let T be a symmetric, 2-covariant tensor field on a Riemannian manifold $(M, \langle , \rangle)$. Assume that, for some continuous functions T_- and T_+ on $\mathbb{R}_0^+ = [0, +\infty)$, the tensor T satisfies the following bounds

$$0 < T_-(r) \leq T(Y, Y) \leq T_+(r) \tag{2.1}$$

for each $Y \in T_xM$, $|Y| = 1$, and every $x \in \partial B_r$, where B_r denotes the geodesic ball of radius r centered at an origin o. Let $\varphi : M \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be such that $\varphi(\cdot, t) \in \mathcal{C}^0(M)$ for each $t \in \mathbb{R}_0^+$, and $\varphi(x, \cdot) \in \mathcal{C}^0(\mathbb{R}_0^+) \cap \mathcal{C}^1(\mathbb{R}^+)$ for each

$x \in M$, $\mathbb{R}^+ = (0, +\infty)$, and

$$\begin{cases} \text{i) } \varphi(x,0) = 0 \text{ for every } x \in M; \\ \text{ii) } \varphi(x,t) > 0 \text{ on } M \times \mathbb{R}^+; \\ \text{iii) } \varphi(x,t) \le A(x)t^\delta \text{ on } M \times \mathbb{R}^+ \end{cases} \tag{2.2}$$

for some $\delta > 0$ and $A(x) \in \mathcal{C}^0(M)$, $A(x) > 0$. Let X be a vector field on M. For $u \in \mathcal{C}^1(M)$ we define

$$Lu = L_{\varphi,T,X}u = \operatorname{div}\left(|\nabla u|^{-1}\varphi(x,|\nabla u|)T(\nabla u,\cdot)^\sharp\right) - \langle X, \nabla u\rangle \tag{2.3}$$

in the weak sense, where $^\sharp : T^*M \to TM$ denotes the musical isomorphism.

Remark 2.1. Note that the LHS inequality in (2.1) and requirement ii) in (2.2) are ellipticity conditions for the operator L. As a matter of fact properties (2.1) and (2.2) will not be used in proving the equivalence in Theorem 2.5 below. On the other hand, they are basic in looking for sufficient conditions to guarantee that the property expressed in Definition 2.2 below holds on the manifold we are considering. See for instance Theorem 2 in [6] or Section 6 in [1], or the other various results reported in [23]. In fact, when $\varphi(x,t) = t$ it is enough to consider $u \in \mathrm{Lip}_{\mathrm{loc}}(M)$; the more restrictive $u \in \mathcal{C}^1(M)$ enables us to deal with the non linear case. Furthermore, for those theorems giving sufficient conditions in terms of the volume growth of geodesic balls we can enlarge the class of admissible solutions to $\mathcal{C}^0(M) \cap W^{1,1+\delta}_{\mathrm{loc}}(M)$. This is due to the fact that the argument of proof of this kind of results is based only on the notion of weak solution.

In what follows we let $q(x) \in \mathcal{C}^0(M)$, $q(x) > 0$.

Definition 2.2. We say that the q-WMP (the q-weak maximum principle) holds on M for the operator L in (2.3) if, for each $u \in \mathcal{C}^1(M)$ with $u^* = \sup_M u < +\infty$ and for each $\gamma \in \mathbb{R}$ with $\gamma < u^*$, we have

$$\inf_{\Omega_\gamma}\{q(x)Lu\} \le 0 \tag{2.4}$$

in the weak sense, where

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\}. \tag{2.5}$$

In case $q(x)$ is a positive constant we will simply say that L satisfies the WMP (the weak maximum principle).

We underline that when $q(x)$ is bounded between two positive constants the validity of the WMP is equivalent to the validity of the q-WMP. In fact, it is easy to see that when $q(x)$ is bounded from below by a positive constant, then the q-WMP implies the WMP, while the converse occurs when $q(x)$ is bounded from above.

Remark 2.3. We recall that (2.4) in the weak sense expresses as follows: for every $\varepsilon > 0$

$$-\int_{\Omega_\gamma} \left(|\nabla u|^{-1}\varphi(x,|\nabla u|)T(\nabla u,\nabla\psi) + \langle X,\nabla u\rangle\psi\right) \leq \int_{\Omega_\gamma} \frac{\varepsilon}{q(x)}\psi, \tag{2.6}$$

for some $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega_\gamma)$, $\psi \geq 0$, $\psi \not\equiv 0$. On the other hand, the strict inequality

$$\inf_{\Omega_\gamma}\{q(x)Lu\} < 0 \tag{2.7}$$

in the weak sense means that for some $\varepsilon > 0$

$$-\int_{\Omega_\gamma} \left(|\nabla u|^{-1}\varphi(x,|\nabla u|)T(\nabla u,\nabla\psi) + \langle X,\nabla u\rangle\psi\right) \leq -\int_{\Omega_\gamma} \frac{\varepsilon}{q(x)}\psi, \tag{2.8}$$

for some $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega_\gamma)$, $\psi \geq 0$, $\psi \not\equiv 0$.

The following fact seems worth mentioning. It extends Proposition 3.4 in [23] to general operators.

Proposition 2.4. *Let $(M,\langle,\rangle_M)$ and $(N,\langle,\rangle_N)$ be non-compact Riemannian manifolds, and assume that there exist compact sets $A \subset M$ and $B \subset N$ and a Riemannian isometry $F : M \setminus A \to N \setminus B$ which preserves divergent sequences in the ambient spaces, that is, $\{x_k\}$ diverges in M if and only if $\{F(x_k)\}$ diverges in N. Let X be a vector field on M, T a symmetric 2-covariant tensor field on M satisfying (2.1) and φ as in (2.2) that define the differential operator $L_{\varphi,T,X}$ on M; let Y, S, ψ be with the same properties on N and define the differential operator $L_{\psi,S,Y}$ on N. Assume that*

$$Y = F_*X, \quad S = F_*T, \quad \psi(y,t) = \varphi(F^{-1}(y),t)$$

on $N \setminus B$. Then the WMP holds on M for the operator $L_{\varphi,T,X}$ if and only if the WMP holds on N for the operator $L_{\psi,S,Y}$.

Observe that the condition that $\{x_k\}$ diverges in M if and only if $\{F(x_k)\}$ diverges in N makes sense for any divergent sequence in M even if F is not globally defined on M because the sequence eventually leaves the compact set A.

Proof of Proposition 2.4. Suppose that the WMP holds on M for the operator $L_{\varphi,T,X}$. Let $v \in \mathcal{C}^1(N)$ with $v^* < +\infty$. Without loss of generality we may assume that v^* is not attained and strictly positive. Consider two relatively compact domains K_1, K_2 in M such that $A \subseteq K_1 \subseteq \overline{K}_1 \subseteq K_2$. Choose a smooth cutoff function $\lambda : M \to [0,1]$ satisfying $\lambda \equiv 0$ on K_1, $\lambda \equiv 1$ on $M \setminus K_2$, and define a function $u \in \mathcal{C}^1(M)$ by

$$u = \begin{cases} \lambda(v \circ F), & \text{on } M \setminus A; \\ 0, & \text{on } A. \end{cases} \tag{2.9}$$

We claim that $v^* = u^*$ and that u^* is not attained. By construction $v^* \leq u^*$. On the other hand, let $\{\overline{y}_k\}$ be a sequence in N such that $v(\overline{y}_k) \nearrow v^*$. Since v does not attain v^* the sequence $\{\overline{y}_k\}$ is divergent, therefore for k sufficiently large $\overline{y}_k$ lies outside B. By the assumption on F, $\{\overline{x}_k\} = \{F^{-1}(\overline{y}_k)\}$ is a divergent sequence in M. Thus $u(\overline{x}_k) = \lambda(\overline{x}_k)(v \circ F(\overline{x}_k)) = v(\overline{y}_k)$ for k sufficiently large, showing that $u(\overline{x}_k) \nearrow v^*$, and $v^* = u^*$. Furthermore, u^* is not attained, indeed, $u(x) = 0$ on A, and $u(x) \leq v(F(x)) < v^* = u^*$ on $M \setminus A$, thus u does not attain u^*, as claimed. Thus we can fix $\gamma < v^*$ sufficiently close to v^* such that

$$\Sigma_\gamma = \{y \in N : v(y) > \gamma\} \subset N \setminus (B \cup F(\overline{K}_2 \setminus A))$$

and consider $F^{-1}(\Sigma_\gamma) = \{x \in M \setminus A : (v \circ F)(x) > \gamma\}$.

Since $v^* = u^* > 0$ we can suppose that $\gamma > 0$ and it follows that

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\} = \{x \in M \setminus A : \lambda(x)(v \circ F)(x) > \gamma\}.$$

In particular $(v \circ F)(x) > \gamma$ so that $\Omega_\gamma \subseteq F^{-1}(\Sigma_\gamma)$.

The validity of the WMP on M, yields that, for each $\varepsilon > 0$ there exist some $\tilde{\psi} \in \mathcal{C}_c^\infty(\Omega_\gamma)$, $\tilde{\psi} \geq 0$, $\tilde{\psi} \not\equiv 0$ such that

$$\begin{aligned}
&\int_{F^{-1}(\Sigma_\gamma)} \varepsilon\tilde{\psi} = \int_{\Omega_\gamma} \varepsilon\tilde{\psi} \geq -\int_{\Omega_\gamma} \Big(|\nabla u|^{-1}\varphi(x, |\nabla u|) T(\nabla u, \nabla\tilde{\psi}) + \langle X, \nabla u\rangle \tilde{\psi} \Big) \\
&= -\int_{F^{-1}(\Sigma_\gamma)} \Big(|\nabla u|^{-1}\varphi(x, |\nabla u|) T(\nabla u, \nabla\tilde{\psi}) + \langle X, \nabla u\rangle \tilde{\psi} \Big) \\
&= -\int_{\Sigma_\gamma} \Big\{ |\nabla(u \circ F^{-1})|^{-1} \varphi(F^{-1}(y), |\nabla(u \circ F^{-1})|) T(\nabla(u \circ F^{-1}), \nabla(\tilde{\psi} \circ F^{-1})) \Big\} \\
&\quad - \int_{\Sigma_\gamma} \Big\{ \langle X \circ F^{-1}, \nabla(u \circ F^{-1})\rangle \tilde{\psi} \circ F^{-1} \Big\}.
\end{aligned}$$

But for $y \in \Sigma_\gamma$, $F^{-1}(y) \in M \setminus K_2$, hence

$$\begin{gathered} u(F^{-1}(y)) = (v \circ F)(F^{-1}(y)) = v(y); \\ \tilde{\varphi} = \tilde{\psi} \circ F^{-1} \in \mathcal{C}_c^\infty(\Sigma_\gamma), \tilde{\varphi} \geq 0, \tilde{\varphi} \not\equiv 0; \\ X \circ F^{-1} = Y \text{ and } T(\nabla(u \circ F^{-1}), \nabla\tilde{\varphi}) = S(\nabla v, \nabla\tilde{\varphi}). \end{gathered}$$

Therefore, being F an isometry,

$$\int_{\Sigma_\gamma} \varepsilon\tilde{\varphi} \geq -\int_{\Sigma_\gamma} \left(|\nabla v|^{-1}\psi(y, |\nabla v|)S(\nabla v, \nabla\tilde{\varphi}) + \langle Y, \nabla v\rangle\tilde{\varphi}\right).$$

This proves that the WMP holds for the operator $L_{\psi,S,Y}$ on N.

Repeating the same argument with M and N interchanged shows that if WMP holds in N, so it holds in M (note that $F^{-1} : N \setminus B \to M \setminus A$ is a Riemannian isometry which maps divergent sequences to divergent sequences). □

We have the following:

Theorem 2.5. *The q-WMP holds on M for the operator L if and only if the open q-WMP holds on M, that is, for each $f \in \mathcal{C}^0(\mathbb{R})$, for each open set $\Omega \subset M$ with $\partial\Omega \neq \emptyset$, and for each $v \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$ satisfying*

$$\begin{cases} i)\ q(x)Lv \geq f(v) \text{ on } \Omega; \\ ii)\ \sup_\Omega v < +\infty, \end{cases} \tag{2.10}$$

we have that either

$$\sup_\Omega v = \sup_{\partial\Omega} v \tag{2.11}$$

or

$$f(\sup_\Omega v) \leq 0. \tag{2.12}$$

Remark 2.6. Observe that the q-WMP on M for the operator L is also equivalent to the following dual statement: The q-WMP holds on M for the operator L if and only if for each $f \in \mathcal{C}^0(\mathbb{R})$, for each open set $\Omega \subset M$ with

$\partial\Omega \neq \emptyset$, and for each $v \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$ satisfying

$$\begin{cases} \text{i) } q(x)Lv \leq f(v) \text{ on } \Omega; \\ \text{ii) } \inf_\Omega v > -\infty, \end{cases} \tag{2.13}$$

we have that either

$$\inf_\Omega v = \inf_{\partial\Omega} v \tag{2.14}$$

or

$$f(\inf_\Omega v) \geq 0. \tag{2.15}$$

Proof of Theorem 2.5. Assume that the q-WMP holds for the operator L on M and let f, v and Ω be as in the statement of the theorem. Suppose that (2.11) is not satisfied, that is

$$\sup_\Omega v > \sup_{\partial\Omega} v. \tag{2.16}$$

Fix $\varepsilon > 0$ sufficiently small that

$$\sup_\Omega v - 2\varepsilon > \sup_{\partial\Omega} v + 2\varepsilon \tag{2.17}$$

and define

$$U_{2\varepsilon} = \{x \in \Omega : v(x) > \sup_\Omega v - 2\varepsilon\}. \tag{2.18}$$

Note that $U_{2\varepsilon} \neq \emptyset$. Moreover, for every $x \in \overline{U}_{2\varepsilon}$ one has from (2.17)

$$v(x) \geq \sup_\Omega v - 2\varepsilon > \sup_{\partial\Omega} v + 2\varepsilon > \sup_{\partial\Omega} v,$$

so that $x \in \Omega$. That is, $\overline{U}_{2\varepsilon} \subset \Omega$, and therefore

$$\overline{U}_\varepsilon \subset U_{2\varepsilon} \subset \overline{U}_{2\varepsilon} \subset \Omega,$$

where U_ε is defined in a way similar to (2.18).

By adding, if necessary, a positive constant to v, we can suppose that $\sup_\Omega v > 2\varepsilon$ and we let $\gamma = \sup_\Omega v - \varepsilon > 0$. Next we choose a smooth cut-off function $\psi : M \to [0,1]$ such that

$$\psi \equiv 1 \text{ on } U_\varepsilon \quad \text{and} \quad \psi \equiv 0 \text{ on } M \setminus U_{2\varepsilon}$$

and we define

$$u(x) = \begin{cases} \psi(x)v(x) \text{ on } \Omega, \\ 0 \text{ on } M \setminus \Omega. \end{cases} \tag{2.19}$$

Then $u \in \mathcal{C}^1(M)$, $u^* < +\infty$ and

$$Lu = Lv \quad \text{on } U_\varepsilon. \tag{2.20}$$

We claim that

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\} = U_\varepsilon = \{x \in \Omega : v(x) > \gamma = \sup_\Omega v - \varepsilon\}. \tag{2.21}$$

Clearly it suffices to show that $\Omega_\gamma \subset U_\varepsilon$. For every $x \in \Omega_\gamma$ one has $u(x) > \gamma > 0$. In particular, by (2.19), it follows that $x \in \Omega$ and $v(x) > 0$, so that

$$v(x) \geq \psi(x)v(x) = u(x) > \gamma = \sup_\Omega v - \varepsilon.$$

Since $x \in \Omega$, this means that $x \in U_\varepsilon$.

Since for any constant $\alpha \in \mathbb{R}$, $L(v + \alpha) = Lv$, using (2.20) and (2.10) we deduce

$$Lu = L(v + \alpha) = Lv \geq \frac{1}{q(x)} f(v) \quad \text{on } \Omega_\gamma.$$

In other words

$$q(x)Lu \geq f(v) \quad \text{on } \Omega_\gamma.$$

Applying the q-WMP to u we infer

$$0 \geq \inf_{\Omega_\gamma} \{q(x)Lu\} \geq \inf_{\Omega_\gamma} f(v).$$

But $\Omega_\gamma = U_\varepsilon$ and thus, letting $\varepsilon \to 0^+$ and using continuity of f we obtain (2.12).

For the converse, assume the validity of the open q-WMP for L. We reason by contradiction and we suppose that the q-WMP is false. Then, there exists $u \in \mathcal{C}^1(M)$ with $u^* < +\infty$, and $\gamma < u^*$ such that

$$\beta = \inf_{\Omega_\gamma}\{q(x)Lu\} > 0. \tag{2.22}$$

This implies that u is non-constant and therefore, since β is increasing with γ, up to choosing γ sufficiently near to u^*, we can suppose that

$$\partial\Omega_\gamma = \{x \in M : u(x) = \gamma\} \neq \emptyset.$$

Set $\Omega = \Omega_\gamma$ and $v = u|_{\overline{\Omega}}$. Because of (2.22) and $u^* < +\infty$ we have

$$\begin{cases} q(x)Lv \geq \beta \text{ on } \Omega, \\ \sup_\Omega v = u^* < +\infty. \end{cases} \tag{2.23}$$

Since $f(v) \equiv \beta > 0$, alternative (2.12) cannot occur. However alternative (2.11) cannot occur either because

$$\sup_\Omega v = u^* > \gamma = \sup_{\partial\Omega} v.$$

This yields the desired contradiction. □

Remark 2.7. Note that the above proof works for any of the choices of the functional classes of the solutions that we have been considering in Remark 2.1. Of course in Definition 2.2 we have to enlarge the functional class accordingly.

A careful reading of the above proof yields the validity of the following form of the theorem useful in applications.

Theorem 2.8. *The q-WMP holds on M for the operator L if and only if for each $\beta \in \mathbb{R}^+$, for each open set $\Omega \subset M$ with $\partial\Omega \neq \emptyset$, and for each $v \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$ satisfying*

$$\begin{cases} i)\ q(x)Lv \geq \beta \text{ on } \Omega; \\ ii)\ \sup_\Omega v < +\infty, \end{cases} \tag{2.24}$$

we have

$$\sup_\Omega v = \sup_{\partial\Omega} v. \tag{2.25}$$

2.1. Parabolicity and the weak maximum principle

The above discussion and the fact that, as explained in some detail in [23], parabolicity for the Laplace-Beltrami operator Δ is equivalent to a certain stronger form of the WMP, suggest to introduce the following

Definition 2.9. We say that on M the operator L as in (2.3) is strongly parabolic (SP) if for each non constant $u \in \mathcal{C}^1(M)$ with $u^* < +\infty$ and for each $\gamma \in \mathbb{R}$ with $\gamma < u^*$ we have

$$\inf_{\Omega_\gamma}\{Lu\} < 0. \tag{2.26}$$

It is immediate to compare this definition with the more familiar

Definition 2.10. We say that on M the operator L is parabolic if each $u \in \mathcal{C}^1(M)$ with $u^* < +\infty$ and satisfying $Lu \geq 0$ on M is constant.

It is clear that strong parabolicity of L implies parabolicity. The converse is also true if we enlarge the functional class to $\mathrm{Lip}_{\mathrm{loc}}(M)$ or $\mathcal{C}^0(M) \cap W^{1,1+\delta}_{\mathrm{loc}}(M)$ and we assume the validity of the following property:

Property 2.11. For every open set $\Omega \subseteq M$, if $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ or $\mathcal{C}^0(\overline{\Omega}) \cap W^{1,1+\delta}_{\mathrm{loc}}(\Omega)$ satisfies $Lu \geq 0$ on Ω then, for each fixed $\alpha \in \mathbb{R}$, the function $v(x) = \max\{u(x), \alpha\}$ satisfies $Lv \geq 0$ on Ω

Indeed, assume Property 2.11 and the validity of Definition 2.10 with $u \in \mathcal{C}^0(M) \cap W^{1,1+\delta}_{\mathrm{loc}}(M)$. To see the validity of Definition 2.9 we reason by contradiction and we suppose the existence of a non-constant u with $u^* < +\infty$ and $\gamma \in \mathbb{R}$, $\gamma < u^*$ such that

$$Lu \geq 0$$

on Ω_γ.

Up to increasing γ we may assume $\partial\Omega_\gamma \neq \emptyset$, because otherwise $\Omega_\gamma = M$ and the result is immediate. Consider the function

$$v(x) = \begin{cases} \max\{u(x), \gamma + \frac{u^*-\gamma}{2}\} \text{ on } \Omega_\gamma, \\ \gamma + \frac{u^*-\gamma}{2} \text{ on } M \setminus \Omega_\gamma. \end{cases}$$

Then $v^* = u^* < +\infty$ and, because of Property 2.11 (on Ω_γ)

$$Lv \geq 0 \quad \text{on} \quad M.$$

By Definition 2.10 v is the constant $\gamma + \frac{u*-\gamma}{2} < u^* = v^*$, contradiction.

We note that Property 2.11 is known to hold, for instance, for the Laplacian and for operators of the form $Lu = \mathrm{div}(T(\nabla u, \quad)^\#) - \langle X, \nabla u\rangle$. In particular for trace operators $Lu = \mathrm{tr}(T \circ \mathrm{hess}(u)) = \mathrm{div}(T(\nabla u, \quad)^\#) - \langle \mathrm{div}(T), \nabla u\rangle$. Furthermore Property 2.11 also holds for the class of operators

$$L_{p,Q}u = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - \langle \nabla Q, \nabla u\rangle$$

with $p \in (1, +\infty)$ and $Q \in C^\infty(M)$ a potential function (see Proposition 7.2 of [8] for a more general result).

Thus Property 2.11 is a sufficient condition for the equivalence between parabolicity and strong parabolicity for the operator L either in the class $\mathrm{Lip}_{\mathrm{loc}}(M)$ or in $\mathcal{C}^0(M) \cap W^{1,1+\delta}_{\mathrm{loc}}(M)$. It is worth wondering if it is also a necessary condition.

As expected we have the following open version of the strong parabolicity for the operator L.

Theorem 2.12. *The strong parabolicity of the operator L as in Definition 2.9 is equivalent to the following open SP: for each $f \in C^0(\mathbb{R})$, for each open set $\Omega \subset M$ with $\partial\Omega \neq \emptyset$ and for each $v \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$, non-constant and satisfying*

$$\begin{cases} Lv \geq f(v) \ \text{on } \Omega, \\ \sup_\Omega v < +\infty \end{cases} \tag{2.27}$$

we have that either

$$\sup_\Omega v = \sup_{\partial\Omega} v \tag{2.28}$$

or, for each $\varepsilon > 0$

$$\inf_{U_\varepsilon} f(v) < 0 \tag{2.29}$$

where

$$U_\varepsilon = \{x \in \Omega : v(x) > \sup_\Omega v - \varepsilon\}.$$

Note the minor, but essential, difference between conclusion (2.29) of Theorem 2.12 and (2.12) of Theorem 2.5. The proof of the above theorem is very similar to that of Theorem 2.5 and it is therefore left to the reader.

As a consequence of Theorem 2.12 we deduce that, if the operator L is SP on M, then for each open set $\Omega \subset M$ with $\partial\Omega \neq \emptyset$ and for each non-constant $v \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$, satisfying

$$\begin{cases} Lu \geq 0 \text{ on } \Omega, \\ \sup_\Omega v < +\infty \end{cases} \tag{2.30}$$

we have

$$\sup_\Omega v = \sup_{\partial\Omega} v. \tag{2.31}$$

Interestingly enough, also the converse is true; that is, calling the above property Ahlfors parabolicity, in strict analogy with Theorem 2.8 we have

Theorem 2.13. *The operator L is SP on M if and only if it is Ahlfors parabolic.*

Proof. We only need to prove that Ahlfors parabolicity implies SP. We reason by contradiction and we suppose the existence of a non-constant $u \in \mathcal{C}^1(M)$ with $u^* < +\infty$ and of $\gamma \in \mathbb{R}$, $\gamma < u^*$ such that $\inf_{\Omega_\gamma} Lu \geq 0$, that is,

$$Lu \geq 0 \quad \text{on } \Omega_\gamma.$$

Since u is non-constant, by possibly increasing γ we can suppose $\partial\Omega_\gamma \neq \emptyset$. Let $v = u|_{\overline{\Omega}_\gamma}$ so that, for $\Omega = \Omega_\gamma$, $v \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$, v is non-constant on Ω and it satisfies (2.30).

Hence, by (2.31),

$$u^* = \sup_\Omega v = \sup_{\partial\Omega} v = \gamma$$

contradiction. ☐

We have provided a number of sufficient conditions to guarantee the strong parabolicity for the general operator L in (2.3). However, since in the sequel we will only consider the case $\varphi(x,t) = t$ we report here the next result valid for the linear case.

Theorem 2.14. *Let $(M, \langle , \rangle)$ be a Riemannian manifold and $L = L_{\varphi,T,X}$ be as in (2.3), with $\varphi(x,t) = t$. Assume the existence of $\gamma \in C^2(M)$ such* *that*

$$\left\{ \begin{array}{l} \gamma(x) \to +\infty \quad \textit{as} \quad x \to \infty, \\ L\gamma \le 0 \quad \textit{outside a compact set.} \end{array} \right. \tag{2.32}$$

Then the operator L is strongly parabolic.

The proof is a minor modification of that of Theorem A in [1], and we omit here the details. See also [9].

Remark 2.15. If $L\gamma \le 0$ in the weak sense outside a compact set for $\gamma \in \mathcal{C}^1(M)$ we need T to be positive definite. If $\gamma = f(r)$ we can use a different method of proof as in [4] but we still need T to be positive definite. In fact in this case the class $\mathrm{Lip}_{\mathrm{loc}}(M)$ would work.

3. Geometric applications to hypersurfaces in warped product spaces

Let $\overline{M} = I \times_\varrho \mathbb{P}^n$ be a warped product manifold, where $I \subseteq \mathbb{R}$ is a open interval, $(\mathbb{P}^n, \langle , \rangle_{\mathbb{P}})$ is a n-dimensional Riemannian manifold and $\varrho : I \to \mathbb{R}^+$ is a smooth function. The product manifold $I \times_\varrho \mathbb{P}^n$ is endowed with the metric

$$\langle , \rangle = dt^2 + \varrho(t)^2 \langle , \rangle_{\mathbb{P}}. \tag{3.1}$$

Each leaf $\mathbb{P}_t = \{t\} \times \mathbb{P}$, called here a slice, of the foliation $t \in I \to \mathbb{P}_t$ of $\overline{M}$ is a totally umbilical hypersurface with constant mean curvature. Its mean curvature vector field H_t is given by

$$H_t = -\mathcal{H}(t)\partial_t \tag{3.2}$$

with $\mathcal{H}(t) = \frac{\varrho'(t)}{\varrho(t)}$, and the higher order mean curvatures $H_k(t)$ with respect to $-\partial_t$ are given by

$$H_k(t) = \mathcal{H}(t)^k, \; k = 1, \dots, n.$$

Given an isometrically immersed hypersurface

$$F : \Sigma^n \to \overline{M} = I \times_\varrho \mathbb{P}^n, \tag{3.3}$$

its height function $h \in \mathcal{C}^\infty(\Sigma)$ is defined as $h = \pi_I \circ F$, where π_I denotes the projection onto the factor $I \subseteq \mathbb{R}$. On I we introduce the function

$$\sigma(t) = \int_{t_0}^{t} \varrho(r)dr \tag{3.4}$$

for some fixed $t_0 \in I$, and, when the hypersurface is two-sided, we let Θ denote the angle function $\Theta : \Sigma \to [-1, 1]$ given by

$$\Theta = \langle N, \partial_t \rangle \tag{3.5}$$

where N denotes the chosen global unit field normal to Σ. A simple computation shows that

$$|\nabla h|^2 = 1 - \Theta^2 \quad \text{on } \Sigma. \tag{3.6}$$

3.1. Graphs in warped product spaces

Given a smooth function $u : \mathbb{P} \to I \subseteq \mathbb{R}$, we consider the immersion $F : \mathbb{P} \to \overline{M} = I \times_\varrho \mathbb{P}^n$ given by the graph of u, that is, $F(x) = (u(x), x)$, and we denote by $\Sigma(u)$ the image $F(\mathbb{P})$. The metric induced on $\mathbb{P}^n$ from the warped metric in the ambient space is given by

$$\langle , \rangle = du^2 + \varrho(u)^2 \langle , \rangle_{\mathbb{P}}$$

and the vector field

$$N = \frac{\varrho(u)}{\sqrt{\varrho(u)^2 + |Du|^2_{\mathbb{P}}}} \left(\frac{1}{\varrho(u)^2} Du - \partial_t \right) \tag{3.7}$$

defines a unit normal to the graph $\Sigma(u)$ satisfying $-1 \leq \langle N, \partial_t \rangle < 0$. The mean curvature function H of $\Sigma(u)$ with respect to this orientation is given by

$$\operatorname{div}_{\mathbb{P}} \left(\frac{Du}{\sqrt{\varrho(u)^2 + |Du|^2_{\mathbb{P}}}} \right) = n\varrho(u) \left(\frac{\varrho'(u)}{\sqrt{\varrho(u)^2 + |Du|^2_{\mathbb{P}}}} - H \right). \tag{3.8}$$

Note that $\operatorname{div}_{\mathbb{P}}$, D and $|\cdot|_{\mathbb{P}}$ are taken here with respect to the original metric $\langle , \rangle_{\mathbb{P}}$ of $\mathbb{P}^n$. We fix an origin $o \in \mathbb{P}^n$ and for $u_0 = u(o)$ we set

$$\phi(t) = \int_{u_0}^{t} \frac{ds}{\varrho(s)}. \tag{3.9}$$

Defining

$$w(x) = \phi(u(x)), \tag{3.10}$$

a computation shows that w satisfies

$$\operatorname{div}_{\mathbb{P}}\left(\frac{Dw}{\sqrt{1+|Dw|^2_{\mathbb{P}}}}\right) = n\varrho(\phi^{-1}(w))\left(\frac{\mathcal{H}(\phi^{-1}(w))}{\sqrt{1+|Dw|^2_{\mathbb{P}}}} - H\right). \tag{3.11}$$

Note that the above change of variable has a geometric interpretation in viewing the warped product metric as a metric conformal to the standard product metric on $J \times \mathbb{P}^n$, where $J \subseteq \mathbb{R}$ is an open interval. See [3, Section 2.3] for complete details. We will refer to the operator in the LHS of (3.11) as to the mean curvature operator on $(\mathbb{P}^n, \langle , \rangle_{\mathbb{P}})$.

We are ready to prove the next

Theorem 3.1. *Let $(\mathbb{P}^n, \langle , \rangle_{\mathbb{P}})$ be an n-dimensional Riemannian manifold and assume that the WMP holds on $\mathbb{P}^n$ for the mean curvature operator. Let $\overline{M} = I \times_\varrho \mathbb{P}^n$ and, for a smooth function $u : \mathbb{P} \to I \subseteq \mathbb{R}$, let $\Sigma(u)$ be a graph in $\overline{M}$ with $H^* = \sup_{\mathbb{P}} H \leq 0$. Assume that u and $|Du|_{\mathbb{P}}$ are bounded above. Then either $\Sigma(u)$ is a slice $\mathbb{P}_{u_0}$ (with $\mathcal{H}(u_0) = H^* \equiv H$) or $\mathcal{H}(u^*) \leq H^*$, with $u^* = \sup_{\mathbb{P}} u$.*

Proof. Assume that $\Sigma(u)$ is not a slice, that is, u is non-constant. We reason by contradiction and we suppose that $\mathcal{H}(u^*) > H^*$. By continuity of $\mathcal{H}(t)$ we can choose a regular value of u, $\gamma < u^*$, sufficiently near to u^* such that $\mathcal{H}(t) > H^*$ for $t \in [\gamma, u^*]$. Next we define w as in (3.10) and we observe that $w^* = \sup_{\mathbb{P}} w = \phi(u^*)$. Furthermore,

$$\Omega_\gamma = \{x \in \mathbb{P} : u(x) > \gamma\} = \Omega = \{x \in \mathbb{P} : w(x) > \phi(\gamma)\}.$$

Since γ is a regular value of u, $\partial\Omega \neq \emptyset$.

The function w satisfies the equation

$$\operatorname{div}_{\mathbb{P}}\left(\frac{Dw}{\sqrt{1+|Dw|^2_{\mathbb{P}}}}\right) = n\varrho(u)\left(\frac{\mathcal{H}(u)}{\sqrt{1+|Dw|^2_{\mathbb{P}}}} - H(x)\right) \quad \text{on } \Omega. \tag{3.12}$$

with

$$u = \phi^{-1}(w).$$

Since $H(x) \leq 0$ we have

$$\frac{H(x)}{\sqrt{1+|Dw|_{\mathbb{P}}^2}} \geq H(x),$$

so that

$$n\varrho(u)\left(\frac{\mathcal{H}(u)}{\sqrt{1+|Dw|_{\mathbb{P}}^2}} - H(x)\right) \geq n\varrho(u)\frac{\mathcal{H}(u) - H(x)}{\sqrt{1+|Dw|_{\mathbb{P}}^2}}.$$

On the other hand, observe that on Ω we have

$$\mathcal{H}(u) > H^* \geq H(x).$$

Since $|Du|_{\mathbb{P}}^2$ and $\varrho(u)$ are bounded on $\overline{\Omega}$, so is

$$|Dw|_{\mathbb{P}}^2 = \frac{|Du|_{\mathbb{P}}^2}{\varrho(u)}.$$

Therefore there exists a positive constant C such that

$$\frac{n\varrho(u)}{\sqrt{1+|Dw|_{\mathbb{P}}^2}} \geq C \qquad \text{on } \Omega,$$

and

$$\begin{cases} \operatorname{div}_{\mathbb{P}}\left(\frac{Dw}{\sqrt{1+|Dw|_{\mathbb{P}}^2}}\right) \geq C(\mathcal{H}(u) - H(x)) \geq C(\mathcal{H}(u) - H^*) > 0 \text{ on } \Omega, \\ \sup_\Omega w = w^* < +\infty. \end{cases}$$

We now apply Theorem 2.5. Since $\mathcal{H}(u^*) - H^* > 0$, alternative (2.12) cannot occur. On the other hand, observe that

$$\sup_\Omega w = w^* > \phi(\gamma) = \sup_{\partial\Omega} w,$$

and the other alternative cannot occur too. This gives the desired contradiction. □

The following result is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let $(\mathbb{P}^n, \langle,\rangle_{\mathbb{P}})$ be an n-dimensional Riemannian manifold and assume that the WMP holds on $\mathbb{P}^n$ for the mean curvature operator.*

Let $\overline{M} = I \times_\varrho \mathbb{P}^n$ and, for a smooth function $u : \mathbb{P} \to I \subseteq \mathbb{R}$, let $\Sigma(u)$ be a minimal graph in $\overline{M}$. Assume that u and $|Du|_\mathbb{P}$ are bounded above. Then either $\Sigma(u)$ is a slice $\mathbb{P}_{u_0}$ (with $\varrho'(u_0) = 0$) or $\varrho'(u^*) \leq 0$, with $u^* = \sup_\mathbb{P} u$.

In the next result we estimate H from below.

Theorem 3.3. *Let $(\mathbb{P}^n, \langle , \rangle_\mathbb{P})$ be an n-dimensional Riemannian manifold and assume that the WMP holds on $\mathbb{P}^n$ for the mean curvature operator, and let $\mathcal{U} \subset \mathbb{P}$ be an open subset with $\partial\mathcal{U} \neq \emptyset$. Let $\overline{M} = I \times_\varrho \mathbb{P}^n$ and for $u \in \mathcal{C}^0(\overline{\mathcal{U}}) \cap \mathcal{C}^\infty(\mathcal{U})$ with $u(\overline{\mathcal{U}}) \subset I$, let $\Sigma(u)$ be a graph in $\overline{M}$ with $\sup_\mathcal{U} H \leq 0$. Assume that u and $|Du|_\mathbb{P}$ are bounded above. If $\sup_\mathcal{U} u > \sup_{\partial\mathcal{U}} u$ then $\mathcal{H}(\sup_\mathcal{U} u) \leq \sup_\mathcal{U} H$.*

Proof. Since $\sup_\mathcal{U} u > \sup_{\partial\mathcal{U}} u$, we know that u is non-constant. We reason by contradiction and we suppose that $\mathcal{H}(\sup_\mathcal{U} u) > \sup_\mathcal{U} H$. Now we proceed as in the proof of Theorem 3.1 by choosing $\gamma < \sup_\mathcal{U} u$, sufficiently near to $\sup_\mathcal{U} u$ such that $\mathcal{H}(t) \leq \sup_\mathcal{U} H$ for $t \in [\gamma, \sup_\mathcal{U} u]$ and $\overline{\Omega}_\gamma \subset \mathcal{U}$, where

$$\Omega_\gamma = \{x \in \mathcal{U} : u(x) > \gamma\}. \qquad \square$$

Corollary 3.4. *Let $(\mathbb{P}^n, \langle , \rangle_\mathbb{P})$ be an n-dimensional Riemannian manifold and assume that the WMP holds on $\mathbb{P}^n$ for the mean curvature operator, and let $\mathcal{U} \subset \mathbb{P}$ be an open subset with $\partial\mathcal{U} \neq \emptyset$. Let $\overline{M} = I \times_\varrho \mathbb{P}^n$ with $\varrho' > 0$, and for $u \in \mathcal{C}^0(\overline{\mathcal{U}}) \cap \mathcal{C}^\infty(\mathcal{U})$ with $u(\overline{\mathcal{U}}) \subset I$, let $\Sigma(u)$ be a minimal graph in $\overline{M}$. Assume that u and $|Du|_\mathbb{P}$ are bounded above. Then $\sup_\mathcal{U} u = \sup_{\partial\mathcal{U}} u$.*

In the results above we have assumed the validity of the WMP for the mean curvature operator. A sufficient condition for this is given by the completeness of the Riemannian manifold $(\mathbb{P}^n, \langle , \rangle_\mathbb{P})$ together with the following volume growth condition

$$\liminf_{r \to +\infty} \frac{\log \mathrm{vol} B_r}{r^2} < +\infty, \tag{3.13}$$

where B_r is the geodesic ball in $\mathbb{P}$ centered at o and with radius r. To see this fact apply Theorem 4.1 in [23] with $\mu = \sigma = 0$, $\delta = 1$ and $\varphi(t) = t/\sqrt{1+t^2}$. Therefore, as another application of Theorem 3.1 we have the following result.

Corollary 3.5. *Let $(\mathbb{P}^n, \langle , \rangle_\mathbb{P})$ be a complete n-dimensional Riemannian manifold. Fix an origin $o \in \mathbb{P}$ and suppose that condition (3.13) holds. Let*

$\overline{M} = \mathbb{R} \times_\varrho \mathbb{P}^n$ and for $u \in \mathcal{C}^\infty(\mathbb{P})$ let $\Sigma(u)$ be a graph in $\overline{M}$ with $H^* = \sup_{\mathbb{P}} H \leq 0$. Assume that u and $|Du|_{\mathbb{P}}$ are bounded above. Then either $\Sigma(u)$ is a slice $\mathbb{P}_{u_0}$ (with $\mathcal{H}(u_0) = H^* \equiv H$) or $\mathcal{H}(u^*) \leq H^*$, with $u^* = \sup_{\mathbb{P}} u$.

See Corollary 1.2 in the Introduction for an interesting application to entire minimal graphs in the hyperbolic space, as well as the interpretation of our results as kind of half-space theorems.

3.2. Hypersurfaces with constant mean curvature or higher order mean curvature

We begin with the case of constant mean curvature. In our next result we only require the validity of the WMP for the Laplace-Beltrami operator, which is equivalent to the stochastic completeness of the manifold [22].

Theorem 3.6. *Let $F : \Sigma \to I \times_\varrho \mathbb{P}^n$ be a stochastically complete, constant mean curvature hypersurface such that, for a correct orientation of the normal N, $H \geq 0$. Suppose that the height function h is bounded above on Σ. If $\tau \in \mathbb{R}$ is such that $\mathcal{H}(\tau) > H$ and $\mathcal{H}'(t) \geq 0$ for $t > \tau$, then $h(x) \leq \tau$ on Σ.*

Proof. We reason by contradiction and suppose that $h^* > \tau$. Observe that h cannot be constant on Σ. Otherwise $h \equiv h^*$ and $F(\Sigma)$ is the slice $\{h^*\} \times \mathbb{P}^n$ with constant mean curvature $H = \mathcal{H}(h^*)$. But $\mathcal{H}$ is non decreasing for $t > \tau$, and $h^* > \tau$ implies $H = \mathcal{H}(h^*) \geq \mathcal{H}(\tau)$, contradicting the hypothesis $H < \mathcal{H}(\tau)$.

Therefore, h is non constant and we can choose a regular value τ_0, with $\tau < \tau_0 < h^*$, so that $\partial\Omega_{\tau_0} \neq \emptyset$, where

$$\Omega_{\tau_0} = \{x \in \Sigma : h(x) > \tau_0\}.$$

From (22) of Proposition 6 in [5] we have

$$\Delta\sigma(h) = n\varrho(h)(\mathcal{H}(h) + \Theta H).$$

Because of the assumptions on $\mathcal{H}$ and $\mathcal{H}'$, on Ω_{τ_0} we have $\varrho(h) \geq \varrho(\tau_0) > 0$ and $\mathcal{H}(h) \geq \mathcal{H}(\tau_0) \geq \mathcal{H}(\tau) > H$. Since $H \geq 0$, then $\Theta H \geq -H$ and

$$\mathcal{H}(h) + \Theta H \geq \mathcal{H}(h) - H \geq \mathcal{H}(\tau_0) - H > 0,$$

so that

$$\Delta\sigma(h) \geq n\varrho(\tau_0)(\mathcal{H}(\tau_0) - H) \quad \text{on } \Omega_{\tau_0}.$$

From (3.4), $\sigma(t)$ is an increasing function and therefore

$$\Lambda_{\sigma(\tau_0)} = \{x \in \Sigma : \sigma(h(x)) > \sigma(\tau_0)\} = \Omega_{\tau_0} \quad \text{and} \quad \partial\Lambda_{\sigma(\tau_0)} = \partial\Omega_{\tau_0}.$$

We set $\Omega = \Lambda_{\sigma(\tau_0)}$ and $v = \sigma(h)|_{\overline{\Omega}}$, so that

$$\Delta v \geq n\varrho(\tau_0)(\mathcal{H}(\tau_0) - H) > 0 \quad \text{on } \Omega$$

and

$$\sup_{\Omega} v = \sigma(h^*) < +\infty.$$

Applying Theorem 2.5, either $\mathcal{H}(\tau_0) - H \leq 0$ or $\sup_\Omega v = \sup_{\partial\Omega} v$. But $\mathcal{H}(\tau_0) \geq \mathcal{H} > H$ and $\sup_\Omega v = \sigma(h^*) > \sigma(\tau_0) = \sup_{\partial\Omega} v$, obtaining the desired contradiction. □

We now focus our attention on higher order mean curvatures. First we recall their definition and some useful facts. We let A denote the Weingarten operator in the direction of N. Its eigenvalues $\lambda_1, \ldots, \lambda_n$ are called the principal curvatures (in the N direction) of the two-sided hypersurface Σ. Their elementary symmetric functions S_k, $k = 0, \ldots, n$, $S_0 \equiv 1$, define the k-mean curvatures of the hypersurface by the formula

$$H_k = \binom{n}{k}^{-1} S_k.$$

Thus $H_1 = H$ is the mean curvature, H_n is the Gauss-Kronecker curvature, and H_2 is, when the ambient space is Einstein, a multiple of the scalar curvature modulo an additive constant. The Newton tensors associated to the immersion are inductively defined by

$$P_0 = I, \quad P_k = S_k I - A \circ P_{k-1},$$

where I is the identity on $T\Sigma$. Note, for further use, that

$$\mathrm{tr} P_k = (n-k)S_k, \quad \mathrm{tr} A \circ P_k = (k+1)S_{k+1}.$$

Associated to each globally defined Newton tensor $P_k : T\Sigma \to T\Sigma$, we consider the second order differential operator $L_k : \mathcal{C}^\infty(\Sigma) \to \mathcal{C}^\infty(\Sigma)$ given by

$$L_k u = \mathrm{tr}(P_k \circ \mathrm{hess}(u)).$$

In particular, L_0 is the Laplace-Beltrami operator. Observe that

$$L_k u = \operatorname{div}(P_k(\nabla u)) - \langle \operatorname{div} P_k, \nabla u \rangle.$$

This implies that L_k is elliptic if and only if P_k is positive definite.

Note that the ellipticity of the operator L_1 is guaranteed by the assumption $H_2 > 0$. Indeed, if this happens the mean curvature does not vanish on Σ, because of the basic inequality $H_1^2 \geq H_2$. Therefore, the immersion is automatically two-sided and furthermore

$$n^2 H_1^2 = \sum_{j=1}^{n} \lambda_j^2 + n(n-1)H_2 > \lambda_i^2$$

for every $i = 1, \dots, n$, and thus the eigenvalues $\mu_{1,i}$ of P_1 satisfy $\mu_{1,i} = nH_1 - \lambda_i > 0$ for every i up to a chosen N so that $H_1 > 0$. This shows ellipticity of L_1. Regarding L_j when $j \geq 2$, we will assume the existence of an elliptic point in Σ, that is, a point $p \in \Sigma$ at which the Weingarten operator A has positive eigenvalues with respect to an appropriate orientation. By Lemma 1 in [19], the existence of p implies that if $H_k > 0$ everywhere on Σ then the same holds for H_j, $j = 1, \dots, k-1$, and

$$H_1 \geq H_2^{1/2} \geq \cdots \geq H_{k-1}^{1/(k-1)} \geq H_k^{1/k} > 0 \quad \text{on } \Sigma, \tag{3.14}$$

with equality at any stage only at umbilical points. In particular, the hypersurface is two-sided. Moreover, by Proposition 3.2 in [7] we also know that each operator L_j is elliptic for $j \leq k-1$. Observe that the existence of an elliptic point is not guaranteed, in general, even in the compact case. For instance, it is clear that totally geodesic spheres and Clifford tori in $\mathbb{S}^{n+1}$ are examples of compact isoparametric hypersurfaces without elliptic points. On the contrary, it is not difficult to see that every compact hypersurface in an open hemisphere has elliptic points.

In what follows with K_M we shall indicate the sectional curvature of a Riemannian manifold M. In order to guarantee the validity of the WMP for the type of operators that we will use in the next result, which are trace type operators that cannot be put in divergence form, one needs to go through the validity of the strong maximum principle for them via a lower bound assumption on K_Σ. This could have been done also in Theorem 3.6 relaxing the assumption on K_Σ to the Ricci curvature of Σ.

Theorem 3.7. *Let $F : \Sigma \to I \times_\varrho \mathbb{P}^n$ be a complete, constant k-mean curvature hypersurface for some $2 \leq k \leq n$ with $\sup_\Sigma |H_1| < +\infty$. Assume the* *existence of an elliptic point on Σ so that, for a correct orientation of the normal N, $H_k > 0$. Suppose*

$$K_\Sigma(x) \geq -G^2(r(x))$$

for some $G \in C^1(\mathbb{R}^+)$ satisfying

$$i) G(0) > 0; \qquad ii) G'(t) \geq 0; \qquad iii) \frac{1}{G(t)} \notin L^1(+\infty).$$

Assume that $\mathcal{H}(t) > 0$ and that there exists $\tau \in \mathbb{R}$ such that $\mathcal{H}(\tau) > H_k^{1/k}$, with $\mathcal{H}' \geq 0$ for $t > \tau$, and let $\Omega_\tau = \{x \in \Sigma : h(x) > \tau\}$. If the height function h is bounded above on Σ, then either $\sup_{\Omega_\tau} \Theta > 0$ or $\Omega_\tau = \emptyset$ (that is, $h(x) \leq \tau$ on Σ).

Proof. Assume that $\Omega_\tau \neq \emptyset$ and by contradiction suppose that $\Theta \leq 0$ on Ω_τ. For the time being, assume the validity of the WMP on Σ for the operator $\tilde{\mathcal{L}}_{k-1}$ defined on $C^2(\Sigma)$ functions by

$$\tilde{\mathcal{L}}_{k-1} u = \operatorname{tr}(\tilde{\mathcal{P}}_{k-1} \circ \operatorname{hess}(u)), \tag{3.15}$$

where

$$\tilde{\mathcal{P}}_{k-1} = \sum_{j=0}^{k-1} \frac{c_{k-1}}{c_j} \mathcal{H}(h)^{k-1-j} |\Theta|^j P_j,$$

with $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$.

Since $\Omega_\tau \neq \emptyset$, then $h^* > \tau$. If h is constant on Σ, then $h \equiv h^*$ and $F(\Sigma)$ is the slice $\{h^*\} \times \mathbb{P}^n$ with k-mean curvature $H_k = \mathcal{H}(h^*)^k$. But $\mathcal{H}$ is non decreasing for $t > \tau$, and $h^* > \tau$ implies

$$H_k^{\frac{1}{k}} = \mathcal{H}(h^*) \geq \mathcal{H}(\tau),$$

which contradicts the hypothesis $\mathcal{H}(\tau) > H_k^{\frac{1}{k}}$. Hence, we can suppose that h is non constant and we therefore can fix a regular value $\tau < \tau_0 < h^*$ for which $\partial\Omega_{\tau_0} \neq \emptyset$. Note that since $\Omega_{\tau_0} \subset \Omega_\tau$ then $\Theta \leq 0$ on Ω_{τ_0}. Hence, on Ω_{τ_0} the operator $\tilde{\mathcal{P}}_{k-1}$ becomes

$$\tilde{\mathcal{P}}_{k-1} = \mathcal{P}_{k-1} = \sum_{j=0}^{k-1} (-1)^j \frac{c_{k-1}}{c_j} \mathcal{H}(h)^{k-1-j} \Theta^j P_j.$$

Therefore, because of the equation following (34) in [5], we have

$$\tilde{\mathcal{L}}_{k-1}\sigma(h) = c_{k-1}\varrho(h)\left(\mathcal{H}(h)^k + (-1)^{k-1}\Theta^k H_k\right) \quad \text{on } \Omega_{\tau_0}. \tag{3.16}$$

Because of the assumptions on $\mathcal{H}$ and $\mathcal{H}'$, on Ω_{τ_0} we have $\varrho(h) \geq \varrho(\tau_0) > 0$ and $\mathcal{H}(h)^k \geq \mathcal{H}(\tau_0)^k \geq \mathcal{H}(\tau)^k > H_k$. Since $H_k > 0$, then $(-1)^{k-1}\Theta^k H_k \geq -H_k$ and

$$\mathcal{H}(h)^k + (-1)^{k-1}\Theta^k H_k \geq \mathcal{H}(h)^k - H_k \geq \mathcal{H}(\tau_0)^k - H_k > 0,$$

so that

$$\tilde{\mathcal{L}}_{k-1}\sigma(h) \geq c_{k-1}\varrho(\tau_0)(\mathcal{H}(\tau_0)^k - H_k) \quad \text{on } \Omega_{\tau_0}.$$

From (3.4), $\sigma(t)$ is an increasing function and therefore

$$\Lambda_{\sigma(\tau_0)} = \{x \in \Sigma : \sigma(h(x)) > \sigma(\tau_0)\} = \Omega_{\tau_0} \quad \text{and} \quad \partial\Lambda_{\sigma(\tau_0)} = \partial\Omega_{\tau_0}.$$

We set $\Omega = \Lambda_{\sigma(\tau_0)}$ and $v = \sigma(h)|_{\overline{\Omega}}$, so that

$$\tilde{\mathcal{L}}_{k-1}v \geq c_{k-1}\varrho(\tau_0)(\mathcal{H}(\tau_0)^k - H_k) > 0 \quad \text{on } \Omega$$

and

$$\sup_{\Omega} v = \sigma(h^*) < +\infty.$$

Applying Theorem 2.5, either $\mathcal{H}(\tau_0)^k - H_k \leq 0$, which is impossible, or $\sup_\Omega v = \sup_{\partial\Omega} v$, which is also impossible because of $\sup_\Omega v = \sigma(h^*) > \sigma(\tau_0) = \sup_{\partial\Omega} v$. This gives the desired contradiction.

It remains to prove the validity of the WMP on Σ for $\tilde{\mathcal{L}}_{k-1}$. In the assumptions of the Theorem we know that

$$H_1 \geq H_j^{1/j} \geq H_K^{1/k} > 0, \quad j = 1, \ldots, k-1. \tag{3.17}$$

Since $\mathcal{H}(h) > 0$ on Σ, $\tilde{\mathcal{P}}_{k-1}$ is positive definite. Furthermore we have

$$\begin{aligned} \operatorname{tr}(\tilde{\mathcal{P}}_{k-1}) &= c_{k-1}\sum_{j=0}^{k-1}|\Theta|^j\mathcal{H}(h)^{k-1-j}H_j \\ &\geq c_{k-1}\mathcal{H}(h)^{k-1} > 0 \quad \text{on } \Sigma, \end{aligned}$$

and

$$\operatorname{tr}(\tilde{\mathcal{P}}_{k-1}) \leq c_{k-1} \sum_{j=0}^{k-1} \mathcal{H}(h)^{k-1-j} H_j^*.$$

Hence from the request

$$\sup_\Sigma |H_1| < +\infty$$

using (3.17) we have

$$0 < \operatorname{tr}(\tilde{\mathcal{P}}_{k-1})(x) \leq \Lambda \tag{3.18}$$

on Σ, for some positive constant Λ.

By the assumption on the sectional curvature of Σ, using Theorem 3 in [4], we deduce that the $\frac{1}{\operatorname{tr}(\tilde{\mathcal{P}}_{k-1}(x))}$-WMP holds on Σ for the operator $\tilde{\mathcal{L}}_{k-1}$. However, because of (3.18), $\frac{1}{\operatorname{tr}(\tilde{\mathcal{P}}_{k-1}(x))}$ is bounded from below by a positive constant and therefore the WMP holds for $\tilde{\mathcal{L}}_{k-1}$. □

Remark 3.8. Theorem 3.7 complements Theorem 6.2 of [5] and it extends the first part of Proposition 4 of [12] to the non-compact case.

4. Geometric applications to hypersurfaces in product spaces

In what follows we shall consider the case of a Riemannian product $\mathbb{R} \times \mathbb{P}^n$. From now on, if the angle function Θ of a two-sided hypersurface does not change sign, the orientation N will be chosen so that $\Theta \leq 0$. Observe that if the hypersurface is a local graph over $\mathbb{P}^n$, then either $\Theta > 0$ or $\Theta < 0$. Thus, requiring Θ not to change sign is an assumption weaker than that of being a local graph.

4.1. Hypersurfaces with constant mean curvature

We begin by considering the case of constant mean curvature. Equation (3.8) of Theorem 3.1 of [2] shows that if $F : \Sigma^n \to \mathbb{R} \times \mathbb{P}^n$ is a two-sided hypersurface with constant mean curvature H and

$$\phi = hH + \Theta \tag{4.1}$$

then

$$\Delta\phi = -\Theta\left(|A|^2 - nH^2 + \operatorname{Ric}_{\mathbb{P}}(\hat{N}, \hat{N})\right). \tag{4.2}$$

Here A is the Weingarten operator of the hypersurface and $\hat{N}$ denotes the projection of N onto the fiber $\mathbb{P}^n$, that is,

$$N = \langle N, \partial_t \rangle \partial_t + \hat{N}.$$

In particular,

$$|\hat{N}|_{\mathbb{P}}^2 = |\nabla h|^2 \leq 1. \tag{4.3}$$

We also have

$$\Delta h = nH\Theta. \tag{4.4}$$

Recalling that the WMP for Δ on Σ is equivalent to the stochastic completeness of $(\Sigma, \langle , \rangle)$, we have

Theorem 4.1. *Let $F : \Sigma^n \to \mathbb{R} \times \mathbb{P}^n$ be a stochastically complete hypersurface with constant mean curvature $H > 0$. Suppose that for some $\alpha > 0$*

$$\mathrm{Ric}_{\mathbb{P}} \geq -n\alpha \tag{4.5}$$

and

$$H^2 > \alpha. \tag{4.6}$$

Let $\Omega \subset \Sigma$ be an open set with $\partial\Omega \neq \emptyset$ for which $F(\Omega)$ is contained in a slab and $F(\partial\Omega) \subset \mathbb{P}_0 = \{0\} \times \mathbb{P}^n$. If

$$\beta = \sup_{\Omega} \Theta < 0 \tag{4.7}$$

then

$$F(\Omega) \subset \left[0, \frac{(1+\beta)H}{H^2 - \alpha}\right] \times \mathbb{P}^n. \tag{4.8}$$

In particular, this happens for every relatively compact set $\Omega \subset \Sigma$ with $\partial\Omega \neq \emptyset$ such that $F(\partial\Omega) \subset \mathbb{P}_0 = \{0\} \times \mathbb{P}^n$ and $\beta = \sup_\Omega \Theta < 0$.

Proof. For any $\delta > 0$ such that

$$\alpha < \alpha + \frac{\delta}{n} \leq H^2$$

let us consider the function

$$\psi = \phi - \frac{\alpha + \delta/n}{H} h = \Theta + \frac{H^2 - \alpha - \delta/n}{H} h. \tag{4.9}$$

Then using (4.2) and (4.4) we obtain

$$\Delta \psi = -\Theta(|A|^2 - nH^2 + \mathrm{Ric}_{\mathbb{P}}(\hat{N}, \hat{N}) + n\alpha + \delta).$$

From (4.5), using also (4.3) and the fact that $\alpha > 0$, we have

$$\mathrm{Ric}_{\mathbb{P}}(\hat{N}, \hat{N}) \geq -n\alpha |\hat{N}|^2_{\mathbb{P}} = -n\alpha |\nabla h|^2 \geq -n\alpha \quad \text{ on } \Sigma.$$

Since $|A|^2 \geq nH^2$, this yields $\Delta\psi \geq -\Theta\delta$ on Σ, and by (4.7) we have

$$\Delta \psi \geq -\Theta\delta \geq -\beta\delta > 0 \quad \text{ on } \Omega.$$

We define $v = \psi|_{\overline{\Omega}}$. Then, since $F(\Omega)$ is contained in a slab we deduce

$$\begin{cases} \Delta v \geq -\beta\delta > 0 \text{ on } \Omega; \\ \sup_\Omega v < +\infty. \end{cases} \tag{4.10}$$

Since Σ is stochastically complete and alternative (2.12) of Theorem 2.5 cannot occur, we obtain

$$\sup_{\Omega} v = \sup_{\partial\Omega} v.$$

But $F(\partial\Omega) \subset \{0\} \times \mathbb{P}^n$ so that $h \equiv 0$ on $\partial\Omega$ and then $v \equiv \psi \equiv \Theta \leq \beta$ on $\partial\Omega$, so that

$$\beta \geq \sup_{\partial\Omega} v = \sup_{\Omega} v.$$

We thus have

$$\beta \geq v = \psi = \Theta + \frac{H^2 - \alpha - \delta/n}{H} h \geq -1 + \frac{H^2 - \alpha - \delta/n}{H} h \quad \text{ on } \Omega.$$

That is,

$$h(x) \leq \frac{(1+\beta)H}{H^2 - \alpha - \delta/n} \quad \text{ on } \Omega$$

for each $\delta > 0$ such that $\alpha < \alpha + \delta/n \leq H^2$. Letting $\delta \to 0^+$ we conclude

$$h(x) \leq \frac{(1+\beta)H}{H^2 - \alpha} \quad \text{on } \Omega. \tag{4.11}$$

On the other hand, from (4.4) and (4.7)

$$\Delta h \leq nH\beta < 0 \quad \text{on } \Omega$$

and since $F(\Omega)$ is contained in a slab, the function $w = h|_{\overline{\Omega}}$ is bounded below. Reasoning as above, using now the dual statement in Remark 2.6, we deduce

$$\inf_{\Omega} w = \inf_{\partial\Omega} w = 0.$$

that is

$$h(x) \geq 0 \quad \text{on } \Omega. \tag{4.12}$$

Putting (4.11) and (4.12) together we obtain (4.8).

The last statement follows from the fact that, being Ω relatively compact, $F(\Omega)$ is contained in a slab. □

In Theorem 4.1 if we assume that Σ is parabolic for the Laplace-Beltrami operator Δ, then (4.7) can be relaxed to

$$\Theta \leq 0 \quad \text{on } \Omega,$$

conclusion (4.8) holding with no changes. To see this, simply observe that since β could be 0, instead of (4.10) we have

$$\begin{cases} \Delta v \geq 0 \text{ on } \Omega; \\ \sup_{\Omega} v < +\infty. \end{cases} \tag{4.13}$$

By Ahlfors parabolicity either

$$\sup_{\Omega} v = \sup_{\partial\Omega} v \tag{4.14}$$

or v is constant on Ω, and in this latter case (4.14) still holds. The rest of the proof is as in Theorem 4.1. The same applies to the reasoning for the lower bound $h(x) \geq 0$.

Next we observe that also the "limit" case $\alpha = 0$, in other words $\mathrm{Ric}_{\mathbb{P}} \geq 0$, can be easily treated. Indeed, fix $\hat{\alpha} > 0$ sufficiently small that (4.6) holds. Then (4.5) is obviously true with $\hat{\alpha}$ instead of α. Applying Theorem 4.1 and letting $\hat{\alpha} \downarrow 0^+$ instead of (4.8) we deduce the improved height estimate

$$F(\Omega) \subset \left[0, (1+\beta)\frac{1}{H}\right] \times \mathbb{P}^n.$$

Thus putting together the above observations, we have that if we strengthen the assumption of stochastic completeness to parabolicity for the operator Δ and we require $\mathrm{Ric}_{\mathbb{P}} \geq 0$ we can get rid of (4.6) and relax assumption (4.7) to $\Theta \leq 0$ on Ω to obtain the height estimate

$$F(\Omega) \subset \left[0, \frac{1}{H}\right] \times \mathbb{P}^n. \tag{4.15}$$

In other words we have

Corollary 4.2. *Let $F : \Sigma^n \to \mathbb{R} \times \mathbb{P}^n$ be a parabolic hypersurface with constant mean curvature $H > 0$ and assume $\mathrm{Ric}_{\mathbb{P}} \geq 0$. Let $\Omega \subset \Sigma$ be an open set with $\partial\Omega \neq \emptyset$ for which $F(\Omega)$ is contained in a slab and $F(\partial\Omega) \subset \mathbb{P}_0 = \{0\} \times \mathbb{P}^n$. If $\Theta \leq 0$ on Ω then*

$$F(\Omega) \subset \left[0, \frac{1}{H}\right] \times \mathbb{P}^n.$$

This result directly compares with the height estimates obtained by Cheng and Rosenberg, for Σ compact, in [10] (see also [15]).

4.2. Some remarks about the stochastic completeness condition and alternative statements of Theorem 4.1

Related to Theorem 4.1, it is worth pointing out that there are geometric conditions that imply the stochastic completeness of Σ^n or, equivalently, the validity of the WMP for the Laplace operator. For instance, as proved by Grigor'yan in [13] (see also [14, Theorem 9.1]), completeness of Σ^n and the volume growth condition

$$\frac{r}{\log \mathrm{vol} B_r} \notin L^1(+\infty), \tag{4.16}$$

where B_r is the geodesic ball in Σ^n centered at a fixed origin and with radius r, imply the validity of the WMP for Δ on Σ^n. Condition (4.16) holds, in particular, if

$$\liminf_{r\to+\infty}\frac{\log \mathrm{vol}B_r}{r^2}<+\infty. \tag{4.17}$$

Therefore, Theorem 4.1 remains true if one changes stochastic completeness by completeness and either condition (4.16) or condition (4.17).

On the other hand, if we assume instead of $\mathrm{Ric}_{\mathbb{P}}\geq -n\alpha$ in Theorem 4.1 that

$$K_{\mathbb{P}}\geq -\alpha \tag{4.18}$$

for some $\alpha>0$, then obviously $\mathrm{Ric}_{\mathbb{P}}\geq -n\alpha$. Moreover, from the Gauss equation for the hypersurface Σ we have that

$$\begin{aligned}K_\Sigma(X,Y)&=\overline{K}(X,Y)+\langle AX,X\rangle\langle AY,Y\rangle-\langle AX,Y\rangle^2\\&\geq \overline{K}(X,Y)-2|A|^2,\end{aligned}$$

where $\{X,Y\}$ is an orthonormal basis for an arbitrary 2-plane tangent to Σ. Here $\overline{K}(X,Y)$ denotes the sectional curvature in the ambient space $\mathbb{R}\times\mathbb{P}^n$ of the 2-plane spanned by $\{X,Y\}$. Observe that

$$\overline{K}(X,Y)=K_{\mathbb{P}}(\hat{X},\hat{Y})|\hat{X}\wedge\hat{Y}|^2,$$

where $\hat{X}$ and $\hat{Y}$ denote the projections of X and Y onto the fiber $\mathbb{P}^n$, respectively, that is,

$$X=\langle X,\partial_t\rangle\partial_t+\hat{X}\quad\text{ and }\quad Y=\langle Y,\partial_t\rangle\partial_t+\hat{Y}.$$

Then, using that

$$|\hat{X}\wedge\hat{Y}|^2\leq |X\wedge Y|^2=1$$

we obtain that $\overline{K}(X,Y)\geq-\alpha$ and hence

$$K_\Sigma(X,Y)\geq -\alpha-2|A|^2. \tag{4.19}$$

Therefore, fixing an origin $o\in\Sigma$ and denoting by $r(x)$ the distance from o in Σ, if

$$|A(x)|\leq G(r(x)) \tag{4.20}$$

we conclude from (4.19) that the radial sectional curvatures from o satisfy

$$K_\Sigma(x)\geq -\alpha-2G(r(x))^2. \tag{4.21}$$

Therefore, if Σ^n is complete and we assume $G \in \mathcal{C}^1([0,+\infty))$ satisfying

$$\text{(i) } G(0) > 0, \quad \text{(ii) } G'(t) \geq 0 \quad \text{and} \quad \text{(iii) } 1/G(t) \notin L^1(+\infty)$$

by Theorem 3 in [4] the Omori-Yau maximum principle (hence the WMP) holds on Σ for Δ. As a consequence, Theorem 4.1 remains true if one changes stochastic completeness and condition (4.5) on the Ricci curvature of $\mathbb{P}^n$ by completeness, condition (4.18) on the sectional curvature of $\mathbb{P}^n$, and condition (4.20) on the growth of the second fundamental form of Σ^n, where G satisfies the above requirements.

Finally, another way to obtain stochastic completeness of Σ^n in Theorem 4.1 is by applying Khas'minskii test [17], which states that a Riemannian manifold is stochastically complete if it supports a $\mathcal{C}^2$ function γ such that $\gamma(x) \to +\infty$ as $x \to \infty$ and satisfying, for some positive constant $\lambda > 0$, $\Delta\gamma \leq \lambda\gamma$ outside a compact subset. It follows from here that Theorem 4.1 remains true if one changes stochastic completeness by the condition that $h : \Sigma^n \to \mathbb{R}$ goes to $+\infty$ as $x \to \infty$ (with no completeness assumption). Actually, since $H > 0$ is constant, it follows from (4.4) that

$$\Delta h \leq nH < +\infty.$$

Therefore, by Khas'minskii test with $\gamma = h$, we derive the stochastic completeness of Σ^n.

4.3. Hypersurfaces with constant higher order mean curvature

Next result extends Theorem 4.1 to higher order mean curvatures.

Theorem 4.3. *Let $F : \Sigma^n \to \mathbb{R} \times \mathbb{P}^n$ be an immersed hypersurface with constant, non-zero, k-mean curvature H_k, for some $k = 2, \ldots, n$ and with an elliptic point. Chosen the normal N so that $H_k > 0$, suppose that for some $\alpha > 0$*

$$K_{\mathbb{P}} \geq -\alpha \tag{4.22}$$

and, having set $H^*_{k-1} = \sup_\Sigma H_{k-1}(x)$,

$$H_k^{\frac{k+1}{k}} > \alpha H^*_{k-1}, \tag{4.23}$$

Suppose that the WMP holds on Σ for the operator L_{k-1}. Let $\Omega \subset \Sigma$ be an open set with $\partial\Omega \neq \emptyset$ for which $F(\Omega)$ is contained in a slab and $F(\partial\Omega) \subset$

$\mathbb{P}_0 = \{0\} \times \mathbb{P}^n$. *If*

$$\beta = \sup_{\Omega} \Theta < 0 \tag{4.24}$$

then

$$F(\Omega) \subset \left[0, \frac{(1+\beta)H_k}{H_k^{\frac{k+1}{k}} - \alpha H_{k-1}^*}\right] \times \mathbb{P}^n. \tag{4.25}$$

Proof. Let us consider the function

$$\phi = H_k^{1/k} h + \Theta.$$

We know from equation (4.1) in Proposition 4.1 in [5] that

$$L_{k-1} h = c_{k-1} H_k \Theta, \tag{4.26}$$

where $c_{k-1} = k\binom{n}{k}$. On the other hand, since H_k is constant from Lemma 7.4 in [5] we also have

$$\begin{aligned} L_{k-1}\Theta = {} & -\Theta\binom{n}{k}(nH_1H_k - (n-k)H_{k+1}) \\ & - \Theta\sum_{i=1}^{n} \mu_{k-1,i} K_{\mathbb{P}}(\hat{E}_i, \hat{N})|\hat{E}_i \wedge \hat{N}|^2. \end{aligned} \tag{4.27}$$

Here the $\mu_{k-1,i}$'s are the eigenvalues of P_{k-1}, $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on Σ diagonalizing A, and the symbol $\hat{}$ denotes projection onto the fiber $\mathbb{P}^n$ of a vector field on the product space $\mathbb{R} \times \mathbb{P}^n$, that is, in our case

$$N = \langle N, \partial_t\rangle \partial_t + \hat{N} \quad \text{and} \quad E_i = \langle E_i, \partial_t\rangle\partial_t + \hat{E}_i, \quad i = 1, \ldots, n.$$

Recall that L_{k-1} is elliptic or, equivalently, the eigenvalues $\mu_{k-1,i}$ are all positive. It follows from (4.26) and (4.27) that

$$\begin{aligned} L_{k-1}\phi = {} & -\Theta\binom{n}{k}(nH_1H_k - (n-k)H_{k+1} - kH_k^{\frac{k+1}{k}}) \\ & - \Theta\sum_{i=1}^{n} \mu_{k-1,i} K_{\mathbb{P}}(\hat{E}_i, \hat{N})|\hat{E}_i \wedge \hat{N}|^2. \end{aligned}$$

Using Garding inequalities,

$$nH_1H_k - kH_k^{\frac{k+1}{k}} \geq nH_k^{\frac{k+1}{k}} - kH_k^{\frac{k+1}{k}} = (n-k)H_k^{\frac{k+1}{k}},$$

hence

$$nH_1H_k - kH_k^{\frac{k+1}{k}} - (n-k)H_{k+1} \geq (n-k)(H_k^{\frac{k+1}{k}} - H_{k+1}) \geq 0.$$

Therefore,

$$L_{k-1}\phi \geq -\Theta \sum_{i=1}^{n} \mu_{k-1,i} K_{\mathbb{P}}(\hat{E}_i, \hat{N})|\hat{E}_i \wedge \hat{N}|^2. \tag{4.28}$$

For any $\delta > 0$ such that

$$\alpha H_{k-1}^* < \alpha H_{k-1}^* + \delta \leq H_k^{\frac{k+1}{k}}$$

let us consider the function

$$\psi = \phi - \frac{\alpha H_{k-1}^* + \delta}{H_k} h = \Theta + \frac{H_k^{\frac{k+1}{k}} - \alpha H_{k-1}^* - \delta}{H_k} h. \tag{4.29}$$

Then using (4.26) and (4.28) we obtain

$$L_{k-1}\psi \geq -\Theta \sum_{i=1}^{n} \mu_{k-1,i} K_{\mathbb{P}}(\hat{E}_i, \hat{N})|\hat{E}_i \wedge \hat{N}|^2 - \Theta c_{k-1}(\alpha H_{k-1}^* + \delta). \tag{4.30}$$

Observe that

$$|\hat{E}_i \wedge \hat{N}|^2 = |\nabla h|^2 - \langle E_i, \nabla h\rangle^2 \leq |\nabla h|^2 \leq 1. \tag{4.31}$$

From (4.22), using also (4.31), the fact that $\alpha > 0$ and each $\mu_{k-1,i} > 0$, we have

$$\begin{aligned}\sum_{i=1}^{n} \mu_{k-1,i} K_{\mathbb{P}}(\hat{E}_i, \hat{N})|\hat{E}_i \wedge \hat{N}|^2 &\geq -\alpha \sum_{i=1}^{n} \mu_{k-1,i} |\nabla h|^2 \geq -\alpha \operatorname{tr}(P_{k-1}) \\ &= -\alpha c_{k-1} H_{k-1} \geq -\alpha c_{k-1} H_{k-1}^*.\end{aligned}$$

That is,

$$\sum_{i=1}^{n} \mu_{k-1,i} K_{\mathbb{P}}(\hat{E}_i, \hat{N}) |\hat{E}_i \wedge \hat{N}|^2 \geq -\alpha c_{k-1} H_{k-1}^*. \tag{4.32}$$

Putting together (4.30) and (4.32), and using (4.24), we finally obtain

$$L_{k-1}\psi \geq -c_{k-1}\Theta\delta \geq -c_{k-1}\beta\delta \quad \text{on } \Omega. \tag{4.33}$$

We define $v = \psi|_{\overline{\Omega}}$. Then, since $F(\Omega)$ is contained in a slab we deduce

$$\begin{cases} L_{k-1}v \geq -c_{k-1}\beta\delta > 0 \text{ on } \Omega; \\ \sup_\Omega v < +\infty. \end{cases}$$

Since the WMP holds on Σ for L_{k-1} and alternative (2.12) of Theorem 2.5 cannot occur we have

$$\sup_\Omega v = \sup_{\partial\Omega} v.$$

But $F(\partial\Omega) \subset \{0\} \times \mathbb{P}^n$ so that $h \equiv 0$ on $\partial\Omega$ and then $v \equiv \psi \equiv \Theta \leq \beta$ on $\partial\Omega$, so that

$$\beta \geq \sup_{\partial\Omega} v = \sup_\Omega v.$$

We thus have

$$\beta \geq v = \psi = \Theta + \frac{H_k^{\frac{k+1}{k}} - \alpha H_{k-1}^* - \delta}{H_k} h \geq -1 + \frac{H_k^{\frac{k+1}{k}} - \alpha H_{k-1}^* - \delta}{H_k} h \quad \text{on } \Omega.$$

that is,

$$h(x) \leq \frac{(1+\beta)H_k}{H_k^{\frac{k+1}{k}} - \alpha H_{k-1}^* - \delta} \quad \text{on } \Omega$$

for each $\delta > 0$ such that $\alpha H_{k-1}^* < \alpha H_{k-1}^* + \delta \leq H_k^{\frac{k+1}{k}}$. Letting $\delta \to 0^+$ we conclude

$$h(x) \leq \frac{(1+\beta)H_k}{H_k^{\frac{k+1}{k}} - \alpha H_{k-1}^*} \quad \text{on } \Omega. \tag{4.34}$$

On the other hand, from (4.24) and (4.26)

$$L_{k-1}h \leq c_{k-1}H_k\beta < 0 \quad \text{on } \Omega,$$

and we conclude as in Theorem 4.1 that

$$h(x) \geq 0 \quad \text{on } \Omega. \tag{4.35}$$

Putting (4.34) and (4.35) together we obtain (4.25). □

The following version of Theorem 4.3 for the "limit" case $\alpha = 0$ can be obtained with a reasoning similar to that used to prove Corollary 4.2.

Corollary 4.4. *Let $F : \Sigma^n \to \mathbb{R} \times \mathbb{P}^n$ be an immersed hypersurface with constant k-mean curvature H_k, for some $k = 2, \ldots, n$ and with an elliptic point (in particular, $H_k \neq 0$), and assume $K_{\mathbb{P}} \geq 0$. Chosen the normal N so that $H_k > 0$, assume that Σ is L_{k-1} parabolic. Let $\Omega \subset \Sigma$ be an open set with $\partial\Omega \neq \emptyset$ for which $F(\Omega)$ is contained in a slab and $F(\partial\Omega) \subset \mathbb{P}_0 = \{0\} \times \mathbb{P}^n$. If $\Theta \leq 0$ on Ω then*

$$F(\Omega) \subset \left[0, \frac{1}{H_k^{\frac{1}{k}}}\right] \times \mathbb{P}^n.$$

4.4. Alternative statements of Theorem 4.3

As in the case of Theorem 4.1, one can give alternative statements of Theorem 4.3 under appropriate geometric conditions which imply the validity of the WMP on Σ for the operator L_{k-1}. For instance, assume that Σ is complete and that, for a fixed origin $o \in \Sigma$,

$$|A(x)| \leq G(r(x)), \tag{4.36}$$

where $r(x)$ denotes the distance from o in Σ and $G \in \mathcal{C}^1([0, +\infty))$ satisfies

$$\text{(i) } G(0) > 0, \quad \text{(ii) } G'(t) \geq 0 \quad \text{and} \quad \text{(iii) } 1/G(t) \notin L^1(+\infty).$$

It then follows (see the proof of (4.19)) that the radial sectional curvatures from o satisfy

$$K_\Sigma(x) \geq -\alpha - 2G(r(x))^2. \tag{4.37}$$

Now observe that (4.37), completeness of Σ and the fact that $H_{k-1}(x) > 0$ on Σ imply, by Theorem 3 in [4], the validity of the q-Omori-Yau maximum principle on Σ for L_{k-1}, with

$$q(x) = \frac{1}{c_{k-1}H_{k-1}(x)}.$$

In particular, the validity of the q-WMP on Σ for L_{k-1}. However, since $H_{k-1}(x)$ is bounded from above on Σ by (4.23), then $q(x)$ is bounded from below by a positive constant, and by the observation after Definition 2.2 this implies that the validity of the WMP for L_{k-1} on Σ. As a consequence, Theorem 4.3 remains true if one replaces the validity of the WMP on Σ for the operator L_{k-1} by the completeness of Σ and condition (4.36) on the growth of its second fundamental form.

On the other hand, the key to give another alternative statement of Theorem 4.1 in subsection 4.2 was to apply Khas'minskii criterium for the stochastic completeness of Σ. In Theorem A of [1] we proved that a similar test yields the validity of the WMP for a wide class of operators including the L_{k-1}'s operators considered above. In particular, Theorem A in [1] with $q(x) \equiv 1$ and $L = L_{k-1}$ states that the WMP holds on Σ for the operator L_{k-1} if Σ supports a $\mathcal{C}^2$ function γ such that $\gamma(x) \to +\infty$ as $x \to \infty$ and $L_{k-1}\gamma \leq B$ outside a compact subset for some constant $B > 0$. It follows from here that Theorem 4.3 remains true if one changes the validity of the WMP on Σ for the operator L_{k-1} by the condition that $h : \Sigma^n \to \mathbb{R}$ goes to $+\infty$ as $x \to \infty$ (with no completeness assumption). In fact, since $H_k > 0$ is constant, it follows from (4.26) that

$$L_{k-1}h \leq c_{k-1}H_k < +\infty.$$

Therefore, choosing $\gamma = h$ we derive the the validity of the WMP on Σ for the operator L_{k-1}

5. Geometric applications to Killing graphs

We now consider the case when that the $(n+1)$-dimensional manifold $\overline{M}$ is endowed with a non-singular Killing vector field Y with complete flow lines and integrable orthogonal distribution. Let $\mathbb{P}$ be a fixed integral leaf. Note that the leaves of the foliation are totally geodesic hypersurfaces of $\overline{M}$. The flow $\Phi : \mathbb{R} \times \mathbb{P} \to \overline{M}$ generated by Y takes isometrically $\mathbb{P} = \mathbb{P}_0$ to the leaf $\mathbb{P}_s = \Phi_s(\mathbb{P})$ for any $s \in \mathbb{R}$, where $\Phi_s = \Phi(s, \cdot)$. We now consider an immersion $F : \mathbb{P} \to \overline{M}$ of the form

$$F(x) = F_u(x) = \Phi(u(x), x) \tag{5.1}$$

for some smooth function $u : \mathbb{P} \to \mathbb{R}$. In this case the hypersurface $F(\mathbb{P})$ is called the Killing graph of u [11]. Since Y is non-singular we can define $\gamma = |Y|^{-2} > 0$. The unit normal to the graph is given by

$$N(x) = \frac{1}{\sqrt{\gamma(x) + |Du|^2(x)}} \left(\gamma(x)Y(x) - \Phi_{u(x)_*}(Du(x))\right), \tag{5.2}$$

where D denotes the covariant derivative on $\mathbb{P}$, and where, for simplicity of notation, we are denoting by γ and Y the restrictions of γ and Y on $\mathbb{P}$ along F. The Killing graph F has constant mean curvature H, in the direction of the normal N, if and only if (see[6])

$$Lu = \mathrm{div}_{\log\sqrt{\gamma}}\left(\frac{Du}{W}\right) = nH \text{ on } \mathbb{P}, \tag{5.3}$$

where,

$$W = \sqrt{\gamma + |Du|^2} \tag{5.4}$$

and L is the operator

$$Lu = \mathrm{div}\left(\frac{Du}{W}\right) - \left\langle \frac{D\gamma}{2\gamma}, \frac{Du}{W} \right\rangle. \tag{5.5}$$

Here div is the divergence on $\mathbb{P}$. We have the following:

Theorem 5.1. *Let $\overline{M}$ be a complete Riemannian manifold endowed with a complete non-singular Killing field Y and let $\mathbb{P}$ be an integral leaf of the Killing foliation. Let $F = F_u : \mathbb{P} \to \overline{M}$ be a Killing graph with constant mean curvature $H \geq 0$. Assume that*

$$\sup_{\mathbb{P}} |Y| < +\infty \tag{5.6}$$

and

$$\liminf_{R\to+\infty} \frac{\log \int_{B_R} |Y|}{R^2} < +\infty, \tag{5.7}$$

where $B_R = B_R(o)$ stands for the geodesic ball in $\mathbb{P}$ centered at a fixed origin o with geodesic radius R.

If there exists a regular value τ of u such that u is bounded above on some connected component of the super level

$$\Omega_\tau = \{x \in \Sigma : u(x) > \tau\}$$

then the Killing graph is minimal.

Proof. First of all, we derive the validity of the WMP for the operator L of (5.5) on $\mathbb{P}$ as an application of Theorem 3.2 of [6]. To apply Theorem 3.2 of [6] we observe that $\mathbb{P}$ is complete and, following the notation of Section 3 of [6], we let $\phi(x) = \log\sqrt{\gamma(x)} = -\log|Y(x)|$ and we choose h to be the metric on $\mathbb{P}$, so that h_- and h_+ are both identically equal to 1. Define

$$\varphi(x,t) = \frac{t}{\sqrt{\gamma(x)+t^2}}.$$

Then φ clearly satisfies i), ii) and iii) in (3.3) of [6] with

$$\delta = 1 \quad \text{and} \quad A(x) = |Y(x)|.$$

Observe that the structure conditions (3.3) of [6] are nothing but our conditions (2.2). Therefore, assumption (3.4) of [6] is guaranteed by (5.6). Following again the notation of Theorem 3.2 of [6], choose $\varsigma = 0$ and $\mu = 0$, so that $\tau = -2$. Then, assumption (3.8) of [6] corresponds to our condition (5.7). Therefore, it follows from Theorem 3.2 of [6] that for each function $u \in \mathcal{C}^2(\mathbb{P})$ such that $u^* = \sup_{\mathbb{P}} u < +\infty$ and each $\gamma < u^*$ we have

$$\inf_{\Omega_\gamma} Lu \leq 0, \quad \text{with} \quad \Omega_\gamma = \{x \in \mathbb{P} : u(x) > \gamma\}.$$

In other words, we obtain the validity of the WMP for the operator L of (5.5) on $\mathbb{P}$.

Let Ω be the connected component of Ω_τ on which u is bounded above. Note that $\emptyset \neq \partial\Omega \subseteq \{x \in \mathbb{P} : u(x) = \tau\}$. Set $v = u|_{\overline{\Omega}}$. By contradiction, suppose $H > 0$. From (5.3)

$$\begin{cases} Lv = nH > 0 \text{ on } \Omega; \\ \sup_\Omega v < +\infty. \end{cases}$$

Applying Theorem 2.5 and noting that, since $H > 0$, alternative (2.12) can not occur, we deduce that

$$\sup_\Omega v = \sup_{\partial\Omega} v = \tau$$

so that $u \equiv \tau$ on Ω. Hence $F_u(x) = \Phi(\tau, x) \subseteq \mathbb{P}_\tau$ on Ω. Thus Ω with the induced metric is isometric to an open set of $\mathbb{P}_\tau$ which is totally geodesic in $\overline{M}$. Therefore $H = 0$, which is a contradiction. $\square$

From the above theorem we deduce the following corollary related to the results given in [6].

Corollary 5.2. *In the assumptions of Theorem 5.1 if u is bounded above then the Killing graph $F_u : \mathbb{P} \to \overline{M}$ is minimal.*

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