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A Picard modular fourfold and the Weyl group $W(E_6)$

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We study the geometry of a Picard modular fourfold which parametrizes abelian fourfolds of Weil type for the field of cube roots of unity. We find a projective model of this fourfold as a singular, degree ten, hypersurface $\mathcal{X}$ in projective 5-space. The Weyl group $W(E_6)$ acts on $\mathcal{X}$ and we provide an explicit description of this action. Moreover, we describe various special subvarieties as well as the boundary of $\mathcal{X}$.

Introduction

The Picard modular fourfold which we consider parametrizes principally polarized abelian varieties of dimension four with an automorphism of order three. The period matrices of these abelian varieties are the fixed points of an element M of order three in $Sp(8, \mathbf{Z})$ which we define in Section 1.1. As far as we know, the elements of finite order in this group have not been classified, and there might, a priori, be other conjugacy classes of elements with properties similar to M. The fixed point locus $\mathcal{H}_4^M$ is a connected submanifold of the Siegel space $\mathcal{H}_4$. The second order thetanulls map the Siegel space to $\mathbf{P}^{15}$ and we show that the image of $\mathcal{H}_4^M$ is a hypersurface in a projective space of dimension five. To find the equation for the image, we use classical relations between thetanulls (and a computer!). It turns out that the closure of the image, denoted by $\mathcal{X}$, is a hypersurface of degree ten.

The elements in $Sp(8, \mathbf{Z})$ which normalize the subgroup generated by M act by projective transformations on $\mathcal{X}$. We show that they generate a subgroup of $Aut(\mathcal{X})$, which is isomorphic to the Weyl group $W(E_6)$ of the root system E_6. Actually the action of $W(E_6)$ on the $\mathbf{P}^5$ is induced from its standard representation.

After having established these basic facts, we consider fixed point loci in $\mathcal{X}$ of elements of $W(E_6)$. These correspond to abelian fourfolds with further automorphisms. To describe these automorphisms, we had to find explicit elements in $Sp(8, \mathbf{Z})$ which represent these automorphisms. Equivalently, we had to find elements, of finite order and up to conjugacy, in the normalizer of M which map to a given subgroup in $W(E_6)$. There seems to be no systematic way to proceed, but in all the examples we succeeded in finding them.

In one case we found a fixed point locus $\mathcal{W}_{10}$ which is the intersection of $\mathcal{X}$ with a hyperplane in $\mathbf{P}^5$. It is defined by a degree ten equation which is the determinant of the 5×5 matrix whose entries are the second partial derivatives of a quartic polynomial. This quartic polynomial defines a threefold known as the Igusa quartic, it is the Satake compactification of the moduli space of abelian surfaces ([vdG], [H]), and $\mathcal{W}_{10}$ is known as the Hessian variety of the Igusa quartic. It was conjectured by Bruce Hunt [H, p.7–8] that $\mathcal{W}_{10}$ is a Shimura variety, and in fact we show that it is the quotient of the Siegel upper half space $\mathcal{H}_2$ by an arithmetic group. We also find a projective model of a moduli space of abelian fourfolds whose endomorphism algebra contains the field of 12-th roots of unity.

The methods we used are based on those from [vG]. The computations involved in this case are however rather more demanding and we used the computer algebra system Magma [M] extensively.

More intrinsically, the Picard modular fourfold we study is the moduli space of abelian fourfolds of Weil type for the field of cube roots of unity. In fact, M induces an automorphism of order three which has two distinct eigenvalues, each with multiplicity two, on the tangent space at the origin of these abelian fourfolds. Such abelian fourfolds are interesting since they have exceptional Hodge classes, shown to be algebraic in this case by C. Schoen [S]. Moreover, the second cohomology group of these abelian fourfolds has a Hodge substructure of K3-type (see [L]). It is not yet known if the Kuga-Satake correspondence is realized by an algebraic cycle. The singular variety $\mathcal{X}$ is a projective model of a Shimura variety associated to the group $SU(2,2)$. It is an interesting problem to find the Hodge numbers $h^{p,0}$, which are birational invariants, for a(ny) desingularization of $\mathcal{X}$. The regular four-forms (if any) on such a desingularization would provide examples of holomorphic modular forms on an arithmetic subgroup of $SU(2,2)$. We hope to return to these matters in the future.

1. Abelian varieties and theta functions

1.1. The automorphism. The group $Sp(8,\mathbf{Z})$ (also written as Γ_4) of 8×8 symplectic matrices with integer coefficients, is defined as

$$Sp(8,\mathbf{Z}) := \{M \in M_8(\mathbf{Z}) : ME{}^tM = E\}, \qquad E := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the 4×4 identity matrix. For $\tau \in \mathcal{H}_4$, the Siegel space of 4×4 complex symmetric matrices with positive definite imaginary part, one defines the principally polarized abelian variety (ppav)

$$A_\tau \cong \mathbf{C}^4/(\mathbf{Z}^8\Omega_\tau), \qquad \Omega_\tau := \begin{pmatrix} \tau \\ I \end{pmatrix}, \quad (\tau \in \mathcal{H}_4),$$

where we consider the elements of $\mathbf{Z}^8$, $\mathbf{C}^4$ as row vectors. The symplectic group $Sp(8,\mathbf{Z})$ acts on $\mathcal{H}_4$, the action of a matrix N with blocks a,b,c,d on $\mathcal{H}_4$ is given by $N\cdot\tau := (a\tau+b)(c\tau+d)^{-1}$ as usual.

To define the abelian fourfolds with an automorphism of order three we introduce the matrices:

$$A := \begin{pmatrix} -I & -I \\ I & 0 \end{pmatrix} \ (\in GL(4,\mathbf{Z})), \qquad M := \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \ (\in Sp(8,\mathbf{Z})),$$

where now I is the 2×2 identity matrix. Both A and M satisfy the equation $x^2+x+1=0$. The fixed point locus of the matrix M above is denoted by

$$\mathcal{H}_4^M = \{\tau \in \mathcal{H}_4 : M\cdot\tau = \tau\} = \left\{ \begin{pmatrix} b+{}^tb & -b \\ -{}^tb & b+{}^tb \end{pmatrix} \in \mathcal{H}_4 \right\},$$

where b is a 2×2 complex matrix, and we used that $M\cdot\tau=\tau$ is equivalent to $A\tau{}^tA = \tau$, which again is equivalent to $A\tau = \tau({}^tA)^{-1}$ and that τ is symmetric. The connectedness of $\mathcal{H}_4^M$ follows from [Fre, Hilfsatz III, 5.14].

For $\tau \in \mathcal{H}_4^M$, the abelian variety A_τ has an automorphism ϕ of order three, in fact $\phi^2+\phi+1=0$, defined by the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}^8 & \xrightarrow{\Omega_\tau} & \mathbf{C}^4 & \longrightarrow & A_\tau & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle {}^tA^{-1}} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \mathbf{Z}^8 & \xrightarrow{\Omega_\tau} & \mathbf{C}^4 & \longrightarrow & A_\tau & \longrightarrow & 0 \end{array} \qquad \left(\text{use } M\Omega_\tau = \Omega_\tau{}^tA^{-1}\right).$$

1.2. Abelian varieties of Weil type and the Hermite upper half space. Identifying $\mathbf{C}^4$ with T_0A_τ, the holomorphic tangent space in 0 to A_τ, we see that the differential of ϕ is given by the matrix ${}^tA^{-1}$. This matrix has eigenvalues ω, $\overline{\omega}$, where $\omega \in \mathbf{C}$ is a primitive cube root of unity, each with multiplicity two. Thus A_τ is an abelian variety of Weil type for the field $\mathbf{Q}(\omega)$. The moduli space of such abelian varieties is isomorphic to $SU(2,2)/S(U(2)\times U(2))$ (cf. [vG, Prop. 5.5]) and this symmetric domain is known as the Hermite upper half space $\mathcal{H}_{2,2}$.

1.3. Remark. In the paper [vG] matrices $M_{p,q}$ were used to investigate projective models of Picard modular varieties. The matrix $M_{2,2}$ is conjugated in $Sp(8,\mathbf{Z})$ to M (where each matrix has 2×2 blocks):

$$M_{2,2} := \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & -I & 0 & I \\ I & 0 & -I & 0 \\ 0 & -I & 0 & 0 \end{pmatrix} = TMT^{-1},$$

$$T := \begin{pmatrix} I & I & 0 & -I \\ I & 0 & -I & I \\ 0 & I & I & -I \\ 0 & -I & 0 & I \end{pmatrix} \quad (\in Sp(8,\mathbf{Z})).$$

Thus T maps $\mathcal{H}_4^M$ to $\mathcal{H}_4^{M_{2,2}}$, and this allows us to apply the results from [vG].

1.4. The theta functions. We introduce the theta functions needed to find the projective model of the Picard modular variety. For $\tau\in\mathcal{H}_4$ and $z\in\mathbf{C}^4$ the classical theta function with characteristics $[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}]$, $\epsilon,\epsilon'\in\{0,1\}^4$, is defined by the series

$$\theta[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}](\tau,z) := \sum_{m\in\mathbf{Z}^4} e^{(m+\epsilon/2)\tau^t(m+\epsilon/2)+2(m+\epsilon/2)^t(z+\epsilon'/2)}.$$

The second order theta functions are the linear combinations of the 16 functions $\theta[\begin{smallmatrix}\epsilon\\ 0\end{smallmatrix}](2\tau,2z)$, $\epsilon\in\{0,1\}^4$. These functions define a holomorphic map $A_\tau\to\mathbf{P}^{15}$ which factors over the Kummer variety $A_\tau/\{\pm1\}$.

We will study the map given by second order thetanulls:

$$\Theta:\mathcal{H}_4\longrightarrow\mathbf{P}^{15},\qquad \tau\longmapsto(\cdots:\theta[\begin{smallmatrix}\epsilon\\ 0\end{smallmatrix}](2\tau,0):\cdots).$$

This map factors over $\mathcal{A}_4(2,4):=\mathcal{H}_4/\Gamma_4(2,4)$, where $\Gamma_4(2,4)$ is a (normal) congruence subgroup of $Sp(8,\mathbf{Z})$ ([I, V.2, p.178]):

$$\Gamma_g(2) := \{M\in Sp(2g,\mathbf{Z}):\ M\equiv I \bmod 2\},$$

$$\Gamma_g(2,4) := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_g(2):\mathrm{diag}(a^tb)\equiv\mathrm{diag}(c^td)\equiv 0 \bmod 4\right\}.$$

The image of Θ is a quasi-projective variety. The map Θ extends to a map from the Satake compactification $\overline{\mathcal{A}_4(2,4)}$ of $\mathcal{A}_4(2,4)$ to $\mathbf{P}^{15}$ whose image is the closure of $\Theta(\mathcal{H}_4)$.

1.5. The classical theta nulls and quadrics. We define quadrics in $\mathbf{P}^{15}$ whose intersection with the Picard modular variety will provide information on special subloci.

The following formula, known as Riemann's bilinear addition formula, relates the $\theta[{}^{\epsilon}_{\epsilon'}]$ to the second order theta functions: (cf. [I, IV.1], [vG, (3.3.2), (3.5.1)]):

$$\theta[{}^{\epsilon}_{\epsilon'}](\tau,z)^2=\sum_{\sigma}(-1)^{(\sigma+\epsilon)^t\epsilon'}\theta[{}^{\sigma}_{0}](2\tau,0)\theta[{}^{\sigma+\epsilon}_{0}](2\tau,2z).$$

Since $\theta[{}^{\epsilon}_{\epsilon'}](\tau,-z)=(-1)^{\epsilon^t\epsilon'}\theta[{}^{\epsilon}_{\epsilon'}](\tau,z)$, the 136 theta functions with $\epsilon^t\epsilon'=0$ are even functions of z, the remaining 120 are odd.

For an even characteristic $[{}^{\epsilon}_{\epsilon'}]$ and for $z=0$, the formula above can be rewritten as

$$\theta[{}^{\epsilon}_{\epsilon'}](\tau,0)^2=Q[{}^{\epsilon}_{\epsilon'}](\ldots,\theta[{}^{\sigma}_{0}](2\tau,0),\ldots),$$

where the functions $\theta[{}^{\epsilon}_{\epsilon'}](\tau,0)$ on $\mathcal{H}_4$ are called thetanulls, and where

$$Q[{}^{\epsilon}_{\epsilon'}]:=\sum_{\sigma}(-1)^{(\sigma+\epsilon)^t\epsilon'}X_\sigma X_{\sigma+\epsilon}$$

is a quadratic polynomial in the 16 variables X_σ, with $\sigma\in(\mathbf{Z}/2\mathbf{Z})^4$. These quadratic polynomials define quadrics $Q[{}^{\epsilon}_{\epsilon'}]=0$ in $\mathbf{P}^{15}$.

The points in the intersection of the closure of $\Theta(\mathcal{H}_4)$ and these quadrics correspond to (limits of) abelian varieties with vanishing thetanulls. The following table summarizes the information we will need in this paper (cf. [vG, 3.6, 3.7, Lemma 3.8] for similar results).

moduli point	# vanishing thetanulls
$E\times A$, E elliptic curve, A abelian 3-fold	28
$B_1\times B_2$, B_i abelian surfaces	36
$(\mathbf{C}^\times)^2\times B$, B abelian surface	96
$(\mathbf{C}^\times)^4$, boundary point	120

2. The Picard Modular variety $\mathcal{X}$

2.1. The action of M on $\mathbf{P}^{15}$. The map Θ is equivariant for the action of M on $\mathcal{H}_4$. In fact, from the definition of the theta functions it is obvious that

$$\theta[{}^{\epsilon}_{0}](2(M\cdot\tau),0)=\theta[{}^{\epsilon A}_{0}](2\tau,0).$$

For $\rho,\sigma\in\mathbf{Z}^4$ one has $\theta[{}^{2\rho+\sigma}_{0}](2\tau,0)=\theta[{}^{\sigma}_{0}](2\tau,0)$. Thus the action of M on these 16 functions is determined by the action of A on $(\mathbf{Z}/2\mathbf{Z})^4$. We will write

$$\begin{aligned} &\theta_i(\tau):=\theta[{}^{\epsilon}_{0}](2\tau,0),\\ \text{with}\quad & i:=\epsilon_1 2^3+\epsilon_2 2^2+\epsilon_3 2+\epsilon_4,\quad \epsilon=(\epsilon_1,\dots,\epsilon_4)\in\{0,1\}^4. \end{aligned}$$

For example θ_{10} has characteristic $\epsilon=(1,0,1,0)$, which we simply write as [1010]. As $\epsilon A=(0,0,1,0)$ mod 2 we see that M maps θ_{10} to θ_2. With this convention, the action of M is as follows: $M\theta_0=\theta_0$, and M has 5 cycles of length three:

$$\begin{array}{ccccccc} \theta_1 & \mapsto & \theta_4 & \mapsto & \theta_5 & \mapsto & \theta_1, \\ \theta_3 & \mapsto & \theta_{12} & \mapsto & \theta_{15} & \mapsto & \theta_3, \\ \theta_7 & \mapsto & \theta_9 & \mapsto & \theta_{14} & \mapsto & \theta_7. \end{array}\qquad \begin{array}{ccccccc} \theta_2 & \mapsto & \theta_8 & \mapsto & \theta_{10} & \mapsto & \theta_2, \\ \theta_6 & \mapsto & \theta_{13} & \mapsto & \theta_{11} & \mapsto & \theta_6, \end{array}$$

Denote the coordinates on $\mathbf{P}^{15}$ by X_ϵ, $\epsilon\in(\mathbf{Z}/2\mathbf{Z})^4$, or by X_i, with the relation between ϵ and i as above. Then we define an action of M on $\mathbf{P}^{15}$ by a projective linear transformation as follows:

$$M:\mathbf{P}^{15}\longrightarrow\mathbf{P}^{15},\qquad M(\cdots:X_\epsilon:\cdots)=(\cdots:X_{\epsilon A}:\cdots),$$

so M permutes the coordinates in the same way as it permutes the theta functions. Obviously, we then have $\Theta(M\tau)=M\Theta(\tau)$, so the map Θ is equivariant for the actions of M on $\mathcal{H}_4$ and $\mathbf{P}^{15}$ respectively.

2.2. The eigenspace $\mathbf{P}^5$. We are now interested in the image of $\mathcal{H}_4^M$ in $\mathbf{P}^{15}$. As the map Θ is equivariant for M, this image must lie in an eigenspace for the action of M on $\mathbf{P}^{15}$. In fact, as $\theta[{}^{\epsilon}_{0}](2(M\cdot\tau))=\theta[{}^{\epsilon A}_{0}](2\tau)$, the image lies in the M-eigenspace, simply denoted by $\mathbf{P}^5$, defined by the linear equations

$$\Theta:\mathcal{H}_4^M\longrightarrow\mathcal{X}\subset\mathbf{P}^5\quad(\subset\mathbf{P}^{15}),\qquad \mathbf{P}^5: X_\epsilon=X_{\epsilon A}\qquad(\epsilon\in(\mathbf{Z}/2\mathbf{Z})^4).$$

This eigenspace $\mathbf{P}^5 \subset \mathbf{P}^{15}$ can thus be parametrized by

$$X \longmapsto (X_0 : X_1 : X_2 : X_3 : X_1 : X_1 : X_6 : X_7 : X_2 : X_7 : \\ X_2 : X_6 : X_3 : X_6 : X_7 : X_3),$$

where $X = (X_0 : X_1 : X_2 : X_3 : X_6 : X_7)$.

2.3. The variety $\mathcal{X}$. The fourfold $\Theta(\mathcal{H}_4^M)$ thus lies in this $\mathbf{P}^5$, and we recall how one can find the equation for its closure $\mathcal{X}$. There are identities, valid for all $\tau \in \mathcal{H}_4$ and for some choice of signs, between even thetanulls of the form

$$\prod_{i=0}^{3} \theta[\begin{smallmatrix}\epsilon_{1i}00\\ \epsilon'_{1i}00\end{smallmatrix}](\tau) \pm \prod_{i=0}^{3} \theta[\begin{smallmatrix}\epsilon_{2i}00\\ \epsilon'_{2i}00\end{smallmatrix}](\tau) \pm \prod_{i=0}^{3} \theta[\begin{smallmatrix}\epsilon_{3i}00\\ \epsilon'_{3i}00\end{smallmatrix}](\tau) \pm \prod_{i=0}^{3} \theta[\begin{smallmatrix}\epsilon_{4i}00\\ \epsilon'_{4i}00\end{smallmatrix}](\tau) = 0$$

for suitable even $g = 2$ characteristics $[\begin{smallmatrix}\epsilon_{ni}\\ \epsilon'_{ni}\end{smallmatrix}]$ (cf. [vG, 4.4],[vGS, 1.8]). Actually the characteristics given in [vGS, 1.8] contain a misprint (one of them is odd!), below are two sets of four $g = 2$ characteristics for which we have identities as above:

$$[\begin{smallmatrix}00\\ 00\end{smallmatrix}],\ [\begin{smallmatrix}00\\ 10\end{smallmatrix}],\ [\begin{smallmatrix}10\\ 00\end{smallmatrix}],\ [\begin{smallmatrix}11\\ 11\end{smallmatrix}], \qquad \text{and} \qquad [\begin{smallmatrix}01\\ 00\end{smallmatrix}],\ [\begin{smallmatrix}01\\ 10\end{smallmatrix}],\ [\begin{smallmatrix}10\\ 00\end{smallmatrix}],\ [\begin{smallmatrix}10\\ 01\end{smallmatrix}].$$

By taking the product of the eight expression on the left hand side for all choices of signs, one obtains a polynomial in the $\theta[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}]^2$. Using the formula from Section 1.5, this can be written as a polynomial of degree 32 in the sixteen second order theta constants $\theta[\begin{smallmatrix}\sigma\\ 0\end{smallmatrix}](2\tau, 0)$. The zero locus of this polynomial in $\mathbf{P}^{15}$ contains the image of $\mathcal{H}_4$, and thus restricting it to the $\mathbf{P}^5$ gives an equation for $\mathcal{X}$. Taking the GCD of two such equations of degree 32, we found that the image is defined by a polynomial F of degree 10. Thus

$$\mathcal{X} = \overline{\Theta(\mathcal{H}_4^M)} = \Theta\big(\overline{\mathcal{A}_4(2,4)}\big) \cap \mathbf{P}^5 = Z(F)$$

where $\overline{\mathcal{A}_4(2,4)}$ is the Satake compactification of $\mathcal{A}_4(2,4) = \mathcal{H}_4/\Gamma_4(2,4)$ and $Z(F)$ is the zero locus of F in $\mathbf{P}^5$.

The polynomial F defining $\mathcal{X}$ is homogeneous of degree 10 in the six variables X_0, X_1, X_2, X_3, X_6, X_7, which give the coordinate functions on $\mathbf{P}^5$, and it has 147 terms. It is symmetric in $X_1, \ldots, X_7$, and can be written as

$$F := F_{10} - X_0X_1X_2X_3X_6X_7F_4$$

where the homogeneous polynomials F_4, F_{10} of degree 4 and 10 respectively, are given by

$$\begin{aligned} F_4 &:= -6S_1^2 + 16S_2 + 4S_1X_0^2 + 2X_0^4, \\ F_{10} &:= S_1S_2^2 - 3S_1^2S_3 + 12S_1S_4 - 48S_5 \\ &\quad + (-S_2^2 + 2S_1S_3 + 4S_4)X_0^2 + S_3X_0^4, \end{aligned}$$

where $S_i(X_1, \ldots, X_7) := s_i(X_1^2, \ldots, X_7^2)$ and s_i is the degree i elementary symmetrical function in the five squares $X_1^2, \ldots, X_7^2$.

2.4. The singular locus of $\mathcal{X}$. The polynomial F defining $\mathcal{X}$ is rather complicated. We relied on Magma to show that the singular locus $Sing(\mathcal{X})$ of $\mathcal{X}$ has dimension two and degree 320. It was then not hard to find 120 quadric surfaces and 80 planes in $Sing(\mathcal{X})$, using Magma again, so we accounted for all two-dimensional components of $Sing(\mathcal{X})$. The general points of these components all correspond to decomposable ppav's, see Propositions 4.3 and 4.5 below. These singular points are in fact quotient singularities in $\mathcal{A}_4(2,4)$. We do not know if there are components of lower dimension in $Sing(\mathcal{X})$.

3. Automorphisms of $\mathcal{X}$

3.1. The Weyl group $W(E_6)$ and $Aut(\mathcal{X})$. The subgroup of $Sp(8, \mathbf{Z})$ of elements which map $\mathcal{H}_4^M$ into itself acts by projective transformations on the eigenspace $\mathbf{P}^5$ of M in $\mathbf{P}^{15}$ and maps $\mathcal{X} = \overline{\Theta(\mathcal{H}_4^M)}$ into itself. We recall the results from [vG] on this subgroup and we show that it acts as the Weyl group of the root system E_6 on $\mathbf{P}^5$.

3.2. Centralizers and normalizers. The normalizer of the subgroup $\langle M \rangle = \{I, M, M^{-1}\}$ in $Sp(8, \mathbf{Z})$ is the subgroup

$$N_M := \{A \in Sp(8, \mathbf{Z}) : AMA^{-1} = M^{\pm 1}\},$$

whereas the centralizer of $\langle M \rangle$ is the subgroup:

$$C_M := \{A \in Sp(8, \mathbf{Z}) : AM = MA\}.$$

An element $A \in N_M$ will either permute M, M^{-1} or it will fix both of them and in that case $M \in C_M$. Thus the index of C_M in N_M is either one or two.

In fact, the index $[N_M : C_M] = 2$ since $M_B M M_B^{-1} = M^{-1}$, where

$$B := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \ (\in GL(4,\mathbf{Z})), \qquad M_B := \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \ (\in Sp(8,\mathbf{Z})),$$

here I is the 2×2 identity matrix.

3.3. The action of the normalizer. Notice that the normalizer N_M maps the Hermite upper half space $\mathcal{H}_4^M$ into itself: if $\tau \in \mathcal{H}_4^M$ then $M\tau = \tau$, hence also $M^{-1}\tau = \tau$ and:

$$M(A\tau) = A(M^{\pm 1}\tau) = A\tau \quad (A \in N_M,\ \tau \in \mathcal{H}_4^M).$$

So we have biholomorphic maps

$$A : \mathcal{H}_4^M \longrightarrow \mathcal{H}_4^M \quad (A \in N_M).$$

In the projective representation of $Sp(8,\mathbf{Z})$ on $\mathbf{P}^{15}$, any $A \in N_M$ will permute the eigenspaces for the action of M on $\mathbf{P}^{15}$. As M has a unique 5-dimensional eigenspace $\mathbf{P}^5$, we must have $A(\mathbf{P}^5) \subset \mathbf{P}^5$ and thus

$$A : \mathbf{P}^5 \longrightarrow \mathbf{P}^5 \quad (A \in N_M).$$

Obviously, $\mathcal{X}$ is mapped into itself and thus we have a homomorphism $N_M \to Aut(\mathcal{X})$.

3.4. A Hermitian form. The action of $M \in Sp(8,\mathbf{Z})$, with $M^2 + M + I = 0$, on $\mathbf{Z}^8$ defines the structure of $\mathbf{Z}[\omega] \cong \mathbf{Z}[x]/(x^2+x+1)$ module on $\mathbf{Z}^8$, where ω acts as M. Using the alternating form on $\mathbf{Z}^8$ defined by the matrix E, which we denote also by E, we define a Hermitian form on $\mathbf{Z}^8$ with values in $\mathbf{Z}[\omega]$ as follows (cf. [vG, Lemma 6.2] and the proof of Proposition 3.5 below):

$$H_M(x,y) := E(x, My) - \omega E(x,y).$$

Let e_i be the i-th standard basis vector of $\mathbf{Z}^8$. Then we have the Gram matrix:

$$\big(H_M(f_i,f_j)\big) = \mathrm{diag}(1,1,-1,-1), \quad f_i := e_i + e_{i+4}, \quad (i,j = 1,\dots,4),$$

so the f_i are an orthogonal $\mathbf{Z}[\omega]$-basis of $\mathbf{Z}^8$ and the signature of H_M is $(2,2)$.

The quotient ring $\mathbf{Z}[\omega]/2\mathbf{Z}[\omega] \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x^2+x+1)$ is isomorphic to the finite field $\mathbf{F}_4$. The Hermitian form H_M on $\mathbf{Z}^8$ defines a Hermitian form on $\mathbf{F}_4^4 \cong (\mathbf{Z}[\omega]/2\mathbf{Z}[\omega])^4$ by reduction modulo 2.

We recall the following results.

3.5. Proposition.

1) *The centralizer* $C_M(\mathbf{R})$ *of* M *in* $Sp(8,\mathbf{R})$ *is isomorphic to* $U(H_M) \cong U(2,2)$*, the unitary group of the (*$\mathbf{R}$*-bilinear extension of the) Hermitian form* H_M *on* $\mathbf{Z}^8 \otimes_{\mathbf{Z}} \mathbf{R}$.

2) *The reduction modulo 2 map induces a surjective homomorphism from* $C_M \subset Sp(8,\mathbf{Z})$ *onto* $U(4,\mathbf{F}_4)$.

3) *The center of* $U(4,\mathbf{F}_4)$ *is cyclic of order* 3 *and is generated by the scalar multiplication* $v \mapsto \omega v$*. The quotient group* $PU(4,\mathbf{F}_4)$ *is a finite simple group of order* 25920.

4) *The reduction modulo 2 map followed by the quotient by* $< M >$*-map is a surjective homomorphism of* N_M *onto* $W(E_6)$*. In particular,* $PU(4, \mathbf{F}_4)$ *is isomorphic to a subgroup of index two of the Weyl group* $W(E_6)$ *of the root system* E_6.

5) *The map* $\Theta : \mathcal{H}_4^M \to \mathbf{P}^5$ *factors over* $\mathcal{H}_4^M / C_M(2)$ *where*

$$C_M(2) := \{A \in C_M : \ A \equiv I \, mod\, 2\} \ = \ C_M \cap \Gamma_g(2).$$

Proof. The isomorphism $C_M(\mathbf{R}) \cong U(H_M)$ is proven in [vG, Prop. 5.5.2]. The reduction map is studied in [vG, Lemma 6.2.3], the structure of $U(4,\mathbf{F}_4)$ and its relation with $W(E_6)$ can be found in the Atlas [At]. The last item is obtained from [vG, Proposition 6.4]. For the sake of completeness we check that H_M is Hermitian. Using that E is alternating, $E(Mx,My) = E(x,y)$, $M^2 = -I - M$, and $1 + \omega = -\omega^2 = -\overline{\omega}$ we have

$$\begin{aligned} H_M(y,x) &= E(y,Mx) - \omega E(y,x) = -E(Mx,y) + \omega E(x,y) \\ &= -E(M^2x, My) + \omega E(x,y) \\ &= E(x,My) + E(Mx,My) + \omega E(x,y) \\ &= E(x,My) - \omega^2 E(x,y) \\ &= \overline{H_M(x,y)} \ . \end{aligned}$$

We also have

$$\begin{aligned} H_M(Mx,y) &= E(Mx,My) - \omega E(Mx,y) = E(x,y) - \omega E(M^2x,My) \\ &= E(x,y) + \omega(E(x,My) + E(Mx,My)) \\ &= -\omega^2 E(x,y) + \omega E(x,My) \\ &= \omega H_M(x,y), \end{aligned}$$

so H_M is indeed $\mathbf{Z}[\omega]$-linear in the first variable. $\square$

3.6. The invariant quadric. The action of N_M on $\mathbf{P}^5$ thus induces an action of $W(E_6)$ on $\mathbf{P}^5$. We will show in Proposition 3.8 that this action is obtained from the standard reflection representation on the root lattice $R(E_6)$ by complexifying and projectivization:

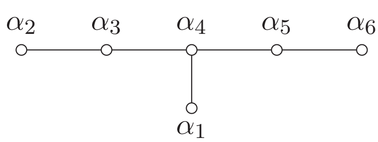
$$\mathbf{P}^5 \cong \mathbf{P}(R(E_6) \otimes_{\mathbf{Z}} \mathbf{C}).$$

Among the 136 quadratic forms Q_m, only one is M-invariant. It is $Q\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = \sum X_\sigma^2$, which restricts to (cf. Section 2.2) $X_0^2 + 3(X_1^2 + X_2^2 + X_3^2 + X_6^2 + X_7^2)$ on $\mathbf{P}^5$. The bilinear form defined by this quadratic form, for convenience multiplied by a scalar, will be denoted by b:

$$b(X,Y) := \tfrac{1}{3}X_0Y_0 + X_1Y_1 + X_2Y_2 + X_3Y_3 + X_6Y_6 + X_7Y_7.$$

We will use b to define the inner product on the root lattice E_6.

3.7. The root system E_6. The root system E_6 is defined by the Dynkin diagram:



The Dynkin diagram of E_6

(so $b(\alpha_i,\alpha_j) = 0, -1, 2$ if α_i and α_j are not connected, are connected or $i = j$ respectively). The following basis of the simple roots of the root system E_6 will be related to action of N_M on $\mathbf{P}^5$:

$$\begin{aligned}
&\alpha_1 := (0,-1,-1,0,0,0), && \alpha_3 := (0,1,-1,0,0,0), && \alpha_5 := (0,0,0,1,-1,0),\\
&\alpha_2 := (3,-1,1,1,1,1)/2, && \alpha_4 := (0,0,1,-1,0,0), && \alpha_6 := (0,0,0,0,1,-1).
\end{aligned}$$

The root lattice of E_6 is $R(E_6) = \oplus_{i=1}^6 \mathbf{Z}\alpha_i$. A root α of E_6 defines a reflection on $\mathbf{C}^6 := R(E_6) \otimes_{\mathbf{Z}} \mathbf{C}$:

$$s_\alpha : \mathbf{C}^6 \longrightarrow \mathbf{C}^6, \qquad s_\alpha(X) = X - \frac{2b(X,\alpha)}{b(\alpha,\alpha)}\alpha \;=\; X - b(X,\alpha)\alpha$$

where we used that $b(\alpha,\alpha) = 2$. The reflections in the simple roots generate the Weyl group $W(E_6)$ of E_6, which is a finite group of order 51840.

The following proposition gives explicit matrices in $N_M \subset Sp(8,\mathbf{Z})$ whose action on $\mathbf{P}^5$ generates the group $W(E_6)$.

3.8. Proposition. *Let $M_B \in N_M$ be as in Section 3.2 and define $M_d, M_e, M_f \in N_M$ by:*

$$M_d := \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_e := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_f := \begin{pmatrix} 0 & B_f \\ -B_f & 0 \end{pmatrix} \quad \textit{with} \quad B_f := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where I is the 2×2 identity matrix.

The action of these elements in N_M on $\mathbf{P}^5$ is induced by the following linear transformations in $W(E_6)$:

$$\begin{aligned} M_B &: (X_0, X_1, X_2, X_3, X_6, X_7) \longmapsto (X_0, X_1, X_2, X_3, X_7, X_6) \\ M_d &: (X_0, X_1, X_2, X_3, X_6, X_7) \longmapsto (X_0, X_1, -X_2, -X_3, -X_6, -X_7), \\ M_e &: (X_0, X_1, X_2, X_3, X_6, X_7) \longmapsto (X_0, X_1, X_2, -X_3, -X_6, X_7), \\ M_f &: X \longmapsto \frac{1}{4} \begin{pmatrix} 1 & 3 & 3 & 3 & 3 & 3 \\ 1 & -1 & 3 & -1 & -1 & -1 \\ 1 & 3 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 3 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 3 \\ 1 & -1 & -1 & -1 & 3 & -1 \end{pmatrix} X, \end{aligned}$$

where $X = (X_0, \ldots, X_7)$ is viewed as a column vector. In particular, M_B acts as s_{α_6}. Moreover, $W(E_6)$ is generated by these four linear transformations on $\mathbf{C}^6$.

Proof. The action of M_B on $\mathbf{P}^{15}$ is very similar to the one of M which we found in Section 2.1, it is simply $M_B : X_\sigma \mapsto X_{\sigma B}$. Thus under the action M_B the coordinates X_0, X_5, X_{10}, X_{15} are fixed and the remaining ten are permuted as follows:

$$X_1 \leftrightarrow X_4, \quad X_2 \leftrightarrow X_8, \quad X_3 \leftrightarrow X_{12},$$
$$X_6 \leftrightarrow X_9, \quad X_7 \leftrightarrow X_{13}, \quad X_{11} \leftrightarrow X_{14}.$$

On the eigenspace $\mathbf{P}^5$ of M, parametrized as in Section 2.2, we then have

$$\begin{aligned} &M_B : \mathbf{P}^5 \longrightarrow \mathbf{P}^5, \\ &M_B : (X_0 : X_1 : X_2 : X_3 : X_6 : X_7) \longmapsto (X_0 : X_1 : X_2 : X_3 : X_7 : X_6), \end{aligned}$$

for example $M_B : X_6 \mapsto X_9$, but on $\mathbf{P}^5$ we have $X_7 = X_9 = X_{14}$.

The action of M_d and M_f is easy to compute using the series expansion of the theta functions. To find the action of M_f, one can use that $M_f = \mathrm{diag}(I, -I, I, -I)E$ where we take 2×2 diagonal blocks and E is as in Section 1.1. The action of E on the second order thetanulls is well-known, on $\mathbf{P}^{15}$ it is given by $X_\sigma \mapsto \sum_\rho (-1)^{\sigma\, {}^t\rho} X_\rho$ (cf. [R, p.60]). By restricting to $\mathbf{P}^5$ we find the matrix in $W(E_6)$. A Magma computation shows that the matrices which define the action on $\mathbf{P}^5$ indeed generate $W(E_6)$. $\square$

3.9. Invariants of $W(E_6)$. The equation F for $\mathcal{X}$, which we determined in Section 2.3, is an invariant for the $W(E_6)$-action. The ring of $W(E_6)$-invariant polynomials in $\mathbf{C}[X_0, \ldots, X_7]$ is generated by invariants of degree k, for $k = 2, 5, 6, 8, 9, 12$ (cf. [C], Chapter 9 and Proposition 10.2.5). Examples of invariants are the I_k defined as the sum of the k-th powers of the hyperplanes perpendicular to the 27 vectors $v_1, \ldots, v_{27}$ in the $W(E_6)$-orbit of the vector $v_1 = (1, 0, \ldots, 0)$, so $I_k = \sum_{i=1}^{27} b(X, v_i)^k$. For example, $I_2 = (3/2)b(X, X)$. A computation shows that, with $c = -2/675$, the polynomial F defining $\mathcal{X}$ can be written as

$$F = c(11520 I_8 I_2 - 4160 I_6 I_2^2 - 4608 I_5^2 + 25 I_2^5).$$

3.10. The quotient $\overline{\mathcal{X}} := \mathcal{X}/W(E_6)$. The quotient variety $\mathbf{P}^5/W(E_6)$ is the weighted projective space $W\mathbf{P}^5 := W\mathbf{P}^5(2, 5, 6, 8, 9, 12)$ and the quotient map is given by the invariants I_k. The projection of $\mathcal{X}/W(E_6) \subset W\mathbf{P}^5$ onto $W\mathbf{P}^4 := W\mathbf{P}^4(2, 6, 8, 9, 12)$ is then a 2:1 branched cover with covering involution given by $I_5 \mapsto -I_5$. Note that the branch locus is reducible, one component is defined by $I_2 = 0$, the other by $11520 I_8 - 4160 I_6 I_2 + 25 I_2^4 = 0$.

4. Decomposable abelian varieties

4.1. The two cases. We consider the case that the abelian fourfold A_τ, for $\tau \in \mathcal{H}_4^M$, is a product of lower dimensional ppav's. We will show the following:

1) The closure of the locus in $\mathcal{X}$ of ppav's which are products of two abelian surfaces consists of 120 (smooth) quadric surfaces (see Section 4.2).
2) The closure of the locus in $\mathcal{X}$ of ppav's which are products of an elliptic curve and an abelian threefold consists of 80 projective planes, we refer to these as Hesse planes (see Section 4.4).

In both cases, the surfaces which parametrize these products lie in $Sing(\mathcal{X})$.

4.2. The products of abelian surfaces. The moduli space of two dimensional ppav's with an automorphism of order three (of type $(1,1)$) and a level two structure has a model which is a $\mathbf{P}^1 \subset \overline{\mathcal{A}_2(2,4)} \cong \mathbf{P}^3$ ([vG, Theorem 8.4]). There are 3 points on this $\mathbf{P}^1$ where the abelian surface degenerates to $(\mathbf{C}^\times)^2$ and there are two points where it decomposes into E_3^2, where the elliptic curve E_3 is defined as

$$E_3 := \mathbf{C}/(\mathbf{Z} + \omega\mathbf{Z}).$$

The product of two such abelian surfaces is a fourfold of the type we consider here, so we expect to see copies of $\mathbf{P}^1 \times \mathbf{P}^1$ inside $\mathcal{X}$. In fact, the quadric Q_{22} in the next proposition parametrizes such products.

The roots $\alpha_5 = (0,0,0,1,-1,0)$, $\alpha_6 = (0,0,0,0,1,-1) \in E_6$ define the hyperplanes $\alpha_5^\perp : X_3 = X_6$ and $\alpha_6^\perp : X_6 = X_7$. Thus the projective 3-space Z in the following proposition is $Z = \alpha_5^\perp \cap \alpha_6^\perp$. These two roots span the root system$\{\pm\alpha_5, \pm\alpha_6, \pm(\alpha_5 + \alpha_6)\}$ in E_6 which is isomorphic to A_2. There are 120 such subsystems and $W(E_6)$ acts transitively on them, so we get 120 quadrics like Q_{22} in $\mathcal{X}$.

4.3. Proposition. *Let $Z \subset \mathbf{P}^5$ be the projective 3-space defined by*

$$Z : \ X_3 = X_6 = X_7 \quad (\subset \mathbf{P}^5).$$

The intersection of Z with $\mathcal{X} \subset \mathbf{P}^5$ has two irreducible components, denoted by Q_{22} and S_{22}, of degree two and six respectively,

$$\mathcal{X} \cap Z = Q_{22} \cup S_{22},$$

and the quadric Q_{22} lies in $Sing(\mathcal{X})$.

The quadric Q_{22} parametrizes products of abelian surfaces, each with an automorphism of order three of type $(1,1)$. The surface S_{22}, which is birationally isomorphic to a K3 surface, parametrizes abelian fourfolds which are isogeneous to a product of abelian surfaces.

Proof. An explicit Magma computation shows that the restriction of F to Z factors as

$$F(X_0, X_1, X_2, X_3, X_3, X_3) = q_{22}^2 f_{22}, \qquad q_{22} := X_0X_3 - X_1X_2.$$

As Q_{22} is a smooth quadric, it is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. A parametrization is given by

$$\begin{aligned} \mathbf{P}^1 \times \mathbf{P}^1 &\longrightarrow Q_{22}\ (\subset Z), \\ ((s:t),(u:v)) &\longmapsto (X_0 : X_1 : X_2 : X_3) = (su : sv : tu : tv). \end{aligned}$$

Substituting this parametrization in the partial derivatives of F with Magma, one finds that they all vanish on Q_{22}, hence $Q_{22} \subset \mathrm{Sing}(\mathcal{X})$. We checked that exactly 36 of the quadrics Q_m vanish in a general point of Q_{22}, so such a point corresponds to a product of two abelian surfaces.

To find a subdomain in $\mathcal{H}_4^M$ which maps to Q_{22}, we consider the following matrix (found by trial and error):

$$M_{pr} := \begin{pmatrix} A_p & 0 \\ 0 & {}^tA_p^{-1} \end{pmatrix} \ (\in Sp(8,\mathbf{Z})), \qquad A_p := \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so A_p acts as A (the $(1,1)$-block of M) on the first and the third coordinates, but it acts as the identity on the second and fourth coordinates. One has $M_{pr}M = MM_{pr}$, so $M_{pr} \in C_M$ acts on the Hermite upper half space $\mathcal{H}_4^M$. The fixed point set of M_{pr} in $\mathcal{H}_4^M$ consists of the block matrices (with indices $1,3$ and $2,4$ respectively) of the period matrices of abelian surfaces with an automorphism of order three. In particular, $\mathcal{H}_4^{M,M_{pr}}$ parametrizes products of ppav's. The map Θ will map this 2-dimensional domain to an eigenspace of the action of M_{pr} on $\mathbf{P}^5$. The action of M_{pr} on $\mathbf{P}^5$ is given by $X_\sigma \mapsto X_{\sigma A_p}$, and one easily checks that M_{pr} fixes Z pointwise and that it has two other isolated fixed points in $\mathbf{P}^5$. For dimension reasons, we get that $\Theta(\mathcal{H}_4^{M,M_{pr}}) \subset Z$, and the closure of the image must be Q_{22}.

Recall that the reflection defined by α_6 is induced by M_B, as in Section 3.8. The matrices M_B and M_{pr} generate a subgroup of order six in

$Sp(8,\mathbf{Z})$ which is isomorphic to $W(A_2)\cong S_3$. One reason for this is that an abelian surface with an automorphism of order three of type $(1,1)$ actually admits S_3 as automorphism group (the case III in [BL, Section 11.7]).

Now we discuss the other component S_{22}. To see a subdomain of the Hermite upper half space which maps to S_{22} we consider the matrix

$$M_{ip}:=\begin{pmatrix} A_p & B_{ip}\\ 0 & {}^tA_p^{-1}\end{pmatrix}\ (\in Sp(8,\mathbf{Z})),\qquad B_{ip}:=\begin{pmatrix} 0 & -2 & 0 & 0\\ 2 & 0 & 0 & 0\\ 0 & 2 & 0 & -2\\ -2 & 0 & 2 & 0\end{pmatrix},$$

where A_p is as above. In particular, M_{pr} and M_{ip} have the same image in $\Gamma_4/\Gamma_4(2,4)$ and thus Z is also an eigenspace of the action of M_{ip} on $\mathbf{P}^5$. The matrices M_{ip} and M_B also generate a subgroup of $Sp(8,\mathbf{Z})$ isomorphic to S_3. However, the fixed point loci $\mathcal{H}_4^{M,M_{pr}}$ and $\mathcal{H}_4^{M,M_{ip}}$ are not conjugate under the action of $Sp(8,\mathbf{Z})$. In fact, consider the sublattices $\Lambda_{pr}:=\ker(M_{pr}-I)$ and $\Lambda_{ip}:=\ker(M_{ip}-I)$, both isomorphic to $\mathbf{Z}^4$. The alternating form E restricts to an alternating form with determinant 1 on Λ_{pr}, but its restriction to Λ_{ip} has determinant 9, which implies the matrices cannot be conjugate in $Sp(8,\mathbf{Z})$. As Θ also maps $\mathcal{H}_4^{M,M_{ip}}$ to Z, the image $\Theta(\mathcal{H}_4^{M,M_{ip}})$ must be the other component S_{22} of the intersection of $\mathcal{X}$ with Z.

The surfaces Q_{22} and S_{22} intersect along a curve of degree 12, which is the union of six lines, each with multiplicity two. These lines are also the singular locus of S_{22}. A better model of S_{22} can be obtained as the image of the birational map

$$S_{22}\longrightarrow\mathbf{P}^4,\qquad (X_0:\cdots:X_3)\longmapsto(X_0q_{22}(X):\cdots:X_3q_{22}(X):r_3(X)),$$

with

$$r_3(X):=X_3(X_0-X_1-X_2+X_3)(X_0+X_1+X_2+X_3).$$

All coordinate functions vanish on the six lines and are homogeneous of degree 3. The birational inverse of this map is induced by the projection on the first four coordinates. The image of this map was found with Magma. It is a complete intersection of a quadratic and a cubic hypersurface. The image is smooth except for 9 ordinary double points. Thus the minimal model of the image, and hence of S_{22}, is a K3 surface. □

4.4. The Hesse planes. In [vG, Theorem 8.5], it was shown that the moduli space of three dimensional ppav's with an automorphism of order three (of type $(2,1)$) and a level two structure has a projective model which is a $\mathbf{P}^2$. Taking the product of such a threefold with the elliptic curve E_3 (see Section 4.2), we obtain an abelian fourfold of the type we consider here. The decomposable ppavs, products of a abelian surface with the elliptic curve E_3, form a configuration of 12 lines. These are the 12 lines in the four reducible curves in the Hesse pencil $x^3 + y^3 + z^3 + \lambda xyz = 0$ (cf. [AD]). A Hesse plane will be a copy of a $\mathbf{P}^2$ with a Hesse pencil. Thus we expect to find such Hesse planes inside $\mathcal{X}$.

4.5. Proposition. *There is a unique conjugacy class C in $W(E_6)$ consisting of 80 elements, each of order three, whose characteristic polynomial in the six dimensional representation is $(x^2 + x + 1)^3$.*

Each $g \in C$ has two eigenspaces in $\mathbf{P}^5$, both are planes $\mathbf{P}^2 \subset \mathcal{X}$. Moreover, each of the two planes lies in $Sing(\mathcal{X})$ and in this way we get 80 *planes in $Sing(\mathcal{X})$. Each plane parametrizes products of an abelian threefold and an elliptic curve.*

Proof. The conjugacy class C can be found from [Fra, Table II, p.104] or [At]. The 80 elements in C come in pairs, g, g^2, which have the same eigenspaces. The group $W(E_6)$ thus acts transitively on the set of $2 \cdot 40 = 80$ eigenspaces of the elements of C.

Magma provided one element $g_3 \in C$ and one of its eigenspaces W_3:

$$g_3 := \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & -3 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$W_3 := \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \omega & -\omega^2 \\ 0 & -\omega^2 & -\omega \\ -1-2\omega & 0 & 0 \end{pmatrix},$$

where the columns of W_3 span the eigenspace of g_3 with eigenvalue ω. Substituting a parametrization of W_3 in the partial derivatives of F, using Magma, we verified that $W_3 \subset Sing(\mathcal{X})$ and similarly that 28 of the quadrics Q_m vanish on W_3. Hence W_3 parametrizes products of an abelian threefold and an elliptic curve. The other quadrics Q_m intersect each plane in two lines. □

4.6. Remark. The centralizer of g_3 in $W(E_6)$ acts on the eigenspace $\mathbf{P}W_3$ as in the proof of the proposition. With Magma we found that it is a group of order 648 which coincides with the group denoted by $\overline{G}_{216}$ in [AD] acting on the Hesse pencil. The $W(E_6)$-invariants I_k, for $k = 2, 5, 8$ restrict to zero on $\mathbf{P}W_3$, whereas I_6, I_9, I_{12} restrict to invariants of $\overline{G}_{216}$. We checked that the restrictions of I_6^2 and I_{12} are linearly independent degree 12 polynomials. The ring of invariants of $\overline{G}_{216}$ is thus generated by the restrictions of I_6, I_9, I_{12} (cf. [AD]).

5. Some fixed point loci in $\mathcal{X}$

5.1. Fixed points. In this section, we consider the fixed point loci in $\mathcal{X}$ of one reflection and of two commuting reflections in $W(E_6)$. The case of two non-commuting reflections was already described in Proposition 4.3. We show that the Hessian variety $\mathcal{W}_{10}$ of Igusa's quartic threefold is an arithmetic quotient. This was conjectured by Hunt ([H, p.7–8]), based on an analogy with the Nieto quintic. We also find a projective model of a two-dimensional moduli space of abelian fourfolds with an automorphism of order 12.

5.2. Proposition. *Let $H_\alpha \subset \mathbf{P}^5$ be the projectivization of the reflection hyperplane defined by a root $\alpha \in E_6$. Then the intersection of H_α with $\mathcal{X}$ is an irreducible 3-fold of degree* 10*, which is $\mathcal{W}_{10}$, the Hessian of the Igusa quartic:*

$$\mathcal{W}_{10} := H_\alpha \cap \mathcal{X}.$$

This 3-fold parametrizes abelian fourfolds of Weil type which are isogeneous to the selfproduct of an abelian surface.

Proof. As $W(E_6)$ acts transitively on the roots of E_6, it suffices to consider the case that $\alpha := (-3, 1, 1, 1, 1, 1)/2$. Then $H_\alpha \subset \mathbf{P}^5$ is defined by the linear equation

$$\begin{aligned} H_\alpha &:= \{\mathcal{X} \in \mathbf{P}^5 : \ b(\alpha, X) = 0 \,\} \\ &= \{X \in \mathbf{P}^5 : \ -X_0 + X_1 + \cdots + X_7 = 0 \,\}. \end{aligned}$$

The equation of the intersection $H_\alpha \cap \mathcal{X}$ is thus $F(X_1 + \cdots + X_7, X_1, \ldots, X_7) = 0 \subset \mathbf{P}^4$. This polynomial is quite complicated, it has 591 terms. However, an explicit computation with Magma shows that it can also be obtained as follows. Let $\mathcal{I}_4$ be Igusa's quartic threefold in $\mathbf{P}^4$, with coordinates X_1, X_2, X_3, X_6, X_7, which is defined by the equation

$$\mathcal{I}_4 : \ G := s_2^2 - 4s_4 = 0, \quad (\subset \mathbf{P}^4)$$

where the s_i are the elementary symmetric functions of degree i in these variables (cf. [vdG, Theorem 5.2, Theorem 4.1], [H, Section 3.3]). Then we have, for a non-zero constant c:

$$F(X_1 + \cdots + X_7, X_1, \ldots, X_7) = c \cdot \det\left(\frac{\partial^2 G}{\partial X_i \partial X_j}\right),$$

(with $i, j \in \{1, 2, 3, 6, 7\}$) so the intersection $H_\alpha \cap \mathcal{X}$ is the Hessian $\mathcal{W}_{10}$ of Igusa's quartic.

The symmetric group $S_6 \cong W(A_5)$ also acts on $\mathcal{W}_{10}$. In fact, one has $b(\alpha, \alpha_i) = 0$ for $i = 2, \ldots, 6$, hence the root system of type A_5 defined by $\alpha_2, \ldots, \alpha_6$ is perpendicular to α. Thus the Weyl group $W(A_5)$ acts on the hyperplane section $\mathcal{W}_{10} = H_\alpha \cap \mathcal{X}$.

Now we find a 3-dimensional subdomain of $\mathcal{H}_4^M$ which maps to $\mathcal{W}_{10}$. As $W(E_6)$ acts transitively on the roots of E_6, so for convenience we may redefine $\alpha := \alpha_6$. The reflection s_α acts as $M_B \in Sp(8, \mathbf{Z})$ (see Section 3.8), thus the fixed points of M_B in $\mathcal{H}_4^M$ map to an eigenspace of s_α in $\mathbf{P}^5$.

The fixed points in $\mathcal{H}_4$ of the involution M_B are the $\tau \in \mathcal{H}_4$ such that $M_B \cdot \tau = \tau$, that is, $B\tau B^{-1} = \tau$, equivalently, $B\tau = \tau B$, so

$$\mathcal{H}_4^{M_B} = \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_1 \end{pmatrix} \in \mathcal{H}_4 : \ \tau_1 = {}^t\tau_1, \ \tau_2 = {}^t\tau_2 \right\}.$$

The intersection of $\mathcal{H}_4^{M, M_B} := \mathcal{H}_4^M \cap \mathcal{H}_4^{M_B}$ is three dimensional, because, as in Section 1.2, we find that a period matrix in $\mathcal{H}_4^{M_B}$ is fixed by M iff $\tau_1 = -(\tau_2 + {}^t\tau_2) = -2\tau_2$. Thus, changing the sign of τ_2, we get

$$\mathcal{H}_4^{M, M_B} = \left\{ \begin{pmatrix} 2\tau_2 & -\tau_2 \\ -\tau_2 & 2\tau_2 \end{pmatrix} \in \mathcal{H}_4 : \ \tau_2 \in \mathcal{H}_2 \right\},$$

in fact, τ_2 must be symmetric and its imaginary part must be positive definite, conversely, given $\tau_2 \in \mathcal{H}_2$ we get an element in $\mathcal{H}_4$ since the matrix with rows $2, -1$; $-1, 2$ (the Cartan matrix of A_2) is positive definite. For $\tau \in \mathcal{H}_4^{M, M_B}$, the abelian fourfold A_τ is isogeneous to 2 copies of the abelian surface A_{τ_2}, since one has, similar to Section 1.1, the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}^4 & \xrightarrow{\Omega_{\tau_2}} & \mathbf{C}^2 & \longrightarrow & A_{\tau_2} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N'} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & \mathbf{Z}^8 & \xrightarrow{\Omega_{\tau}} & \mathbf{C}^4 & \longrightarrow & A_{\tau} & \longrightarrow & 0 \end{array} \qquad \left(\text{so } N\Omega_\tau = \Omega_{\tau_2}N'\right),$$

where the maps are defined by matrices with 2×2 blocks and the vectors are row vectors:

$$N := \begin{pmatrix} I & I & 0 & 0 \\ 0 & 0 & I & I \end{pmatrix}, \quad \Omega_\tau := \begin{pmatrix} 2\tau_2 & -\tau_2 \\ -\tau_2 & 2\tau_2 \\ I & 0 \\ 0 & I \end{pmatrix},$$

$$\Omega_{\tau_2} := \begin{pmatrix} \tau_2 \\ I \end{pmatrix}, \quad N' := \begin{pmatrix} I & I \end{pmatrix}.$$

This shows that there is a non-trivial holomorphic map $\psi : A_{\tau_2} \to A_\tau$. Applying the automorphism ϕ of order three of A_τ to the image of ψ, one obtains another copy (up to isogeny) of the abelian surface A_{τ_2} in the fourfold A_τ and thus A_τ is isogeneous to $A_{\tau_2}^2$. $\square$

5.3. Another reducible section of $\mathcal{X}$. In Proposition 4.3 we showed that $\mathcal{X} \cap Z$, where $Z \cong \mathbf{P}^3$ is subspace of $\mathbf{P}^5$ perpendicular to a root subsystem of E_6 of type A_2, was reducible. Now we consider the intersection of $\mathcal{X}$ with a $\mathbf{P}^3$ which is perpendicular to two orthogonal roots, so a root system of type A_1^2.

Up to the action of $W(E_6)$ such a subsystem is unique, and there are 270 of these. We take the perpendicular roots $(0,0,0,0,1,1)$ and $(0,0,0,0,1,-1)$, then the $\mathbf{P}^3$ perpendicular to both roots is defined by $X_6 = X_7 = 0$.

5.4. Proposition. *Let $W \subset \mathbf{P}^5$ be the projective 3-space defined by*

$$W: \; X_6 = X_7 = 0 \quad (\subset \mathbf{P}^5).$$

The intersection of W with $\mathcal{X} \subset \mathbf{P}^5$ has two irreducible components, $\mathcal{X} \cap W = Q_{67} \cup S_{67}$, where Q_{67} is a smooth quadric and S_{67} is a degree 8 (singular) rational surface.

The quadric Q_{67} parametrizes abelian fourfolds whose endomorphism algebra contains the field of 12-th roots of unity. The surface S_{67} parametrizes abelian fourfolds each of which is isogeneous to a product of elliptic curves.

Proof. We found by explicit Magma computation that

$$\mathcal{X} \cap W \;=\; Q_{67} \cup S_{67}, \qquad Q_{67}:\; X_0^2 - X_1^2 - X_2^2 - X_3^2 = 0,$$

and the degree 8 surface S_{67} is defined by

$$\begin{aligned} X_0^2X_1^2X_2^2X_3^2 - X_1^4X_2^4 - X_1^4X_3^4 - X_2^4X_3^4& \\ + X_1^4X_2^2X_3^2 + X_1^2X_2^4X_3^2 + X_1^2X_2^2X_3^4 &= 0. \end{aligned}$$

The intersection of these two surfaces consists of 8 conics. These conics are also the intersection of Q_{67} with the planes $X_0 = \pm X_1 \pm X_2 \pm X_3$, for any choice of signs.

The singular locus of S_{67} consists of the three lines $X_i = X_j = 0$ for $i, j \in \{1, 2, 3\}$. Let S be the double cover of $\mathbf{P}^2$ (with coordinates Z_0, Z_1, Z_2) given by

$$S:\; T^2 = Z_0^4 + Z_1^4 + Z_2^4 - Z_0^2Z_1^2 - Z_0^2Z_2^2 - Z_1^2Z_2^2 \quad (S \subset W\mathbf{P}(1,1,1,2)).$$

Then there is a birational isomorphism between S_{67} and S given by

$$S_{67} \dashrightarrow S, \qquad (Z_0 : Z_1 : Z_2 : T) := (X_2X_3 : X_1X_3 : X_1X_2 : X_0X_1X_2X_3),$$

with birational inverse

$$S \dashrightarrow S_{67}, \qquad (X_0 : X_1 : X_2 : X_3) := (T : Z_1Z_2 : Z_0Z_2 : Z_0Z_1).$$

Notice that the branch locus of $S \to \mathbf{P}^2$ is reducible:

$$\begin{aligned} &Z_0^4 + Z_1^4 + Z_2^4 - Z_0^2Z_1^2 - Z_0^2Z_2^2 - Z_1^2Z_2^2 \\ &= (Z_0^2 + \omega^2Z_1^2 + \omega Z_2^2)(Z_0^2 + \omega Z_1^2 + \omega^2 Z_2^2). \end{aligned}$$

The conics defined by the two factors intersect in the four points $(1 : \pm1 : \pm1)$. It is now easy to see that S_{67} is rational: the inverse image of a general line through the point $(1 : 1 : 1)$ is again isomorphic to $\mathbf{P}^1$ and each conic gives a (ramification) point on this double cover. Thus one can parametrize the double cover.

We now show that the quadric Q_{67} parametrizes abelian fourfolds X whose endomorphism algebra contains the field of 12-roots of unity, $\mathbf{Q}(\zeta_{12}) \subset End_{\mathbf{Q}}(X)$.

For this, it is convenient to change the pair of perpendicular roots to $\alpha_3 = (0, 1, -1, 0, 0, 0)$ and $\alpha_6 = (0, 0, 0, 0, 1, -1)$. The projective 3-space W'

perpendicular to both of these roots is defined by $X_1 = X_2$, $X_6 = X_7$. We write

$$\mathcal{X} \cap W' = Q_{36} \cup S_{36}$$

where Q_{36} and S_{36} are surfaces of degree 2 and 8 respectively.

To find a subdomain of $\mathcal{H}_4^M$ which maps to Q_{36}, we define an element $M_C \in Sp(8, \mathbf{Z})$ as a block-matrix with four diagonal 2×2 blocks $C = {}^t C^{-1}$:

$$C := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\in GL(2, \mathbf{Z})),$$
$$M_C := \mathrm{diag}(C, C, C, C) \quad (\in Sp(8, \mathbf{Z})).$$

One easily verifies that $M_C M = M M_C$, so M_C lies in C_M, the centralizer of M. Thus M_C maps $\mathcal{H}^M$ into itself. As M has order 3 and M_C has order four (in fact $M_C^2 = -I$ so M_C has eigenvalues $\pm i$ with $i^2 = -1$), the matrix $M_{12} := M M_C$ is an element of order twelve in $Sp(8, \mathbf{Z})$. As $M_{12}^4 = M$, $M_{12}^3 = M_C$, we get

$$(\mathcal{H}^M)^{M_C} = \mathcal{H}^{M_{12}}, \qquad M_{12} := M M_C.$$

Using a commutative diagram as in Section 1.1, one finds that for $\tau \in \mathcal{H}^{M_{12}}$ the matrix M_{12} induces an automorphism ϕ_{12} of order 12 the abelian variety A_τ. Thus the field $\mathbf{Q}(\zeta_{12})$, where ζ_{12} is a primitive 12-root of unity, is contained in the endomorphism algebra of A_τ for any $\tau \in \mathcal{H}^{M_{12}}$. The eigenvalues of ϕ_{12} on the tangent space $T_0 A_\tau = \mathbf{C}^4$ are $\zeta_{12}, \zeta_{12}^5, \zeta_{12}^7, \zeta_{12}^{11}$.

From [BL, Section 9.6] one then obtains (with $d = 1$, $e_0 = 2$, $m = 2$ and $(r_\nu, s_\nu) = (1, 1)$ for $\nu = 1, 2$ and notice that their disc $\mathcal{H}_{1,1}$ is biholomorphic to $\mathcal{H}_1$) that abelian fourfolds with such an automorphism are parametrized by $\mathcal{H}_1^2$. In particular, $\dim \mathcal{H}^{M_{12}} = 2$. From [BL, Exercise 9.10 (4)] it follows that for $\tau \in \mathcal{H}^{M_{12}}$ the endomorphism algebra of A_τ contains an indefinite quaternion algebra over the field $\mathbf{Q}(\sqrt{3})$.

Now we show that the closure of the image of $\mathcal{H}^{M_{12}}$ under $\Theta : \mathcal{H}_4^M \to \mathbf{P}^5$ is the quadric $Q_{36} \subset Z'$. The image is contained in the fixed points of M_C acting on $\mathbf{P}^5 \subset \mathbf{P}^{15}$. The action of M_C on $\mathbf{P}^{15}$ can be found as we did in Section 2.1 for M (now with A replaced by $\mathrm{diag}(C, C)$), one finds that the coordinates X_0, X_3, X_{12}, X_{15} are fixed and that M_C interchanges $X_1 \leftrightarrow X_2$, $X_4 \leftrightarrow X_8$, $X_5 \leftrightarrow X_{10}$, $X_6 \leftrightarrow X_9$, $X_7 \leftrightarrow X_{11}$, $X_{13} \leftrightarrow X_{14}$. Thus:

$$M_C : \mathbf{P}^5 \longrightarrow \mathbf{P}^5,$$
$$(X_0 : X_1 : X_2 : X_3 : X_6 : X_7) \longmapsto (X_0 : X_2 : X_1 : X_3 : X_7 : X_6),$$

hence $(\mathcal{H}^M)^{M_C} = \mathcal{H}^{M_{12}}$ maps to the subspace $W' = (\mathbf{P}^5)^{M_C} \subset \mathbf{P}^5$.

The image of $\mathcal{H}^{M_{12}}$ is thus Q_{36} or S_{36}. We checked that none of the quadrics Q_m's is identically zero on W' and that there are exactly six which vanish on Q_{36} (and as S_{36} has degree 8, none of the Q_m can vanish on it). Now we show that six of the Q_m vanish on the image of $\mathcal{H}^{M_{12}}$, hence Q_{36} is the closure of the image of $\mathcal{H}^{M_{12}}$.

For this we use the action of M_C on the $\theta[{\epsilon \atop \epsilon'}]$, similar to [vG, Proposition 10.7.3]. Using the series defining these theta constants and the fact that $C^tC = I$ one has, with now $A = \mathrm{diag}(C, C)$:

$$\begin{aligned}\theta[{\epsilon \atop \epsilon'}](M_C \cdot \tau) = \theta[{\epsilon A \atop \epsilon' A}](\tau) &= \theta \begin{bmatrix} -\epsilon_2 & \epsilon_1 & -\epsilon_4 & \epsilon_3 \\ -\epsilon_2' & \epsilon_1' & -\epsilon_4' & \epsilon_3' \end{bmatrix} (\tau) \\ &= (-1)^{\epsilon_2\epsilon_2' + \epsilon_4\epsilon_4'} \theta \begin{bmatrix} \epsilon_2 & \epsilon_1 & \epsilon_4 & \epsilon_3 \\ \epsilon_2' & \epsilon_1' & \epsilon_4' & \epsilon_3' \end{bmatrix} (\tau).\end{aligned}$$

It easily follows that if $M_C \cdot \tau = \tau$, then there are six $\theta_m(\tau)$ which satisfy $\theta_m(M_C \cdot \tau) = -\theta_m(\tau)$ and thus they vanish. Therefore six Q_m vanish on the image of $\Theta(\mathcal{H}^{M_{12}}) \subset Q_m$.

Finally we identify the period matrices mapping to the octic surface S_{36}. We define an element $M_D \in Sp(8, \mathbf{Z})$ as a block-matrix with four diagonal 2×2 blocks $D = {}^tD^{-1}$:

$$D := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\in GL(2, \mathbf{Z})),$$

$$M_D := \mathrm{diag}(D, D, D, D) \quad (\in Sp(8, \mathbf{Z})).$$

One easily verifies that $M_D M = M M_D$, so M_D lies in C_M, the centralizer of M. Thus M_D maps $\mathcal{H}^M$ into itself. Moreover, $M_B M_D = M_D M_B$, thus M_D maps $\mathcal{H}_4^{M,M_B}$ into itself.

The fixed point set $\mathcal{H}_4^{M,M_B,M_D}$ has dimension two because it consists of the matrices $\tau = \tau(\tau_2) \in \mathcal{H}_4^{M,M_B}$ as in the proof of Proposition 5.2, with $\mathrm{diag}(D, D)\tau = \tau \mathrm{diag}(D, D)$, so

$$\mathcal{H}_4^{M,M_B,M_D} = \left\{ \begin{pmatrix} 2\tau_2 & -\tau_2 \\ -\tau_2 & 2\tau_2 \end{pmatrix} \in \mathcal{H}_4 : \ \tau_2 = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{11} \end{pmatrix} \in \mathcal{H}_2 \right\}.$$

In particular, $\dim \mathcal{H}_4^{M,M_B,M_D} = 2$. As $M_D \equiv M_C \bmod \Gamma_4(2,4)$, these matrices act in the same way on $\mathbf{P}^{15}$ and thus $\Theta(\mathcal{H}_4^{M,M_B,M_D})$ is a surface in W'.

For $\tau \in \mathcal{H}_4^{M,M_B}$, the abelian variety A_τ is isogeneous to a selfproduct Y_τ^2 where Y_τ is an abelian surface. The map M_D induces an involution on the abelian variety Y_τ, and thus Y_τ is isogeneous to a product $E_1 \times E_2$. Therefore if $\tau \in \mathcal{H}_4^{M,M_B,M_D}$, the abelian variety A_τ is isogeneous to a product

$(E_1 \times E_2)^2$, where the E_i are elliptic curves depending on τ and, for dimension reasons, the moduli of these two elliptic curves vary independently over $\mathcal{H}_1$. This implies that $End_{\mathbf{Q}}(A_\tau) = M_2(\mathbf{Q})^2$ for a general $\tau \in \mathcal{H}_4^{M,M_B,M_D}$. The minimal polynomial of an element in this $\mathbf{Q}$-algebra cannot be irreducible of degree 4, and thus $\mathbf{Q}(\zeta_{12}) \not\subset End_{\mathbf{Q}}(A_\tau)$. Therefore the closure of $\Theta(\mathcal{H}_4^{M,M_B,M_D})$ is not Q_{36}, but it is the octic surface S_{36}. $\square$

6. The boundary of $\mathcal{X}$

6.1. The boundary components. The quotient of the Satake compactification $\overline{\mathcal{A}_4(2,4)}$ of $\mathcal{A}_4(2,4)$ by the group $\Gamma_4 := Sp(8,\mathbf{Z})$ is the Satake compactification $\overline{\mathcal{A}}_4 = \overline{\mathcal{H}_4/\Gamma_4}$. This variety has one boundary component of dimension $k(k+1)/2$ for $k = 0,1,2,3$, whose closure is isomorphic to the Satake compactification $\overline{\mathcal{A}}_k := \overline{\mathcal{H}_k/\Gamma_k}$. Therefore the group Γ_4 acts transitively on the boundary components of a given dimension of $\overline{\mathcal{A}_4(2,4)}$. The boundary of $\mathcal{X} = \Theta\big(\overline{\mathcal{A}_4(2,4)}\big) \cap \mathbf{P}^5$, is, by definition, the intersection of $\mathbf{P}^5$ with the boundary of $\Theta(\overline{\mathcal{A}_4(2,4)})$.

6.2. Proposition.

1) *The boundary of $\mathcal{X}$ consists of 45 lines, which we call the boundary lines. They are the $W(E_6)$-orbit of the line*

$$l: \ X_2 = X_3 = X_6 = X_7 = 0.$$

2) *There are 27 points of intersection of the boundary lines. These points are called the cusps of $\mathcal{X}$. The group $W(E_6)$ acts transitively on the cusps.*

3) *The cusps correspond to degenerate ppav's $(\mathbf{C}^\times)^4$. A point on a boundary line which is not a cusp corresponds to a degenerate ppav $(\mathbf{C}^\times)^2 \times B$, where B is an abelian surface with an automorphism of order three.*

4) *Each boundary line contains 3 cusps and each cusp is on 5 of the boundary lines. The 3 cusps on l are:*

$$(1:0:0:0:0:0), \quad (1:1:0:0:0:0), \quad (1:-1:0:0:0:0).$$

Proof. We recall some facts on the action of Γ_4 on $\mathbf{P}^{15}$. The normal subgroup $\Gamma_4(2)/\Gamma_4(2,4) \cong (\mathbf{Z}/2\mathbf{Z})^8$ of $Sp(8,\mathbf{Z})/\Gamma_4(2,4)$ is generated by block matrices with $a, d = I$, and $c = 0$ and b a diagonal matrix with even entries or the transposed of such a matrix. These matrices act as elements of a finite

Heisenberg group on $\mathbf{P}^{15}$. The matrix $M_{\beta,0}$ with $c=0$ and $b=\mathrm{diag}(2\beta)$ acts as $X_\sigma \mapsto (-1)^{\beta^t\sigma}X_\sigma$ and $M_{0,\gamma}$, with $b=0$ and $c=\mathrm{diag}(2\gamma)$, acts as $X_\sigma \mapsto X_{\sigma+\gamma}$.

From [vG, section 3.7] (but note that we interchanged the diagonal blocks in $\tau(t)$) it follows that a $k(k+1)/2$-dimensional boundary component of $\Theta(\overline{\mathcal{A}_4(2,4)})$ is contained in the linear subspace defined by $X_\sigma=0$ for those $\sigma\in(\mathbf{Z}/2\mathbf{Z})^4$ with $(\sigma_{4-k},\ldots,\sigma_4)\neq(0,\ldots,0)$. This $\mathbf{P}^{2^k-1}$ is a common eigenspace of the elements in the Heisenberg group with $c=0$. Thus to find all boundary components of $\mathcal{X}$ one determines the intersection of $\mathcal{X}$ with the eigenspaces of the elements of the Heisenberg group.

For $k=0$, one finds a zero dimensional boundary component of the image of $\overline{\mathcal{A}_4(2,4)}$ in $\mathbf{P}^{15}$, it is the point $p:=(1:0:\cdots:0)$, so only $X_0\neq 0$. This point is fixed under the action of the subgroup in the Heisenberg group of matrices with $c=0$. The point p actually lies in $\mathcal{X}\subset\mathbf{P}^5$ and we checked that all 0-dimensional boundary components of $\mathcal{X}$ are in the $W(E_6)$-orbit of p, which has 27 elements.

In terms of the E_6 root system, p can be described as follows. The fundamental weight λ_i of E_6 is defined by the equations $\bar{B}_0(\lambda_i,\alpha_j)=\delta_{ij}$, where δ_{ij} is Kronecker's delta. It is easy to check that $\lambda_2=(2,0,0,0,0,0)$, thus p is the image of this fundamental weight in $\mathbf{P}^5=\mathbf{P}(R(E_6)\otimes\mathbf{C})$.

The $\mathbf{P}^3\subset\mathbf{P}^{15}$ defined by by $X_{abcd}=0$ if $(c,d)\neq(0,0)\in\mathbf{F}_2^2$ is the closure of a 3-dimensional boundary component of $\Theta(\overline{\mathcal{A}_4(2,4)})$. Using the action of Γ_4, one finds that also the $\mathbf{P}^3$ defined by $X_{abcd}=0$ if $(a,c)\neq(0,0)\in\mathbf{F}_2^2$ is the closure of a boundary component. On this $\mathbf{P}^3$, only the coordinates X_0,X_1,X_4,X_5 are non-zero and the other 12 are zero. Using the results from Section 2.1, one finds that this $\mathbf{P}^3$ intersects the eigenspace $\mathbf{P}^5$ in the line l defined by $X_2=X_3=X_6=X_7=0$.

By restricting the quadrics Q_m to the line l, one finds that there are only three points in which 120 of them vanish, thus there are exactly three cusps, the ones listed in the proposition, on l. The other results stated in the proposition follow from further Magma computations. □

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