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Good degeneration of Quot-schemes and coherent systems

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We construct good degenerations of Quot-schemes and coherent systems using the stack of expanded degenerations. We show that these good degenerations are separated and proper DM stacks of finite type. Applying to the projective threefolds, we derive degeneration formulas for DT-invariants of ideal sheaves and PT stable pair invariants.

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1. Introduction

Good degenerations are a class of degenerations suitable to study the geometry of moduli spaces via degenerations. Successful applications include the degeneration formula of Gromov-Witten invariants [Li01, Li02]. In this paper, we will construct the good degenerations of Hilbert schemes, of

Grothendieck's Quot-schemes, and of the moduli of coherent systems introduced by Le Potier [LP93]. As applications, we obtain the degeneration formulas of Donaldson-Thomas invariants of ideal sheaves, and of invariants of PT stable pairs of threefolds.

The degenerations we study in this paper are simple degenerations $\pi : X \to C$ over pointed smooth curves $0 \in C$.

Definition 1.1. We say $\pi : X \to C$ is a simple degeneration if

1) X is smooth, π is projective, π has smooth fiber over $c \neq 0 \in C$;
2) the central fiber X_0 has normal crossing singularity and the singular locus D of X_0 is smooth;
3) let Y be the normalization of X_0 and $\tilde{D} = Y \times_{X_0} D \subset Y$, then $\tilde{D} \to D$ is isomorphic to a union of two copies of D.

We denote the two copies of $\tilde{D} \to D$ by D_- and D_+. We call $(Y, D_\pm)$ the relative pair associated with X_0.

We fix a relatively ample line bundle H on X/C, and a polynomial $P(v)$; we form the Hilbert scheme $\mathrm{Hilb}^P_{X_c}$ of closed subschemes $Z \subset X_c$ with Hilbert polynomial $\chi^H_{\mathcal{O}_Z}(v) := \chi(\mathcal{O}_Z \otimes H^{\otimes v}) = P(v)$. We will use the technique developed by the first named author in [Li01] to find a good degeneration of the relative Hilbert scheme (denoting $X^* = X - X_0$ and $C^* = C - 0$)

$$\mathrm{Hilb}^P_{X^*/C^*} = \coprod_{c \in C^*} \mathrm{Hilb}^P_{X_c}.$$

To fill in the central fiber of this family over $0 \in C$, we consider closed subschemes in $X[n]_0$ that are normal to the singular loci of $X[n]_0$; where $X[n]_0$ is obtained by inserting a chain of n-copies of the ruled variety (over D)

$$\Delta = \mathbb{P}_D(1 \oplus N_{D_+/Y})$$

to D in X_0, ($X[n]_0$ is constructed in the next section,) and normal to the singular loci means that it is flat along the normal direction to the singular loci of $X[n]_0$.

The central fiber of the good degeneration has set-theoretic description

$$\left\{ Z \subset X[n]_0 \;\middle|\; \begin{array}{l} n \geq 0,\ Z \text{ is normal to the singular loci} \\ \text{of } X[n]_0,\ \mathrm{Aut}_{\mathfrak{X}}(Z) \text{ is finite},\ \chi^H_{\mathcal{O}_Z}(v) = P(v). \end{array} \right\} \Big/ \cong .$$

Here the equivalence and the automorphism group are defined in the next section. In case D is irreducible, it has a simple description: two closed subschemes $Z_1, Z_2 \subset X[n]_0$ are equivalent if there is an isomorphism $\sigma : X[n]_0 \to X[n]_0$ preserving the projections $X[n]_0 \to X_0$ such that $\sigma(Z_1) = Z_2$. The self-equivalences of a $Z \subset X[n]_0$ form a group, which we denote by $\mathrm{Aut}_{\mathfrak{X}}(Z)$. We call Z stable if $\mathrm{Aut}_{\mathfrak{X}}(Z)$ is finite. Finally, $\chi^H_{\mathcal{O}_Z}(v) = \chi(\mathcal{O}_Z \otimes p^* H^{\otimes v})$, where $p : X[n]_0 \to X_0$ is the projection by contracting the fibers of Δ.

Constructing the stack structure of this set-theoretic description of the central fiber, and fitting it into the family $\mathrm{Hilb}^P_{X^*/C^*}$, is achieved by working with the stack $\mathfrak{X} \to \mathfrak{C}$ of expanded degenerations. Using $\mathfrak{X} \to \mathfrak{C}$, we prove that the set-theoretic description of good degeneration is a Deligne-Mumford stack. The first part of this paper is devoted to prove

Theorem 1.2. *Let $\pi : X \to C$ be a simple degeneration, H be relative ample on $X \to C$, and P be a polynomial. Then the good degeneration described is a Deligne-Mumford stack proper and separated over C; it is of finite type.*

Similar results hold for good degenerations of Grothendieck's Quot-schemes and of coherent systems of Le Potier.

The primary goal to construct such a good degeneration is to derive a degeneration formula of Donaldson-Thomas invariants and PT stable pair invariants of threefolds. For simplicity, we only state the degeneration formula in case Y is a union of two irreducible complements $Y = Y_- \cup Y_+$, and D is connected. We let $D_\pm = Y_\pm \cap \tilde{D}$.

Let Λ^{spl}_P be the set of splittings $\delta = (\delta_\pm, \delta_0)$ of P, (i.e. $\delta_+ + \delta_- - \delta_0 = P$.) For each $\delta \in \Lambda^{\mathrm{spl}}_P$, we construct the moduli stack $\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}$ of relative ideal sheaves of $(Y_\pm, D_\pm)$. This moduli space is constructed using the stack $\mathfrak{D}_\pm \subset \mathfrak{Y}_\pm$ of expanded pairs of $D_\pm \subset Y_\pm$. Closed points of this moduli space consists of ideal sheaves $\mathcal{I}_Z$ of $Y_\pm[n_\pm]$ relative to $D_\pm$, meaning that Z is normal to the singular loci of $Y_\pm[n_\pm]$ and to $D_\pm$. This moduli space is also a proper and separated Deligne-Mumford stack of finite type. Furthermore, we have a natural morphisms

$$\mathrm{ev}_\pm : \mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond} \longrightarrow \mathrm{Hilb}^{\delta_0}_D,$$

to the Hilbert scheme of ideal sheaves on D of Hilbert polynomial δ_0, defined via restricting ideal sheaves on $Y_\pm[n_\pm]$ to its relative divisor $D_\pm$.

Using the evaluation morphisms, we form the fiber product

$$\mathfrak{I}^\delta_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} = \mathfrak{I}^{\delta_-,\delta_0}_{\mathfrak{Y}_-/\mathfrak{A}_\diamond} \times_{\mathrm{Hilb}^{\delta_0}_D} \mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond}.$$

Each $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ is a closed substack of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$, and is indeed a "virtual" Cartier divisor.

Theorem 1.3. *Let $\pi : X \to C$ be a simple degeneration of projective threefolds such that $X_0 = Y_- \cup Y_+$ is a union of two smooth irreducible components. Let $[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} \in A_*\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$ be the virtual class of the good degeneration, and let $\triangle$ be the diagonal morphism $\mathrm{Hilb}^{\delta_0}_D \to \mathrm{Hilb}^{\delta_0}_D \times \mathrm{Hilb}^{\delta_0}_D$. Then $i^!_c[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} = [\mathfrak{I}^P_{X_c}]^{\mathrm{vir}}$ for $c \neq 0 \in C$, and*

$$i^!_0 \left[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}\right]^{\mathrm{vir}} = \sum_{\delta\in\Lambda^{\mathrm{spl}}_P} \triangle^! \left([\mathfrak{I}^{\delta_-,\delta_0}_{\mathfrak{Y}_-/\mathfrak{A}_\diamond}]^{\mathrm{vir}} \times [\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond}]^{\mathrm{vir}}\right). \tag{1.1}$$

Using the Chern characters of the universal ideal sheaves, we also obtain the numerical version of the Donaldson-Thomas invariant and its degeneration, first introduced in the work of Maulik, Nekrasov, Okounkov and Pandharipande [MNOP06]. For a smooth projective threefold X and a polynomial $P(v) = d \cdot v + n$, we let $\mathfrak{I}^P_X$ ($\cong \mathrm{Hilb}^P_X$ canonically) be the moduli of ideal sheaves of curves $\mathcal{I}_Z \subset \mathcal{O}_X$ with Hilbert polynomial P; and let $\mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_{X\times\mathfrak{I}^P_X}$ be its universal family. For any $\gamma \in H^l(X,\mathbb{Z})$, we define

$$\mathrm{ch}_{k+2}(\gamma) : H_*\left(\mathfrak{I}^P_X, \mathbb{Q}\right) \longrightarrow H_{*-2k+2-l}\left(\mathfrak{I}^P_X, \mathbb{Q}\right),$$

via

$$\mathrm{ch}_{k+2}(\gamma)(\xi) = \pi_{2*}\left(\mathrm{ch}_{k+2}(\mathcal{I}_{\mathcal{Z}}) \cdot \pi_1^*(\gamma) \cap \pi_2^*(\xi)\right),$$

where π_1 and π_2 are the first and second projection of $X\times\mathfrak{I}^P_X$. The Donaldson-Thomas invariants (in short DT-invariants) with descendent insertions are the degree of the following cycle class

$$\left\langle \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i) \right\rangle_X^P = \left[\prod_{i=1}^r (-1)^{k_i+1}\mathrm{ch}_{k_i+2}(\gamma_i) \cdot \left[\mathfrak{I}^P_X\right]^{\mathrm{vir}}\right]_0,$$

where γ_i are cohomology classes of pure degree l_i, and $[\cdot]_0$ is taking the dimension zero part of the term inside the bracket. The partition function is

$$Z_d\left(X; q \left| \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i)\right.\right) = \sum_{n\in\mathbb{Z}} \deg \left\langle \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i) \right\rangle_X^{d\cdot v+n} q^n.$$

The commonly used form of DT-invariants as introduced in [MNOP06], uses the moduli $I_n(X,\beta)$ of ideal sheaves of subschemes $Z \subset X$ with fixed curve class $\beta = [Z]$. In this paper we package the DT-invariant using the moduli $\mathfrak{I}_X^P$ of ideal sheaves with fixed Hilbert polynomial. This enables us to avoid the technical issue of decomposing curve classes during degenerations. In explicit application, one should be able to derive the general case after analyzing this issue in details.

Next, we let $\beta_1, \dots, \beta_m$ be a basis of $H^*(D, \mathbb{Q})$. Let $\{C_\eta\}_{|\eta|=k}$ be a Nakajima basis of the cohomology of Hilb_D^k (where η is a cohomology weighted partition w.r.t. β_i). The relative DT-invariants with descendent insertions [MNOP06] are the degree of

$$\left\langle \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i) \,\middle|\, \eta \right\rangle_{\mathfrak{Y}_\pm}^{\delta_\pm} = \left[\prod_{i=1}^r (-1)^{k_i+1} \mathrm{ch}_{k_i+2}(\gamma_i) \cap \mathrm{ev}_\pm^*(C_\eta) \cdot \left[\mathfrak{I}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}^{\delta_\pm, \delta_0} \right]^{\mathrm{vir}} \right]_0$$

which form a partition function

$$Z_{d_\pm, \eta}\left(Y_\pm, D_\pm; q \,\middle|\, \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i) \right) = \sum_{n \in \mathbb{Z}} \deg \left\langle \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i) \,\middle|\, \eta \right\rangle_{\mathfrak{Y}_\pm}^{d_\pm \cdot v + n} q^n.$$

Using the cycle version of the degeneration formula in Theorem 1.3, we verify the following form of degeneration formula

Theorem 1.4 ([MNOP06]). *Fix a basis $\beta_1, \dots, \beta_m$ of $H^*(D, \mathbb{Q})$. Let γ_i be cohomology classes of X of pure degree l_i. The degeneration formula of Donaldson-Thomas invariants has the following form*

$$\begin{aligned} Z_d\left(X_c; q \,\middle|\, \prod_{i=1}^r \tilde{\tau}_0(i_c^* \gamma_i) \right) = \sum_{\substack{d_-, d_+; \eta \\ d = d_- + d_+}} & \frac{(-1)^{|\eta| - l(\eta)} \mathfrak{z}(\eta)}{q^{|\eta|}} \\ & \cdot Z_{d_-, \eta}\left(Y_-, D_-; q \,\middle|\, \prod_{i=1}^r \tilde{\tau}_0(i_-^* \gamma_i) \right) \\ & \cdot Z_{d_+, \eta^\vee}\left(Y_+, D_+; q \,\middle|\, \prod_{i=1}^r \tilde{\tau}_0(i_+^* \gamma_i) \right) \end{aligned}$$

where $i_c : X_c \to X$, $i_\pm : Y_\pm \to X$ are the inclusions, η are cohomology weighted partitions w.r.t. β_i, and $\mathfrak{z}(\eta) = \prod_i \eta_i |\mathrm{Aut}(\eta)|$.

Comments. In this paper, parallel results on the PT stable pairs invariants are proved. The PT stable pair invariant was introduced by Pandharipande and Thomas in [PT09]. Their degeneration was essentially proved in [MPT10], though in a special form. For future reference, we include the statement and the necessary constructions that lead to a proof of the degeneration of PT stable pair invariants in this paper.

The notion of relative ideal sheaves was developed through email communication between Pandharipande and the first named author. The technical part of this paper is the proof of the properness and boundedness of good generations of Grothendieck's Quot-schemes. The parallel results for PT stable pairs are simpler. The part on perfect obstruction theory necessary for proving the degenerations of invariants are taken from the work [MPT10].

The good degeneration of ideal sheaves for threefolds was constructed by the second named author in his thesis [Wu07]. The properness, separatedness and the boundedness were proved there. The proofs in this paper for Grothendieck's Quot-schemes are new.

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2. The stack of expanded degenerations

We work with a fixed algebraically closed field $\mathbf{k}$ of characteristic 0. We denote $G_{\mathrm{m}} = GL(1, \mathbf{k})$. Let $\pi\colon X \to C$, $0 \in C$, be a simple degeneration; let Y be the normalization of X_0; let $\tilde{D} \subset Y$ be the preimage of $D \subset X_0$, and fix $\tilde{D} = D_- \cup D_+$, as defined in Definition 1.1. In this paper, we call $(Y, D_\pm)$ the relative pair associated with X_0.

In [Li01] and [Li02], the first named author proved the degeneration of Gromov-Witten invariants of a simple degeneration in case Y is a union of two irreducible components $Y = Y_- \cup Y_+$ and D is connected. Often, one needs to deal with simple degeneration $X \to C$ when Y is irreducible or contains more than two connected components, or D is not connected. In this paper, we will construct good degenerations of moduli spaces for general simple degenerations.

In this section, we review the construction of the stack of expanded degenerations and its family $\mathfrak{X} \to \mathfrak{C}$: presented in the survey article [Li10]. Some formulation of the stack $\mathfrak{X}$ is new; however, the proofs of the results listed follow directly from the arguments in [Li01].

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mathfrak{p}} & X \\ \downarrow & & \downarrow{\scriptstyle\pi} \\ \mathfrak{C} & \longrightarrow & C \end{array} \tag{2.1}$$

2.1. The stack $\mathfrak{C}$

We consider $\mathbb{A}^{n+1}$ with the group action

$$(t_1, \dots, t_{n+1})^\sigma = (\sigma_1 t_1, \sigma_1^{-1}\sigma_2 t_2, \dots, \sigma_{n-1}^{-1}\sigma_n t_n, \sigma_n^{-1} t_{n+1}), \quad \sigma \in G_{\mathrm{m}}^n.$$

This group action generates a class of equivalence relations on $\mathbb{A}^{n+1}$.

We need another class of equivalences. We fix the convention on indices. We denote by $[n+1] = \{1, \dots, n+1\}$; for any $I \subset [n+1]$, we let $I^\circ = [n+1] - I$ be its complement. For $|I| = m+1$, we let

$$\mathrm{ind}_I \colon [m+1] \to I \subset [n+1] \quad \text{and} \quad \mathrm{ind}_{I^\circ} \colon [n-m] \to I^\circ \subset [n+1]$$

be the order preserving isomorphisms; let

$$\mathbb{A}^{n+1}_{U(I)} = \{(t) \in \mathbb{A}^{n+1} \mid t_i \neq 0,\ i \in I^\circ\} \subset \mathbb{A}^{n+1}. \tag{2.2}$$

We let

$$\tilde{\tau}_I : \mathbb{A}^{m+1} \times G_{\mathrm{m}}^{n-m} \xrightarrow{\cong} \mathbb{A}^{n+1}_{U(I)} \tag{2.3}$$

be defined by the rule

$$(t'_1, \dots, t'_{m+1}; \sigma_1, \dots, \sigma_{n-m}) \mapsto (t_1, \dots, t_{n+1}), \quad \begin{cases} t_k = t'_l, & \text{if } k = \mathrm{ind}_I(l); \\ t_k = \sigma_l, & \text{if } k = \mathrm{ind}_{I^\circ}(l). \end{cases}$$

Restricting to $(\sigma_1, \dots, \sigma_{n-m}) = (1)$, it defines

$$\tau_I : \mathbb{A}^{m+1} \longrightarrow \mathbb{A}^{n+1}, \quad (t'_1, \dots, t'_{m+1}) \mapsto \tilde{\tau}_I(t'_1, \dots, t'_{m+1}, 1, \dots, 1). \tag{2.4}$$

We call such τ_I standard embeddings. Given two $I, I' \subset [n+1]$ of same cardinalities, we define the isomorphism

$$\tilde{\tau}_{I,I'} = \tilde{\tau}_I \circ \tilde{\tau}_{I'}^{-1} : \mathbb{A}^{n+1}_{U(I')} \longrightarrow \mathbb{A}^{n+1}_{U(I)}. \tag{2.5}$$

Next, we let $\mathbb{A}^{n+1} \to \mathbb{A}^{n+2}$ be the closed immersion τ_I using $I = [n+1] \subset [n+2]$. Let $G_{\mathrm{m}}^n \to G_{\mathrm{m}}^{n+1}$ be the homomorphism defined via $(\sigma_1, \ldots, \sigma_n) \mapsto (\sigma_1, \ldots, \sigma_n, 1)$. Via this homomorphism, and viewing $\mathbb{A}^{n+1}$ as scheme over $\mathbb{A}^1$ via $(t) \mapsto t_1 \cdots t_{n+1}$, the morphism

$$\tau_I : \mathbb{A}^{n+1} \to \mathbb{A}^{n+2} \tag{2.6}$$

is a G_{m}^n equivariant $\mathbb{A}^1$-morphism with G_{m}^n acting on $\mathbb{A}^1$ trivially.

Further, for general $I, I' \subset [n+1]$ of $|I| = |I'|$, the equivalence $\tilde{\tau}_{I,I'}$ of $\mathbb{A}^{n+1}$ is the restriction of the equivalence $\tilde{\tau}_{I,I'}$ of $\mathbb{A}^{n+2}$, by considering I, I' as subsets in $[n+2]$ via $I, I' \subset [n+1] \subset [n+2]$.

Definition 2.1. We define $\mathfrak{A}_n$ be the quotient $[\mathbb{A}^{n+1}/\sim]$ by the equivalences generated by the G_{m}^n action and by the equivalences $\tilde{\tau}_{I,I'}$ for all pairs $I, I' \subset [n+1]$ with $|I| = |I'|$. The morphism (2.6) defines an open immersion $\mathfrak{A}_n \to \mathfrak{A}_{n+1}$. We define $\mathfrak{A}$ be the direct limit $\mathfrak{A} = \varinjlim \mathfrak{A}_n$.

Note that the tautological $\mathbb{A}^{n+1} \to \mathfrak{A}_n$ is a surjective smooth chart; the collection $\{\mathbb{A}^{n+1} \to \mathfrak{A}\}_{n \geq 0}$ forms a smooth atlas of $\mathfrak{A}$.

Now let $0 \in C$ be the pointed smooth affine curve given. Without loss of generality, we assume there is an étale morphism $C \to \mathbb{A}^1$ so that the inverse image of $0 \in \mathbb{A}^1$ is the distinguished point $0 \in C$. We define

$$\mathfrak{C} = C \times_{\mathbb{A}^1} \mathfrak{A}.$$

It is clear that $\mathfrak{C}$ does not depend on the choice of $C \to \mathbb{A}^1$, and is covered by smooth charts

$$C[n] := C \times_{\mathbb{A}^1} \mathbb{A}^{n+1} \longrightarrow \mathfrak{C} = C \times_{\mathbb{A}^1} \mathfrak{A}.$$

Let $o_n \in \mathfrak{A}$ be the image of $0 \in \mathbb{A}^{n+1}$ under the tautological $\mathbb{A}^{n+1} \to \mathfrak{A}$. By abuse of notation, we denote by the same $o_n \in \mathfrak{C}$ the lift of $o_n \in \mathfrak{A}$ and $0 \in C$. By construction, o_n has automorphism group G_{m}^n; and $\overline{o_k} = \{o_{k'} : k' \geq k\}$.

2.2. The stack $\mathfrak{X}$

We begin with describing $\mathfrak{X} \times_C 0$. We keep the decomposition $\tilde{D} = D_- \cup D_+$ specified at the beginning of this section. Let $N_\pm$ be the normal line bundles of $D_\pm$ in Y. Since π is a simple degeneration, $N_- \otimes N_+ \cong \mathcal{O}_D$. (Here and later we implicitly identify $D_\pm$ with D using $D_- \cup D_+ = \tilde{D} \to D$.)

We introduce the ruled variety

$$\Delta = \mathbb{P}_D(N_+ \oplus 1);$$

it is a $\mathbb{P}^1$-bundle over D coming with two distinguished sections $D_+ = \mathbb{P}(1)$ and $D_- = \mathbb{P}(N_+)$. For any $\sigma \in G_{\mathrm{m}}$, the G_{m}-action on $N_+ \oplus 1$ via $(a, b)^\sigma = (\sigma \cdot a, b)$ defines a G_{m}-action

$$\sigma : \Delta \longrightarrow \Delta, \quad [a, b]^\sigma = [\sigma a, b], \tag{2.7}$$

called the tautological G_{m}-action on Δ. This action fixes D_- and $D_+ \subset \Delta$.

We now construct $X[n]_0$. We take n copies of Δ, indexed by $\Delta_1, \ldots, \Delta_n$, and form a new scheme $X[n]_0$ according to the following rule: we identify $D_- \subset Y$ with $D_+ \subset \Delta_1$, ($D_- \cong D_+$ is via the isomorphism $D_\pm \to D$;) identify $D_- \subset \Delta_i$ with $D_+ \subset \Delta_{i+1}$, and identify $D_- \subset \Delta_n$ with $D_+ \subset Y$. We denote

$$X[n]_0 = Y \sqcup \Delta_1 \sqcup \cdots \sqcup \Delta_n \sqcup (Y).\ ^1 \tag{2.8}$$

We denote $D_i \subset X[n]_0$ be D_- in Δ_{i-1}, which is also the $D_+ \subset \Delta_i$.[2] The singular locus of $X[n]_0$ is the union of $D_1, \ldots, D_{n+1}$.

Figure 1: The two ends are the same Y, in the middle a chain of n Δ's are inserted; the D_- of Y is glued to D_+ of Δ_1, which is named D_1.

Because the inserted Δ_i intersects the remainder components along D_i and $D_{i+1} \subset \Delta_i$, the tautological G_{m}-action on Δ_i (cf. (2.7)) lifts to an automorphism of $X[n]_0$ that acts trivially on all other $\Delta_{j\neq i}$. We let G_{m}^n acts on $X[n]_0$ so that its i-th factor acts on $X[n]_0$ via the tautological G_{m}-action on Δ_i and trivially on $\Delta_{j\neq i}$. Let $p\colon X[n]_0 \longrightarrow X_0$ be the projection contracting all inserted components $\Delta_1, \ldots, \Delta_n$; it is G_{m}^n-equivariant with the trivial action on X.

We now construct the family $\mathfrak{X} \to \mathfrak{C}$ associated with $X \to C$. Let $0 \in C[n]$ be the preimage of $0 \in \mathbb{A}^{n+1}$ in $C[n]$. We denote $C^* = C - 0$ and let $C[m]^* = C[m] \times_C C^*$.

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[1] Here $\sqcup$ means that we identify $D_- \subset \Delta_i$ with $D_+ \subset \Delta_{i+1}$, agreeing that $Y = \Delta_0 = \Delta_{n+1}$; we put the further right Y in parenthesis indicating that it is the same Y appearing in the further left.

[2] Thus $D_+ \subset \Delta_i$ is $\Delta_{i-1} \cap \Delta_i$ and $D_- \subset \Delta_i$ is $\Delta_i \cap \Delta_{i+1}$.

Lemma 2.2. *We let $X[n]$ be the small resolution $X[n] \to X \times_C C[n]$, coupled with the projection $p\colon X[n] \to X$ induced from $X \times_C C[n] \to X$. It is characterized by the properties:*

1) *$X[n]$ is smooth;*
2) *the central fiber $(X[n] \times_{C[n]} 0, p)$ is the $(X[n]_0, p)$ constructed;*
3) *let $\bar\tau_I\colon C[m] \to C[n]$ be a morphism induced by $\tau_I : \mathbb{A}^{m+1} \to \mathbb{A}^{n+1}$ (cf. (2.4)); then the induced family $(\bar\tau_I^* X[n], \bar\tau_I^* p)$ is isomorphic to $(X[m], p)$ as families over $C[m]$, extending the identity map*
$$\bar\tau_I^* X[n]|_{C[m]^*} = X[m]|_{C[m]^*} = X \times_C C[m]^*;$$
4) *let ℓ_l be the l-th coordinate line of $\mathbb{A}^{n+1}$; let $L_l = C[n] \times_{\mathbb{A}^{n+1}} \ell_l$, and let $\iota_l\colon L_l \to C[n]$ be the inclusion; then the induced family $\iota_l^* X[n]$ smooths the l-th singular divisor D_l of $X[n]_0$.*

Because of (2), we will view $X[n]_0$ as the central fiber $X[n] \times_{C[n]} 0$.

Lemma 2.3. *The G_{m}^n action on $C[n]$ with the trivial action on X lifts to a unique G_{m}^n-action on $X[n]$. The induced G_{m}^n action on $X[n]_0$ is the action described before Lemma 2.2. For $I, I' \subset [n+1]$ of identical cardinalities, the equivalence $\tilde\tau_{I,I'}$ in (2.5) lifts to a C-isomorphism*

$$\tilde\tau_{I,I',X} : X[n] \times_{C[n]} C[n]_{U(I')} \cong X[n] \times_{C[n]} C[n]_{U(I)}, \tag{2.9}$$

where $C[n]_{U(I)} = C[n] \times_{\mathbb{A}^{n+1}} \mathbb{A}^{n+1}_{U(I)}$.

As an illustration, let $C = \mathbb{A}^1$, and $X/\mathbb{A}^1$ is a smoothing of $X_0 = Y_1 \sqcup Y_2$ with a single node D. Then $C[1] = \mathbb{A}^2$; the central fiber $X[1]_0 = Y_1 \sqcup \Delta \sqcup Y_2$, $\Delta = \mathbb{P}^1$, has two singular divisors $D_1 = Y_1 \cap \Delta$ and $D_2 = \Delta \cap Y_2$. Restricting $X[1]$ to the first coordinate line $\mathbb{A}^2_1$, we obtain a family that smoothes $D_1 \subset X[1]_0$ but not D_2; restricting to the second coordinate line $\mathbb{A}^2_2$ the family smoothes D_2 but not D_1.

Definition 2.4. We define $\mathfrak{X}_n$ be $[X[n]/\sim]$, where $\sim$ are equivalence relations generated by the G_{m}^n action and the equivalences $\tilde\tau_{I,I',X}$ for all $I, I' \subset [n+1]$ of $|I| = |I'|$. We let $\mathfrak{p}_n\colon \mathfrak{X}_n \to X$ be the morphism induced by the tautological projection $p : X[n] \to X$.

The quotient is an Artin stack; it is over C since the G_{m}^n action and the equivalence $\tilde\tau_{I,I',X}$ are defined over C.

Figure 2: It shows that D_1 is smoothed over $\mathbb{A}^2_1$; D_2 smoothed over $\mathbb{A}^2_2$.

Using the inclusion $[n+1] \subset [n+2]$, the induced $\mathbb{A}^{n+1} \to \mathbb{A}^{n+2}$ in (2.6) and the induced $C[n] \to C[n+1]$, we have tautological immersion of stacks

$$\mathfrak{X}_n \longrightarrow \mathfrak{X}_{n+1} \tag{2.10}$$

that commute with the projections $\mathfrak{p}_n$ and $\mathfrak{p}_{n+1}$.

Definition 2.5. We define $\mathfrak{X} = \varinjlim \mathfrak{X}_n$; we define $\mathfrak{p}\colon \mathfrak{X} \to X$ be the induced projection.

Theorem 2.6. *The morphisms $X[n] \to C[n]$ induce a representable C-morphism $\mathfrak{X} \to \mathfrak{C}$. It fits into the commutative square* (2.1)*.*

We call $(\mathfrak{X} \to \mathfrak{C}, \mathfrak{p})$ the stack of expanded degenerations of $X \to C$. For any C-scheme S, we call $\mathfrak{X} \times_{\mathfrak{C}} S \to S$ an S-family of expanded degenerations.

2.3. The stack $\mathfrak{D}_\pm \subset \mathfrak{Y}$

We now construct the stack

$$\mathfrak{D}_\pm \subset \mathfrak{Y} \longrightarrow \mathfrak{A}_\diamond \tag{2.11}$$

of expanded pairs of $(Y, D_\pm)$.

We fix the convention on indexing $\mathbb{A}^{n_-+n_+}$ and $G_{\mathrm{m}}^{n_-+n_+}$. In this paper, whenever we see product of $n_- + n_+$ copies, we index the individual factor by indices $-n_-, \ldots, -1, 1, \ldots, n_+$. (Note that index 0 is skipped.) Thus the $(-n_-)$-th coordinate line of $\mathbb{A}^{n_-+n_+}$ is $(t, 0, \ldots, 0)$, and the n_+-th coordinate line is $(0, \ldots, 0, t)$. The same convention applies to indexing factors of

$G_{\mathrm{m}}^{n_-+n_+}$. We let $G_{\mathrm{m}}^{n_-+n_+}$ acts on $\mathbb{A}^{n_-+n_+}$ via the traditional convention

$$(t_{-n_-},\ldots,t_{-1},t_1,\ldots,t_{n_+})^\sigma=(\sigma_{-n_-}t_{-n_-},\ldots,\sigma_{-1}t_{-1},\sigma_1t_1,\ldots,\sigma_{n_+}t_{n_+}).$$

We then construct

$$D[n_-]_-,\ D[n_+]_+\subset Y[n_-,n_+]\longrightarrow\mathbb{A}^{n_-+n_+},\quad p:Y[n_-,n_+]\longrightarrow Y, \tag{2.12}$$

inductively by the rule:

1) $(Y[0,0],D[0]_\pm)=(Y,D_\pm)$;
2) $Y[n_-,n_++1]$ is the blow-up of $Y[n_-,n_+]\times\mathbb{A}^1$ along $D[n_+]_+\times 0$, and $D[n_-]_-$ and $D[n_++1]_+$ are the proper transforms of $D[n_-]_-\times\mathbb{A}^1$ and $D[n_+]_+\times\mathbb{A}^1$, respectively;
3) $Y[n_-+1,n_+]$ is the blow-up of $\mathbb{A}^1\times Y[n_-,n_+]$ along $0\times D[n_-]_-$, and $D[n_-+1]_-$ and $D[n_+]_+$ are the proper transforms of $\mathbb{A}^1\times D[n_-]_-$ and $\mathbb{A}^1\times D[n_+]_+$, respectively;
4) $p:Y[n_-,n_+]\to Y$ is the one induced by the identity $Y\to Y$.

Following the convention, the extra copy of $\mathbb{A}^1$ added to the right in item (2) is the (n_++1)-th factor of $\mathbb{A}^{n_-+(n_++1)}$; the copy $\mathbb{A}^1$ added to the left in item (3) is the $(-n_--1)$-th copy in $\mathbb{A}^{(n_-+1)+n_+}$.

The central fiber of (2.12) is easily described. We let $N_\pm$ be the normal line bundle of $D_\pm$ in Y; let $\Delta=\mathbb{P}_D(N_+\oplus 1)$ with distinguished divisors $D_+=\mathbb{P}(1)$ and $D_-=\mathbb{P}(N)$. Then

$$Y[n_-,n_+]_0=Y[n_-,n_+]\times_{\mathbb{A}^{n_-+n_+}}0\quad\text{and}\quad D[n_\pm]_{\pm,0}=D[n_\pm]_\pm\times_{\mathbb{A}^{n_-+n_+}}0$$

are

$$\begin{aligned}&Y[n_-,n_+]_0\\&=\Delta_{-n_-}\sqcup\cdots\sqcup\Delta_{-1}\sqcup Y\sqcup\Delta_1\sqcup\cdots\sqcup\Delta_{n_+},\quad n_-,\,n_+\geq 0,\end{aligned}\tag{2.13}$$

where the square cup "$\sqcup$" means that we identify the divisor $D_-\subset\Delta_i$ with $D_+\subset\Delta_{i+1}$, understanding that $\Delta_0=Y$, and $\Delta_i=\Delta$ for $i\neq 0$; $D[n_-]_{-,0}$ is the divisor D_+ in Δ_{-n_-}, and $D[n_+]_{+,0}$ is the divisor $D_-\subset\Delta_{n_+}$.

We let $p:Y[n_-,n_+]_0\to Y$ be induced by $p:Y[n_-,n_+]\to Y$ (cf. item (4)); it is by contracting all $\Delta_{i\neq 0}$. The scheme $Y[n_-,n_+]_0$ has simple normal crossing singularities when $(n_-,n_+)\neq(0,0)$.

We call

$$(Y[n_-,n_+]_0, D[n_\pm]_{\pm,0}) \quad \text{with} \quad p : Y[n_-,n_+]_0 \to Y \tag{2.14}$$

and the $G_{\mathrm{m}}^{n_-+n_+}$-action an expanded relative pair of $(Y, D_\pm)$.

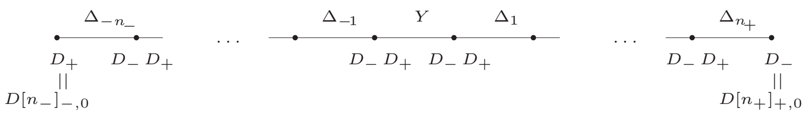

Figure 3: The Y, Δ's glue to form $Y[n_-,n_+]_0$; the two end divisors are the new relative divisors of $Y[n_-,n_+]_0$.

The families $Y[n_-,n_+] \to \mathbb{A}^{n_-+n_+}$ has the following additional properties:

5) let $\ell_l \to \mathbb{A}^{n_-+n_+}$ be the l-th coordinate line of $\mathbb{A}^{n_-+n_+}$, $-n_- \le l \le n_+$, $l \ne 0$, then the restriction $Y[n_-,n_+] \times_{\mathbb{A}^{n_-+n_+}} \ell_l$ smoothes the divisor $D_l = \Delta_{l-1} \cap \Delta_l$ if $l > 0$, of $D_l = \Delta_l \cap \Delta_{l+1}$ if $l < 0$.

(Notice that $Y[n_-,n_+]_0$ has singular divisors D_l, $-n_- \le l \le n_+$ and $l \ne 0$.)

The family (2.12) and the pair (2.14) are $G_{\mathrm{m}}^{n_-+n_+}$-equivariant. The k-th factor of the G_{m} in $G_{\mathrm{m}}^{n_-+n_+}$ acts trivially on all Δ_i except Δ_k; on Δ_k the action is the tautological G_{m}-action of (2.7).

Like the stack $\mathfrak{X} \to \mathfrak{C}$, the stack (2.11) we aim to construct will be the limit of the quotients of (2.12) by $G_{\mathrm{m}}^{n_-+n_+}$ and another class of equivalences associated to subsets

$$I \subset [-n_-, n_+] - \{0\}. \tag{2.15}$$

(We define its complement $I^\circ = [-n_-, n_+] - I \cup \{0\}$.)

Given an I as in (2.15), we define $\mathbb{A}_{U(I)}^{n_-+n_+} \subset \mathbb{A}^{n_-+n_+}$ be as in (2.2). Like (2.3), letting $m_\pm = |I \cap \mathbb{Z}_\pm|$, we have an isomorphism

$$\tilde{\tau}_I : \mathbb{A}^{m_-+m_+} \times G_{\mathrm{m}}^{(n_--m_-)+(n_+-m_+)} \longrightarrow \mathbb{A}_{U(I)}^{n_-+n_+}, \tag{2.16}$$

and for any I' as in (2.15) with

$$m_\pm = |I \cap \mathbb{Z}_\pm| = |I' \cap \mathbb{Z}_\pm|, \tag{2.17}$$

the pair (I, I') defines an isomorphism

$$\tilde{\tau}_{I,I'} = \tilde{\tau}_I \circ \tilde{\tau}_{I'}^{-1} : \mathbb{A}_{U(I')}^{n_-+n_+} \longrightarrow \mathbb{A}_{U(I)}^{n_-+n_+}. \tag{2.18}$$

As before, we let

$$\tau_I : \mathbb{A}^{m_-+m_+} \longrightarrow \mathbb{A}^{n_-+n_+} \tag{2.19}$$

be $\tilde{\tau}_I$ restricting to $\mathbb{A}^{m_-+m_+} \times \{1\}$, where $1 \in G_{\mathrm{m}}^{(n_--m_-)+(n_+-m_+)}$ is the identity element.

Following the construction, one checks that for any I as in (2.15), we have a canonical isomorphism

$$\tau_{I,Y} : Y[m_-, m_+] \longrightarrow \tau_I^* Y[n_-, n_+],$$

lifting the τ_I in (2.19); for any pair (I, I') of subsets in (2.15) satisfying (2.17), we have a canonical isomorphism

$$\tilde{\tau}_{I,I',Y} : Y[n_-, n_+] \times_{\mathbb{A}^{n_-+n_+}} \mathbb{A}^{n_-+n_+}_{U(I')} \longrightarrow Y[n_-, n_+] \times_{\mathbb{A}^{n_-+n_+}} \mathbb{A}^{n_-+n_+}_{U(I)},$$

lifting the $\tilde{\tau}_{I,I'}$ in (2.18).

Definition 2.7. We define $\mathfrak{A}_{\diamond,n_-+n_+}$ be the quotient $[\mathbb{A}^{n_-+n_+}/\!\sim]$, quotient by the equivalence relations generated by the $G_{\mathrm{m}}^{n_-+n_+}$-action and the equivalences $\tilde{\tau}_{I,I'}$ for all allowable pairs (I, I') in (2.15); using (2.19), for $m_\pm \le n_\pm$, we have open immersion $\mathfrak{A}_{\diamond,m_-+m_+} \to \mathfrak{A}_{\diamond,n_-+n_+}$; we define $\mathfrak{A}_\diamond = \lim_{n_-,n_+} \mathfrak{A}_{\diamond,n_-+n_+}$. $\mathfrak{A}_\diamond$ is an Artin stack.

We define $\mathfrak{D}_{n_\pm,\pm} \subset \mathfrak{Y}_{n_-+n_+}$ be the quotient of $D[n_\pm]_\pm \subset Y[n_-, n_+]$ by $G_{\mathrm{m}}^{n_-+n_+}$ and the equivalences $\tilde{\tau}_{I,I',Y}$ for all pairs (I, I') satisfying (2.17); we define $\mathfrak{D}_\pm \subset \mathfrak{Y}$ be the limit of $\mathfrak{D}_{n_\pm,\pm} \subset \mathfrak{Y}_{n_-+n_+}$ as $n_-, n_+ \to +\infty$. We let $\mathfrak{p} : \mathfrak{Y} \to Y$ be the projection induced by the tautological $Y[n_-, n_+] \to Y$.

Theorem 2.8. *The projections $Y[n_-, n_+] \to \mathbb{A}^{n_-+n_+}$ induce a representable morphism $\mathfrak{D}_\pm \subset \mathfrak{Y} \to \mathfrak{A}_\diamond$.*

We call $\mathfrak{D}_\pm \subset \mathfrak{Y} \to \mathfrak{A}_\diamond$ with $\mathfrak{p} : \mathfrak{Y} \to Y$ the stack of expanded relative pairs of $(Y, D_\pm)$. Using $(\mathfrak{D}_\pm \subset \mathfrak{Y} \to \mathfrak{A}_\diamond, \mathfrak{p})$, we define the collection $\mathfrak{Y}(S)$ of expanded families of pair $(Y, D_\pm)$ over a scheme S be

$$\mathfrak{D}_\pm \times_{\mathfrak{A}_\diamond} S \subset \mathfrak{Y} \times_{\mathfrak{A}_\diamond} S, \quad S \to \mathfrak{A}_\diamond.$$

In case $Y = Y_- \cup Y_+$ is a union of two connected components, we use $D_\pm = \tilde{D} \cap Y_\pm$. We define the pair of stack

$$\mathfrak{D}_+ \subset \mathfrak{Y}_+ := \mathfrak{Y} \times_Y Y_+. \tag{2.20}$$

Or $\mathfrak{Y}_+$ can be defined as in Definition 2.7 with Y replaced by Y_+, $n_- = 0$ and $D_- = \emptyset$. The pair $\mathfrak{D}_- \subset \mathfrak{Y}_-$ is defined similarly.

2.4. Decomposition of degenerations I

To state the decomposition of good degenerations, we introduce the stack of node-marking objects in $\mathfrak{X}_0 := \mathfrak{X} \times_C 0$. This construction was first introduced in [KL07].

Definition 2.9. A node-marking of $X[n]_0$ is a marking of one of the singular divisor D_k of $X[n]_0$. A node-marking of a family $\mathcal{X} \to S$ in $\mathfrak{X}_0(S)$ is an S-morphism $\eta : D \times S \to \mathcal{X}$ so that for any closed $s \in S$, $\eta(D \times s) \subset \mathcal{X}_s$ is a node-marking of $\mathcal{X}_s$.

An arrow between two $\mathcal{X}$ and $\mathcal{X}'$ in $\mathfrak{X}_0(S)$ with node-markings η and η' is an arrow $\rho : \mathcal{X} \to \mathcal{X}'$ in $\mathfrak{X}_0(S)$ so that for any closed $s \in S$, $\rho \circ \eta(D \times s) = \eta'(D \times s)$.

Proposition 2.10. *The collection of families in $\mathfrak{X}_0$ with node-markings form an Artin stack, denoted by $\mathfrak{X}_0^\dagger$. Forgetting the node-marking defines a morphism*

$$\mathfrak{X}_0^\dagger \longrightarrow \mathfrak{X}_0.$$

Proof. The smooth chart $X[n] \to \mathfrak{X}$ induces a smooth chart $X[n] \times_C 0 \to \mathfrak{X}_0$. We denote $\mathbb{A}^{n+1}_{t_k=0} = \{(t) \in \mathbb{A}^{n+1} \mid t_k = 0\}$. Then $\mathbb{A}^{n+1} \times_{\mathbb{A}^1} 0 = \bigcup_{k=1}^{n+1} \mathbb{A}^{n+1}_{t_k=0}$. Further,

$$X[n]_{t_k=0} := X[n] \times_{\mathbb{A}^{n+1}} \mathbb{A}^{n+1}_{t_k=0}$$

has normal crossing singularity and its singular divisor is the image of the $X \times \mathbb{A}^{n+1}_{t_k=0}$-morphism

$$\eta_k : D \times \mathbb{A}^{n+1}_{t_k=0} \longrightarrow X[n]_{t_k=0}. \tag{2.21}$$

According to Definition 2.9, one checks that (2.21) is a node-marking of $X[n]_{t_k=0}$; thus

$$(X[n]_{t_k=0}, \eta_k) \in \mathfrak{X}_0^\dagger(\mathbb{A}^{n+1}_{t_k=0}). \tag{2.22}$$

The disjoint union of (2.22) for all $1 \le k \le n+1$ form a smooth atlas of $\mathfrak{X}_0^\dagger$. This proves that $\mathfrak{X}_0^\dagger$ is an Artin stack. $\square$

It will be useful to construct a stack $\mathfrak{C}_0^\dagger$ and an arrow $\mathfrak{C}_0^\dagger \to \mathfrak{C}$ that fits into a Cartesian product

$$\begin{array}{ccc} \mathfrak{X}_0^\dagger & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathfrak{C}_0^\dagger & \longrightarrow & \mathfrak{C}. \end{array} \tag{2.23}$$

We construct $\mathfrak{C}_0^\dagger$ as follows. For a pair of integers $1 \le k \le n+1$, we let G_{m}^n acts on $\mathbb{A}^{n+1}_{t_k=0}$ via the G_{m}^n action on $\mathbb{A}^{n+1}$ and the inclusion $\mathbb{A}^{n+1}_{t_k=0} \subset \mathbb{A}^{n+1}$. Such action generates equivalence relation on $\mathbb{A}^{n+1}_{t_k=0}$.

For any $I \subset [n+1]$ and k an integer, we denote $I_{<k} = \{i \in I \mid i < k\}$; similarly for $I_{>k}$. Let $k \in I \subset [n+1]$ and $k' \in I' \subset [n+1]$ such that

$$|I_{<k}| = |I'_{<k}| \quad \text{and} \quad |I_{>k}| = |I'_{>k}|. \tag{2.24}$$

The equivalence $\tilde{\tau}_{I,I'}$ of (2.5) restricted to $\mathbb{A}^{n+1}_{t_k=0} \cap \mathbb{A}^{n+1}_{U(I')}$ defines

$$\tau_{(I,k),(I',k')} : \mathbb{A}^{n+1}_{k'^c} \cap \mathbb{A}^{n+1}_{U(I')} \xrightarrow{\cong} \mathbb{A}^{n+1}_{k^c} \cap \mathbb{A}^{n+1}_{U(I)}. \tag{2.25}$$

These isomorphisms generate equivalence relations too.

We define the closed immersion

$$\tau_{+1} : \mathbb{A}^{n+1}_{t_k=0} \longrightarrow \mathbb{A}^{n+2}_{k^c}, \quad (z) \mapsto (z,1). \tag{2.26}$$

Definition 2.11. We define $\mathfrak{C}^\dagger_{n,0}$ be the quotient $[\coprod_{k=1}^{n+1} \mathbb{A}^{n+1}_{t_k=0}/\sim]$, where $\sim$ is the equivalence generated by the G_{m}^n action on $\mathbb{A}^{n+1}_{t_k=0}$ and by $\tau_{(I,k),(I',k')}$ for all pairs $k \in I$ and $k' \in I'$ satisfying (2.24); we define open immersions $\mathfrak{C}^\dagger_{n,0} \to \mathfrak{C}^\dagger_{n+1,0}$ using (2.26); we define $\mathfrak{C}_0^\dagger = \varinjlim \mathfrak{C}^\dagger_{n,0}$.

Proposition 2.12. *The morphisms $X[n]_{t_k=0} \to \mathbb{A}^{n+1}_{t_k=0}$, where $X[n]_{t_k=0}$ is with the node-marking* (2.22)*, induce a morphism $\mathfrak{X}_0^\dagger \to \mathfrak{C}_0^\dagger$ that fits into the Cartesian product* (2.23)*.*

As $\coprod \mathbb{A}^{n+1}_{t_k=0} \to \mathfrak{C}^\dagger_{n,0}$ is a smooth chart of $\mathfrak{C}^\dagger_{n,0}$, and the former is the normalization of $\mathbb{A}^{n+1} \times_{\mathbb{A}^1} 0$, the morphism $\mathfrak{C}_0^\dagger \to \mathfrak{C}_0$ is a normalization. It is fitting to call $\mathfrak{X}_0^\dagger \to \mathfrak{X}_0$ the decomposition of locally complete intersection singularity of $\mathfrak{X}_0$.

The final step of the decomposition is the following isomorphism result.

Proposition 2.13. *There is a canonical isomorphism $\mathfrak{C}_0^\dagger \cong \mathfrak{A}_\diamond$ so that $\mathfrak{X}_0^\dagger$ is derived from $\mathfrak{Y}$ by identifying the stacks $\mathfrak{D}_-$ with $\mathfrak{D}_+$ via the isomorphisms $\mathfrak{D}_- \cong D \times \mathfrak{A}_\diamond \cong \mathfrak{D}_+$, and declaring the identifying loci the node-marking.*

Proof. We define $\mathbb{A}^{n_- + n_+} \to \mathbb{A}^{n+1}_{t_k=0}$, $k = n_- + 1,\ n = n_- + n_+$, via

$$(t_{-n_-}, \ldots, t_{-1}, t_1, \ldots, t_{n_+}) \mapsto (t_{-1}, \ldots, t_{-n_-}, 0, t_{n_+}, \ldots, t_1).$$

This is G_{m}^n equivariant via a homomorphism $G_{\mathrm{m}}^n \to G_{\mathrm{m}}^{n+1}$, and induces a morphism $\mathfrak{A}_\diamond \to \mathfrak{C}_0^\dagger$. The remainder of the proof is straightforward. □

2.5. Decomposition of degenerations II

This decomposition works for the case $Y = Y_- \cup Y_+$ is the union of two irreducible components; we let $D_\pm = \tilde{D} \cap Y_\pm$ and define $\mathfrak{D}_\pm \subset \mathfrak{Y}_\pm$ as in (2.20).

We fix an additive group Λ. Using $Y = Y_- \cup Y_+$, we index the irreducible components of $X[n]_0$ as $\Delta_0 = Y_-$, $\Delta_{n+1} = Y_+$, and other Δ_i are as usual.

Definition 2.14. A weight assignment of $X[n]_0$ is a function

$$w : \{\Delta_0, \ldots, \Delta_{n+1}, D_1, \ldots, D_{n+1}\} \longrightarrow \Lambda$$

that assigns weights in Λ to Δ_i and D_j in $X[n]_0$. A weight assignment of X_t, $t \neq 0$, is a single value assignment $w(X_t) \in \Lambda$. A weight assignment w of $\mathcal{X} \in \mathfrak{X}(S)$ is a collection $\{w_s \mid s \in S\}$ of weight assignments w_s of $\mathcal{X}_s$.

We make sense of continuous weight assignments of families. For any subchain $\Delta_{[l,l']} := \Delta_l \cup \cdots \cup \Delta_{l'}$ we define its weight to be (recall $D_i = \Delta_{i-1} \cap \Delta_i$)

$$w(\Delta_{[l,l']}) = \sum_{l \le i \le l'} w(\Delta_i) - \sum_{l < i \le l'} w(D_i).$$

Let $s_0 \in S$ be an irreducible curve, and let w be a weight assignment of $\mathcal{X} \in \mathfrak{X}(S)$. Suppose $\mathcal{X}_{s_0} \cong X[n]_0$ and $\mathcal{X}_s \cong X[m]_0$ for a general $s \in S$, Then $m \le n$, and there are

$$k_0 = 0 < k_1 < \cdots < k_{m+1} < k_{m+2} = n + 2 \tag{2.27}$$

so that the $\Delta_i \subset \mathcal{X}_s$ specializes to the chain $\Delta_{[k_i, k_{i+1}-1]} \subset \mathcal{X}_{s_0}$, (i.e. the singular divisors $D_{k_i} \subset \mathcal{X}_{s_0}$ are not smoothed in the family $\mathcal{X}$.) The total weight of w is $w(X[n]_0)$.

Definition 2.15. Let $s_0 \in S$ be an irreducible curve, and $\mathcal{X} \in \mathfrak{X}(S)$ be as stated. We say a weight assignment w of $\mathcal{X}$ is continuous at s_0 if the followings hold:

1) In case for a general $s \in S$ we have $\mathcal{X}_s = X[m]_0$, letting k_i be as in (2.27), then $w_s(\Delta_i) = w_{s_0}(\Delta_{[k_i,k_{i+1}-1]})$ and $w_s(D_i) = w_{s_0}(D_{k_i})$.
2) In case for a general $s \in S$ we have $\mathcal{X}_s = X_t$ for a $t \neq 0 \in C$, then $w_s(\mathcal{X}_s) = w_{s_0}(\mathcal{X}_{s_0})$.

In general, a weight assignment of $\mathcal{X} \in \mathfrak{X}(S')$ is continuous if for any irreducible curve $s_0 \in S$ and $S \to S'$, the pull back family $\mathcal{X} \times_{S'} S$ with the induced weight assignment is continuous at s_0.

Example 2.16. Suppose $\dim X/C = 1$. In case there is a locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, assigning each $\Delta_k \subset \mathcal{X}_s$ the degree of $\mathcal{E}|_{\Delta_k}$ and assigning each $D_l \subset \mathcal{X}_s$ zero is a continuous weight assignment taking values in $\mathbb{Z}$.

We define the stack of weighted expanded degenerations $\mathfrak{X}^\beta$.

Definition-Proposition 2.17. *Given a $\beta \in \Lambda$, we define the groupoid $\mathfrak{X}^\beta(S)$ be the collections of pairs $(\mathcal{X}, w)$, where $\mathcal{X} \in \mathfrak{X}(S)$ and w is a continuous weight assignment of $\mathcal{X}$ of total weights β. An arrow between $(\mathcal{X}, w)$ and $(\mathcal{X}', w') \in \mathfrak{X}^\beta(S)$ consists of an arrow $\rho : \mathcal{X} \to \mathcal{X}'$ in $\mathfrak{X}(S)$ that preserves the weights w and w'. The groupoid $\mathfrak{X}^\beta$ is an Artin stack.*

By forgetting the weights, we obtain the forgetful morphism $\mathfrak{X}^\beta \to \mathfrak{X}$. We claim that there is a weighted stack $\mathfrak{C}^\beta$ together with a forgetful morphism $\mathfrak{C}^\beta \to \mathfrak{C}$ so that $\mathfrak{X}^\beta$ is the Cartesian product

$$\begin{array}{ccc} \mathfrak{X}^\beta & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathfrak{C}^\beta & \longrightarrow & \mathfrak{C}. \end{array} \tag{2.28}$$

The easiest way to do this is to define a weight assignment of a $t \in C[n]$ be a weight of $X[n]_t$. Or a weight of $S \to \mathfrak{C}$ is a weight of $\mathfrak{X} \times_{\mathfrak{C}} S$. We then define $\mathfrak{C}^\beta$ to be the groupoid consisting of $(S \to \mathfrak{C}, w)$, where w is a weight assignment of $S \to \mathfrak{C}$, etc.

Proposition 2.18. *The groupoid $\mathfrak{C}^\beta$ is an Artin stack, together with a tautological morphism $\mathfrak{X}^\beta \to \mathfrak{C}^\beta$; the forgetful morphism $\mathfrak{C}^\beta \to \mathfrak{C}$ is étale and fits into the Cartesian square (2.28).*

Replacing $\mathfrak{X}/\mathfrak{C}$ by $\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger$, we obtain a pair

$$\mathfrak{X}_0^{\dagger,\beta} \longrightarrow \mathfrak{C}_0^{\dagger,\beta},$$

where closed points in $\mathfrak{X}_0^{\dagger,\beta}$ are $(X[n]_0, D_k, w)$ of which $D_k \subset X[n]_0$ are node-markings and w are weight assignments of $X[n]_0$ of total weights β. We define $\mathfrak{C}_0^{\dagger,\beta}$ parallelly, combining the construction of $\mathfrak{C}_0^\dagger$ and $\mathfrak{C}^\beta$.

The pair $\mathfrak{X}_0^{\dagger,\beta} \longrightarrow \mathfrak{C}_0^{\dagger,\beta}$ is a disjoint union of open and closed substacks indexed by the set of splittings of β. We let

$$\Lambda_\beta^{\mathrm{spl}} = \{\delta = (\delta_\pm, \delta_0) \mid \delta_-, \delta_+, \delta_0 \in \Lambda,\ \delta_- + \delta_+ - \delta_0 = \beta\}.$$

For each $\delta \in \Lambda_\beta^{\mathrm{spl}}$, we define $\mathfrak{X}_0^{\dagger,\delta}(\mathbf{k})$ be the collection of those $(X[n]_0, D_k, w) \in \mathfrak{X}_0^{\dagger,\beta}(\mathbf{k})$ such that

$$w(\Delta_{[0,k-1]}) = \delta_-, \quad w(\Delta_{[k,n+1]}) = \delta_+ \quad \text{and} \quad w(D_k) = \delta_0.$$

It is both open and closed in $\mathfrak{X}_0^{\dagger,\beta}(\mathbf{k})$; thus defines an open and closed substack $\mathfrak{X}_0^{\dagger,\delta} \longrightarrow \mathfrak{X}_0^{\dagger,\beta}$.

Accordingly, we can form the stack $\mathfrak{C}_0^{\dagger,\delta}$ and a morphism $\mathfrak{C}_0^{\dagger,\delta} \to \mathfrak{C}_0^{\dagger,\beta}$ that fits into a Cartesian product

$$\begin{tikzcd} \mathfrak{X}_0^{\dagger,\delta} \arrow[r] \arrow[d] & \mathfrak{X}_0^{\dagger,\beta} \arrow[d] \\ \mathfrak{C}_0^{\dagger,\delta} \arrow[r, "\Phi_\delta^\dagger"] & \mathfrak{C}_0^{\dagger,\beta} \end{tikzcd}$$

We let

$$\Phi_\delta : \mathfrak{C}_0^{\dagger,\delta} \longrightarrow \mathfrak{C}^\beta \tag{2.29}$$

be $\Phi_\delta^\dagger$ composed with the forgetful morphism $\mathfrak{C}_0^{\dagger,\beta} \to \mathfrak{C}^\beta$. The following Proposition says that they are Cartier divisors.

Proposition 2.19. *There are canonical line bundles with sections (L_δ, s_δ) on $\mathfrak{C}^\beta$, indexed by $\delta \in \Lambda_\beta^{\mathrm{spl}}$, such that*

1) *let* $t \in \Gamma(\mathcal{O}_{\mathbb{A}^1})$ *be the standard coordinate function and* $\pi : \mathfrak{C}^\beta \to \mathbb{A}^1$ *be the tautological projection, then*

$$\bigotimes_{\delta \in \Lambda_\beta^{\mathrm{spl}}} L_\delta \cong \mathcal{O}_{\mathfrak{C}^\beta} \quad \text{and} \quad \prod_{\delta \in \Lambda_\beta^{\mathrm{spl}}} s_\delta = \pi^* t;$$

2) *the morphism* Φ_δ *factors through* $s_\delta^{-1}(0) \subset \mathfrak{C}^\beta$ *and effects an isomorphism* $\mathfrak{C}_0^{\dagger,\delta} \cong s_\delta^{-1}(0)$.

The proof of this decomposition is essentially given in [Li02]. Note that this Proposition states that $\mathfrak{C}_0^\beta \subset \mathfrak{C}^\beta$ is a complete intersection substack, and the disjoint union of $\mathfrak{C}_0^{\dagger,\delta}$ is its normalization.

We complete the weighted decomposition by introducing the stack of weighted relative pairs. We define a weight assignment of $(Y_+[n], D_+[n])$ be a function w that assigns values in Λ to the irreducible components of $Y_+[n]$, of its D_k's, and of $D_+[n]$. We define the continuous weight assignments of $(\mathcal{Y}_+, \mathcal{D}_+) \in \mathfrak{Y}_+(S)$ parallel to Definition 2.15.

For a $\delta \in \Lambda_\beta^{\mathrm{spl}}$, we define the stack $\mathfrak{Y}_+^{\delta_+,\delta_0}$ so that $\mathfrak{Y}_+^{\delta_+,\delta_0}(S)$ consists of data $(\mathcal{Y}_+, \mathcal{D}_+, w)$, where $(\mathcal{Y}_+, \mathcal{D}_+) \in \mathfrak{Y}_+(S)$ and w are weight assignments of $(\mathcal{Y}_+, \mathcal{D}_+)$, so that for any closed $s \in S$, $w_s(\mathcal{D}_{+,s}) = \delta_0$ and the total weights $w_s(\mathcal{Y}_{+,s}) = \delta_+$. The case for (Y_-, D_-) and similar objects are defined with "$+$" replaced by "$-$".

We let $\mathfrak{A}_\diamond^{\delta_\pm,\delta_0}$ be the stack defined similarly so that we have Cartesian product

$$\begin{array}{ccc} \mathfrak{Y}_\pm^{\delta_\pm,\delta_0} & \longrightarrow & \mathfrak{Y}_\pm \\ \downarrow & & \downarrow \\ \mathfrak{A}_\diamond^{\delta_\pm,\delta_0} & \longrightarrow & \mathfrak{A}_\diamond \end{array}$$

By gluing the two relative divisors $\mathcal{D}_-$ and $\mathcal{D}_+$ of $(\mathcal{Y}_\pm, \mathcal{D}_\pm, w_\pm) \in \mathfrak{Y}_\pm^{\delta_\pm,\delta_0}(S)$ and combining the weights w_- and w_+, we obtain the following commutative square of morphisms

$$\begin{array}{ccc} \mathfrak{Y}_-^{\delta_-,\delta_0} \cup \mathfrak{Y}_+^{\delta_+,\delta_0} & \longrightarrow & \mathfrak{X}_0^{\dagger,\delta} \\ \downarrow & & \downarrow \\ \mathfrak{A}_\diamond^{\delta_-,\delta_0} \times \mathfrak{A}_\diamond^{\delta_+,\delta_0} & \xrightarrow{\Psi_\delta} & \mathfrak{C}_0^{\dagger,\delta} \end{array}$$

where $\cup$ is the usual gluing along the substack $\mathfrak{D}$.

Proposition 2.20. *The morphism* Ψ_δ *is an isomorphism.*

3. Admissible coherent sheaves

We develop necessary technical results on admissible coherent sheaves on singular schemes. In this paper, we adopt the convention that for any closed or open $V \subset W$ and $\mathcal{F}$ a sheaf of $\mathcal{O}_W$-modules, we denote $\mathcal{F}|_V = \mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{O}_V$.

3.1. Coherent sheaves normal to a closed subscheme

Let W be a noetherian scheme and $D \subset W$ be a closed subscheme.

Definition 3.1. Let $\mathcal{F}$ be a coherent sheaf on W. We say $\mathcal{F}$ is normal to D if $\mathrm{Tor}_1^{\mathcal{O}_W}(\mathcal{F}, \mathcal{O}_D) = 0$.

In this paper, we are interested in two situations. One is when $D \subset W$ is a Cartier divisor; the other is when $W = W_1 \cup W_2$ is a union of subschemes W_1 and $W_2 \subset W$ that intersect transversally along a Cartier divisor $D = W_1 \cap W_2$.

To study flat families of coherent sheaves, we quote the following known fact.

Lemma 3.2. *Let $(A, \mathfrak{m})$ be a noetherian local ring with residue field $\mathbf{k}$, and B a finitely generated A-algebra, flat over A. Let M be a finitely generated B-module. Then $\mathrm{Tor}_1^B(M, B/\mathfrak{m}B) = 0$ if and only if M is flat over A.*

Proof. Since M is a finitely generated B-module, it fits into an exact sequence

$$0 \longrightarrow M' \longrightarrow B^{\oplus n} \longrightarrow M \longrightarrow 0.$$

Tensoring with $B/\mathfrak{m}B$, we know $\mathrm{Tor}_1^B(M, B/\mathfrak{m}B) = 0$ if and only if $M'/\mathfrak{m}M' = M' \otimes_A \mathbf{k} \to (B/\mathfrak{m}B)^{\oplus n}$ is injective. On the other hand, applying $\otimes_A \mathbf{k}$ to the above exact sequence, we obtain

$$\mathrm{Tor}_1^A(B^{\oplus n}, \mathbf{k}) \longrightarrow \mathrm{Tor}_1^A(M, \mathbf{k}) \longrightarrow M' \otimes_A \mathbf{k} \longrightarrow B^{\oplus n} \otimes_A \mathbf{k} = (B/\mathfrak{m}B)^{\oplus n}.$$

Since B is A-flat, $\mathrm{Tor}_1^A(B^{\oplus n}, \mathbf{k}) = 0$. Thus the last arrow is injective if and only if $\mathrm{Tor}_1^A(M, \mathbf{k}) = 0$. By local criterion of flatness [Mat80, Theorem 49], this is equivalent to M being A-flat. This proves the Lemma. $\square$

For the case where $D \subset W$ is a Cartier divisor in a smooth W, a coherent sheaf $\mathcal{F}$ on W normal to D is equivalent to that $\mathcal{F}$ is flat along the "normal direction" of $D \subset W$. To make this precise, we assume W is affine and pick a regular $z \in \Gamma(\mathcal{O}_W)$ so that $D = (z = 0)$. We define $\tau : W \to \mathbb{A}^1 = \operatorname{Spec} \mathbf{k}[u]$ via $\tau^*(u) = z$. For any scheme S, we denote by $\pi_S \colon W \times S \to S$ the projection and view $W \times S$ as a family over $\mathbb{A}^1 \times S$ via

$$(\tau, \pi_S) : W \times S \longrightarrow \mathbb{A}^1 \times S. \tag{3.1}$$

Proposition 3.3. *Let $D \subset W$, S and (3.1) be as stated. Suppose $\mathcal{F}$ an S-flat family of coherent sheaves on $W \times S$, and $s \in S$ is a closed point so that $\mathcal{F}_s = \mathcal{F} \otimes_{\mathcal{O}_S} \mathbf{k}(s)$ is normal to D. Then there is an open subset $(0, s) \in U \subset \mathbb{A}^1 \times S$ so that the sheaf $\mathcal{F}|_U$ is flat over U.*

Conversely, let $U \subset \mathbb{A}^1 \times S$ be an open subset such that $\mathcal{F}$ is flat over U, then for $(0, s) \in U$, $\mathcal{F}_s$ is normal to D.

Proof. We let

$$U = \{x \in \mathbb{A}^1 \times S \mid \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{A}^1 \times S}} \mathcal{O}_{\mathbb{A}^1 \times S, x} \text{ is } \mathcal{O}_{\mathbb{A}^1 \times S, x}\text{-flat}\}.$$

By [Mat80, Theorem 53], U is an open subset of $\mathbb{A}^1 \times S$ (possibly empty) and $\mathcal{F}|_U$ is flat over U.

To prove the Proposition, we only need to show that $(0, s) \in U$. But this is a direct application of Lemma 3.2. We let

$$A = \mathcal{O}_{\mathbb{A}^1 \times S, (0,s)}, \quad B = \Gamma(\mathcal{O}_{W \times S} \otimes_{\mathcal{O}_{\mathbb{A}^1 \times S}} A), \quad M = \Gamma(\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{A}^1 \times S}} A).$$

Since the assumption that $\mathcal{F}_s$ is normal to D implies that $\operatorname{Tor}_1^B(M, B/\mathfrak{m}B) = 0$, Lemma 3.2 implies that M is flat over A, that is, $(0, s) \in U$.

For the converse, given $(0, s) \in U$, by the base change property of flatness, $\mathcal{F}_s = \mathcal{F}|_{W \times s}$ is flat over $U_s = U \cap (\mathbb{A}^1 \times s)$. Since $(0, s) \in U$, we have $0 \in U_s$. By Lemma 3.2, $\operatorname{Tor}_1^{\mathcal{O}_W}(\mathcal{F}_s, \mathcal{O}_D) = 0$; by Definition 3.1, $\mathcal{F}_s$ is normal to D. □

Corollary 3.4. *Let the situation be as in Proposition 3.3 and let $\mathcal{F}$ be an S-flat family of coherent sheaves on $W \times S$. Then the set $V = \{s \in S \mid \mathcal{F}_s \text{ is normal to } D\}$ is open in S, and $\mathcal{F}|_{D \times V}$ is a V-flat family of coherent sheaves on $D \times V$.*

Proof. Let U be the open subset introduced in the proof of Proposition 3.3. Then $U \cap (0 \times S) \subset S$ is exactly the locus where $\mathcal{F}_s$ is normal to D.

By Proposition 3.3, we know that there exists an open subset $U \subset \mathbb{A}^1 \times S$, so that $0 \times V \subset U$ and $\mathcal{F}|_U$ is flat over U. Thus, by the base change property of flatness, $\mathcal{F}|_{D \times V}$ is V-flat. This proves the second part of the Corollary. □

Now we move to the second case where $W = W_1 \cup W_2$ is a union of two smooth schemes W_1 and W_2 intersecting transversally along a Cartier divisor $D = W_1 \cap W_2$ (in W_1 and W_2). Assume W is affine; we find $z_i \in \Gamma(\mathcal{O}_W)$ so that $W_1 = (z_2 = 0)$ and $W_2 = (z_1 = 0)$, thus $D = (z_1 = z_2 = 0)$. We let

$$T = \operatorname{Spec} \mathbf{k}[u_1, u_2]/(u_1 u_2),$$

and let $\xi\colon W \to T$ be defined by $\xi^*(u_i) = z_i$. As before, since the fiber of $W \to T$ over $0 \in T$ is D, which is smooth, by shrinking W if necessary, we can assume that ξ is smooth.

Now let S be any scheme, $\pi_S\colon W \times S \to S$ be the projection. We will view $W \times S$ as a family over $T \times S$ via

$$(\xi, \pi_S) : W \times S \longrightarrow T \times S. \tag{3.2}$$

By our choice, it is smooth.

Proposition 3.5. *Proposition 3.3 and Corollary 3.4 hold with the family* (3.1) *replaced by the family* (3.2).

Proof. The proof is exactly the same. □

Proposition 3.6. *Let the situation be as in* (3.2). *Let $\mathcal{F}$ be an S-flat family of coherent sheaves on $W \times S$. Suppose for any $s \in S$ the sheaf $\mathcal{F}_s$ is normal to D. Then $\mathcal{F}_i = \mathcal{F}|_{W_i \times S}$ is an S-flat family of coherent sheaves each of its members normal to D.*

Proof. We prove the case $i = 1$. Since this is a local problem, we assume W is affine. We pick the morphism in (3.2). Applying Proposition 3.5, we can find an open $D \times S \subset U \subset W \times S$ so that $\mathcal{F}|_U$ is flat over $T \times S$. By the base change property of flatness, $\mathcal{F}|_{U \cap W_1 \times S}$ is flat over $T_1 \times S$, where $T_1 = (u_2 = 0)$. Since $D \times S \subset U$, $\mathcal{F}_1 = \mathcal{F}|_{W_1 \times S}$ is flat over S near $D \subset W_1 \times S$. Since $W_1 - D$ is open in W and $\mathcal{F}|_{W_1 - D} = \mathcal{F}_1|_{W_1 - D}$, $\mathcal{F}_1$ is flat over S.

Finally, because $\mathcal{F}_s$ is normal to D, $\mathcal{F}_1|_{W_1 \times s}$ is normal to D as well. This proves the Proposition for $i = 1$. The case $i = 2$ is the same. □

We also have the converse.

Lemma 3.7. *Let $\mathcal{F}$ be a sheaf on W in the situation* (3.2). *Then $\mathcal{F}$ is normal to $D \subset W$ if and only if both $\mathcal{F}|_{W_i}$, $i = 1, 2$, are normal to $D \subset W_i$.*

Proof. Let $T_1 = (u_2 = 0)$ and $T_2 = (u_1 = 0) \subset T$. It is proved in Proposition 3.6 that $\mathcal{F}$ normal to D implies that both $\mathcal{F}|_{W_i}$ are normal to D. Suppose both $\mathcal{F}|_{W_i}$ are normal to D. Then both $\mathcal{F}|_{W_i}$ are flat over T_i near $0 \in T_i$. We prove that $\mathcal{F}$ is flat over T near $0 \in T$. Since $\hat{\mathcal{O}}_{T,0} = \mathbf{k}[[u_1, u_2]]/(u_1 u_2)$, each ideal $I \subset \hat{\mathcal{O}}_{T,0}$ is either principal or has the form $I = (u_1^{a_1}, u_2^{a_2})$. We show that $I \otimes_{\hat{\mathcal{O}}_{T,0}} \mathcal{F} \to I\mathcal{F}$ is injective. Assume that $I = (u_1^{a_1}, u_2^{a_2})$; (for I principal, the argument is the same.) Let $\alpha_i \in \mathcal{F}$ so that

$$u_1^{a_1} \otimes \alpha_1 + u_2^{a_2} \otimes \alpha_2 \mapsto 0 \in \mathcal{F}.$$

Since $\mathcal{O}_T \to \mathcal{O}_W$ is defined by $u_i \mapsto z_i$. Using $z_1 z_2 = 0$, we get $z_1^{a_1+1}\alpha_1 = 0$. Because $\mathcal{F}|_{W_1}$ is flat over $0 \in T_1$, this is possible only if $\alpha_1 = z_2\beta$ for some $\beta \in \mathcal{F}$. Then $u_1^{a_1} \otimes \alpha_1 = u_1^{a_1} u_2 \otimes \beta = 0$. For the same reason, $u_2^{a_2} \otimes \alpha_2 = 0$. Hence $I \otimes_{\hat{\mathcal{O}}_{T,0}} \mathcal{F} \to I\mathcal{F}$ is injective. This proves that $\mathcal{F}$ is flat over T near 0. □

We have a parallel result.

Lemma 3.8. *Let $D_1, D_2 \subset X$ be smooth divisors intersecting transversally in a smooth variety. Suppose a sheaf $\mathcal{F}$ is normal to D_1 and D_2, then it is normal to the union $D_1 \cup D_2$.*

Proof. The proof is similar, and will be omitted. □

3.2. Admissible coherent sheaves

We shall study coherent sheaves on a simple degeneration $\pi\colon X \to C$.

Definition 3.9. We call a coherent sheaf $\mathcal{F}$ on $X[n]_0$ admissible if it is normal to all $D_i \subset X[n]_0$. Let $(\mathcal{X}, p)$ be an S-family of expanded degenerations. Let $\mathcal{F}$ be an S-flat family of coherent sheaves on $\mathcal{X}$. We say $\mathcal{F}$ is an S-family of admissible coherent sheaves if $\mathcal{F}_s := \mathcal{F}|_{\mathcal{X}_s}$ is admissible for every closed $s \in S$.

We agree that any coherent sheaf on a smooth X_t is admissible.

Proposition 3.10. *Let $\mathcal{F}$ be an S-flat family of coherent sheaves on $\mathcal{X}$. Then the set $\{s \in S \mid \mathcal{F}_s$ is admissible$\}$ is open in S.*

Proof. This follows directly from Proposition 3.5. □

Similarly, we have the relative version. We agree that for $(\mathcal{Y}, \mathcal{D}_\pm)$ an S-family of relative pairs and $s \in S$ a closed point, we denote $\mathcal{Y}_s = \mathcal{Y} \times_S s$ and $\mathcal{D}_{\pm,s} = \mathcal{D}_\pm \times_S s$.

Definition 3.11. We call a coherent sheaf $\mathcal{F}$ on $Y[n_-, n_+]_0$ relative to $D[n_\pm]_{\pm,0}$ if it is normal to all $D_i \in Y[n_-, n_+]$ and is normal to the distinguished divisor $D[n_\pm]_{\pm,0}$. Let $(\mathcal{Y}, \mathcal{D}_\pm)$ be an S-family of relative pairs. We say an S-flat sheaf $\mathcal{F}$ on $\mathcal{Y}$ relative to $\mathcal{D}_\pm$ if for every closed $s \in S$, $\mathcal{F}_s$ is a sheaf on $\mathcal{Y}_s$ relative to $\mathcal{D}_{\pm,s}$.

Proposition 3.12. *Let $\mathcal{F}$ be an S-flat family of coherent sheaves on $\mathcal{Y}$. Then the set $\{s \in S \mid \mathcal{F}_s$ is relative to $\mathcal{D}_{\pm,s}\}$ is open in S.*

Proof. This follows directly from Corollary 3.4 and Proposition 3.5. $\square$

For later study, we show that the failure of admissible property of a class of G_{m}-equivariant quotient sheaves are constant in t. Since this is a local study, we work with modules. We let B be an integral $\mathbf{k}$-algebra of finite type; let A be the G_{m} $\mathbf{k}$-algebra

$$\begin{aligned} &A = B[z_1, z_2, t]/(z_1 z_2); \\ &z_1^\sigma = \sigma^a z_1,\ z_2^\sigma = z_2,\ t^\sigma = \sigma^b t; \quad a \in \mathbb{Z}_+,\ b \in \mathbb{Z}_-. \end{aligned} \tag{3.3}$$

We let $R = A^{\oplus m}$ be an A-module with the G_{m}-action acting on individual factors as in (3.3).

Given an A-module M, for $f \in M$ we denote by $\mathrm{ann}(f) \subset A$ the annihilator of f: $\mathrm{ann}(f) = \{a \in A \mid af = 0\}$. Let

$$I = (z_1, z_2) \subset A$$

be the ideal generated by z_1 and z_2. We define $M_I = \{f \in M \mid \mathrm{ann}(f) \supset I^k$ for some $k \in \mathbb{N}\}$. Namely, M_I consists of elements annihilated by I^k for some k.

We use the G_{m}-spectral decomposition to study G_{m}-sheaves. Given a G_{m}-module M, we let $M_{[\ell]} = \{v \in M \mid v^\sigma = \sigma^\ell v\}$. Since G_{m} is reductive and commutative, we have direct sum decomposition $M = \oplus_{\ell \in \mathbb{Z}} M_{[\ell]}$. We call an element $v \in M$ of weight ℓ if $v \in M_{[\ell]}$. By the weight assignments of z_i and t, we see that for an element $f \in A$ of weight $\ell \geq 0$ and is divisible by t, then f is divisible by z_1.

Let $A_0 = A/(t)$ be the quotient ring. For any R-module M, we denote $M_0 = M \otimes_A A_0$. Let $R_0 = A_0^{\oplus m} = R \otimes_A A_0$, and $I_0 = (z_1, z_2) \subset A_0$.

Lemma 3.13. *Let $\varphi : R \to M$ be a G_{m}-equivariant quotient A-module. Suppose M is $\mathbf{k}[t]$-flat, then the natural homomorphism $M_I \otimes_A A_0 \to (M_0)_{I_0}$ is an isomorphism.*

We next study the failure of the flatness of M over $T = \mathbf{k}[z_1, z_2]/(z_1 z_2)$. We let $A^- = A/(z_2)$, let $M^- = M \otimes_A A^-$, $R^- = R \otimes_A A^-$, and define $K^- = \ker\{R^- \to M^-\}$. We consider the localization $K^-_{(t)}$ of K^- by the ideal (t);[3] consider its further localization by (z_1), its intersection with $R^-_{(t)}$, and the quotient:

$$\Big((K^-_{(t)})_{(z_1)} \cap R^-_{(t)}\Big) \otimes_{A^-_{(t)}} A^-_{(t)}/(z_1) \subset B[t, t^{-1}]^{\oplus m}. \tag{3.4}$$

By the construction, the inclusion is G_{m}-invariant, thus the $B[t, t^{-1}]$-submodule is generated by elements in $B^{\oplus m}$. In other words, there is a B-submodule $C_{\mathrm{gen}} \subset B^{\oplus m}$ such that as submodules of $B[t, t^{-1}]^{\oplus m}$,

$$C_{\mathrm{gen}} \otimes_B B[t, t^{-1}] = \Big((K^-_{(t)})_{(z_1)} \cap R^-_{(t)}\Big) \otimes_{A^-_{(t)}} A^-_{(t)}/(z_1).$$

Applying the same construction to the module $K_0^- = \ker\{R_0^- \to M_0^-\}$, where $R_0^- = R^0 \otimes_{A_0} A_0^-$, where $A_0^- = A_0/(z_2)$, and same for M_0^-, we obtain a submodule $C_0 \subset B^{\oplus m}$ such that as submodules of $B^{\oplus m}$,

$$C_0 = \Big((K_0^-)_{(z_1)} \cap R_0^-\Big) \otimes_{A_0^-} A_0^-/(z_1).$$

Lemma 3.14. *Let the situation be as in Lemma 3.13. Then as B-modules, $C_0 \subset B^{\oplus m}$ coincide with $C_{\mathrm{gen}} \subset B^{\oplus m}$.*

The proofs will be given in Appendix.

3.3. Numerical criterion

We introduce numerical criterion to measure the failure of a coherent sheaf normal to a closed subscheme. This will be used to prove the properness of moduli spaces.

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[3]Since $(K^-)_{(t)} = (K_{(t)})^-$, there is no confusion using $K^-_{(t)}$ to denote either.

Let $I_l \subset \mathcal{O}_{X[n]_0}$ be the ideal sheaf of $D_l \subset X[n]_0$. For a sheaf $\mathcal{F}$ on $X[n]_0$, as in the previous subsection, we define

$$\mathcal{F}_{I_l} = \{v \in \mathcal{F} \mid \operatorname{ann}(v) \subset I_l^k \text{ for some } k \in \mathbb{Z}_+\}.$$

We define

$$\mathcal{F}^{\text{t.f.}} = \mathcal{F}/(\oplus_{l=1}^{n+1} \mathcal{F}_{I_l}).$$

It is the sheaf $\mathcal{F}$ quotient out its subsheaf supported on a sufficiently thickening of the singular loci of $X[n]_0$. We then denote $(\mathcal{F}^{\text{t.f.}})_l = \mathcal{F}^{\text{t.f.}}|_{\Delta_l}$, and form

$$(\mathcal{F}^{\text{t.f.}})_{l,I_l} := \big((\mathcal{F}^{\text{t.f.}})_l\big)_{I_l} \quad \text{and} \quad (\mathcal{F}^{\text{t.f.}})_{l,I_{l+1}} := \big((\mathcal{F}^{\text{t.f.}})_l\big)_{I_{l+1}};$$

they are subsheaves of $(\mathcal{F}^{\text{t.f.}})_l$ supported along D_l and D_{l+1} respectively.

Example. We give an example of non-admissible quotient sheaf of $\mathcal{O}_X$. For simplicity, we consider the affine case where $Y = \Delta_1 \cap \Delta_2 \subset \mathbb{A}^4$ is defined via $\Delta_1 = \{(z_i) | z_2 = 0\}$ and $\Delta_2 = \{(z_i) | z_1 = 0\}$. We let

$$\mathcal{F}_1 = \mathcal{O}_{\Delta_1}/\big(z_4, z_3^3, z_3^2 z_1\big), \quad \text{and} \quad \mathcal{F}_2 = \mathcal{O}_{\Delta_2}/\big(z_3, z_4^3, z_4^2 z_2\big).$$

Let $\iota_i : \Delta_i \to Y$ be the inclusion. We define $\mathcal{F} = \ker\{\iota_{1*}\mathcal{F}_1 \oplus \iota_{2*}\mathcal{F} \to k(0)\}$, where $k(0)$ is the structure sheaf of the origin $0 \in \mathbb{A}^4$. Then $\mathcal{F}_1^{\text{t.f.}} = \mathcal{O}_{\Delta_1}/(z_4, z_3^2)$, and $\mathcal{F}_2^{\text{t.f.}} = \mathcal{O}_{\Delta_2}/(z_3, z_4^2)$ (cf. (3.6) below); further

$$\mathcal{F}^{\text{t.f.}} = \ker\{\iota_{1*}\mathcal{F}_1^{\text{t.f.}} \oplus \iota_{2*}\mathcal{F}_2^{\text{t.f.}} \to k(0)\}, \quad \operatorname{length}\big(\mathcal{F}/\mathcal{F}^{\text{t.f.}}\big) = 2,$$

and $\mathcal{F}^{\text{t.f.}}|_{\Delta_1}$ has a dimension zero element support at 0.

For an integer v, we continue to denote by $\mathcal{F}(v) = \mathcal{F} \otimes p^* H^{\otimes v}$, where $p : X[n]_0 \to X$ is the projection.

Definition 3.15. We define the l-th error of $\mathcal{F}$ be

$$\operatorname{Err}_l \mathcal{F} = \chi(\mathcal{F}_{I_l}(v)) + \frac{1}{2}\chi\big((\mathcal{F}^{\text{t.f.}})_{l,I_l}(v)\big) + \frac{1}{2}\chi\big((\mathcal{F}^{\text{t.f.}})_{l-1,I_l}(v)\big); \tag{3.5}$$

we define the total error of $\mathcal{F}$ be $\operatorname{Err}\mathcal{F} = \displaystyle\sum_{l=0}^{n+1} \operatorname{Err}_l \mathcal{F}$.

Lemma 3.16. *A sheaf $\mathcal{F}$ on $X[n]_0$ is admissible along D_l if and only if all $\mathcal{F}_{I_l}$, $(\mathcal{F}^{\text{t.f.}})_{l,I_l}$ and $(\mathcal{F}^{\text{t.f.}})_{l-1,I_l}$ are zero.*

Proof. This is a local problem. We pick an affine open $W \subset X[n]_0$ so that $W \subset \Delta_{l-1} \cup \Delta_l - D_{l-1} \cup D_{l+1}$. We let $W_1 = W \cap \Delta_{l-1}$ and $W_2 = W \cap \Delta_l$. We form $\xi : W \to T$ as in (3.2) so that for $T_i \subset T$ the lines $\mathbb{A}^1 \cong T_i \subset T$, we have $W_i = W \times_T T_i$; thus $\xi^{-1}(0) = D_l \cap W$.

By Proposition 3.3 and Lemma 3.7, $\mathcal{F}|_W$ is admissible if and only if $\mathcal{F}|_{W_i}$ are flat over T_i near 0. Let J (resp. J_i) be the ideal sheaf of $W_1 \cap W_2 \subset W$ (resp. $W_1 \cap W_2 \subset W_i$); let $(\mathcal{F}|_W)_J$ be the torsion subsheaf of $\mathcal{F}|_W$ supported along $W_1 \cap W_2$, and let $\mathcal{F}|_W^{\mathrm{t.f.}} = (\mathcal{F}|_W)/(\mathcal{F}|_W)_J$. By the flatness criterion, this is true if and only if $(\mathcal{F}|_W)_J = 0$ and $\big((\mathcal{F}|_W^{\mathrm{t.f.}})|_{W_i}\big)_{J_i} = 0$ for $i = 1, 2$. This proves that $\mathcal{F}|_W$ is admissible if and only if all $\mathcal{F}_{I_l}|_W$, $(\mathcal{F}^{\mathrm{t.f.}})_{l,I_l}|_W$ and $(\mathcal{F}^{\mathrm{t.f.}})_{l-1,I_l}|_W$ are zero. Going over a covering of $D_l \subset X[n]_0$, the lemma follows. □

There is a useful identity expressing $\chi(\mathcal{F}(v))$ in terms of $\mathrm{Err}\mathcal{F}$ and the Hilbert polynomial of

$$\mathcal{F}_l^{\mathrm{t.f.}} := \mathcal{F}|_{\Delta_l}/(\mathcal{F}_{l,I_l} \oplus \mathcal{F}_{l,I_{l+1}}). \tag{3.6}$$

(It is $\mathcal{F}|_{\Delta_l}$ quotient out its subsheaf support along $D_l \cup D_{l+1} \subset \Delta_l$.)

Lemma 3.17. *Let*

$$\delta_{l,i} = \chi(\mathcal{F}_l^{\mathrm{t.f.}}(v)) + \chi\big((\mathcal{F}^{\mathrm{t.f.}})_{l,I_{l+i}}(v)\big) - \chi\big(\mathcal{F}^{\mathrm{t.f.}}|_{D_{l+i}}(v)\big), \quad i = 0, 1.$$

Then we have the identity

$$\chi(\mathcal{F}(v)) = \mathrm{Err}\mathcal{F} + \frac{1}{2}\sum_{l=0}^{n+1}\big(\delta_{l,0} + \delta_{l,1}\big). \tag{3.7}$$

Proof. Since $\mathcal{F}^{\mathrm{t.f.}} = \mathcal{F}/\mathcal{F}_I$,

$$\chi\left(\mathcal{F}(v)\right) = \chi\big(\mathcal{F}^{\mathrm{t.f.}}(v)\big) + \chi(\mathcal{F}_I(v)). \tag{3.8}$$

For $\mathcal{F}^{\mathrm{t.f.}}$, we have the exact sequence

$$0 \longrightarrow \mathcal{F}^{\mathrm{t.f.}} \longrightarrow \bigoplus_{l=0}^{n+1} \mathcal{F}^{\mathrm{t.f.}}|_{\Delta_l} \longrightarrow \bigoplus_{l=1}^{n+1} \mathcal{F}^{\mathrm{t.f.}}|_{D_l} \longrightarrow 0. \tag{3.9}$$

(Here we view both $\mathcal{F}^{\mathrm{t.f.}}|_{\Delta_l}$ and $\mathcal{F}^{\mathrm{t.f.}}|_{D_l}$ as sheaves of $\mathcal{O}_{X[n]_0}$-modules.) Using

$$\chi\big(\mathcal{F}^{\mathrm{t.f.}}|_{\Delta_l}(v)\big) = \chi\big(\mathcal{F}_l^{\mathrm{t.f.}}(v)\big) + \chi\big((\mathcal{F}^{\mathrm{t.f.}})_{l,I_l}(v)\big) + \chi\big((\mathcal{F}^{\mathrm{t.f.}})_{l,I_{l+1}}(v)\big)$$

and (3.9), after regrouping, we conclude

$$\chi(\mathcal{F}(v))=\left(\chi(\mathcal{F}_I(v))+\frac{1}{2}\sum_{l=0}^{n+1}\sum_{i=0}^{1}\chi\big((\mathcal{F}^{\text{t.f.}})_{l,I_{l+i}}(v)\big)\right)+\frac{1}{2}\sum_{l=0}^{n+1}(\delta_{l,0}+\delta_{l,1}).$$

This proves the lemma. □

We have the following positivity in case $\mathcal{F}$ is a quotient sheaf of $p^*\mathcal{V}$ for a locally free sheaf $\mathcal{V}$ on X.

Lemma 3.18. *Suppose $\mathcal{F}$ is a quotient sheaf of $p^*\mathcal{V}$. For $1\le l\le n$, the leading coefficients of $\delta_{l,0}$ and $\delta_{l,1}$ are non-negative; moreover, $\delta_{l,0}$ is zero if and only if $p^*\mathcal{V}|_{\Delta_l}\to\mathcal{F}_l^{\text{t.f.}}$ is a pull back of a quotient $p^*\mathcal{V}|_{D_l}\to\mathcal{E}$ of sheaves on D_l via the projection $\pi_l:\Delta_l\to D_l$.*

Proof. Let $1\le l\le n$. The quotient $p^*\mathcal{V}\to\mathcal{F}$ induces a quotient homomorphism $p^*\mathcal{V}|_{\Delta_l}\to\mathcal{F}_l^{\text{t.f.}}$. We let $\mathcal{K}$ be its kernel, which fits into the exact sequence

$$0\longrightarrow\mathcal{K}\longrightarrow p^*\mathcal{V}|_{\Delta_l}\longrightarrow\mathcal{F}_l^{\text{t.f.}}\longrightarrow 0.$$

Let $\pi_l:\Delta_l\to D_l$ be the projection. We claim $R^1\pi_{l*}\mathcal{F}_l^{\text{t.f.}}=0$. Indeed, since π_l is a $\mathbb{P}^1$-bundle, $R^{\ge 2}\pi_{l*}\mathcal{K}=0$. By base change, $R^1\pi_{l*}(p^*\mathcal{V}|_{\Delta_l})=0$ since for all closed $x\in D_l$, $H^1(\pi_l^{-1}(x),p^*\mathcal{V}|_{\pi_l^{-1}(x)})=0$. Applying π_{l*} to the above exact sequence, by the induced long exact sequence, we conclude that $R^1\pi_{l*}\mathcal{F}_l^{\text{t.f.}}=0$. Therefore, since $p^*H|_{D_l}$ is ample, for large v,

$$\chi\big(\mathcal{F}^{\text{t.f.}}(v)\big)=\chi\big(\pi_{l*}\mathcal{F}^{\text{t.f.}}(v)\big)=\chi\big((\pi_{l*}\mathcal{F}^{\text{t.f.}})(v)\big).$$

On the other hand, the surjective homomorphisms $p^*\mathcal{V}|_{\Delta_l}\to\mathcal{F}^{\text{t.f.}}\to\mathcal{F}^{\text{t.f.}}|_{D_l}$ induces a surjective $\pi_{l*}\mathcal{F}^{\text{t.f.}}\to\mathcal{F}^{\text{t.f.}}|_{D_l}$. This implies that the leading coefficient of $\chi((\pi_{l*}\mathcal{F}^{\text{t.f.}})(v))-\chi(\mathcal{F}^{\text{t.f.}}|_{D_l}(v))$ is non-negative; and is zero if and only if $\pi_{l*}\mathcal{F}^{\text{t.f.}}=\mathcal{F}^{\text{t.f.}}|_{D_l}$.

Finally, we suppose $\delta_{l,0}=0$. Then $\pi_{l*}\mathcal{F}^{\text{t.f.}}=\mathcal{F}^{\text{t.f.}}|_{D_l}$. Using $\pi_l^*\pi_{l*}\mathcal{F}^{\text{t.f.}}\to\mathcal{F}^{\text{t.f.}}$, we obtain a homomorphism $\pi_l^*(\mathcal{F}^{\text{t.f.}}|_{D_l})\to\mathcal{F}^{\text{t.f.}}$. As this homomorphism is an isomorphism when restricted to D_l, it is injective. Suppose it has nontrivial cokernel, then $\chi(\mathcal{F}^{\text{t.f.}}(v))\ne\chi(\pi_l^*\mathcal{F}^{\text{t.f.}}|_{D_l}(v))=\chi(\mathcal{F}^{\text{t.f.}}|_{D_l}(v))$, a contradiction. This proves the lemma. □

A parallel result holds for coherent sheaves on $Y[n_-,n_+]_0$. For the singular divisor $D_l\subset Y[n_-,n_+]_0$, we define $\mathrm{Err}_l\mathcal{F}$ be as in (3.5). For the relative divisor $D[n_\pm]_{\pm,0}$, we let $I_\pm$ be the ideal sheaf of $D[n_\pm]_{\pm,0} \subset Y[n_-,n_+]_0$, and define $\mathrm{Err}_\pm\mathcal{F} = \chi(\mathcal{F}_{I_\pm}(v))$. We define

$$\mathrm{Err}\mathcal{F} = \sum_{-n_-\le l\le n_+} \mathrm{Err}_l\mathcal{F} + \mathrm{Err}_-\mathcal{F} + \mathrm{Err}_+\mathcal{F}. \tag{3.10}$$

Lemma 3.19. *A coherent sheaf* $\mathcal{F}$ *on* $Y[n_-,n_+]_0$ *is relative to* $D[n_\pm]_{\pm,0}$ *if and only if* $\mathrm{Err}\mathcal{F} = 0$.

4. Degeneration of Quot schemes and coherent systems

We construct good degenerations of Quot schemes and moduli spaces of certain types of coherent systems. We shall focus on the case of Quot schemes. For coherent systems, we will comment on the modification needed at the end of the section.

4.1. Stable admissible quotients

We let $\pi : X \to C$ be a simple degeneration. We fix a relative ample line bundle H on X/C, and fix a locally free sheaf $\mathcal{V}$ on X.

We begin with admissible quotients on $X[n]_0$. Let $p : X[n]_0 \to X$ be the projection.

Definition 4.1. We call a quotient (sheaf) $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ on $X[n]_0$ admissible if $\mathcal{F}$ is admissible.

For two quotients $\phi_1\colon p^*\mathcal{V} \to \mathcal{F}_1$ and $\phi_2\colon p^*\mathcal{V} \to \mathcal{F}_2$ on $X[n]_0$, an equivalence between them consists of a pair (σ,ψ), where $\sigma\colon X[n]_0 \to X[n]_0$ is an automorphism induced from the canonical G_{m}^n action on $X[n]_0$, and $\psi\colon \mathcal{F}_1 \cong \sigma^*\mathcal{F}_2$ is an isomorphism, so that the following square is commutative:

$$\begin{array}{ccc} p^*\mathcal{V} & \xrightarrow{\lambda_1} & \mathcal{F}_1 \\ {\scriptstyle\sigma^\natural}\downarrow & & \downarrow{\scriptstyle\psi} \\ p^*\mathcal{V} \cong \sigma^*p^*\mathcal{V} & \xrightarrow{\sigma^*\lambda_2} & \sigma^*\mathcal{F}_2. \end{array} \tag{4.1}$$

Here the isomorphism $p^*\mathcal{V} \cong \sigma^*p^*\mathcal{V}$ is the (unique) one whose restriction to $\Delta_0 \cup \Delta_{n+1}$ is the identity map.

Suppose (σ,ψ_1) and (σ,ψ_2) are autoequivalences of a quotient $\phi\colon p^*\mathcal{V} \to \mathcal{F}$, then $\psi_2^{-1}\circ\psi_1$ is an automorphism of $\phi\colon p^*\mathcal{V} \to \mathcal{F}$, which is identity.

Therefore $\psi_1 = \psi_2$. It follows that the group $\mathrm{Aut}_{\mathfrak{X}}\phi$ of autoequivalences of $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ is a subgroup of G_{m}^n.

Definition 4.2. We say a quotient $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ on $X[n]_0$ is stable if it is admissible and $\mathrm{Aut}_{\mathfrak{X}}\phi$ is finite.

Let $(\mathcal{X}, p) \in \mathfrak{X}(S)$ be an S-family of expanded degenerations, let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$ and $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ be a quotient. We call $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ an S-flat family of stable quotients if $\mathcal{F}$ is flat over S, and for every closed point $s \in S$ the restriction $\phi_s\colon p^*\mathcal{V}|_{\mathcal{X}_s} \to \mathcal{F}|_{\mathcal{X}_s}$ (of ϕ to $\mathcal{X}_s$) is stable.

Lemma 4.3. *Let $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ be an S-flat family of quotients on $(\mathcal{X}, p) \in \mathfrak{X}(S)$. Then the set $\{s \in S \mid \phi_s\colon p^*\mathcal{V}|_{\mathcal{X}_s} \to \mathcal{F}|_{\mathcal{X}_s}$ is stable$\}$ is an open subset of S.*

Proof. Because automorphism groups being finite is an open condition, the Lemma follows from Proposition 3.10. ☐

We define the category $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$ of families of stable quotients. For any scheme S over C, we define $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(S)$ be the set of all $(\phi; \mathcal{X}, p)$ so that $(\mathcal{X}, p) \in \mathfrak{X}(S)$ and $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ is an S-flat family of stable quotients on $\mathcal{X}$. An arrow between $(\phi_1; \mathcal{X}_1, p)$ and $(\phi_2; \mathcal{X}_2, p)$ in $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(S)$ is an arrow $\sigma\colon \mathcal{X}_1 \to \mathcal{X}_2$ in $\mathfrak{X}(S)$ so that $\phi_1 \cong \sigma^*\phi_2$. For $\rho : S \to T$, the map $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(\rho) : \mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(T) \to \mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(S)$ is defined by pull back.

Sending $(\phi; \mathcal{X}, p) \in \mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$ to the base scheme of $\mathcal{X}$ defines $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$ as a groupoid over C.

Proposition 4.4. *$\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$ is a Deligne-Mumford stack locally of finite type.*

Proof. First we show that $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$ is a stack. We let Sch_C be the category of schemes over C. For any S in Sch_C and two families ϕ_1, ϕ_2 in $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(S)$, we define a functor

$$\underline{\mathrm{Isom}}_S(\phi_1, \phi_2) : \mathrm{Sch}_C \to (\mathrm{Sets})$$

that associates to any morphism $\rho : S' \to S$ the set of isomorphisms in $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(S')$ between $\rho^*\phi_1$ and $\rho^*\phi_2$. Since stable quotients have finite automorphism groups, by a standard argument, $\underline{\mathrm{Isom}}_S(\phi_1, \phi_2)$ is represented by a finite group scheme over S. An application of descent theory shows that $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$ is a stack.

Now we show that $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$ admits an étale cover by a Deligne-Mumford stack locally of finite type. Let $p\colon X[n]\to X$ be the projection; let $\mathrm{Quot}^{p^*\mathcal{V}}_{X[n]/C[n]}$ be the Quot scheme on $X[n]/C[n]$ of $p^*\mathcal{V}$. We form the subset $\mathrm{Quot}^{p^*\mathcal{V},\mathrm{st}}_{X[n]/C[n]} \subset \mathrm{Quot}^{p^*\mathcal{V}}_{X[n]/C[n]}$ of stable quotients as in Definition 4.2. By Lemma 4.3, it is open in $\mathrm{Quot}^{p^*\mathcal{V}}_{X[n]/C[n]}$. Since G_{m}^n acts on $X[n]/C[n]$, it acts on $\mathrm{Quot}^{p^*\mathcal{V}}_{X[n]/C[n]}$, and then on $\mathrm{Quot}^{p^*\mathcal{V},\mathrm{st}}_{X[n]/C[n]}$. By the stable assumption, G_{m}^n acts with finite stabilizers on $\mathrm{Quot}^{p^*\mathcal{V},\mathrm{st}}_{X[n]/C[n]}$, thus the quotient stack $[\mathrm{Quot}^{p^*\mathcal{V},\mathrm{st}}_{X[n]/C[n]}/G_{\mathrm{m}}^n]$ is a Deligne-Mumford stack.

Let

$$F_n : \left[\mathrm{Quot}^{p^*\mathcal{V},\mathrm{st}}_{X[n]/C[n]}/G_{\mathrm{m}}^n\right] \longrightarrow \mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$$

be the morphism induced by the universal family over $\mathrm{Quot}^{p^*\mathcal{V},\mathrm{st}}_{X[n]/C[n]}$. By construction F_n is étale. Hence, the induced

$$\coprod_{n\geq 0} F_n : \coprod_{n\geq 0} \left[\mathrm{Quot}^{p^*\mathcal{V},\mathrm{st}}_{X[n]/C[n]}/G_{\mathrm{m}}^n\right] \longrightarrow \mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$$

is étale and surjective. This proves the Proposition. □

We define relative stable quotients on an expanded pair in the same way by replacing $X[n]_0$ with $(Y[n_-,n_+]_0, D[n_\pm]_{\pm,0})$. Let

$$\mathcal{V}_0 = \mathcal{V}\otimes_{\mathcal{O}_X}\mathcal{O}_Y,$$

where $Y\to X$ is induced by the normalization $Y\to X_0\subset X$. Let

$$p\colon (Y[n_-,n_+]_0, D[n_\pm]_{\pm,0}) \to (Y, D_\pm)$$

be the projection. For any quotient $\phi : p^*\mathcal{V}_0\to\mathcal{F}$ on Y, the group $\mathrm{Aut}_{\mathfrak{Y}}\phi$ is defined in the same way as that of $\mathrm{Aut}_{\mathfrak{X}}\phi$, which is a subgroup of G_{m}^n.

Definition 4.5. Let $(Y[n_-,n_+]_0, D[n_\pm]_{\pm,0})$ be a relative pair. A relative quotient $\phi : p^*\mathcal{V}_0\to\mathcal{F}$ on $(Y[n_-,n_+]_0, D[n_\pm]_{\pm,0})$ is a quotient so that $\mathcal{F}$ is admissible and is normal to $D[n_\pm]_{\pm,0}$. We call $\phi : p^*\mathcal{V}_0\to\mathcal{F}$ stable if in addition $\mathrm{Aut}_{\mathfrak{Y}}\phi$ is finite.

We define families of relative quotients on $(\mathcal{Y},\mathcal{D}_\pm,p)\in(\mathfrak{D}_\pm\subset\mathfrak{Y})(S)$ similarly. We have

Proposition 4.6. *Let $\phi : p^*\mathcal{V}_0\to\mathcal{F}$ be an S-flat family of relative quotients on $(\mathcal{Y},\mathcal{D}_\pm)$. Then the restriction $\phi_{\mathcal{D}_\pm} : p^*\mathcal{V}_0|_{\mathcal{D}_\pm}\to\mathcal{F}|_{\mathcal{D}_\pm}$ is an S-flat family of quotients on $\mathcal{D}_\pm$.*

Proof. This follows from Corollary 3.4. □

We remark that Lemma 4.3 still holds after replacing families $\mathcal{X}/S$ by families $\mathcal{Y}/S$. We define the category $\mathfrak{Quot}^{\mathcal{V}_0}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$ of families of relative stable quotients accordingly.

Proposition 4.7. *$\mathfrak{Quot}^{\mathcal{V}_0}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$ is a Deligne-Mumford stack locally of finite type.*

Proof. The proof is parallel to that of Proposition 4.4. □

4.2. Coherent systems

Coherent systems we will consider are sheaf homomorphisms

$$\varphi\colon \mathcal{O}_{X[n]_0} \to \mathcal{F}$$

(or on $Y[n_-, n_+]_0$) so that $\mathcal{F}$ is pure of dimension one and φ has finite cokernel. Since an automorphism of $\varphi\colon \mathcal{O}_{X[n]_0} \to \mathcal{F}$ is a sheaf isomorphism $\sigma\colon \mathcal{F} \cong \mathcal{F}$ so that $\sigma \circ \varphi = \varphi$, that $\mathcal{F}$ is pure of dimension one and cokerφ is finite imply that σ is the identity map. We define the group $\mathrm{Aut}_{\mathfrak{X}}\varphi$ be the collection of pairs (σ, ξ) so that $\sigma \in G_{\mathrm{m}}^n$ and ξ form an isomorphism of $\varphi : \mathcal{O}_{X[n]_0} \to \mathcal{F}$ with $\sigma^*\varphi : \mathcal{O}_{X[n]_0} = \sigma^*\mathcal{O}_{X[n]_0} \to \sigma^*\mathcal{F}$; in other words, such pairs (σ, ξ) consist of commutative diagrams as in (4.1). Obviously $\mathrm{Aut}_{\mathfrak{X}}\varphi$ is a subgroup of G_{m}^n.

Definition 4.8. We say a coherent system $\varphi\colon \mathcal{O}_{X[n]_0} \to \mathcal{F}$ admissible if both $\mathcal{F}$ and cokerφ are admissible. We say it is stable if it is admissible and $\mathrm{Aut}_{\mathfrak{X}}\varphi$ is finite.

Since cokerφ has dimension zero and $\mathcal{F}$ is pure, φ is admissible implies that cokerφ is away from the singular locus of $X[n]_0$. We adopt the convention that any coherent system on a smooth X_t is admissible and stable. We define families of stable coherent systems in the same way as families of stable quotients. We have

Proposition 4.9. *Let $\varphi\colon \mathcal{O}_{\mathcal{X}} \to \mathcal{F}$ be an S-flat family of coherent systems on an expanded degeneration $(\mathcal{X}, p) \in \mathfrak{X}(S)$. Then the set $\{s \in S \mid \varphi_s\colon \mathcal{O}_{\mathcal{X}_s} \to \mathcal{F}_s$ is stable$\}$ is an open subset of S.*

We form the category $\mathfrak{P}_{\mathfrak{X}/\mathfrak{C}}$ of families of stable coherent systems. We have

Proposition 4.10. $\mathfrak{P}_{\mathfrak{X}/\mathfrak{C}}$ *is a Deligne-Mumford stack locally of finite type.*

Accordingly, we have the following relative version.

Definition 4.11. We say a coherent system $\varphi\colon \mathcal{O}_{Y[n_-,n_+]_0} \to \mathcal{F}$ relative if both $\mathcal{F}$ and cokerφ are admissible, and cokerφ is normal to $D_\pm[n_\pm]_0$. We say it is stable if it is admissible and $\mathrm{Aut}_{\mathfrak{Y}}\varphi$ is finite.

Proposition 4.12. *Let $\varphi\colon \mathcal{O}_{\mathcal{Y}} \to \mathcal{F}$ be an S-flat family of relative coherent systems on $(\mathcal{Y}, \mathcal{D}_\pm)$. Then the restriction $\varphi_{\mathcal{D}_+} : \mathcal{O}_{\mathcal{D}_+} \to \mathcal{F}|_{\mathcal{D}_+}$ and $\varphi_{\mathcal{D}_-} : \mathcal{O}_{\mathcal{D}_-} \to \mathcal{F}|_{\mathcal{D}_-}$ are S-flat families of quotient sheaves on $\mathcal{D}_+$ and $\mathcal{D}_-$.*

Proof. This is because for a family of relative coherent systems $\varphi\colon \mathcal{O}_{\mathcal{Y}} \to \mathcal{F}$, coker$\varphi$ is away from $\mathcal{D}_+$ and $\mathcal{D}_-$. Therefore, the restrictions $\varphi_{\mathcal{D}_+} : \mathcal{O}_{\mathcal{D}_+} \to \mathcal{F}|_{\mathcal{D}_+}$ and $\varphi_{\mathcal{D}_-} : \mathcal{O}_{\mathcal{D}_-} \to \mathcal{F}|_{\mathcal{D}_-}$ are surjective. The flatness follows from Corollary 3.4. □

We form the stack $\mathfrak{P}_{\mathfrak{D}_\pm \subset \mathfrak{Y}}$ of families of relative coherent systems. Analogue to Proposition 4.10, we have

Proposition 4.13. $\mathfrak{P}_{\mathfrak{D}_\pm \subset \mathfrak{Y}}$ *is a Deligne-Mumford stack locally of finite type.*

4.3. Components of the moduli stack

The moduli stacks $\mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$ and $\mathfrak{P}_{\mathfrak{X}/\mathfrak{C}}$ can be decomposed into disjoint pieces according to the topological invariants of the sheaves. We will discuss the case for Quot scheme; it is the same for the moduli of coherent systems.

We use Hilbert polynomials to keep track of the topological data of quotients. For any coherent sheaf $\mathcal{F}$ on an $(\mathcal{X}, p) \in \mathfrak{X}(S)$, and for a closed $s \in S$, denote $\mathcal{F}_s = \mathcal{F}|_{\mathcal{X}_s}$ and define

$$\chi^H_{\mathcal{F}_s}(v) = \chi(\mathcal{F}_s \otimes p^* H^{\otimes v}), \quad p : \mathcal{X}_s \longrightarrow X, \quad v \in \mathbb{Z}.$$

Let $P(v)$ be a fixed polynomial. We define $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k}) \subset \mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k})$ be the subset consisting of $[\varphi\colon \mathcal{O}_{X[n]_0} \to \mathcal{F}] \in \mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k})$ so that $\chi^H_{\mathcal{F}} = P$. Since the Hilbert polynomials of a flat family of sheaves are locally constant in their parameter space, $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k}) \subset \mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k})$ is both open and closed. Thus it defines an open and closed substack $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}} \subset \mathfrak{Quot}^{\mathcal{V}}_{\mathfrak{X}/\mathfrak{C}}$.

Similarly, we let $q : Y \to X$ and $p : Y[n_-, n_+]_0 \to Y$ be the projections; for a sheaf $\mathcal{F}$ on $Y[n_-, n_+]_0$, we denote $\chi^H_{\mathcal{F}}(v) = \chi(\mathcal{F} \otimes p^* q^* H^{\otimes v})$. We define the open and closed substack $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}} \subset \mathfrak{Quot}^{\mathcal{V}_0}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$ be so that $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}(\mathbf{k})$ consists of relative stable quotients $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ such that $\chi^H_{\mathcal{F}} = P$.

For moduli of coherent systems, following the same procedure, we have open and closed substacks $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$ of $\mathfrak{P}_{\mathfrak{X}/\mathfrak{C}}$ and $\mathfrak{P}^P_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$ of $\mathfrak{P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$.

We state the main theorems of the first part of this paper whose proofs will occupy the next section.

Theorem 4.14. *The Deligne-Mumford stacks $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}$ and $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$ are separated, proper over C, and of finite type.*

Theorem 4.15. *The Deligne-Mumford stacks $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$ and $\mathfrak{P}^P_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$ are separated, proper and of finite type.*

5. Properness of the moduli stacks

We apply the valuative criterion to prove Theorems 4.14 and 4.15. We let S be an affine scheme such that $\Gamma(\mathcal{O}_S)$ is a discrete valuation $\mathbf{k}$-algebra; let η and $\eta_0 \in S$ be its generic and closed point. We will often denote by $S' \to S$ a finite base change; in this case we denote by η' and η'_0 its generic and closed points.

For any quotient homomorphism $\phi : p^*\mathcal{V} \to \mathcal{F}$ on $(\mathcal{X}, p) \in \mathfrak{X}(S)$, we denote by ϕ_η and ϕ_{η_0} the restriction of ϕ to $\mathcal{X}_\eta = \mathcal{X} \times_S \eta$ and $\mathcal{X}_{\eta_0}$, respectively.

Proposition 5.1. *Let (S,η,η_0) be as stated. Given any $(\phi_\eta,\mathcal{X}_\eta) \in \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(\eta)$, we can find a finite base change $S' \to S$ so that $(\phi_\eta, \mathcal{X}_\eta) \times_\eta \eta' \in \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(\eta')$ extends to a family in $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(S')$. Further, the same conclusion holds for $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$.*

Proposition 5.2. *Let (S, η, η_0) be as stated. Given $(\phi_1, \mathcal{X}_1)$, $(\phi_2, \mathcal{X}_2) \in \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(S)$, any isomorphism $(\phi_1, \mathcal{X}_1) \times_S \eta \cong (\phi_2, \mathcal{X}_2) \times_S \eta$ in $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(\eta)$ extends to an isomorphism $(\phi_1, \mathcal{X}_1) \cong (\phi_2, \mathcal{X}_2)$ in $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(S)$. Further, the same conclusion holds for $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$.*

We need an ordering on a set of polynomials.

Definition 5.3. We let $\mathcal{A}^* \subset \mathbb{Q}[k]$ be the set of polynomials whose leading terms are of the form $a_r\frac{k^r}{r!}$ with $a_r \in \mathbb{Z}_+$; let $\mathcal{A} = \mathcal{A}^* \cup \{0\}$. For any $f(k) = a_r\frac{k^r}{r!} + \cdots$ and $g(k) = b_s\frac{k^s}{s!} + \cdots$ in $\mathcal{A}^*$, we say $f(k) \prec g(k)$ if either $r < s$, or $r = s$ and $a_r < b_s$; we say $f(k) \approx g(k)$ if $r = s$ and $a_r = b_s$. We agree that 0 is $\prec$ to all other elements.

For convenience, we use $\preccurlyeq$ to denote $\prec$ or $\approx$.

Lemma 5.4. *The set $\mathcal{A}$ satisfies the descending chain condition.*

Proof. For any sequence $f_1(k) \succcurlyeq f_2(k) \succcurlyeq \cdots$, since 0 is the minimal element in $\mathcal{A}$, we can assume $f_i(k) \neq 0$ for all i. By Definition 5.3, we know the pairs (r, a_r) of the degrees and the leading coefficients of polynomials $f_i(k)$ decrease according to the lexicographic order. Since the pairs consist of non-negative integers, we can find an integer n, so that $f_n(k) \approx f_{n+1}(k) \approx \cdots$. □

5.1. The completeness I

Let (S, η, η_0) be as stated in the beginning of this section, and $S \to C$ be a scheme over C; let $(\phi_\eta : p_\eta^* \mathcal{V} \to \mathcal{F}_\eta) \in \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(\eta)$ be a quotient on $(\mathcal{X}_\eta, p_\eta) \in \mathfrak{X}(\eta)$. In this subsection, we assume $\mathcal{X}_\eta$ is smooth. Since the case where $S \to C$ sends η_0 to $C - 0$ is trivially true, (following from the properness of Quot-schemes,) we assume it sends η_0 to $0 \in C$.

Lemma 5.5. *We can extend ϕ_η to a family of S-flat quotient $\phi : p^* \mathcal{V} \to \mathcal{F}$ on an $(\mathcal{X}, p) \in \mathfrak{X}(S)$ such that $\mathrm{Aut}_{\mathfrak{X}} \phi_{\eta_0}$ is finite.*

Proof. Since $\mathcal{X}_\eta$ is smooth, $S \to C$ sends $\eta \in S$ to a point in $C - 0$. Using that S is a C-scheme, we define $\mathcal{X} = X \times_C S$, and denote $p : \mathcal{X} \to X$ the projection. Because Grothendieck's quot-scheme is proper, the quotient ϕ_η on $\mathcal{X}_\eta$ extends to a quotient $\phi : p^* \mathcal{V} \to \mathcal{F}$, flat over S. Since $\mathcal{X}_{\eta_0}$ has no added Δ_l, $\mathrm{Aut}_{\mathfrak{X}} \phi_{\eta_0}$ is $\{e\}$. □

We will show that by varying the extensions $(\mathcal{X}, p) \in \mathfrak{X}(S)$ of $\mathcal{X}_\eta$, we can decrease $\mathrm{Err}\mathcal{F}_{\eta_0}$ while keeping $\mathrm{Aut}_{\mathfrak{X}} \phi_{\eta_0}$ finite. By the descending chain condition, this implies that we can find an extension with stable quotient at the special fiber.

Lemma 5.6. *Let $\phi_\eta : p_\eta^* \mathcal{V} \to \mathcal{F}_\eta$ be a quotient as in Lemma 5.5, and let $\phi : p^* \mathcal{V} \to \mathcal{F}$ be an S-flat quotient that extends ϕ_η with $\mathrm{Aut}_{\mathfrak{X}} \phi_{\eta_0}$ finite. Suppose $\mathrm{Err}\mathcal{F}_{\eta_0} \neq 0$, then we can find a finite base change $S' \to S$, an S'-flat quotients $\phi' : p'^* \mathcal{V} \to \mathcal{F}'$ on $(\mathcal{X}', p') \in \mathfrak{X}(S')$ such that*

1) *$\mathcal{X}'_{\eta'} \cong \mathcal{X}_\eta \times_\eta \eta' \in \mathfrak{X}(\eta')$, and under this isomorphism $\phi'_{\eta'} = \phi_\eta \times_\eta \eta'$;*

2) $\mathrm{Aut}_{\mathfrak{X}}(\phi'_{\eta'_0})$ *is finite, and*

3) $\mathrm{Err}\mathcal{F}'_{\eta'_0} \prec \mathrm{Err}\mathcal{F}_{\eta_0}$.

We prove the Lemma by proving a sequence of lemmas. Since S is local,

$$\mathcal{X} = X[n] \times_{C[n]} S \quad \text{for a} \ \ \xi : S \to C[n]$$

such that $\xi(\eta_0) = 0 \in C[n]$. We let u be a uniformizing parameter of S. Denoting by $\pi_n : C[n] \to \mathbb{A}^{n+1}$ the projection, we express

$$\pi_n \circ \xi = \big(c_1 u^{e_1}, \ldots, c_{n+1} u^{e_{n+1}}\big), \quad c_i \in \Gamma(\mathcal{O}_S)^*. \tag{5.1}$$

($\Gamma(\mathcal{O}_S)^*$ are the invertible elements in $\Gamma(\mathcal{O}_S)$.) Since $\xi(\eta_0) = 0$, all $e_i > 0$. Since $\mathrm{Err}\mathcal{F}_{\eta_0} \neq 0$, we pick an $1 \leq l \leq n$ so that

$$\deg \mathrm{Err}_l \mathcal{F}_{\eta_0} = \deg \mathrm{Err}\mathcal{F}_{\eta_0}.$$

We let

$$\tau_l : C[n] \times G_{\mathrm{m}} \to C[n+1]$$

be induced from the $\mathbb{A}^{n+1} \times G_{\mathrm{m}} \to \mathbb{A}^{n+2}$:

$$(t_1, \ldots, t_{n+1}, \sigma) \mapsto (t_1, \ldots, t_{l-1}, \sigma^{e_l}, \sigma^{-e_l} t_l, t_{l+1}, \ldots, t_{n+1}). \tag{5.2}$$

We then introduce

$$\xi_l = \tau_l \circ (\xi, \mathrm{id}) : S \times G_{\mathrm{m}} \longrightarrow C[n] \times G_{\mathrm{m}} \longrightarrow C[n+1],$$

and let $\mathcal{X}' := \xi_l^* X[n+1]$ over $S \times G_{\mathrm{m}}$ be the pull back family. Because of the canonical isomorphism $\tau_l^* X[n+1] \cong X[n] \times G_{\mathrm{m}}$ as families over $C[n] \times G_{\mathrm{m}}$,

$$\mathcal{X}' \cong \xi^* X[n] \times G_{\mathrm{m}} = \mathcal{X} \times G_{\mathrm{m}}.$$

We let $p' : \mathcal{X}' \to X$ and $\pi_1 : \mathcal{X}' \to \mathcal{X}$ be the projections.

We let $\phi' = \pi_1^* \phi : p'^* \mathcal{V} \to \mathcal{F}'$ be the pullback quotient sheaves (of ϕ). Since $(\mathcal{X}', p')$ is induced by $\xi_l : S \times G_{\mathrm{m}} \to C[n+1]$, the family of quotients ϕ' induces a $C[n+1]$-morphism

$$f_l : S \times G_{\mathrm{m}} \longrightarrow \mathrm{Quot}^{p^*\mathcal{V}, P}_{X[n+1]/C[n+1]}. \tag{5.3}$$

For simplicity, we abbreviate $\mathfrak{Q} = \mathrm{Quot}^{p^*\mathcal{V}, P}_{X[n+1]/C[n+1]}$.

We now construct a regular G_{m}-surface V and G_{m}-morphisms that fit into the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "\bar{\jmath}"] & \mathfrak{Q} := \mathrm{Quot}^{p^*\mathcal{V},P}_{X[n+1]/C[n+1]} \arrow[d, "\pi"] \\ S \times G_{\mathrm{m}} \arrow[u, "j"] \arrow[ur, "f_l"] \arrow[r, "\xi_l"] & C[n+1] \end{tikzcd}$$

so that $\pi \circ \bar{\jmath} : V \to C[n+1]$ is proper.

We first loot at the composite

$$\xi_l \circ \pi_n : S \times G_{\mathrm{m}} \longrightarrow C[n+1] \longrightarrow \mathbb{A}^{n+2}; \tag{5.4}$$

it is given by

$$\xi_l \circ \pi_n(u,t) = (c_1 u^{e_1}, \ldots, c_{l-1} u^{e_{l-1}}, t^{e_l}, c_l t^{-e_l} u^{e_l}, c_{l+1} u^{e_{l+1}}, \ldots, c_{n+1} u^{e_{n+1}}).$$

We embed $S \times G_{\mathrm{m}} \subset S \times \mathbb{A}^1$ via the embedding $G_{\mathrm{m}} \subset \mathbb{A}^1$ so that the induced G_{m}-action on $\mathbb{A}^1$ is $t^\sigma = \sigma t$. We then blow up $S \times \mathbb{A}^1$ at $(\eta_0, 0) \in S \times \mathbb{A}^1$, let $\widetilde{S}$ be the proper transform of $S \times 0$, and form

$$V' = \mathrm{bl}_{(\eta_0,0)} S \times \mathbb{A}^1 - \widetilde{S}.$$

Note that $V' \subset S \times \mathbb{A}^1 \times \mathbb{A}^1$ is defined via $u = vt$, where v is the standard coordinate of the last $\mathbb{A}^1$-factor.

By construction, (5.4) extends to a $V' \to \mathbb{A}^{n+2}$, in the form

$$(v,t) \mapsto (c_1 u^{e_1}, \ldots, c_{l-1} u^{e_{l-1}}, t^{e_l}, c_l v^{e_l}, c_{l+1} u^{e_{l+1}}, \ldots, c_{n+1} u^{e_{n+1}}),\ u = vt. \tag{5.5}$$

Because $C[n+1] \to \mathbb{A}^{n+2}$ is proper over a neighborhood of $0 \in \mathbb{A}^{n+2}$, and because all $e_i > 0$, (cf. (5.1)), $V' \to \mathbb{A}^{n+2}$ lifts to a unique

$$\xi_l' : V' \longrightarrow C[n+1],$$

extending $\xi_l : S \times G_{\mathrm{m}} \to C[n+1]$.

We let G_{m} acts on $S \times \mathbb{A}^1 \times \mathbb{A}^1$ be $(u,t,v)^\sigma = (u, \sigma t, \sigma^{-1} v)$. It leaves $V' \subset S \times \mathbb{A}^1 \times \mathbb{A}^1$ invariant, thus induces a G_{m}-action on V'. We let $E \subset V'$ be the exceptional divisor of $V' \to S \times \mathbb{A}^1$; let $E' \subset V'$ be the proper transform of $\eta_0 \times \mathbb{A}^1$. In coordinates, $E = (t = 0)$ and $E' = (v = 0)$.

By construction, f_l is a morphism from $V' - E$ to $\mathfrak{Q}$. Since $\mathfrak{Q}$ is proper over $C[n+1]$, f_l extends to $\tilde{f}_l : U \to \mathfrak{Q}$ for an open $U \subset V'$ that contains

$V' - E$ and the generic point of E. On the other hand, since all schemes and morphisms are G_{m}-equivariant, $U \subset V'$ can be made G_{m}-invariant. Therefore, either $U = V'$ or $U = V' - \{o\}$, where $\{o\} = E \cap E'$.

We now consider the case $U = V' - \{o\}$. Since $\mathfrak{Q}$ is proper over $C[n+1]$, after successive blowing up, say

$$b : V \longrightarrow V',$$

we can extend $\tilde{f}_l : V' - \{o\} \to \mathfrak{Q}$ to a morphism

$$\bar{\jmath} : V \to \mathfrak{Q}.$$

Since all the relevant schemes and morphisms are G_{m}-equivariant, we are able to make the blowing-up $V \to V'$ G_{m}-equivariant and the extension $\bar{\jmath}$ G_{m}-equivariant.

Since $V \to V'$ is a G_{m}-equivariant blowing up, and since the G_{m}-action on the tangent space of the (only) fixed point $o \in V'$ has weights e_l and $-e_l$, the exceptional divisor of $V \to V'$ can be made a chain of rational curves $\Sigma_1, \ldots, \Sigma_k$. We let $\Sigma_0 \subset V$ (resp. $\Sigma_{k+1} \subset V$) be the proper transform of $E' \subset V'$ (resp. $E \subset V'$); then possibly after reindexing,

$$\Sigma := \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_k \cup \Sigma_{k+1}$$

forms a connected chain of rational curves; namely, $\Sigma_i \cap \Sigma_{i+1} \neq \emptyset$, for $0 \leq i \leq k$. Using the explicit expression (5.5), we conclude that under the morphism

$$\pi_n \circ \xi'_l \circ b : V \longrightarrow \mathbb{A}^{n+2}, \tag{5.6}$$

$\Sigma_1, \ldots, \Sigma_k$ are mapped to $0 \in \mathbb{A}^{n+2}$, and Σ_0 (resp. Σ_{k+1}) is mapped to the line $\ell_l = \{t_i = 0, i \neq l\} \subset \mathbb{A}^{n+2}$ (resp. $\ell_{l+1} \subset \mathbb{A}^{n+2}$). (Recall Σ_0 is the proper transform of $(v = 0)$ and Σ_{k+1} of $(t = 0)$.)

The proof of Lemma 5.6 will be carried out by studying the pull back of the universal family of $\mathfrak{Q}$ via $\bar{\jmath} : V \to \mathfrak{Q}$. We fix our convention on this pull back family. In the remainder of this subsection, we denote

$$\big(\tilde{p} : \widetilde{\mathcal{X}} = X[n+1] \times_{C[n+1]} V \longrightarrow X\big) \in \mathfrak{X}(V); \tag{5.7}$$

we denote Φ the universal family on $\mathfrak{Q}$ and denote $\tilde{\phi} = \bar{\jmath}^* \Phi$:

$$\tilde{\phi} : \tilde{p}^* \mathcal{V} \longrightarrow \widetilde{\mathcal{F}} \quad \text{on} \quad \tilde{p} : \widetilde{\mathcal{X}} \to X. \tag{5.8}$$

For any closed subscheme $A \subset V$, we use $\tilde{\phi}_A$ to denote the restriction of $\tilde{\phi}$ to $\widetilde{\mathcal{X}}_A := \widetilde{\mathcal{X}} \times_V A$:

$$\tilde{\phi}_A : \tilde{p}_A^* \mathcal{V} \longrightarrow \widetilde{\mathcal{F}}_A \quad \text{on} \quad \tilde{p}_A : \widetilde{\mathcal{X}}_A \to X.$$

Lemma 5.7. *The family $\tilde{\phi}$ is G_{m}-equivariant, where the G_{m}-action is the one induced from the G_{m}-morphism $\bar{\jmath}$. The chain of rational curves Σ is G_{m}-invariant, and the G_{m}-fixed points of Σ_i are $q_i = \Sigma_i \cap \Sigma_{i-1}$ and q_{i+1}.*

Proof. The first part follows from that $\bar{\jmath}$ is G_{m}-equivariant. The second part follows from that $V \to V'$ is a successive G_{m}-equivariant blowing up, and that G_{m} acts on the tangent space $T_0 V'$ with weights e_l and $-e_l$. $\square$

Lemma 5.8. *The fiber of $\widetilde{\mathcal{X}}_{\Sigma_0}$ over $a \neq q_1 \in \Sigma_0$ (resp. $a = q_1$) is $X[n]_0$ (resp. $X[n+1]_0$); the family $\widetilde{\mathcal{X}}_{\Sigma_0}$ is a smoothing of the divisor $D_l \subset \widetilde{\mathcal{X}}_{q_1} \cong X[n+1]_0$. The G_{m}-action on $\widetilde{\mathcal{X}}_{q_1} \cong X[n+1]_0$ leaves all $\Delta_i \subset X[n+1]_0$ except Δ_l fixed, and acting on Δ_l with fixed loci $D_l \cup D_{l+1}$.*

Proof. By the construction of $X[n+1] \to C[n+1]$, for the l-th coordinate line $\ell_l \subset \mathbb{A}^{n+2}$, $X[n+1] \times_{\mathbb{A}^{n+2}} \ell_l$ is a family over ℓ_l whose fiber over $a \neq 0 \in \ell_l$ is isomorphic to $X[n]_0$, and whose fiber over $0 \in \ell_l$ is isomorphic to $X[n+1]_0$; the family is a smoothing of the l-th singular divisor $D_l \subset X[n+1]_0$.

Applying this to the Lemma, knowing that $\Sigma_0 \to \mathbb{A}^{n+2}$ (cf. (5.6)) is mapped onto the coordinate line ℓ_l, the first part of the lemma follows immediately.

For the second part, we need to understand the G_{m}-action on

$$X[n+1]_{\ell_l} := X[n+1] \times_{\mathbb{A}^{n+2}} \ell_l.$$

Recall the G_{m}-action on $\mathbb{A}^{n+2}$ is via

$$(z)^\sigma = (z_1, \ldots, z_{l-1}, \sigma^{e_l} z_l, \sigma^{-e_l} z_{l+1}, z_{l+2}, \ldots, z_{n+2}).$$

By the construction of $X[n+1]/C[n+1]$, this G_{m}-action on $X[n+1]_0$ leaves $\Delta_i \subset X[n+1]_0$ except Δ_l fixed, and leaves Δ_l invariant with fixed loci $D_l \cup D_{l+1}$. (This can be seen using explicit description of $X[n+1]$; it is also apparent in case $n = 0$, since then $l = 1$ and the G_{m}-action on Δ_0 can only be trivial.) This proves the second part of the lemma. $\square$

We have a parallel Lemma.

Lemma 5.9. *The fiber of $\widetilde{\mathcal{X}}_{\Sigma_{k+1}}$ over $a \neq q_{k+1} \in \Sigma_{k+1}$ (resp. $a = q_{k+1}$) is $X[n]_0$ (resp. $X[n+1]_0$); the family $\widetilde{\mathcal{X}}_{\Sigma_{k+1}}$ is a smoothing of the divisor $D_{l+1} \subset \widetilde{\mathcal{X}}_{q_{k+1}} \cong X[n+1]_0$. The G_{m}-action on $\widetilde{\mathcal{X}}_{q_{k+1}}$ leaves all $\Delta_i \subset \widetilde{\mathcal{X}}_{q_{k+1}}$ except Δ_l fixed, and acting on Δ_l with fixed loci $D_l \cup D_{l+1}$.*

Using that the families over Σ_i, $1 \le i \le k$ are all pull backs of the central fiber $X[n+1]_0$ over $0 \in C[n+1]$, and combining with the results proved in the previous two Lemmas, we have

Lemma 5.10. *For $1 \le i \le k$, $\widetilde{\mathcal{X}}_{\Sigma_i} \cong X[n+1]_0 \times \Sigma_i$; the G_{m}-action on $\widetilde{\mathcal{X}}_{\Sigma_i}$ are the product action of the G_{m}-action on Σ_i, and the G_{m}-action on $\widetilde{\mathcal{X}}_{q_1}$, (which is identical to that on $\widetilde{\mathcal{X}}_{q_{k+1}}$).*

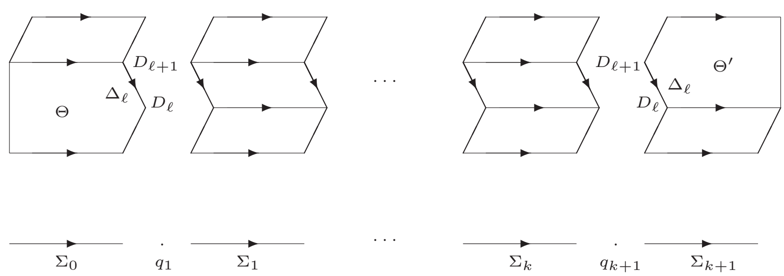

Figure 4: In the figure, the slated lines represent Δ_i; the horizontal lines represent $\Delta_i \times \Sigma_j$; the arrows represent the G_{m}-action; lines w/o arrows are fixed by G_{m}.

In the figure, the left column represents $\widetilde{\mathcal{X}}_{\Sigma_0}$, of which only $\Delta_{l+1} \times \Sigma_0$ (the top parallelgram) and the Θ are shown. The piece Θ is the blowing up of $\Delta_{l-1} \times \Sigma_0$ along $D_l \times q_1$, where $\Delta_{l-1} \subset X[n]_0$. We endow Θ with the G_{m}-action induced by the product action on $X[n]_0 \times \Sigma_0$, where G_{m}-acts on Δ_{l-1} trivially, and acts on Σ_0 by that induced from the G_{m}-action on V. The family $\widetilde{\mathcal{X}}_{\Sigma_0}$ is by replacing $\Delta_{l-1} \times \Sigma_0 \subset X[n]_0 \times \Sigma_0$ with Θ.

The right column represents $\widetilde{\mathcal{X}}_{\Sigma_{k+1}}$. The piece $\Theta' \subset \widetilde{\mathcal{X}}_{\Sigma_{k+1}}$ is constructed similarly: it is the blowing up of $\Delta_l \times \Sigma_{k+1}$ along $D_l \times q_{k+1}$; the total family $\widetilde{\mathcal{X}}_{\Sigma_{k+1}}$ is by replacing $\Delta_l \times \Sigma_{k+1}$ in $X[n]_0 \times \Sigma_{k+1}$ by Θ'. The G_{m}-action is the one induced from the product action on $X[n]_0 \times \Sigma_{k+1}$, where the action on $X[n]_0$ is via the trivial action, and on Σ_{k+1} is via the one induced from that on V.

The next lemma explains the role of the families $\widetilde{\mathcal{X}}_{\Sigma_i}$ in our proof of Lemma 5.6

Lemma 5.11. *For $a \in \Sigma_0 - q_1$ or $a \in \Sigma_{k+1} - q_{k+1}$, $\tilde{\phi}_a : \tilde{p}_a^* \mathcal{V} \to \widetilde{\mathcal{F}}_a$ on $\widetilde{\mathcal{X}}_a$ is isomorphic to $\phi_{\eta_0} : p_{\eta_0}^* \mathcal{V} \to \mathcal{F}_{\eta_0}$.*

Proof. We comment that since $C[n+1] = C \times_{\mathbb{A}^1} \mathbb{A}^{n+2}$, a morphism $h : S \to C[n+1]$ is given by a pair of morphisms $h' : S \to C$ and $h'' : S \to \mathbb{A}^{n+2}$ so that their corresponding compositions $S \to C \to \mathbb{A}^1$ and $S \to \mathbb{A}^{n+2} \to \mathbb{A}^1$ coincide.

We pick a morphism $\varphi_1 : S \to V$ that is the lift of $S = S \times 1 \xrightarrow{\subset} S \times G_{\mathrm{m}}$. By the description of $V \to V' \to \mathbb{A}^{n+2}$ (cf. (5.5)), we see that $\varphi_1(\eta_0) \in V$ lies over $(\ldots, 0, 1, 0, \ldots) \in \mathbb{A}^{n+2}$, thus $\varphi_1(\eta_0) \in \Sigma_{k+1} - q_{k+1}$.

By the construction of φ_1, we see that the composite $\bar{\jmath} \circ \varphi_1 : S \to V \to \mathfrak{Q}$ coincide with the restriction of f_l (cf. (5.3)) to $S \times 1$: $\jmath \circ \varphi_1 = f_l|_{S \times 1}$. Since f_l is induced by the family ϕ, we obtain

$$\phi \cong (\bar{\jmath} \circ \varphi_1)^* \Phi \cong \varphi_1^* \tilde{\phi},$$

where Φ is the universal family of $\mathfrak{Q}$. Let $a' = \varphi_1(\eta_0)$; this proves $\tilde{\phi}_{a'} \cong \phi_{\eta_0}$. Finally, since all points in $\Sigma_{k+1} - q_{k+1}$ form a single G_{m}-orbit, for $a \in \Sigma_{k+1} - q_{k+1}$, $\tilde{\phi}_a \cong \tilde{\phi}_{a'} \cong \phi_{\eta_0}$. This prove the part of the Lemma for the case $\Sigma_{k+1} - q_{k+1}$.

For the other case, we let $\varphi_2 : S \to V$ be the lift of $(1_S, \rho) : S \to S \times \mathbb{A}^1$, where $\rho : S \to \mathbb{A}^1$ is via $\rho^*(t) = u$. By the construction, we see that $\varphi_2(\eta_0) \in \Sigma_0 - q_1$.

We let $h_i = \pi \circ \bar{\jmath} \circ \varphi_i : S \to C[n+1]$ be the composite of φ_i with the tautological $V \to C[n+1]$. By inspection, we see that the composites of h_1 and h_2 with $C[n+1] \to C$ are identical, and their composites with $\mathbb{A}^{n+2} \to \mathbb{A}^1$ are of the form

$$h_1''(u) = (\ldots, 1, c_l u^{e_l}, \ldots) \quad \text{and} \quad h_2''(u) = (\ldots, u^{e_l}, c_l, \ldots).$$

Here the expressed terms are in the l and $(l+1)$-th places, and the omitted terms of h_1'' and h_2'' are identical.

We let $\widetilde{\mathcal{X}}_i := X[n+1] \times_{h_i} S$. Using the isomorphism $\tilde{\tau}_{I,I',X}$ in (2.9) with $I = [n+2] - \{l\}$ and $I' = [n+2] - \{l+1\}$, we conclude that

1) the generic points $h_1(\eta)$ and $h_2(\eta)$ lie in the same G_{m}^{n+1}-orbit;
2) there is an isomorphism $\widetilde{\mathcal{X}}_1 \cong \widetilde{\mathcal{X}}_2$ extending the isomorphism $\widetilde{\mathcal{X}}_1 \times_S \eta \cong \widetilde{\mathcal{X}}_2 \times_S \eta$ given by the G_{m}^{n+1}-action in (1).

Let $\varphi_2^* \tilde{\phi}$ be the pull back of $\tilde{\phi}$ via $\varphi_2 : S \to V$; it is an S-flat family of quotient sheaves on $\widetilde{\mathcal{X}}_2$. Since $\varphi_1(\eta)$ and $\varphi_2(\eta)$ lie in the same G_{m}-orbit in

V, (following from the construction,) we have an induced isomorphism

$$(\varphi_1^*\tilde{\phi})_\eta \cong (\varphi_2^*\tilde{\phi})_\eta. \tag{5.9}$$

(Recall $(\varphi_1^*\tilde{\phi})_\eta = (\varphi_1^*\tilde{\phi}) \times_S \eta$.) As the G_{m}^{n+1}-action on $\mathfrak{Q}$ is induced by the G_{m}^{n+1}-action on $X[n+1]/C[n+1]$, the isomorphism (5.9) is compatible with the isomorphism $\widetilde{\mathcal{X}}_1 \times_S \eta \cong \widetilde{\mathcal{X}}_2 \times_S \eta$ in (1).

Finally, using $\widetilde{\mathcal{X}}_1 \cong \widetilde{\mathcal{X}}_2$ given by (2), we pull back the family ϕ on $\widetilde{\mathcal{X}}_1 \cong \mathcal{X}$ to a quotient family $\bar{\phi}$ on $\widetilde{\mathcal{X}}_2$; knowing that the isomorphism $\widetilde{\mathcal{X}}_1 \cong \widetilde{\mathcal{X}}_2$ extends the isomorphism $(\widetilde{\mathcal{X}}_1)_\eta \cong (\widetilde{\mathcal{X}}_2)_\eta$ given by (5.9), the isomorphism (5.9) gives an isomorphism $(\bar{\phi})_\eta \cong (\varphi_2^*\tilde{\phi})_\eta$.

Let $\tilde{p}_2 : \widetilde{\mathcal{X}}_2 \to X$ be the projection. Since both $\bar{\phi}$ and $\varphi_2^*\tilde{\phi}$ are S-flat family of quotient sheaves of $\tilde{p}_2^*\mathcal{V}$, and are isomorphic as quotient sheaves over the generic fiber of $\widetilde{\mathcal{X}}_2/S$, by that $\mathfrak{Q}$ is separated, we conclude $\bar{\phi} \cong \varphi_2^*\tilde{\phi}$. This implies

$$(\varphi_2^*\tilde{\phi})_{\eta_0} \cong (\bar{\phi})_{\eta_0} \cong (\varphi_1^*\tilde{\phi})_{\eta_0} \cong \phi_{\eta_0}$$

as quotient sheaves on $X[n]_0$. In the end, using that $\Sigma_0 - q_1$ is a single G_{m}-orbit, $\tilde{\phi}_a \cong \phi_{\eta_0}$ for all $a \in \Sigma_0 - q_1$; the Lemma follows. □

Lemma 5.12. *The sheaf $\widetilde{\mathcal{F}}_{q_1}$ (resp. $\widetilde{\mathcal{F}}_{q_{k+1}}$) is normal to D_l (res. D_{l+1}). Let $\Delta_l^* = \Delta_l - D_l \cup D_{l+1}$; the restriction $\tilde{\phi}_{q_1}|_{\Delta_l^*}$ (resp. $\tilde{\phi}_{q_{k+1}}|_{\Delta_l^*}$) is G_{m}-invariant.*

Proof. We prove the case for $\widetilde{\mathcal{F}}_{q_1}$. We consider the $\Theta \subset \widetilde{\mathcal{X}}_{\Sigma_0}$ mentioned before Lemma 5.11. Let $\Theta^* = \Theta - \mathrm{closure}(\widetilde{\mathcal{X}}_{\Sigma_0} - \Theta)$. We let $\mathrm{bl} : \Theta \to \Delta_{l-1} \times \Sigma_0$ be the blowing up morphism, and g be the composite

$$g : \Theta^* \xrightarrow{\subset} \Theta \xrightarrow{\mathrm{bl}} \Delta_{l-1} \times \Sigma_0 \xrightarrow{\mathrm{pr}} \Delta_{l-1}.$$

Let $p_{l-1} : \Delta_{l-1} \to X$ be the tautological projection, let $\mathcal{F}^{\mathrm{t.f.}}_{\eta_0,l-1}$ be $\mathcal{F}_{\eta_0}|_{\Delta_{l-1}}$ quotient by its subsheaf supported along $D_l \cup D_{l-1}$. By Lemma 3.2 and Proposition 3.3, $\mathcal{F}^{\mathrm{t.f.}}_{\eta_0,l-1}$ is normal to both D_l and D_{l-1}.

We consider the quotient on Δ_{l-1} induced by $\phi_{\eta_0}|_{\Delta_{l-1}}$:

$$\phi^{\mathrm{t.f.}}_{l-1} : p_{l-1}^*\mathcal{V} \longrightarrow \mathcal{F}^{\mathrm{t.f.}}_{\eta_0,l-1}.$$

We claim

$$g^*\phi^{\mathrm{t.f.}}_{l-1} \cong \tilde{\phi}_{\Sigma_0}|_{\Theta^*}. \tag{5.10}$$

First, we know that Θ is a blowing up of $\Delta_{l-1} \times \Sigma_0$ along $D_l \times 0$, and that G_{m}-acts on Θ via the trivial action on Δ_{l-1} and that on Σ_0 with the only fixed point q_1. Second, we know that $\tilde{\phi}_{\Sigma_0}:\tilde{p}^*_{\Sigma_0}\mathcal{V}\to\widetilde{\mathcal{F}}_{\Sigma_0}$ is G_{m}-equivariant, and for an $a\in\Sigma_0-q_1$, $\tilde{\phi}_a\cong\phi_{\eta_0}$. From these two, we conclude

$$g^*\phi^{\mathrm{t.f.}}_{l-1}|_{\Theta^*-\widetilde{\mathcal{X}}_{q_1}}\cong\tilde{\phi}_{\Sigma_0}|_{\Theta^*-\widetilde{\mathcal{X}}_{q_1}}. \tag{5.11}$$

To conclude the claim, we notice that the isomorphism

$$g^*p^*_{l-1}\mathcal{V}|_{\Theta^*-\widetilde{\mathcal{X}}_{q_1}}\cong\tilde{p}^*_{\Sigma_0}\mathcal{V}|_{\Theta^*-\widetilde{\mathcal{X}}_{q_1}}, \tag{5.12}$$

which is part of the isomorphism (5.11), is the identity map of the pull back of $\mathcal{V}$ via the tautological projection; $\Theta^*-\widetilde{\mathcal{X}}_{q_1}\to X$. Thus (5.12) extends to a

$$g^*p^*_{l-1}\mathcal{V}|_{\Theta^*}\cong\tilde{p}^*_{\Sigma_0}\mathcal{V}|_{\Theta^*}. \tag{5.13}$$

On the other hand, the family $\tilde{p}^*_{\Sigma_0}\mathcal{V}|_{\Theta^*}\to\widetilde{\mathcal{F}}_{\Sigma_0}|_{\Theta^*}$ is flat over Σ_0. By the uniqueness of flat completion of quotient sheaves, the claim follows if we can show that $g^*p^*_{l-1}\mathcal{V}\to g^*\mathcal{F}^{\mathrm{t.f.}}_{l-1}$ is flat over Σ_0.

Since $\mathcal{F}^{\mathrm{t.f.}}_{l-1}$ is normal to D_l, by Proposition 3.3, $\mathcal{F}^{\mathrm{t.f.}}_{l-1}$ is flat along the normal direction of $D_l\subset\Delta_{l-1}$. Thus $g^*\mathcal{F}^{\mathrm{t.f.}}_{l-1}$ is flat along the normal direction of the exceptional divisor of $\Theta^*\to\Delta_{l-1}\times\Sigma_0$. Applying Proposition 3.3, we conclude that it is flat over Σ_0, and in addition, $g^*\mathcal{F}^{\mathrm{t.f.}}_{l-1}|_{\Theta^*\cap\widetilde{\mathcal{X}}_{q_1}}$ is admissible.

This proves that $\widetilde{\mathcal{F}}_{q_1}$ is normal to D_l. It is G_{m}-equivariant because $\tilde{\phi}_{\Sigma_0}$ is G_{m}-equivariant. □

Lemma 5.13. *For all $1\le i\le k$, we have*

$$\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i}+\mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_i}=\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_{i+1}}+\mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_{i+1}}. \tag{5.14}$$

Suppose for an $1\le i\le k$, $\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i}\prec\mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_i}$ and $\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_{i+1}}\succcurlyeq\mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_{i+1}}$, then for $a\in\Sigma_i-\{q_i,q_{i+1}\}$,

$$\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i}+\mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_i}\succ\mathrm{Err}_l\widetilde{\mathcal{F}}_a+\mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_a. \tag{5.15}$$

Proof. Since $\widetilde{\mathcal{F}}$ is flat over Σ, we get $\chi(\widetilde{\mathcal{F}}_{q_i}(v))=\chi(\widetilde{\mathcal{F}}_{q_{i+1}}(v))$ for all $1\le i\le k$. Moreover, since G_{m} leaves Δ_j fixed for $j\ne l$, we know the restriction of $\widetilde{\mathcal{F}}$ to $(X[n+1]_0-\Delta_l)\times\Sigma_i$ is a constant family of sheaves parameterized by Σ_i for all $1\le i\le k$. Therefore, for any $j\ne l,l+1$, the quantities $\mathrm{Err}_j\widetilde{\mathcal{F}}_a$ are the same for all $a\in\Sigma_i$. If we let $\delta^a_{j,i}$ be the quantities associated to sheaf $\widetilde{\mathcal{F}}_a$ defined as $\delta_{l,i}$ in Lemma 3.17, then for $j\ne l$, $\delta^a_{j,0}$ (resp. $\delta^a_{j,1}$) are the same for all $a\in\Sigma_i$.

Applying identity (3.7) in Lemma 3.17, and subtracting these identical quantities from the right hand side of (3.7), we conclude that

$$\mathrm{Err}_l\widetilde{\mathcal{F}}_a + \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_a + \frac{1}{2}(\delta^a_{l,0} + \delta^a_{l,1}) \tag{5.16}$$

have the same values for all $a \in \Sigma_1 \cup \cdots \cup \Sigma_k$.

Since by Lemma 5.7 and 5.10, $q_i, q_{i+1} \in \Sigma_i$ are G_{m}-fixed points of Σ_i, and G_{m} acts linearly on Δ_l with fixed locus $D_l \cup D_{l+1}$, we know the restriction of $\tilde{\phi}_{q_i}$ to Δ_l^* is G_{m}-invariant. Moreover, for $1 \le l \le k$,

$$\phi^{\mathrm{t.f.}}_{q_i,l} : p_l^*\mathcal{V} \longrightarrow \mathcal{F}^{\mathrm{t.f.}}_{q_i,l}$$

is the pull back of a quotient sheaf on D_l via the projection $\Delta_l \to D_l$. Applying Lemma 3.18, $\delta^{q_i}_{l,0} = \delta^{q_i}_{l,1} = 0$ for $1 \le l \le k$. For the same reason, we have $\delta^{q_i}_{l+1,0} = \delta^{q_i}_{l+1,1} = 0$ for $2 \le i \le k+1$. The identity (5.14) follows from that (5.16) takes same values for $a = q_1, \ldots, q_{k+1}$.

Next we prove (5.15). By (5.16) and the previous argument, for any $a \in \Sigma_i$,

$$\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i} + \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_i} = \mathrm{Err}_l\widetilde{\mathcal{F}}_a + \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_a + \frac{1}{2}(\delta^a_{l,0} + \delta^a_{l,1}) \tag{5.17}$$

Applying Lemma 3.13 and 3.14, for $a \in \Sigma_i - \{q_i, q_{i+1}\}$, we have

$$\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i} = \mathrm{Err}_l\widetilde{\mathcal{F}}_a \quad \text{and} \quad \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_{i+1}} = \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_a.$$

Therefore, (5.17) gives us

$$\mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_i} = \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_{i+1}} + \frac{1}{2}(\delta^a_{l,0} + \delta^a_{l,1}), \quad a \in \Sigma_i - \{q_i, q_{i+1}\}.$$

Applying (5.14), we also have

$$\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_{i+1}} = \mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i} + \frac{1}{2}(\delta^a_{l,0} + \delta^a_{l,1}).$$

Now suppose for a $1 \le i \le k$, $\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i} \prec \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_i}$ and $\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_{i+1}} \succcurlyeq \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_{i+1}}$. Then $\deg(\delta^a_{l,0} + \delta^a_{l,1}) = \deg \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_i} \ge \deg \mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i}$. Therefore, in the identity (5.17), the degree of the left hand side is equal to the degree of the last term on the right hand side; because of the weak positivity of $\delta^a_{l,0} + \delta^a_{l,1}$ proved in Lemma 3.18, (5.15) follows. □

Proof of Lemma 5.6. For any quotient $\phi_\eta : p_\eta^*\mathcal{V} \to \mathcal{F}_\eta$ and its extension to an S-flat quotient $\phi : p^*\mathcal{V} \to \mathcal{F}$ such that $\mathrm{Err}\mathcal{F}_{\eta_0} \neq 0$ as stated in the Lemma, according to our construction, we pick $1 \le l \le n$ so that

$$\deg \mathrm{Err}_l\mathcal{F}_{\eta_0} = \deg \mathrm{Err}\mathcal{F}_{\eta_0},$$

and form a regular G_{m}-surface V, together with a family $\tilde{p} : \widetilde{\mathcal{X}} \to X$ in $\mathfrak{X}(V)$ and a G_{m}-equivariant quotient $\tilde{\phi} : \tilde{p}^*\mathcal{V} \to \widetilde{\mathcal{F}}$ on $\widetilde{\mathcal{X}}$.

We further find a connected chain of rational curves $\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_{k+1}$ in V so that the restriction of $\tilde{\phi}$ to Σ satisfies the properties stated in Lemmas 5.11 and 5.12;

According to Lemma 5.12, we know

$$0 = \mathrm{Err}_l\widetilde{\mathcal{F}}_{q_1} \prec \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_1} = \mathrm{Err}_l\mathcal{F}_{\eta_0} \neq 0$$

and $0 = \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_{k+1}} \prec \mathrm{Err}_l\widetilde{\mathcal{F}}_{q_{k+1}} = \mathrm{Err}_l\mathcal{F}_{\eta_0} \neq 0$. By (5.14) in Lemma 5.13, we can find a Σ_i, so that the assumptions in Lemma 5.13 $\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i} \prec \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_i}$ and $\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_{i+1}} \succcurlyeq \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_{i+1}}$ are satisfied. For such i,

$$\mathrm{Err}_l\widetilde{\mathcal{F}}_{q_i} + \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_{q_i} \succ \mathrm{Err}_l\widetilde{\mathcal{F}}_a + \mathrm{Err}_{l+1}\widetilde{\mathcal{F}}_a, \quad a \in \Sigma_i - \{q_i, q_{i+1}\}.$$

Moreover, $\widetilde{\mathcal{F}}_a|_{\Delta_l^*}$ is not G_{m}-invariant by the non-vanishing of its associated quantity $\delta_{l,0}^a + \delta_{l,1}^a$ via Lemma 3.18. By our choice of l, we conclude that $\mathrm{Err}\widetilde{\mathcal{F}}_{q_i} \succ \mathrm{Err}\widetilde{\mathcal{F}}_a$. Combined with $\mathrm{Err}\widetilde{\mathcal{F}}_{q_i} = \mathrm{Err}\mathcal{F}_{\eta_0}$, we have the $\mathrm{Err}\widetilde{\mathcal{F}}_a \prec \mathrm{Err}\mathcal{F}_{\eta_0}$, and $\dim \mathrm{Aut}_{\mathfrak{X}}(\tilde{\phi}_a) \le \dim \mathrm{Aut}_{\mathfrak{X}}(\phi_{\eta_0})$.

Finally we find the desired curve $S' \subset V$. Because V is smooth at the point a, and $b : V \to V'$ is a sequence of blow-ups whose exceptional divisor contains a, we can find a smooth curve $S' \subset V'$ that contains a, and that the composition $S' \to V' \to S \times \mathbb{A}^1 \to S$ is non constant, thus branched at $\eta_0' = a \in S'$, and $S' \to S$ is finite. Furthermore, we can take such S' so that its image in V' is not contained in $E' \cup E \subset V'$. For such an $S' \subset V$, the induced family of quotients $\phi' = \tilde{\phi}_{S'}$ on $p' : \mathcal{X}' = \widetilde{\mathcal{X}} \times_V S' \to X$ satisfies the properties stated in Lemma 5.6. □

5.2. The completeness II

We complete the proof of Theorems 4.14 and 4.15 by working out the remainder cases.

Let (S, η, η_0) be as stated in the beginning of this section. We prove a Lemma analogous to Lemma 5.6 for $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$.

Lemma 5.14. *Let* $(\phi_\eta, \mathcal{Y}_\eta) \in \mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}(\eta)$*, and let* $\phi : p^*\mathcal{V}_0 \to \mathcal{F}$ *be an* S*-flat extension of* ϕ_η *over* $(\mathcal{Y}, p) \in \mathfrak{Y}(S)$*. Suppose* $\mathcal{Y}_\eta$ *is smooth,* $\mathrm{Aut}_{\mathfrak{Y}}\phi_{\eta_0}$ *is finite, and* $\mathrm{Err}\mathcal{F}_{\eta_0} \neq 0$*. Then we can find a finite base change* $S' \to S$*, an* S'*-flat quotients* $\phi' \colon p'^*\mathcal{V}_0 \to \mathcal{F}'$ *on* $(\mathcal{Y}', p') \in \mathfrak{Y}(S')$ *such that*

1) $\mathcal{Y}'_{\eta'} \cong \mathcal{Y}_\eta \times_\eta \eta' \in \mathfrak{Y}(\eta')$*, and under this isomorphism* $\phi'_{\eta'} = \phi_\eta \times_\eta \eta'$*;*
2) $\mathrm{Aut}_{\mathfrak{Y}}(\phi'_{\eta'_0})$ *is finite, and*
3) $\mathrm{Err}\mathcal{F}'_{\eta'_0} \prec \mathrm{Err}\mathcal{F}_{\eta_0}$*.*

Proof. We follow the same strategy used to prove Lemma 5.6. Since S is local, we can find a $\xi : S \to \mathbb{A}^{n_-+n_+}$ so that $\xi(\eta_0) = 0$ and $\mathcal{Y} \cong \xi^*Y[n_-, n_+]$. Since $\mathrm{Err}\mathcal{F}_{\eta_0} \neq 0$, we pick an l so that as polynomials,

$$\deg \mathrm{Err}_l\mathcal{F}_{\eta_0} = \deg \mathrm{Err}\mathcal{F}_{\eta_0}, \quad -n_- - 1 \le l \le n_+ + 1,\ l \neq 0$$

Here we agree that $\mathrm{Err}_{-n_--1} = \mathrm{Err}_-$ and $\mathrm{Err}_{n_++1} = \mathrm{Err}_+$ (cf. (3.10)). Without loss of generality, we assume $l > 0$.

We let u be a uniformizing parameter of S, and express

$$\xi = \bigl(c_{-n_-}u^{e_{-n_-}}, \dots, c_{n_+}u^{e_{n_+}}\bigr), \quad c_i \in \Gamma(\mathcal{O}_S)^*. \tag{5.18}$$

Since $\xi(\eta_0) = 0$, all $e_i \ge 0$. We let

$$\tau_l : \mathbb{A}^{n_-+n_+} \times G_{\mathrm{m}} \to \mathbb{A}^{n_-+n'_+}, \quad n'_+ = n_+ + 1,$$

be defined by

$$(t_{-n_-}, \dots, t_{n_+}; \sigma) \mapsto (t_{-n_-}, \dots, t_{l-1}, \sigma^{-e_l}t_l, \sigma^{e_l}, t_{l+1}, \dots, t_{n_+}). \tag{5.19}$$

(In case $l = 1$, we replace $t_{l-1} = t_0$ by t_{-1}.) We then introduce

$$\xi_l = \tau_l \circ (\xi, \mathrm{id}) : S \times G_{\mathrm{m}} \longrightarrow \mathbb{A}^{n_-+n_+} \times G_{\mathrm{m}} \longrightarrow \mathbb{A}^{n_-+n'_+},$$

and let $\mathcal{Y}' := \xi_l^*Y[n_-, n'_+]$ over $S \times G_{\mathrm{m}}$ be the pull back family. By the construction of $Y[n_-, n_+]$, we have $\mathcal{Y}' \cong \xi^*Y[n_-, n_+] \times G_{\mathrm{m}} = \mathcal{Y} \times G_{\mathrm{m}}$. We let $p' : \mathcal{Y}' \to Y$ and $\pi_1 : \mathcal{Y}' \to \mathcal{Y}$ be the projections.

We let $\phi' = \pi_1^*\phi : p'^*\mathcal{V} \to \mathcal{F}'$ be the pullback quotient sheaves. By the universal property of Grothendieck's Quot-scheme, the family ϕ' induces an

$\mathbb{A}^{n_-+n'_+}$-morphism

$$f_l : S \times G_{\rm m} \longrightarrow \mathrm{Quot}^{p^*\mathcal{V}_0,P}_{Y[n_-,n'_+]/\mathbb{A}^{n_-+n'_+}}. \tag{5.20}$$

Mimic the proof of Lemma 5.6, We construct a regular $G_{\rm m}$-surface V and $G_{\rm m}$-morphisms that fit into the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "\bar{\jmath}"] & \mathrm{Quot}^{p^*\mathcal{V}_0,P}_{Y[n_-,n'_+]/\mathbb{A}^{n_-+n'_+}} \arrow[d, "\pi"] \\ S \times G_{\rm m} \arrow[u, "j"] \arrow[ur, "f_l"] \arrow[r, "\xi_l"] & \mathbb{A}^{n_-+n'_+} \end{tikzcd}$$

so that $\pi \circ \bar{\jmath} : V \to \mathbb{A}^{n_-+n'_+}$ is proper.

Once we have the surface the pull back family over V from $\bar{\jmath}$, we can repeat the proof of Lemma 5.6 line by line to conclude the existence of $S' \subset S$ that satisfies the requirement of the Lemma. Since the proof is a mere repetition, we omit the details. This completes the proof. □

Proof of Proposition 5.1. We first prove the Proposition for $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$. Let $(\phi_\eta : p_\eta^*\mathcal{V}_0 \to \mathcal{F}_\eta) \in \mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}(\eta)$ be a quotient on $(\mathcal{Y}_\eta, p_\eta) \in \mathfrak{Y}(\eta)$. Then $\mathcal{Y}_\eta = Y[n_-,n_+]_0 \times \eta$ for some $n_-, n_+ \geq 0$. Following the convention (2.13),

$$Y[n_-,n_+]_0 = \Delta_{-n_-} \cup \cdots \cup \Delta_0 \cup \cdots \cup \Delta_{n_+},$$

where $\Delta_0 = Y$.

In the remainder of this proof, we adopt the convention that $W_l = \Delta_l$ for $-n_- \leq l \leq n_+$; following the rule specified after (2.14) we endow W_l the relative divisors $E_{l,-}$ and $E_{l,+}$ by the rule: for $l > -n_-$, $E_{l,+} = \Delta_{l-1} \cap \Delta_l$; for $l < n_+$, $E_{l,-} = \Delta_l \cap \Delta_{l+1}$; $E_{-n_-,+} = D[n_-]_{-,0}$ and $E_{n_+,-} = D[n_+]_{+,0}$, where $D[n_-]_{-,0}$ and $D[n_+]_{+,0}$ are the two relative divisors of $Y[n_-,n_+]_0$.

We let $W_{l,\eta} = W_l \times \eta \subset \mathcal{Y}_\eta$; we let $E_{l,\pm,\eta} = E_{l,\pm} \times \eta \subset W_{l,\eta}$, let $p_{l,\eta} : W_{l,\eta} \to X$ be the tautological projection, and let $G_{m,\eta} = G_{\rm m} \times \eta$. We adopt the same convention when η is replaced by η_0 or S.

We consider

$$\phi_{l,\eta} := \phi_\eta|_{W_{l,\eta}} : p_{l,\eta}^*\mathcal{V} \longrightarrow \mathcal{F}_{l,\eta} := \mathcal{F}_\eta|_{W_{l,\eta}} \tag{5.21}$$

Since ϕ_η is stable, $\mathcal{F}_{l,\eta}$ is normal to the relative divisors $E_{l,\pm,\eta}$ of $W_{l,\eta}$. Because the Grothendieck's Quot-scheme is proper, we can extend $\phi_{l,\eta}$ to an S-flat quotient family $\tilde{\phi}_l : p_l^*\mathcal{V}_0 \to \tilde{\mathcal{F}}_l$ on $W_{l,S} = W_l \times S$.

In the ideal case where all $\tilde{\mathfrak{F}}_{l,\eta_0} = \tilde{\mathfrak{F}}_l|_{W_{l,\eta_0}}$ are normal to $E_{l,\pm,\eta_0}$, then we will show that we can patch $\tilde{\phi}_l$ to a quotient family $\tilde{\phi}$ on $Y[n_-, n_+]_0 \times S$ whose quotient sheaf is admissible. Suppose further that its automorphism group $\mathrm{Aut}_{\mathfrak{Y}}(\tilde{\phi}|_{\eta_0})$ is finite, this family will be the desired family that proves Proposition 5.1.

In general, we divide the proof into several steps. We first take care of the automorphism groups caused by the G_{m}-action on W_l, $l \neq 0$. Suppose at least one of n_- and n_+ is positive. For any $n_- \leq l \neq 0 \leq n_+$, suppose $\tilde{\phi}_l|_{W_{l,\eta_0}}$ is not invariant under the tautological G_{m} action on W_{l,η_0} and $p^*_{l,\eta_0}\mathcal{V}_0$, we do nothing. Suppose it is invariant under G_{m}. We claim that $\tilde{\phi}_l|_{W_{l,\eta_0}}$ is not a pull back quotient sheaf from $W_{l,\eta_0} \to D \times \eta_0$. Suppose it is a pull back quotient sheaf, then $\tilde{\mathfrak{F}}_l$ is flat over S implies that $\tilde{\mathfrak{F}}_l$ is a pull back sheaf from $W \times S \to D \times S$; in particular $\tilde{\mathfrak{F}}_l|_{W_{l,\eta}} = \mathfrak{F}_{l,\eta}$ is a pull back sheaf from $W_{l,\eta} \to D \times \eta$. But this is impossible since ϕ_η is stable implies that $\phi_{l,\eta}$ is not invariant under the G_{m}-action, a contraction.

We continue to suppose $\tilde{\phi}_l|_{W_{l,\eta_0}}$ is G_{m}-invariant. This invariance together with that $\tilde{\phi}_l|_{W_{l,\eta_0}}$ is not a pull back sheaf from $D \times \eta_0$ implies that $\tilde{\mathfrak{F}}_l|_{W_{l,\eta_0}}$ is not normal to at least one of $E_{l,\pm,\eta_0} \subset W_{l,\eta_0}$. Therefore, by repeating the proof of Lemma 5.6, and possibly after a base change, we can find a $\xi_l : \eta \to G_{m,\eta}$ so that under

$$\psi_l : W_{l,\eta} \xrightarrow{(1,\bar{\xi}_l)} W_{l,\eta} \times_\eta G_{m,\eta} \xrightarrow{\times} W_{l,\eta}$$

where $\bar{\xi} : W_{l,\eta} \to G_{m,\eta}$ is via $W_{l,\eta} \xrightarrow{\mathrm{pr}} \eta \xrightarrow{\xi_l} G_{m,\eta}$, and the second arrow is the $G_{m,\eta}$-action on $W_{l,\eta}$, the pull back family $\psi_l^*(\phi_{l,\eta})$ extends to a new S-flat family $\tilde{\phi}_l$ (denoted by the same $\tilde{\phi}_l$) on $W_l \times S$ so that $\tilde{\phi}_l|_{W_{l,\eta_0}}$ is not invariant under G_{m}.

For the modified families $\tilde{\phi}_l$, $n_- \leq l \leq n_+$, our next step is to modify them so that they are normal to $E_{l,\pm,\eta_0} \subset W_{l,\eta_0}$. We let $\mathfrak{E}_{l,\pm} \subset \mathfrak{W}_l$ over $\mathfrak{A}_\diamond$ be the stack of expanded relative pairs of $E_{l,\pm} \subset W_l$. (Like $\mathfrak{D}_\pm \subset \mathfrak{Y}$ with $D_\pm \subset Y$ replaced by $E_{l,\pm} \subset W_l$.) Then $\tilde{\phi}_{l,\eta} \in \mathfrak{Quot}^{p_l^*\mathcal{V}_0}_{\mathfrak{W}_l/\mathfrak{A}_\diamond}(\eta)$. In case $\tilde{\mathfrak{F}}_{l,\eta_0} = \tilde{\mathfrak{F}}_l|_{W_{l,\eta_0}}$ is normal to $E_{l,\pm,\eta_0}$, which is equivalent to $\mathrm{Err}\mathfrak{F}_{\eta_0} = 0$ by the criteria Lemma 3.19, $\tilde{\mathfrak{F}}_{l,\eta_0}$ is admissible and $\tilde{\phi}_{l,\eta_0}$ is stable. Otherwise, by Lemma 5.14, we can find a finite base change $S_l' \to S$ and an S_l'-flat family of quotients ϕ_l' on $(\mathcal{W}_l', p_l')$ so that, letting η_0' and η' be the closed and the generic points of S_l',

1) $\mathcal{W}'_{l,\eta'} \cong \mathcal{W}_{l,\eta} \times_\eta \eta'$, that under this isomorphism $\phi'_{l,\eta'} = \phi_{l,\eta} \times_\eta \eta'$;
2) $\mathrm{Aut}_{\mathfrak{W}_l}(\phi'_{l,\eta_0'})$ is finite;

3) $\mathrm{Err}\mathcal{F}'_{\eta'_0} \prec \mathrm{Err}\mathcal{F}_{\eta_0}$.

If $\mathrm{Err}\mathcal{F}'_{\eta'_0}$ is still nonzero, we repeat this process. By Lemma 5.4 on descending chain, this process terminates at finitely many steps. Thus we obtain an S'_l-family of quotient family ϕ'_l satisfying (1) and (2) above together with (3) replaced by $\mathrm{Err}\mathcal{F}'_{l,\eta'_0} = 0$. Namely, $\{\phi'_l : p'^*_l \mathcal{V}_0 \longrightarrow \mathcal{F}'_l\} \in \mathfrak{Quot}^{p^*_l\mathcal{V}_0}_{\mathfrak{W}_l/\mathfrak{A}_\diamond}(S'_l)$.

In case $l \neq 0$, we can say more of the symmetry of ϕ'_{l,η'_0}. When $l \neq 0$, $\mathcal{W}'_{l,\eta'_0} \cong \Delta \cup \cdots \cup \Delta$, is the union of a chain of, say n_l copies, of Δ. We define

$$\mathrm{Aut}_{\mathfrak{W}_l,G_\mathrm{m}}(\phi'_{l,\eta'_0}) = \big\{g \in G_\mathrm{m}^{\times n_l} \mid g \cdot (\phi'_{l,\eta'_0}) \cong \phi'_{l,\eta'_0}\big\}. \tag{5.22}$$

Here $g \cdot (\phi'_{l,\eta'_0})$ is the pull back family of ϕ'_{l,η'_0} under the $G_\mathrm{m}^{\times n_l}$ action $\mathcal{W}'_{l,\eta'_0} \xrightarrow{\cdot g} \mathcal{W}'_{l,\eta'_0}$, and $g \cdot (\phi'_{l,\eta'_0}) \cong \phi'_{l,\eta'_0}$ is the isomorphism as quotient families, using that $p'^*_l \mathcal{V}_0|_{\mathcal{W}_{l,\eta'_0}}$ is invariant under $G_\mathrm{m}^{\times n_l}$. It follows from the construction of ϕ'_l and the proof of Lemma 5.14 that $\mathrm{Aut}_{\mathfrak{W}_l,G_\mathrm{m}}(\phi'_{l,\eta'_0})$ is finite.

By replacing each S'_l by the fiber product of all S'_l over S, we can assume all $S'_l = S'$ for a single finite base change $S' \to S$. Let η' be the generic point of S'. We now show that we can glue the families ϕ'_l to a family $\phi' \in \mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm \subset \mathfrak{Y}}(S')$ that extends $\phi_\eta \times_\eta \eta'$. Let $\mathcal{W}_l$ over S' be the underlying family of ϕ'_l. Since ϕ'_l is an extension of $\phi_{l,\eta} \times_\eta \eta'$, we have $\mathcal{W}_l \times_{S'} \eta' = W_{l,\eta} \times_\eta \eta'$. We let $\mathcal{E}_{l,\pm} \subset \mathcal{W}_l$ be the closure of $E_{l,\pm,\eta} \times_\eta \eta' \subset W_{l,\eta} \times_\eta \eta'$; $\mathcal{E}_{l,\pm} \subset \mathcal{W}_l$ is the the pair of relative divisors of $\mathcal{W}_l \in \mathfrak{W}_l(S')$. Thus, they are smooth divisor in $\mathcal{W}_l$ and $\mathcal{E}_{l,\pm} \cong E_{l,\pm} \times S'$ canonically.

We then form the union $\cup_{l=-n_-}^{n_+} \mathcal{W}_l$; using the canonical isomorphism $E_{l,-} \cong E_{l+1,+}$, we identify $\mathcal{E}_{l,-} \subset \mathcal{W}_l$ with $\mathcal{E}_{l+1,+} \subset \mathcal{W}_{l+1}$ for $n_- < l < n_+$, resulting a family, denoted by $\mathcal{Y}' \to S'$. Let $p' : \mathcal{Y}' \to Y$ be the projection induced by $p'_l : \mathcal{W}_l \to Y$, which exists. In conclusion, our construction of $\mathfrak{W}$ (or $\mathfrak{Y}$) ensures that $(\mathcal{Y}', p') \in \mathfrak{Y}(S')$.

We let $\iota_l : \mathcal{W}_l \to \mathcal{Y}'$ be the tautological closed immersion. We claim that we can find a quotient family $\phi' : p'^*\mathcal{V}_0 \to \mathcal{F}'$ so that $\iota_l^*\phi' \cong \phi'_l$. Indeed, since $p'_l : \mathcal{W}_l \to Y$ is equal to $p' \circ \iota_l : \mathcal{W}_l \to \mathcal{Y}' \to Y$, we have canonical isomorphism $\iota_l^* p'^* \mathcal{V}_0 \cong p'^*_l \mathcal{V}_0$. Hence, using the canonical $p'^*\mathcal{V}_0 \to \iota_{l*}\iota_l^* p'^*\mathcal{V}_0 \cong \iota_{l*} p'^*_l \mathcal{V}_0$, we obtain quotient sheaf $p'^*\mathcal{V}_0 \to \iota_{l*}\mathcal{F}'_l$. We now verify that as quotient sheaves

$$\begin{aligned} &\big(p'^*\mathcal{V}_0 \to \iota_{l-1*}\mathcal{F}'_{l-1}\big) \otimes_{\mathcal{O}_{\mathcal{Y}'}} \mathcal{O}_{\iota_{l-1}(\mathcal{E}_{l-1,+})} \\ \cong &\big(p'^*\mathcal{V}_0 \to \iota_{l*}\mathcal{F}'_l\big) \otimes_{\mathcal{O}_{\mathcal{Y}'}} \mathcal{O}_{\iota_l(\mathcal{E}_{l,-})}. \end{aligned} \tag{5.23}$$

(Note $\iota_{l-1}(\mathcal{E}_{l-1,+}) = \iota_l(\mathcal{E}_{l,-}) \subset \mathcal{Y}'$.) First, the above two sides are canonically isomorphic after restricting to fibers over $\eta' \in S'$; this is true because the two sides of (5.23) restricted to fiber over η' are the quotient ϕ_η restricted to $E_{l-1,+} \times \eta = E_{l,-} \times \eta \subset \mathcal{Y}_\eta$. On the other hand, since both ϕ'_{l-1} and ϕ'_l are families of stable quotients, by Corollary 3.4, both sides of (5.23) are flat over S'. Therefore, by the separatedness of Grothendieck's Quot-scheme, (5.23) holds. Consequently, the desired quotient family ϕ' exists.

Finally, we check that ϕ' is a family in $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}(S')$. The fact that ϕ' is admissible follows from Lemma 3.7; that $\mathrm{Aut}_{\mathfrak{Y}}(\phi'_{\eta'_0})$ is finite follows from that $\mathrm{Aut}_{\mathfrak{W}_l}(\phi'_{l,\eta'_0})$ is finite for $l = 0$ and (5.22) is finite for $l \neq 0$. This shows that $\phi' \in \mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}(S')$. This completes the proof of Proposition 5.1 for the stack $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$.

The proof for $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}$ is exactly the same. In case $\phi \in \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(\eta)$ has its underlying scheme $\mathcal{X}_\eta$ smooth, then the existence of its extension to an $\phi' \in \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(S')$ for a finite base change $S' \to S$ follows from Lemma 5.5 and 5.6. In case $\mathcal{X}_\eta$ is singular, then it is isomorphic to $X[n]_0 \times \eta$. Like in the proof of the previous case, we split $X[n]_0$ as union of smooth Δ_i and Y; study the extension problem for the restriction of ϕ to $\Delta_i \times \eta$ and $Y \times \eta$, and glue them to form a desired extension. The proof is exactly the same to the first part of the proof. This proves the Proposition. $\square$

5.3. The separatedness

We show the separatedness part in Theorems 4.14 and 4.15. By valuative criteria, this is equivalent to show that the extension of ϕ_η to ϕ constructed in the previous subsections is unique.

We prove Proposition 5.2 for smooth generic fibers, the others are the same.

Proof of Proposition 5.2. Let $(\phi_1,\mathcal{X}_1)$ and $(\phi_2,\mathcal{X}_2) \in \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(S)$ be two families of quotients, where S is as before, such that there is a $\rho_\eta : \mathcal{X}_{1,\eta} \to \mathcal{X}_{2,\eta}$ in $\mathfrak{X}(\eta)$ such that $\phi_{1,\eta} = \rho_\eta^*\phi_{2,\eta}$.

Suppose $\rho_\eta : \mathcal{X}_{1,\eta} \to \mathcal{X}_{2,\eta}$ extends to $\rho : \mathcal{X}_1 \to \mathcal{X}_2$, then $\rho^*\phi_2$ is a family of stable quotient sheaves. By the separatedness of the Quot-schemes, we have $\rho^*\phi_2 \cong \phi_1$. Adding that $(\rho^*\phi_2)_{\eta_0}$ is stable, we conclude that $\rho : \mathcal{X}_1 \to \mathcal{X}_2$ is an isomorphism, and the Proposition is done.

Suppose such an extension ρ does not exist. Instead, we will construct $\bar{\mathcal{X}}_i \in \mathfrak{X}(S)$, and morphisms $h_i : \bar{\mathcal{X}}_i \to \mathcal{X}_i$ so that $h_{i,\eta} : \bar{\mathcal{X}}_{i,\eta} \cong \mathcal{X}_{i,\eta}$ and the arrow $h_{2,\eta}^{-1} \circ \rho_\eta \circ h_{1,\eta} : \mathcal{X}_{1,\eta} \to \mathcal{X}_{2,\eta}$ extends to an arrow $h : \bar{\mathcal{X}}_1 \cong \bar{\mathcal{X}}_2$.

We express $\mathcal{X}_i$ as $\xi_i^* X[n_i]$ induced by $\xi_i : S \to C[n_i]$ with $\xi_i(\eta_0) = 0$. Let u be a uniformizing parameter of S; we express

$$\pi_{n_i} \circ \xi_i = \big(c_{i,1}u^{e_{i,1}}, \ldots, c_{i,n_i+1}u^{e_{i,n_i+1}}\big)$$

as in (5.1). Because $\mathcal{X}_{1,\eta} = \mathcal{X}_{2,\eta} \in \mathfrak{X}(\eta)$, we have

$$n := \sum_{j=1}^{n_1+1} e_{1,j} = \sum_{j=1}^{n_2+1} e_{2,j}. \tag{5.24}$$

We then define ξ_i' and $\bar{\xi}_i : S \to C[n]$ by the rule

$$\pi_{[n]} \circ \xi_i' = \big(c_{i,1}u^{e_{i,1}}, 1 \ldots, 1, c_{i,2}u^{e_{i,2}}, 1, \ldots, 1, c_{i,n_i+1}u^{e_{i,n_i+1}}, 1, \ldots, 1\big), \tag{5.25}$$

where after each term $c_{i,j}u^{e_{i,j}}$ we repeat 1 exactly $e_{i,j} - 1$ times, and by

$$\pi_{[n]} \circ \bar{\xi}_i = \big(c_{i,1}u, u, \ldots, u, c_{i,2}u, u, \ldots, u, c_{i,n_i+1}u, u, \ldots, u\big), \tag{5.26}$$

where after each term $c_{i,j}u$ we repeat u exactly $e_{i,j} - 1$ times.

We let $\mathcal{X}_i' = \xi_i'^* X[n]$ and let $\bar{\mathcal{X}}_i = \bar{\xi}_i^* X[n]$. We describe the relations between these families. First, since (5.25) has the form of the standard embedding defined in (2.4), the families $\mathcal{X}_i' \cong \mathcal{X}_i \in \mathfrak{X}(S)$. Next, we let $\sigma_{i,\eta} : \eta \to G_{\mathrm{m}}^n$ be defined via

$$\sigma_{i,\eta}(u) = \big(u^{e_{i,1}-1}, u^{e_{i,1}-2}, \ldots, 1, u^{e_{i,2}-1}, u^{e_{i,2}-2}, \ldots,\\ 1, u^{e_{i,n_i+1}-1}, u^{e_{i,n_i+1}-2}, \ldots 1\big),$$

then $\xi_i' = (\bar{\xi}_i)^{\sigma_{i,\eta}}$. Lastly, because $c_{i,j}$ are elements in $\Gamma(\mathcal{O}_S)^*$, from the expression (5.26), there is a $\sigma : S \to G_{\mathrm{m}}^n$ so that $\bar{\xi}_1 = (\bar{\xi}_2)^\sigma$, which induces an isomorphism $h : \bar{\mathcal{X}}_1 \cong \bar{\mathcal{X}}_2$.

Moreover, because in the coordinate expression of the morphism $\sigma_{i,\eta} : \eta \to G_{\mathrm{m}}^n$, all powers of u are nonnegative, the isomorphisms $\bar{\mathcal{X}}_{i,\eta} \cong \mathcal{X}_{i,\eta}$ induced by $\sigma_{i,\eta}$ and the standard embedding (5.25) extend to morphisms $h_i : \bar{\mathcal{X}}_i \to \mathcal{X}_i$, and the restriction of h_i to η_0, $h_{i,\eta_0} : \bar{\mathcal{X}}_{i,\eta_0} \to \mathcal{X}_{i,\eta_0}$, is a contraction of all components $\Delta_j \subset \bar{\mathcal{X}}_{i,\eta_0}$ except $\Delta_0, \Delta_{e_{i,1}}, \Delta_{e_{i,1}+e_{i,2}}, \ldots, \Delta_{e_{i,1}+\cdots+e_{i,n_i+1}}$.

We now show that the isomorphism $\phi_{1,\eta} = \rho_\eta^* \phi_{2,\eta}$ extends to $(\phi_1, \mathcal{X}_1) \cong (\phi_2, \mathcal{X}_2)$. We first prove $e_{1,j} = e_{2,j}$ for all j. Indeed, using isomorphism $\bar{\mathcal{X}}_{i,\eta} \cong \mathcal{X}_{i,\eta}$ we define $\bar{\phi}_{i,\eta}$ be the pull back of $\phi_{i,\eta}$ to $\bar{\mathcal{X}}_{i,\eta}$. Let $\bar{\phi}_i$ on $\bar{\mathcal{X}}_i$ be the S-flat completion of $\bar{\phi}_{i,\eta}$. Such completion exists since the relative Quot-scheme $\mathrm{Quot}^P_{\mathcal{X}_i/S}$ is proper over S.

Since ϕ_{i,η_0} is stable, in particular admissible, one checks that the pull back of ϕ_i via $h_i : \bar{\mathcal{X}}_i \to \mathcal{X}_i$ is flat over S. Then by the separatedness of the relative Quot-scheme, $\bar{\phi}_i = h_i^* \phi_i$.

Then since $\bar{\phi}_{1,\eta} = h^* \bar{\phi}_{2,\eta}$ under the isomorphism $h : \bar{\mathcal{X}}_1 \to \bar{\mathcal{X}}_2$, we must have $\bar{\phi}_1 = h^* \bar{\phi}_2$. This implies $e_{1,1} = e_{2,1}$, $e_{1,1} + e_{1,2} = e_{2,1} + e_{2,2}$, etc. Thus combined with identity (5.24), we conclude $n_1 = n_2$ and $e_{1,j} = e_{2,j}$ for all j. This implies that the arrow $\mathcal{X}_{1,\eta} \cong \mathcal{X}_{2,\eta}$ in $\mathfrak{X}(\eta)$ extends to an arrow $\mathcal{X}_1 \cong \mathcal{X}_2$ in $\mathfrak{X}(S)$. By the separatedness of the Quot-scheme, we get $(\phi_1, \mathcal{X}_1) \cong (\phi_2, \mathcal{X}_2)$ in $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(S)$. This proves that $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}$ is separated. $\square$

5.4. For the stable pairs

We now investigate the properness and separatedness of $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$ and $\mathfrak{P}^P_{\mathfrak{D}_\pm \subset \mathfrak{Y}}$. Let $S = \operatorname{Spec} R \to C$ with η_0 and $\eta \in S$ be as in the statement of Proposition 5.1. Let $\phi_\eta : \mathcal{O}_{\mathcal{X}_\eta} \to \mathcal{F}_\eta$ be an element in $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}(\eta)$. We indicate how to find a finite base change $S' \to S$ and a $\phi' : \mathcal{O}_{\mathcal{X}'} \to \mathcal{F}'$ in $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}(S')$ so that $\phi' \times_{S'} \eta' = \phi_\eta \times_\eta \eta'$.

By definition, $\phi' \in \mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}(S')$ if the following hold:

1) $\mathcal{F}'$ is a flat S'-family of pure one-dimensional sheaves; cokerϕ' has relative dimension at most zero;
2) cokerϕ' is away from the singular divisor of $\mathcal{X}_{\eta_0}$;
3) $\mathcal{F}'$ is normal to the singular divisor of $\mathcal{X}_{\eta_0}$;
4) $\operatorname{Aut}_{\mathfrak{X}}(\phi'_{\eta_0})$ is finite.

Let $\mathcal{K}_\eta = \operatorname{coker}(\phi_\eta)$ and let $E_\eta \subset \mathcal{Y}_\eta$ be its support. We first study the case where $\mathcal{X}_\eta$ is smooth. In this case, following the proof in [Li02], possibly after a finite base change of S, which by abuse of notation we still denote by S, we can find an $\mathcal{X} \in \mathfrak{X}(S)$ that extends $\mathcal{X}_\eta$ so that

(a) the closure $\overline{E}_\eta$ of E_η in $\mathcal{X}$ is disjoint from the singular divisors of $\mathcal{X}_{\eta_0}$;
(b) for any added $\Delta \subset \mathcal{X}_{\eta_0}$, we have $\Delta \cap \overline{E}_\eta \neq \emptyset$.

Since the moduli of stable pairs over a projective scheme is projective (cf. [LP93]), we can extend ϕ_η to a $\phi : \mathcal{O}_{\mathcal{Y}} \to \mathcal{F}$ that satisfies (1); because of (b), (4) holds as well. Suppose (2) is violated for the family ϕ, then by repeating the argument in Subsection 5.1, we conclude that by a further finite base change, which we still denote by S, we can find an extension ϕ of ϕ_η that satisfies (1), (2) and (4). In case the extension ϕ does not satisfies (3).

Then because of (2), $\phi : \mathcal{O}_{\mathcal{X}} \to \mathcal{F}$ near where $\mathcal{F}$ is not normal to the singular divisor of $\mathcal{X}_{\eta_0}$ is a quotient homomorphism. Thus we can apply the result in Subsection 5.1 directly to conclude that we can find a finite base change $S' \to S$ and an extension $\phi' \in \mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}(S')$ of ϕ_η as desired.

The general case for $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$ and $\mathfrak{P}^P_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$ is similar to the proof developed in Section 5. Since it is merely a duplication of the previous argument, we will not repeat it here. This completes the proof of the separatedness and the properness of Theorems 4.14 and 4.15.

5.5. The boundedness

We prove the boundedness part in Theorems 4.14, 4.15.

Proposition 5.15. *The set $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}_0/\mathfrak{C}_0}(\mathbf{k})$ is bounded.*

We quote the following known result (cf. [HL97]).

Proposition 5.16. *A set of isomorphism classes of coherent sheaves on a projective scheme is bounded if and only if the set of their Hilbert polynomials is finite, and there is a coherent sheaf $\mathcal{F}$ so that every sheaf in this set is a quotient sheaf of $\mathcal{F}$.*

These two Propositions imply that $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k})$ is bounded. We prove Proposition 5.15 by induction on the degree of the polynomial $P(v)$. To carry out the induction, we need the following lemma. For simplicity, in the remainder part of this Section, we assume that H on $X \to C$ is sufficiently ample.

Lemma 5.17. *Let W be either $X[n]_0$ or $Y[n_-, n_+]_0$, and let $p : W \to X_0$ be the projection. For any coherent sheaf $\mathcal{F}$ on W, there is an open dense subset $U \subset |p^*H|$ such that each divisor $V \in U$ has normal crossing singularity; is smooth away from the singular locus of W, and $\mathcal{F}$ is normal to V. Moreover, if $\mathcal{F}$ is normal to the singular divisors of W (resp. the distinguished divisor of $Y[n_-, n_+]_0$), so does $\mathcal{F}|_V$, viewed as a sheaf on W.*

Proof. Given $\mathcal{F}$, we can find a finite length filtration

$$0 \subset \mathcal{F}_{\leq 0} \subset \mathcal{F}_{\leq 1} \subset \cdots \subset \mathcal{F}_{\leq d} = \mathcal{F},$$

where $\mathcal{F}_{\leq k}$ is the subsheaf of $\mathcal{F}$ consisting of elements of dimension at most k. Let Z_k be the support of $\mathcal{F}_{\leq k}$; it is closed in W. Because H is sufficiently ample, $|H|$ is base point free.

We then let $U \subset |p^*H|$ be the open subset of those divisors $V \in U$ that have normal crossing singularities; are smooth away from the singular locus of W, and do not contain any irreducible component of Z_k for all k. By Bertini's theorem, U is open and non-empty. For $V \in U$, by Definition 3.1, $\mathcal{F}$ is normal to V if and only if no element of $\mathcal{F}_{\leq k}$ is supported entirely in V. Because of the construction, U satisfies the requirement of the lemma.

By the same reason, if $\mathcal{F}$ is normal, we can choose U so that in addition to the requirement stated, we have that for every $V \in U$, all $\mathcal{F}$ and $\mathcal{F}|_{D_i}$ are normal to V. Therefore, $\mathcal{F}|_V$ is normal to D_i for all $V \in U$. $\square$

Remark 5.18. Following the proof of the Lemma, one sees that the set U in the Lemma covers every $D_i \subset X[n]_0$ (of $Y[n_-, n_+]_0$) up to finite points in that

$$\dim\big(D_i - \cup_{S\in U} S \cap D_i\big) = 0.$$

We state the following lemma due to Grothendieck [HL97].

Lemma 5.19. *Let W be a projective scheme with an ample line bundle h. Let $\mathcal{V}$ be a fixed coherent sheaf on W. Let $\mathfrak{S}$ be the set of those quotients $\phi : \mathcal{V} \to \mathcal{F}$ so that $\mathcal{F}$ is pure of dimension d. Suppose there is a constant N so that for any $\mathcal{F} \in \mathfrak{S}$, its Hilbert polynomial*

$$\chi^h_{\mathcal{F}}(v) = a_d v^d + a_{d-1} v^{d-1} + \cdots ,$$

satisfies $|a_d| \leq N$ and $a_{d-1} \leq N$. Then $\mathfrak{S}$ is bounded.

Here we use $\chi^h_{\mathcal{F}}(v) = \chi(\mathcal{F} \otimes h^{\otimes v})$ to indicate the dependence on the polarization h of the Hilbert polynomial of $\mathcal{F}$. Also we use $(\phi, \mathcal{F})$ to abbreviate a quotient sheaf $\phi : \mathcal{V} \to \mathcal{F}$ when $\mathcal{V}$ is understood.

Corollary 5.20. *Let W and $\mathcal{V}$ be as in Lemma 5.19, and let N and $d \geq 0$ be two integers. Let $\mathfrak{S}$ be a set of quotient sheaves $\phi : \mathcal{V} \to \mathcal{F}$ on W. Suppose for any $(\phi, \mathcal{F}) \in \mathfrak{S}$, every subsheaf of $\mathcal{F}$ has dimension $\geq d$, and the Hilbert polynomial $\chi^h_{\mathcal{F}}(v) = a_m v^m + \cdots + a_0$ satisfies $|a_i| \leq N$ for $i \geq d$ and $a_{d-1} \leq N$. Then $\mathfrak{S}$ is bounded.*

Proof. We let $\mathfrak{S}_k = \{(\phi, \mathcal{F}) \mid \deg \chi^h_{\mathcal{F}} \leq k,\ (\phi, \mathcal{F}) \in \mathfrak{S}\}$. We prove that $\mathfrak{S}_k$ are bounded by induction on k.

When $k = d$, every sheaf $\mathcal{F}$ in the $\mathfrak{S}_d$ is of pure dimension d. The result then follows from Lemma 5.19. We now suppose the statement is true for a $k \geq d$; we will show that it is true for $k + 1$.

For any quotient $(\phi, \mathcal{F}) \in \mathfrak{S}_{k+1}$, we let $\mathcal{F}_{\le k} \subset \mathcal{F}$ be the maximal subsheaf of dimension at most k. Since $\mathcal{F}$ has dimension at most $k+1$, $\mathcal{F}_{>k} := \mathcal{F}/\mathcal{F}_{\le k}$ is either zero or is pure of dimension $k+1$. Also, the quotient homomorphism $\phi : \mathcal{V} \to \mathcal{F}$ induces a quotient $\phi_{>k} : \mathcal{V} \to \mathcal{F}_{>k}$; we let $\mathfrak{S}'$ be the set $\{(\phi_{>k}, \mathcal{F}_{>k}) \mid (\phi, \mathcal{F}) \in \mathfrak{S}_{k+1}\}$, and let $\mathfrak{T} = \{(\phi_{\le k} : \ker \phi_{>k} \to \mathcal{F}_{\le k}) \mid (\phi, \mathcal{F}) \in \mathfrak{S}_{k+1}\}$, where $\phi_{\le k}$ is induced from $\phi_{>k}$. Let

$$\mathfrak{S}_{k+1} \longrightarrow \mathfrak{S}' \times \mathfrak{T}, \quad \phi \mapsto (\phi_{>k}, \phi_{\le k}),$$

which is injective.

Since $\mathcal{F}_{\le k}$ has dimension at most k, its Hilbert polynomial $\chi^h_{\mathcal{F}_{\le k}}(v) = b_k v^k + \cdots$ has $b_k \ge 0$. Since

$$\chi^h_{\mathcal{F}_{>k}}(v) = \chi^h_{\mathcal{F}}(v) - \chi^h_{\mathcal{F}_{\le k}}(v) = a_{k+1} v^{k+1} + (a_k - b_k) v^k + \cdots ,$$

and by assumption a_{k+1} and a_k are bounded, we see that $a_k - b_k$ is bounded from above. Applying Lemma 5.19, we see that $\mathfrak{S}'$ is bounded. It also implies that $\{\ker \phi_{>k} \mid (\phi, \mathcal{F}) \in \mathfrak{S}_{k+1}\}$ is bounded.

Finally, we consider the quotients $(\phi_{\le k} : \ker \phi_{>k} \to \mathcal{F}_{\le k}) \in \mathfrak{T}$. Since $\mathcal{F}_{\le k}$ has dimension at most k, and since the collection $\{\ker \phi_{>k} \mid (\phi, \mathcal{F}) \in \mathfrak{S}_{k+1}\}$ is bounded, we can apply the induction hypothesis to obtain the boundedness of the set $\mathfrak{T}$. Therefore, $\mathfrak{S}_{k+1}$ is bounded since $\mathfrak{S}_{k+1} \to \mathfrak{S}' \times \mathfrak{T}$ is injective. □

For any polynomial $f(v)$, we denote $[f(v)]_{>0} = f(v) - f(0)$, which is $f(v)$ taking out the constant term.

Lemma 5.21. *Let $D \subset W$ be a divisor in a smooth variety W; h be an ample line bundle on W; $\mathcal{U}$ be a coherent sheaf on W, and B be a finite set of polynomials in v. Let $\mathfrak{S}$ be a set of quotients $\phi : \mathcal{U} \to \mathcal{E}$ so that for any $(\phi, \mathcal{E}) \in \mathfrak{S}$, $\mathcal{E}$ is normal to D and $[\chi^h_{\mathcal{E}}(v)]_{>0} \in B$. Then $\mathfrak{S}_D = \{(\phi|_D, \mathcal{E}|_D) \mid (\phi, \mathcal{E}) \in \mathfrak{S}\}$ is bounded. Further, suppose $\{\chi(\mathcal{E}) \mid (\phi, \mathcal{E}) \in \mathfrak{S}\}$ is finite, then $\mathfrak{S}$ is bounded.*

Proof. For $(\phi, \mathcal{E}) \in \mathfrak{S}$, we denote by $\phi_{>1} : \mathcal{U} \to \mathcal{E}_{>1}$ the induced quotient homomorphism. We claim that $\mathfrak{S}' = \{(\phi_{>1}, \mathcal{E}_{>1}) \mid (\phi, \mathcal{E}) \in \mathfrak{S}\}$ is bounded. Indeed, since B is finite, there is a constant M so that for any $(\phi, \mathcal{E}) \in \mathfrak{S}$, the coefficients of $\chi^h_{\mathcal{E}}(v) = a_n v^n + \cdots + a_0$ satisfy $|a_i| \le M$ for $i \ge 1$. Since $\mathcal{E}_{\le 1}$ has dimension ≤ 1, $\chi^h_{\mathcal{E}_{\le 1}}(v) = b_1 v + b_0$ has $b_1 \ge 0$. Then

$$\chi^h_{\mathcal{E}_{>1}}(v) = \chi^h_{\mathcal{E}}(v) - \chi^h_{\mathcal{E}_{\le 1}}(v) = a_n v^n + \cdots + a_2 v^2 + (a_1 - b_1)v + (a_0 - b_0)$$

has $|a_i| \le M$ for $i \ge 2$ and $a_1 - b_1 \le M$. Applying Corollary 5.20, we conclude that $\mathfrak{S}'$ is bounded. Thus, $a_1 - b_1$ is bounded; thus by replace M by a larger constant if necessary, we have $|b_1| \le M$.

We now study $\mathcal{E}|_D$. As $(\phi, \mathcal{E}) \in \mathcal{F}$, $\mathcal{E}$ is normal to D, thus both $\mathcal{E}_{\le 1}$ and $\mathcal{E}_{>1}$ are normal to D; therefore

$$0 \longrightarrow \mathcal{E}_{\le 1}|_D \longrightarrow \mathcal{E}|_D \longrightarrow \mathcal{E}_{>1}|_D \longrightarrow 0 \tag{5.27}$$

is exact. Since $\mathfrak{S}'$ is bounded, the set $\{(\phi_{>1}|_D, \mathcal{E}_{>1}|_D) \mid (\phi, \mathcal{E}) \in \mathfrak{S}\}$ is bounded. On the other hand, since the leading coefficients b_1 of $\chi^h_{\mathcal{E}_{\le 1}}(v)$ for $(\phi, \mathcal{E}) \in \mathfrak{S}$ satisfy $b_1 \le M$, using that the set of effective one-dimensional cycles in W of bounded degree is bounded, we conclude that the restrictions $\mathcal{E}_{\le 1}|_D$ form a set of zero dimensional sheaves of bounded length. Therefore, the set $\{\mathcal{E}_{\le 1}|_D \mid (\phi, \mathcal{E}) \in \mathfrak{S}\}$ is bounded. By (5.27), together with that $\{(\phi_{>1}|_D, \mathcal{E}_{>1}|_D) \mid (\phi, \mathcal{E}) \in \mathfrak{S}\}$ is bounded, we conclude that $\mathfrak{S}_D = \{(\phi|_D, \mathcal{E}|_D) \mid (\phi, \mathcal{E}) \in \mathfrak{S}\}$ is bounded.

Finally, assuming $\{\chi(\mathcal{E}) \mid (\phi, \mathcal{E}) \in \mathfrak{S}\}$ is finite, then B finite implies that $\{\chi^h_{\mathcal{E}}(v) \mid (\phi, \mathcal{E}) \in \mathfrak{S}\}$ is finite. Since h is ample, by Proposition 5.16, we conclude that $\mathfrak{S}$ is bounded. □

Let $p : \Delta \to D$ be the ruled variety over D used to construct $X[n]_0$; let $D_\pm \subset \Delta$ be its two distinguished sections. Denote $h = p^*(H|_D)$, where H is sufficiently ample on X, we form $L = h(D_+)$, which is ample. Let $\mathcal{V}$ be a locally free sheaf on X as before, and we denote $p^*\mathcal{V} = p^*(\mathcal{V}|_D)$. Let $\mathfrak{B}$ be a bounded set of sheaves of $\mathcal{O}_\Delta$-modules, and let B be a finite set of polynomials. For $S \in |h|$, we denote by $\iota_S : S \to \Delta$ the embedding.

Lemma 5.22. *Let $\mathfrak{R}$ be a set of quotients $\phi : p^*\mathcal{V} \to \mathcal{E}$ on Δ. Suppose every $\mathcal{E} \in \mathfrak{R}$ is normal to D_+, $\chi^h_{\mathcal{E}|_{D_+}}(v) \in B$, and there is a smooth $S \in |h|$ so that $\iota_{S*}(\mathcal{E}|_S) \in \mathfrak{B}$. Then the set $\{[\chi^L_{\mathcal{E}}(v)]_{>0} \mid (\phi, \mathcal{E}) \in \mathfrak{R}\}$ is finite. Moreover, if there is an N so that $\chi(\mathcal{E}) \le N$ for all $(\phi, \mathcal{E}) \in \mathfrak{R}$, then $\mathfrak{R}$ is bounded.*

Proof. Let $(\phi, \mathcal{E}) \in \mathfrak{R}$. By the proof of Lemma 5.17, we can find a smooth $S \in |h|$ so that $\mathcal{E}$ is normal to S. Since $\mathcal{E}$ is normal to D_+, $\mathcal{E}$ is normal to the divisor $S + D_+$. We can also require that $\mathcal{E}|_S$ is normal to D_+. Using that $L \cong \mathcal{O}_\Delta(S + D_+)$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{E} \otimes L^{-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_{S+D_+} \longrightarrow 0.$$

It follows that

$$\chi^L_{\mathcal{E}|_{S+D_+}}(v) = \chi^L_{\mathcal{E}}(v) - \chi^L_{\mathcal{E}}(v-1). \tag{5.28}$$

Using the exact sequence

$$0 \longrightarrow \mathcal{E}|_{D_+}(-S \cap D_+) \longrightarrow \mathcal{E}|_{S+D_+} \longrightarrow \mathcal{E}|_S \longrightarrow 0,$$

and $\iota_{S*}(\mathcal{E}|_S) \in \mathfrak{B}$, which is bounded, and $\chi^h_{\mathcal{E}|_{D_+}}(v) \in B$, by the standard argument used in Corollary 5.20, the set of quotients $\{p^*\mathcal{V} \to \mathcal{E}_{S+D_+}\}$ induced from $(p^*\mathcal{V} \to \mathcal{E}) \in \mathfrak{R}$ is bounded. Therefore, the set of polynomials $\{\chi^L_{\mathcal{E}|_{S+D_+}}(v) \mid (\phi, \mathcal{E}) \in \mathfrak{R}\}$ is finite. By (5.28), the set $\{[\chi^L_{\mathcal{E}}(v)]_{>0} \mid (\phi, \mathcal{E}) \in \mathfrak{R}\}$ is finite. This proves the first part of the lemma.

Moreover, when $\chi(\mathcal{E}) \leq N$ for all $(\phi, \mathcal{E}) \in \mathfrak{R}$, Corollary 5.20 implies that $\mathfrak{R}$ is bounded. $\square$

Lemma 5.23. *Let $\phi : p^*\mathcal{V} \to \mathcal{E}$ be a quotient sheaf on Δ, and $\mathcal{E}$ is normal to both D_+ and D_-. Suppose there is an open subset $U \subset |h|$ such that every $V \in U$ has the following property: V is smooth; $\mathcal{E}$ is normal to V; $\dim\big(D_- - \cup_{V \in U} D_- \cap V\big) = 0$, and the restriction $\phi|_V : p^*\mathcal{V}|_V \to \mathcal{E}|_V$ is G_{m}-invariant. Then*

$$\chi^h_{\mathcal{E}|_{D_-}}(v) = \chi^h_{\mathcal{E}|_{D_+}}(v).$$

Proof. As before, we let $\mathcal{E}_{\leq 1} \subset \mathcal{E}$ be the subsheaf of elements of dimension at most 1, and form the quotient sheaf $\mathcal{E}_{>1} = \mathcal{E}/\mathcal{E}_{\leq 1}$. Let $\phi_{>1} : p^*\mathcal{V} \to \mathcal{E}_{>1}$ be the induced quotient homomorphism. We claim that the tautological $p^*p_*\mathcal{E}_{>1} \to \mathcal{E}_{>1}$ is an isomorphism.

Since $\mathcal{E}$ is normal to D_-, $\mathcal{E}_{>1}$ is normal to D_-. Thus we have

$$0 \longrightarrow \mathcal{E}_{>1}(-D_-) \longrightarrow \mathcal{E}_{>1} \longrightarrow \mathcal{E}_{>1}|_{D_-} \longrightarrow 0.$$

Applying p_*, we obtain

$$0 \longrightarrow p_*(\mathcal{E}_{>1}(-D_-)) \longrightarrow p_*\mathcal{E}_{>1} \longrightarrow p_*(\mathcal{E}_{>1}|_{D_-}) \longrightarrow R^1p_*(\mathcal{E}_{>1}(-D_-)) = 0.$$

Here the last term is zero because $\mathcal{E}_{>1}$ is a quotient sheaf of $p^*\mathcal{V}$. We claim that $p_*(\mathcal{E}_{>1}(-D_-)) = 0$. Suppose not, then it is supported on a positive dimensional subset since $\Delta \to D$ has dimension one fibers. Let $A \subset D$ be an irreducible positive dimensional component of the support of $p_*(\mathcal{E}_{>1}(-D_-))$. Because $\dim\big(D_- - \cup_{V\in U} D_- \cap V\big) = 0$, the union $\cup\{p^{-1}(A)\cap V \mid V \in U\}$ is dense in $p^{-1}(A)$. Therefore, for an open $S \subset D$ such that $S \cap A \neq \emptyset$, we have that $\mathcal{E}|_{p^{-1}(S)} \cong p^*(\mathcal{E}|_{D_-\cap S})$. Thus $p_*(\mathcal{E}_{>1}(-D_-))|_S = 0$, contradicting to $S \cap A \neq \emptyset$. This proves $p_*(\mathcal{E}_{>1}(-D_-)) = 0$; consequently, $p^*p_*\mathcal{E}_{>1} \cong \mathcal{E}_{>1}$, and

$$\chi^h_{\mathcal{E}_{>1}|_{D_-}}(v) = \chi^h_{p_*\mathcal{E}_{>1}}(v) = \chi^h_{\mathcal{E}_{>1}|_{D_+}}(v).$$

Repeating the same argument, we conclude that $\mathcal{E}_{\leq 1}$ is supported at finite fibers of $p : \Delta \to D$. Since $\mathcal{E}$ is normal to D_- and D_+, $\mathcal{E}_{\leq 1}$ is normal to D_- and D_+ too. Thus $\chi(\mathcal{E}_{\leq 1}|_{D_-}) = \chi(\mathcal{E}_{\leq 1}|_{D_+})$. Therefore

$$\chi^h_{\mathcal{E}|_{D_-}}(v) = \chi^h_{\mathcal{E}_{>1}|_{D_-}}(v) + \chi(\mathcal{E}_{\leq 1}|_{D_-}) = \chi^h_{\mathcal{E}_{>1}|_{D_+}}(v) + \chi(\mathcal{E}_{\leq 1}|_{D_+}) = \chi^h_{\mathcal{E}|_{D_+}}(v).$$

This proves the Lemma. $\square$

In the remainder of this Section, we abbreviate $\mathfrak{Q}_P := \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}_0/\mathfrak{C}_0}(\mathbf{k})$.

Proof of Proposition 5.15. We prove that $\mathfrak{Q}_P$ is bounded by induction on the degree of the polynomial $P(v)$.

Suppose $P(v) = c$ is a constant. Let $(\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P$. Then $\mathcal{F}$ is a zero dimensional sheaf such that its support is away from the singular locus of $X[n]_0$ and its length is c. The stability of $\mathcal{F}$ implies that $\mathcal{F}|_{\Delta_i}$ is nonzero for every $1 \leq i \leq n$. Therefore, $n \leq \mathrm{length}(\mathcal{F}) = c$. Applying Proposition 5.16, we conclude that $\mathfrak{Q}_P$ is bounded in this case.

Next we assume that for an integer d, $\mathfrak{Q}_P$ is bounded when $\deg P(v) \leq d-1$. We show that $\mathfrak{Q}_P$ is bounded when $P(v)$ has degree d.

Let P be a polynomial of degree d, and let $(\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P$. By Lemma 5.17, we can find an $S \in |p^*H|$ so that it has normal crossing singularity; is smooth away from the singular locus of $X[n]_0$; that $\mathcal{F}$ is normal to S, and the restriction $\mathcal{F}|_S$ is normal to the singular divisor of S.

Let $\mathcal{F}' = \iota_{S*}(\mathcal{F}|_S)$ and $\phi' : p^*\mathcal{V} \to \mathcal{F}'$ be the quotient homomorphism induced by ϕ. We have $\chi^{p^*H}_{\mathcal{F}'}(v) = P(v) - P(v-1)$. By our choice of S, $\mathcal{F}'$ is admissible but not necessary stable. We let $\Lambda_\phi \subset \{1, \ldots, n\}$ be the subset of indices k so that $\phi'|_{\Delta_k}$ is *not* G_{m}-invariant; we let $n_\phi = \#\Lambda_\phi \geq 0$, and let

$$I_\phi : \{1, \ldots, n_\phi\} \to \Lambda_\phi$$

be the order-preserving isomorphism. Let $\Lambda^{\complement}_\phi$ be the complement of Λ_ϕ. We then contract all $\Delta_i \subset X[n]_0$, $i \in \Lambda^{\complement}_\phi$, to obtain $p_\phi : X[n]_0 \to X[n_\phi]_0$. Let

$p' : X[n_\phi]_0 \to X_0$ be the projection. Since ϕ' is admissible, and $\phi'|_{\Delta_i}$ is G_{m}-invariant for $i \in \Lambda_\phi^{\complement}$, there is a quotient

$$(\phi')^{\mathrm{st}} : p'^*\mathcal{V} \longrightarrow \mathcal{F}'^{\mathrm{st}} \quad \text{such that} \quad \phi' = p_\phi^*(\phi')^{\mathrm{st}}.$$

Then $\big((\phi')^{\mathrm{st}}, \mathcal{F}'^{\mathrm{st}}, X[n_\phi]_0\big) \in \mathfrak{Q}_{P_1}$, where $P_1(v) = P(v) - P(v-1)$. By the induction hypothesis, $\mathfrak{Q}_{P_1}$ is bounded. Therefore, there is an N depending on P only so that

$$n_\phi \le N. \tag{5.29}$$

To proceed, we let $p_\Delta : \Delta \to D$ be the ruled variety used to construct $X[n]_0$ with distinguished sections $D_\pm \subset \Delta$. Let $h = p_\Delta^*(H|_D)$, where H is sufficiently ample on X (using $H^{\otimes m}$ if necessary), and form $L = h(D_+)$, which is ample. Let $H_i = p^*H|_{\Delta_i}$; and let $L_i = H_i(D_i)$, $i > 0$. We fix the tautological isomorphisms

$$\rho_i : \Delta \cong \Delta_i, \quad \text{so that } h = \rho_i^* H_i, \quad L = \rho_i^* L_i, \tag{5.30}$$

for all intermediate components $\Delta_1, \ldots, \Delta_n$ of $X[n]_0$.

Sublemma 1. *The set* $\{\chi^{H_i}_{\mathcal{F}|_{D_i}}(v) \mid (\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P,\ i \le n+1\}$ *is finite.*

Proof. Let N be as specified in (5.29). We first construct a finite sequence of finite sets $B_1, B_2, \ldots, B_{N+1}$ and show that for any $(\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P$, and any $1 \le i \le n+1$, we have $\chi^{H_i}_{\mathcal{F}|_{D_i}}(v) \in B_k$ for some k. This will prove the Sublemma.

Let $B_1 = \{\chi^{H_0}_{\mathcal{F}|_{D_1}}(v) \mid (\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P\}$. We prove that B_1 is a finite set. Indeed, by induction, we can find $S \in |p^*H|$ so that $\mathcal{F}' = \iota_{S*}(\mathcal{F}|_S)$ is admissible, and $\chi^{p^*H}_{\mathcal{F}'}(v) = P(v) - P(v-1)$. Restricting to $\Delta_0 = Y$, since $(\mathcal{F}')^{\mathrm{st}}|_{\Delta_0} = \mathcal{F}'|_{\Delta_0}$, the induction hypothesis that $\mathfrak{Q}_{P_1}$ is bounded implies that $\{\chi^{H_0}_{\mathcal{F}'|_{\Delta_0}}(v) \mid ((\phi')^{st}, (\mathcal{F}')^{\mathrm{st}}, X[n_\phi]_0) \in \mathfrak{Q}_{P_1}\}$ is finite. Therefore,

$$\{[\chi^{H_0}_{\mathcal{F}|_{\Delta_0}}(v)]_{>0} \mid (\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P\} \quad \text{is finite.} \tag{5.31}$$

Since H_0 is ample on Δ_0, using Lemma 5.21, we know that $\{\mathcal{F}|_{D_1} \mid (\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P\}$ is bounded. Therefore, B_1 is finite.

We define $B_{i\geq 2}$ inductively. Suppose we have defined B_k. Using the isomorphisms (5.30), we define a set of quotient homomorphisms on Δ:

$$\mathfrak{R}_k = \cup_{i\geq 1}\bigl\{\rho_i^*(\phi|_{\Delta_i}) \mid (\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P,\ \chi^h_{\rho_i^*(\mathcal{F}|_{D_i})}(v) \in B_k\bigr\}.$$

(Recall that $D_i \subset \Delta_i$ is identified with $D_+ \subset \Delta$ under ρ_i (cf. (2.8)).) We apply the first assertion of Lemma 5.22 to $B = B_k$ and $\mathfrak{B} = \cup_{i\geq 1}\{\rho_i^*(\phi|_{\Delta_i}) \mid (\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_{P_1}\}$ to conclude that the set $\{[\chi^L_{\mathcal{F}}(v)]_{>0} \mid (\phi, \mathcal{F}) \in \mathfrak{R}_k\}$ is finite. Then applying Lemma 5.21 to $D_- \subset \Delta$, we conclude that $\{(\phi, \mathcal{F})|_{D_-} \mid (\phi, \mathcal{F}) \in \mathfrak{R}_k\}$ is bounded, which implies that $B_{k+1} = \{\chi^H_{\mathcal{F}|_{D_-}}(v) \mid (\phi, \mathcal{F}) \in \mathfrak{R}_k\}$ is finite.

For any $(\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P$ and $1 \leq i \leq n+1$, we claim that $\chi^{H_i}_{\mathcal{F}|_{D_i}}(v) \in B_k$ for some $k \leq N+1$. To show this, we consider the sequence of polynomials

$$\chi^{H_1}_{\mathcal{F}|_{D_1}}(v), \ldots, \chi^{H_{n+1}}_{\mathcal{F}|_{D_{n+1}}}(v). \tag{5.32}$$

By Lemma 5.23, for $i \in \Lambda_\phi^{\complement}$, $\chi^{H_i}_{\mathcal{F}|_{D_i}}(v) = \chi^{H_{i+1}}_{\mathcal{F}|_{D_{i+1}}}(v)$; for $i = I_\phi(k) \in \Lambda_\phi$ for some k, $\chi^{H_{i+1}}_{\mathcal{F}|_{D_{i+1}}}(v) \in B_{k+1}$. Since $\#\Lambda_\phi \leq N$, we have $\chi^{H_i}_{\mathcal{F}|_{D_i}}(v) \in \cup_{k=1}^{N+1} B_k$. Since each B_k is finite, the Sublemma follows. $\square$

Sublemma 2. *There is a constant $M > 0$ so that for any $(\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P$, then*

1) *for $i \in \Lambda_\phi^{\complement}$, we have $\chi(\mathcal{F}|_{\Delta_i}(-D_i)) \geq 1$;*
2) *$\chi(\mathcal{F}|_{\Delta_0}(-D_{n+1})) \geq -M$.*
3) *for $i = I_\phi(k) \in \Lambda_\phi$, $\chi(\mathcal{F}|_{\Delta_i}(-D_i)) \geq -M$.*

Proof. We first prove item (1). Let $(\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P$ and let $i \in \Lambda_\phi^{\complement}$. We let $S \in |p^*H|$ and $\phi' : p^*\mathcal{V} \to \mathcal{F}' = \iota_{S*}(\mathcal{F}|_S)$ be as the quotient sheaf constructed at the beginning of the this proof (of Proposition 5.15). By the construction of $\Lambda_\phi^{\complement}$, we know that the restriction (to Δ_i) $(\phi'|_{\Delta_i}, \mathcal{F}'|_{\Delta_i})$ is G_{m}-invariant. By Lemma 3.18, $\chi^{H_i}_{\mathcal{F}'|_{\Delta_i}}(v) - \chi^{H_i}_{\mathcal{F}'|_{D_i}}(v) = 0$. Since

$$\chi^{H_i}_{\mathcal{F}'|_{\Delta_i}}(v) = \chi^{H_i}_{\mathcal{F}|_{\Delta_i}}(v) - \chi^{H_i}_{\mathcal{F}|_{\Delta_i}}(v-1) \quad \text{and}$$
$$\chi^{H_i}_{\mathcal{F}'|_{D_i}}(v) = \chi^{H_i}_{\mathcal{F}|_{D_i}}(v) - \chi^{H_i}_{\mathcal{F}|_{D_i}}(v-1),$$

the polynomial $f(v) = \chi^{H_i}_{\mathcal{F}|_{\Delta_i}}(v) - \chi^{H_i}_{\mathcal{F}|_{D_i}}(v)$ then satisfies $f(v) = f(v-1)$, which makes it a constant equal to $\chi(\mathcal{F}|_{\Delta_i}(-D_i))$. Since $\mathcal{F}|_{\Delta_i}$ is not G_{m}-invariant, by Lemma 3.18, $\chi(\mathcal{F}|_{\Delta_i}(-D_i)) \geq 1$.

We now prove item (2). Suppose the lower bound does not exist. Then there is a sequence $(\phi_k, \mathcal{F}_k, X[n_k]_0) \in \mathfrak{Q}_P$

$$\chi(\mathcal{F}_k|_{\Delta_0}(-D_{n_k+1})) \to -\infty, \quad \text{when } k \to +\infty. \tag{5.33}$$

But by (5.31) and Corollary 5.20, we know that $\{\mathcal{F}_k|_{\Delta_0}\}_{k\geq 1}$ is bounded; contradicts to (5.33). Thus item (2) holds.

Suppose item (3) does not hold, then there is a sequence $(\phi_k, \mathcal{F}_k, X[n_k]_0) \in \mathfrak{Q}_P$ and a sequence $1 \leq i_k \leq n_k$ such that

$$\chi(\mathcal{F}_k|_{\Delta_{i_k}}(-D_{i_k})) \to -\infty, \quad \text{when } k \to +\infty. \tag{5.34}$$

Using isomorphisms (5.30), we introduce $\bar{\mathcal{F}}_k = \rho^*_{i_k}(\mathcal{F}_k|_{\Delta_{i_k}})$. Tensoring $\bar{\mathcal{F}}_k$ with $\mathcal{O}_\Delta(-D_+)$, we obtain a sequence of quotients $\bar{\phi}_k : \bar{\mathcal{V}}(-D_+) \to \bar{\mathcal{F}}_k(-D_+)$, where $\bar{\mathcal{V}} = p^*_\Delta \mathcal{V}|_D$, $p_\Delta : \Delta \to D$. By construction, $\chi(\bar{\mathcal{F}}_k(-D_+)) \to -\infty$. In particular $\chi(\bar{\mathcal{F}}_k(-D_+))$ is bounded from above.

We claim that the set of polynomials $\{[\chi^L_{\bar{\mathcal{F}}_k(-D_+)}(v)]_{>0}\}_{k\geq 1}$ is finite. Once this is proved, then applying Corollary 5.20 we conclude that $\{\bar{\phi}_k\}_{k\geq 1}$ is bounded, which contradicts to $\chi(\bar{\mathcal{F}}_k(-D_+)) \to -\infty$.

We prove the claim. By Sublemma 1, there is a finite set B so that $\chi^{H_k}_{\mathcal{F}_k|_{D_{i_k}}}(v) \in B$. Using isomorphism (5.30), we obtain $\chi^{H}_{\bar{\mathcal{F}}_k|_{D_+}}(v) \in B$. Applying the first assertion of Lemma 5.22, we conclude that $\{[\chi^L_{\bar{\mathcal{F}}_k}(v)]_{>0}\}_{k\geq 1}$ is finite. Restricting to D_+, Lemma 5.21 implies that $\{\chi^L_{\bar{\mathcal{F}}_k|_{D_+}}(v)\}_{k\geq 1}$ is finite. The claim then follows from $[\chi^L_{\bar{\mathcal{F}}_k(-D_+)}(v)]_{>0} = [\chi^L_{\bar{\mathcal{F}}_k}(v)]_{>0} - [\chi^L_{\bar{\mathcal{F}}_k|_{D_+}}(v)]_{>0}$. □

We now complete the proof of Proposition 5.15. Let $(\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P$. Since $\mathcal{F}$ is normal to all D_i,

$$\chi(\mathcal{F}) = \chi(\mathcal{F}|_{\Delta_0}(-D_{n+1})) + \chi(\mathcal{F}|_{\Delta_1}(-D_1)) + \cdots + \chi(\mathcal{F}|_{\Delta_n}(-D_n)). \tag{5.35}$$

For $i \in \Lambda_\phi^{\complement}$, we have $\chi(\mathcal{F}|_{\Delta_i}(-D_i)) \geq 1$; for $i \in \Lambda_\phi \cup \{0\}$, by Sublemma 2, we have $\chi(\mathcal{F}|_{\Delta_i}(-D_i)) \geq -M$ $(D_0 = D_{n+1})$. Since $n_\phi \leq N$, we obtain $\chi(\mathcal{F}) \geq (N+1)(-M) + (n - \#\Lambda_\phi)$, which implies

$$n \leq \chi(\mathcal{F}) + (N+1)M + N. \tag{5.36}$$

The identity (5.35) and Sublemma 2 also gives the bound,

$$\chi(\mathcal{F}|_{\Delta_i}(-D_i)) \le \chi(\mathcal{F}) + (N+1)M + N, \quad 0 \le i \le n.$$

Therefore, applying Lemmas 5.21 and 5.22, we conclude that for each i, the set $\{\mathcal{F}|_{\Delta_i} \mid (\phi, \mathcal{F}, X[n]_0) \in \mathfrak{Q}_P\}$ is bounded. This together with the bound (5.36) implies that $\mathfrak{Q}_P$ is bounded. $\square$

By a parallel argument, we have

Proposition 5.24. *The set* $\mathfrak{Quot}^{\mathcal{V}_0,P}_{\mathfrak{D}_\pm\subset\mathfrak{Y}}(\mathbf{k})$ *is bounded.*

5.6. The moduli of stable pairs

We prove the boundedness of the moduli $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$ and $\mathfrak{P}^P_{\mathfrak{D}_\pm\subset\mathfrak{Y}}$. Here $P(v)$ is a degree one polynomial.

Proposition 5.25. *The set* $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k})$ *and* $\mathfrak{P}^P_{\mathfrak{D}_\pm\subset\mathfrak{Y}}(\mathbf{k})$ *are bounded.*

Proof. We work with the case $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k})$. The other is the same. Let $P(v) = av + b$. Let $(\varphi, \mathcal{F}, X[n]_0) \in \mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k})$, let $\mathcal{F}_i = \mathcal{F}|_{\Delta_i}$ and $H_i = p^*H|_{\Delta_i}$. Then each $\chi^{H_i}_{\mathcal{F}_i}(v) = a_i v + b_i$ has $a_i \ge 0$, and

$$a = a_0 + a_1 + \cdots + a_n. \tag{5.37}$$

Let Λ_φ be the set of those $k \ge 1$ so that $\chi^{H_k}_{\mathcal{F}_k}(v)$ has positive degree. Then by (5.37), $n_\varphi = \#\Lambda_\varphi \le a$. Let $\Lambda^{\complement}_\varphi = \{1, \ldots, n\} - \Lambda_\varphi$.

First, we show that for each $i \in \Lambda^{\complement}_\varphi$, $\chi(\mathcal{F}_i(-D_i)) \ge 1$. Let $\varphi_i : \mathcal{O}_{\Delta_i} \to \mathcal{F}_i$ be the restriction of φ to Δ_i. Since cokerφ has zero dimensional support, $\chi(\text{coker}\varphi) \ge 0$. Hence $\chi(\mathcal{F}_i(-D_i)) \ge \chi(\text{Im}\varphi_i(-D_i))$.

For Imφ_i, we have the induced quotient homomorphism $\varphi'_i : \mathcal{O}_{\Delta_i} \to$ Imφ_i. Applying Lemma 3.18 to φ'_i, we get $\chi(\text{Im}\varphi_i(-D_i)) \ge 0$. Since φ_i is not G_{m}-invariant, either $\chi(\text{coker}\varphi) > 0$ or $\chi(\text{Im}\varphi_i(-D_i)) > 0$. Thus $\chi(\mathcal{F}_i(-D_i)) \ge 1$.

Next, we let $I_\varphi : \{1, \ldots, n_\varphi\} \to \Lambda_\varphi$ be the order-preserving isomorphism. We form

$$\Xi_k = \bigl\{\chi(\mathcal{F}_j(-D_j)) \mid (\varphi, \mathcal{F}, X[n]_0) \in \mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k}),\, j = I_\varphi(k)\bigr\}.$$

(For $k = 0$, we agree $I_\varphi(0) = 0$ and $D_0 = D_{n+1}$.) Applying the same argument as in Sublemma 2 of the proof of Proposition 5.15 to φ'_k, we conclude that there is an $M > 0$ so that for each $k \le a$, $\inf\{\chi \in \Xi_k\} \ge -M$. (Note that by the bound $\#\Lambda_\varphi \le a$, $\Xi_k = \emptyset$ if $k > a$.)

Lastly, since $\mathcal{F}$ is normal to D_i, we have

$$\chi(\mathcal{F}) = \chi(\mathcal{F}_0(-D_{n+1})) + \chi(\mathcal{F}_1(-D_1)) + \cdots + \chi(\mathcal{F}_n(-D_n)). \tag{5.38}$$

Repeating the argument following (5.35), we prove the boundedness of $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k})$. $\square$

5.7. Decomposition of the central fiber

In this subsection, we assume that Y is a disjoint union of two smooth components Y_- and Y_+. We introduce a canonical decomposition of the central fiber of the moduli stacks $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}$ and $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$ over C. We shall focus on $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}$ and omit the details for $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$.

Let

$$\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}_0/\mathfrak{C}_0} = \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}} \times_C 0$$

be the central fiber of $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}$ over C. We denote $\mathfrak{C}^P$ be the weighted stack of weights in $\Lambda = \mathbb{Q}[m]$ (polynomials in m) and of total weight P (cf. Section 2.5). For each stable quotient $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ in $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(\mathbf{k})$, where $\mathcal{F}$ is a sheaf on $X[n]_0$, it assigns a weight w to $X[n]_0$ by assigning each irreducible $\Delta_l \subset X[n]_0$ (resp. divisor $D_l \subset X[n]_0$) the polynomial $\chi^H_{\mathcal{F}_{\Delta_l}}$ (resp. $\chi^H_{\mathcal{F}_{D_l}}$). Since $\mathcal{F}$ is admissible, this rule applied to $(\phi, \mathcal{X}) \in \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}(S)$ defines a continuous weight assignment of the family $\mathcal{X}/S$. In particular, the morphism $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}$ factors through

$$\pi_P : \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}} \longrightarrow \mathfrak{C}^P. \tag{5.39}$$

We now form the set of splittings of P: Λ^{spl}_P, which is the set of triples $\delta = (\delta_\pm, \delta_0)$ in Λ so that $\delta_- + \delta_+ - \delta_0 = P$. We follow the notation developed in Subsection 2.5. For any $\delta \in \Lambda^{\mathrm{spl}}_P$, we form the moduli of stable relative quotients on $\mathfrak{D}_\pm \subset \mathfrak{Y}_\pm$ over $\mathfrak{A}_\diamond$: for any scheme S, we define $\mathfrak{Quot}^{\delta_-,\delta_0}_{\mathfrak{Y}_-/\mathfrak{A}_\diamond}(S)$ be the collection of $(\phi; \mathcal{Y}, \mathcal{D})$, where $(\mathcal{Y}, \mathcal{D}) \in \mathfrak{Y}_-(S)$ and $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ is an S-flat family of stable relative quotients on the pair $\mathcal{D} \subset \mathcal{Y}$ such that for any closed $s \in S$, $\chi^H_{\mathcal{F}_s} = \delta_-$ and $\chi^H_{\mathcal{F}_s|_{\mathcal{D}_s}} = \delta_0$. We form $\mathfrak{Quot}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond}$ similarly. By Theorem 4.15, we have

Proposition 5.26. *The groupoids $\mathfrak{Quot}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}$ are Deligne-Mumford stacks, proper and separated, and of finite type.*

Using $\delta \in \Lambda_P^{\rm spl}$, we form the stack $\mathfrak{C}_0^{\dagger,\delta}$, according to the rule specified in Section 2. We define

$$\mathfrak{Quot}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} = \mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}} \times_{\mathfrak{C}^P} \mathfrak{C}_0^{\dagger,\delta}.$$

It parameterizes stable quotients $\phi\colon p^*\mathcal{V} \to \mathcal{F}$ on $X[n]_0$ with a node-marking $D_k \subset X[n]_0$ so that the Hilbert polynomials of $\mathcal{F}$ restricted to $\cup_{i<k}\Delta_i$, to $\cup_{i\geq k}\Delta_i$ and to D_k are δ_-, δ_+ and δ_0, respectively.

For each $\delta \in \Lambda_P^{\rm spl}$, like the case of stable morphisms, we have the gluing morphism that factors through $\mathfrak{Quot}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{A}_0^\dagger}$ (it originally maps to $\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}} \times_C 0$):

$$\Phi_\delta : \mathfrak{Quot}^{\delta_-,\delta_0}_{\mathfrak{Y}_-/\mathfrak{A}_\diamond} \times_{\mathrm{Quot}_D^{\mathcal{V}_D,\delta_0}} \mathfrak{Quot}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond} \longrightarrow \mathfrak{Quot}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{A}_0^\dagger}, \tag{5.40}$$

where $\mathrm{Quot}_D^{\mathcal{V}_D,\delta_0}$ is the Grothendieck's Quot-scheme of quotient sheaves $\mathcal{V}_D = \mathcal{V}|_D \to \mathcal{E}$ with $\chi^H_{\mathcal{E}}(v) = \delta_0$.

Using the collection of pairs of line bundles and sections (L_δ, s_δ) for $\delta \in \Lambda_P^{\rm spl}$ constructed in Proposition 2.19, and let π_P be as in (5.39), we have

Theorem 5.27. *Let (L_δ, s_δ) and the notation be as in Proposition 2.19. Then*

1) $\otimes_{\delta\in\Lambda_P^{\rm spl}} \pi_P^* L_\delta \cong \mathcal{O}_{\mathfrak{Quot}^{\mathcal{V},P}_{\mathfrak{X}/\mathfrak{C}}}$, *and* $\prod_{\delta\in\Lambda_P^{\rm spl}} \pi_P^* s_\delta = \pi_P^*\pi^* t$;
2) *as closed substacks,* $\mathfrak{Quot}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{A}_0^\dagger} = (\pi_P^* s_\delta = 0)$;
3) *The morphism Φ_δ in* (5.40) *is an isomorphism of Deligne-Mumford stacks.*

For the case of coherent systems, like Quot-schemes, the morphism $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}$ factors through

$$\pi_P : \mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}} \longrightarrow \mathfrak{C}^P. \tag{5.41}$$

For any $\delta \in \Lambda_P^{\rm spl}$, we define the moduli of relative stable pairs on $\mathfrak{D}_\pm \subset \mathfrak{Y}_\pm$ over $\mathfrak{A}_\diamond$:

$$\mathfrak{P}^{\delta_-,\delta_0}_{\mathfrak{Y}_-/\mathfrak{A}_\diamond} \quad \text{and} \quad \mathfrak{P}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond}.$$

They are again Deligne-Mumford stacks, proper and separated, and of finite type; and they both admit an evaluation morphism to the Hilbert scheme $\mathrm{Hilb}_D^{\delta_0}$ via restriction.

Accordingly, for $\delta \in \Lambda_P^{\mathrm{spl}}$, we define

$$\mathfrak{P}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} = \mathfrak{P}^{P}_{\mathfrak{X}/\mathfrak{C}} \times_{\mathfrak{C}^P} \mathfrak{C}_0^{\dagger,\delta}.$$

We have a glueing morphism

$$\Phi_\delta : \mathfrak{P}^{\delta_-,\delta_0}_{\mathfrak{Y}_-/\mathfrak{A}_\diamond} \times_{\mathrm{Hilb}^{\delta_0}_D} \mathfrak{P}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond} \longrightarrow \mathfrak{P}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}. \tag{5.42}$$

Theorem 5.28. *Let (L_δ, s_δ) and the notation be as in Proposition 2.19. Then*

1) $\otimes_{\delta\in\Lambda_P^{\mathrm{spl}}} \pi_P^* L_\delta \cong \mathcal{O}_{\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}}$, *and* $\prod_{\delta\in\Lambda_P^{\mathrm{spl}}} \pi_P^* s_\delta = \pi_P^*\pi^* t$;
2) *as closed substacks,* $\mathfrak{P}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} = (\pi_P^* s_\delta = 0)$;
3) *The morphism Φ_δ in (5.42) is an isomorphism of Deligne-Mumford stacks.*

6. Virtual cycles and their degenerations

Let $\pi\colon X \to C$ and H ample on X be a simple degeneration of projective threefolds. We fix a degree one polynomial $P(v)$. Applying Theorem 4.14, we form the good degeneration $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} := \mathfrak{Quot}^{\mathcal{O}_X,P}_{\mathfrak{X}/\mathfrak{C}}$ of Hilbert scheme of subschemes of X/C, of Hilbert polynomial P.

In this section, we construct the virtual class of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$, and use this class to prove a degeneration formula of the Donaldson-Thomas invariants of ideal sheaves. For notational simplicity, we only treat the case where the central fiber X_0 is the union of two irreducible components and their intersection $D \subset X_0$ is connected. Our construction of perfect relative obstruction theory of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}^P$ is based on the work of Huybrechts-Thomas on Atiyah class [HT10]; our proof of degeneration formula follows the proof of a similar degeneration formula by Maulik, Pandharipande and Thomas in [MPT10]; the formulation of degeneration based on Chern characters follows the work of Maulik, Nekrasov, Okounkov and Pandharipande in [MNOP06].

As X_0 is assumed to have two irreducible components, the normalization $q : Y \to X_0$ has two connected components

$$Y = Y_- \cup Y_+, \quad \text{and} \quad D_\pm = Y_\pm \cap q^{-1}(D).$$

6.1. Virtual cycle of the total space

We first construct the relative obstruction theory of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}^P$ (cf. (2.28)). We let

$$\pi : \mathcal{X} = \mathfrak{X} \times_{\mathfrak{C}} \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \longrightarrow \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}, \quad \text{and} \quad \mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_{\mathcal{X}}$$

be the universal underlying family and the universal ideal sheaf of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$. We form the traceless part of the derived homomorphism of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules:

$$E = R\pi_* R\mathcal{H}om(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})_0[1]. \tag{6.1}$$

Since $\mathcal{X} \to \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$ is a family of l.c.i. schemes, and $\mathcal{I}_{\mathcal{Z}}$ is admissible and of rank one, by Serre duality, locally E is a two-term perfect complex concentrated at $[0,1]$.

Let

$$L_{\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}/\mathfrak{C}^P} = \tau^{\geq -1}\mathbb{L}_{\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}/\mathfrak{C}^P}$$

be the truncated relative cotangent complex of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}^P$.

Proposition 6.1 ([MPT10, Prop 10]). *The Atiyah class constructed in [HT10] defines a perfect relative obstruction theory*

$$\phi : E^\vee \longrightarrow L_{\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}/\mathfrak{C}^P}. \tag{6.2}$$

We let $[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} \in A_*\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$ be the associated virtual class.

Proposition 6.2. *Let $c \neq 0 \in C$, and let $i^!_c : A_*\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to A_{*-1}\mathfrak{I}^P_{X_c}$ be the Gysin map associate to the divisor $c \in C$. Then $i^!_c[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} = [\mathfrak{I}^P_{X_c}]^{\mathrm{vir}}$.*

Proof. This is because the obstruction theory of $\mathfrak{I}^P_{X_c}$ is the pull back of the relative obstruction theory of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}^P$ via $c \in C$ (cf. [BF97]). $\square$

Next we construct the virtual class of the relative Hilbert schemes. In the subsequent discussion, we use that $Y = Y_- \cup Y_+$ is the union of Y_- and Y_+. We let $\delta = \{(\delta_+, \delta_0), (\delta_-, \delta_0)\}$ be two pairs of polynomials.

We denote by $\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond}$ the moduli of stable relative ideal sheaves on $\mathfrak{D}_+ \subset \mathfrak{Y}_+$ of pair Hilbert polynomial (δ_+, δ_0). For simplicity, we abbreviate it to $\mathcal{M}^\delta_+$. Let $L_{\mathcal{M}^\delta_+/\mathfrak{A}^{\delta_+,\delta_0}_\diamond}$ be the truncated relative cotangent complex of

$\mathcal{M}^{\delta}_{+} = \mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond} \to \mathfrak{A}^{\delta_+,\delta_0}_{\diamond}$. Let

$$\pi_+ : \mathcal{Y}_+ \longrightarrow \mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond} \quad \text{and} \quad \mathcal{I}_{\mathcal{Z}_+} \subset \mathcal{O}_{\mathcal{Y}_+}$$

be the universal underlying family and the universal ideal sheaf of $\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond}$.

Proposition 6.3 ([MPT10]). *The Atiyah class in [HT10] defines a perfect relative obstruction theory*

$$\phi_+ : E_+^\vee := R\pi_{+*}R\mathcal{H}om(\mathcal{I}_{\mathcal{Z}_+}, \mathcal{I}_{\mathcal{Z}_+})_0[1]^\vee \longrightarrow L_{\mathcal{M}^{\delta}_{+}/\mathfrak{A}^{\delta_+,\delta_0}_{\diamond}}. \tag{6.3}$$

The obstruction theory defines its virtual class $[\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond}]^{\mathrm{vir}} \in A_*\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond}$. By replacing the subscript "+" with "−", we obtain a parallel theory for $\mathcal{M}^{\delta}_{-} := \mathfrak{I}^{\delta_-,\delta_0}_{\mathfrak{Y}_-/\mathfrak{A}_\diamond}$.

6.2. Decomposition of the virtual cycle

We study the decomposition of the virtual cycles of the central fiber $\mathfrak{I}^P_{\mathfrak{X}_0/\mathfrak{C}_0} := \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \times_C 0$.

We let Λ^{spl}_P be the collection of triples $\delta = (\delta_-, \delta_+, \delta_0)$ of polynomials in $\mathcal{A}$ so that $\delta_+ + \delta_- - \delta_0 = P$. Following the notation developed in Section 5, the morphism $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}$ lifts to $\pi_P : \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}^P$. Fixing a splitting data $\delta \in \Lambda^{\mathrm{spl}}_P$, we define the closed substack $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ via the Cartesian diagram

$$\begin{array}{ccc} \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} := \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \times_{\mathfrak{C}^P} \mathfrak{C}_0^{\dagger,\delta} & \longrightarrow & \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \\ \downarrow & & \downarrow{\scriptstyle \pi_P} \\ \mathfrak{C}_0^{\dagger,\delta} & \longrightarrow & \mathfrak{C}^P. \end{array} \tag{6.4}$$

We denote by (L_δ, s_δ) the pair of the line bundle and the section for $\delta \in \Lambda^{\mathrm{spl}}_P$ constructed in Proposition 2.19. Then $\mathfrak{C}_0^{\dagger,\delta} = (s_\delta = 0) \subset \mathfrak{C}^P$; and by Theorem 5.27, $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} = (\pi_P^* s_\delta = 0)$. We define

$$c_1^{\mathrm{loc}}(L_\delta, s_\delta) : A_*\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \longrightarrow A_{*-1}\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$$

be the localized first Chern class of (L_δ, s_δ).

We define the perfect relative obstruction theory of $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} \to \mathfrak{C}_0^{\dagger,\delta}$ by pulling back the relative obstruction theory (6.2) of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}^P$ via the diagram (6.4):

$$\phi_\delta : E_\delta^\vee := R\pi_{\delta *} R\mathcal{H}om(\mathcal{I}_{\mathcal{Z}_\delta}, \mathcal{I}_{\mathcal{Z}_\delta})_0[1]^\vee \longrightarrow L_{\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}/\mathfrak{C}_0^{\dagger,\delta}}, \tag{6.5}$$

where

$$\pi_\delta : \mathcal{X}_\delta \to \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} \quad \text{and} \quad \mathcal{I}_{\mathcal{Z}_\delta} \subset \mathcal{O}_{\mathcal{X}_\delta}$$

is the universal family of $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$, which is also the pull back of $(\mathcal{X}, \mathcal{I}_{\mathcal{Z}})$ to $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ via the arrow in (6.4).

Applying [BF97], we get

$$[\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}]^{\mathrm{vir}} = c_1^{\mathrm{loc}}(L_\delta, s_\delta)[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}}. \tag{6.6}$$

Proposition 6.4. *Let $\iota_\delta : \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} \to \mathfrak{I}^P_{\mathfrak{X}_0/\mathfrak{C}_0}$ be the inclusion. We have an identity of cycle classes*

$$i_0^![\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} = \sum_{\delta \in \Lambda_P^{\mathrm{spl}}} \iota_{\delta *}[\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}]^{\mathrm{vir}}. \tag{6.7}$$

Proof. This follows from item (1) of Theorem 5.27 and the identity (6.6). $\square$

To reinterpret the terms in the summation of (6.7), we will express them in terms of the virtual class of relative Hilbert schemes. For this, we will use the Cartesian product (keeping the abbreviation $\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond} = \mathcal{M}^\delta_\pm$)

$$\begin{array}{ccc} \mathcal{M}^\delta_- \times_{\mathrm{Hilb}_D^{\delta_0}} \mathcal{M}^\delta_+ & \xrightarrow{\ u\ } & \mathcal{M}^\delta_- \times \mathcal{M}^\delta_+ \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{ev}_-, \mathrm{ev}_+)} \\ \mathrm{Hilb}_D^{\delta_0} & \xrightarrow{\ \triangle\ } & \mathrm{Hilb}_D^{\delta_0} \times \mathrm{Hilb}_D^{\delta_0}, \end{array} \tag{6.8}$$

where $\mathrm{ev}_\pm$ are the evaluation morphisms and $\triangle$ is the diagonal morphism, and use the isomorphism (cf. Theorem 5.27)

$$\Phi_\delta : \mathcal{M}^\delta_- \times_{\mathrm{Hilb}_D^{\delta_0}} \mathcal{M}^\delta_+ \longrightarrow \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}. \tag{6.9}$$

Note that the relative obstruction theory of $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} \to \mathfrak{C}_0^{\dagger,\delta}$ endows $\mathcal{M}^\delta_- \times_{\mathrm{Hilb}_D^{\delta_0}} \mathcal{M}^\delta_+ \to \mathfrak{C}_0^{\dagger,\delta}$ a perfect relative obstruction theory; also

$$\mathcal{M}^\delta_- \times \mathcal{M}^\delta_+ \longrightarrow \mathfrak{A}_\diamond^{\delta_-,\delta_0} \times \mathfrak{A}_\diamond^{\delta_+,\delta_0} \tag{6.10}$$

has a perfect relative obstruction theory induced from that of its factors. We will compare these two obstruction theories.

We continue to denote by $\mathcal{X}_\delta \to \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ with $\mathcal{I}_\delta \subset \mathcal{O}_{\mathcal{X}_\delta}$ (resp. $\mathcal{Y}_\pm \to \mathcal{M}^\delta_\pm$ with $\mathcal{I}_\pm \subset \mathcal{O}_{\mathcal{Y}_\pm}$) the universal family of $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ (resp. $\mathcal{M}^\delta_\pm$). We let

$$\tilde{\mathcal{Y}}_\pm = \mathcal{Y}_\pm \times_{\mathcal{M}^\delta_\pm} \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} \quad \text{and} \quad \tilde{\mathcal{I}}_\pm = \mathcal{I}_\pm \otimes_{\mathcal{O}_{\mathcal{Y}_\pm}} \mathcal{O}_{\tilde{\mathcal{Y}}_\pm},$$

where $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger} \to \mathcal{M}^\delta_\pm$ is the composite of Φ_δ^{-1} (cf. (6.9)) with the projection; we let $\mathcal{D}_\delta \subset \mathcal{X}_\delta$ be the total space of the distinguished (marked) divisor (of $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$). We have the short exact sequence

$$0 \longrightarrow \mathcal{I}_\delta \longrightarrow \tilde{\mathcal{I}}_+ \oplus \tilde{\mathcal{I}}_- \xrightarrow{(1,-1)} \tilde{\mathcal{I}}_0 \longrightarrow 0, \tag{6.11}$$

where $\tilde{\mathcal{I}}_0 := \mathcal{I}_\delta \otimes_{\mathcal{O}_{\mathcal{X}_\delta}} \mathcal{O}_{\mathcal{D}_\delta}$. Because of the admissible requirement, $\tilde{\mathcal{I}}_0$ is an ideal sheaf of $\mathcal{O}_{\mathcal{D}_\delta}$, and via the f in (6.8), we have isomorphism as ideal sheaves of $\mathcal{O}_{\mathcal{D}_\delta}$:

$$\tilde{\mathcal{I}}_0 \cong \mathcal{I}_{\mathcal{Z}_D} \otimes_{\mathcal{O}_{\mathrm{Hilb}^{\delta_0}_D}} \mathcal{O}_{\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}}, \tag{6.12}$$

where $\mathcal{Z}_D \subset D \times \mathrm{Hilb}^{\delta_0}_D$ is the universal family of $\mathrm{Hilb}^{\delta_0}_D$.

Let $\pi_\delta : \mathcal{X}_\delta \to \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$, let $\tilde{\pi}_\pm : \tilde{\mathcal{Y}}_\pm \to \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ and $\tilde{\pi}_0 : \mathcal{D}_\delta \to \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ be the corresponding projections. According to [MPT10, p.961], we have the following commutative diagram of derived objects

$$\begin{array}{ccccc}
L^\vee_{\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}/\mathfrak{C}_0^{\dagger,\delta}}[-1] & \longrightarrow & L^\vee_{\mathcal{M}^\delta_+/\mathfrak{A}^{\delta_+,\delta_0}_\diamond \times \mathcal{M}^\delta_-/\mathfrak{A}^{\delta_-,\delta_0}_\diamond}[-1] & \longrightarrow & L^\vee_{\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}/\mathcal{M}^\delta_+\times\mathcal{M}^\delta_-} \\
\downarrow & & \downarrow & & \downarrow \\
R\pi_{\delta*}R\mathcal{H}om(\mathcal{I}_\delta,\mathcal{I}_\delta)_0 & \longrightarrow & \bigoplus_{-,+} R\tilde{\pi}_{\pm*}R\mathcal{H}om(\tilde{\mathcal{I}}_\pm,\tilde{\mathcal{I}}_\pm)_0 & \longrightarrow & R\tilde{\pi}_{0*}R\mathcal{H}om(\tilde{\mathcal{I}}_0,\tilde{\mathcal{I}}_0)_0,
\end{array}$$

where the vertical arrows are the dual of the perfect obstruction theories, and the lower sequence is part of the distinguished triangle induced by (6.11).

We claim that, under the morphism f in (6.8),

$$R\tilde{\pi}_{0*}R\mathcal{H}om(\tilde{\mathcal{I}}_0,\tilde{\mathcal{I}}_0)_0^\vee \cong f^*L_\triangle, \quad L_\triangle := L_{\mathrm{Hilb}^{\delta_0}_D/\mathrm{Hilb}^{\delta_0}_D\times\mathrm{Hilb}^{\delta_0}_D}, \tag{6.13}$$

and via this isomorphism the last vertical arrow in the above diagram is identical to the canonical arrow

$$L^\vee_{\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}/\mathcal{M}^\delta_+\times\mathcal{M}^\delta_-} \longrightarrow f^*L^\vee_\triangle. \tag{6.14}$$

Indeed, since $\mathrm{Hilb}_D^{\delta_0}$ is smooth, and the conormal bundle of $\mathrm{Hilb}_D^{\delta_0}$ in $\mathrm{Hilb}_D^{\delta_0} \times \mathrm{Hilb}_D^{\delta_0}$ via the diagonal Δ is isomorphism to the cotangent sheaf $\Omega_{\mathrm{Hilb}_D^{\delta_0}}$, we have $L_\triangle \cong \Omega_{\mathrm{Hilb}_D^{\delta_0}}[1]$.

Next, we let $\pi_H : D \times \mathrm{Hilb}_D^{\delta_0} \to \mathrm{Hilb}_D^{\delta_0}$ be the projection. Then by the deformation of ideal sheaves of smooth surfaces, the derived objects

$$R\tilde{\pi}_{H*} R\mathcal{H}om(\tilde{\mathfrak{I}}_{\mathcal{Z}_D}, \tilde{\mathfrak{I}}_{\mathcal{Z}_D})_0^\vee \cong \Omega_{\mathrm{Hilb}_D^{\delta_0}}[1].$$

By the isomorphism (6.12), we have canonical isomorphism

$$R\tilde{\pi}_{0*} R\mathcal{H}om(\tilde{\mathfrak{I}}_0, \tilde{\mathfrak{I}}_0)_0 \cong f^* R\tilde{\pi}_{H*} R\mathcal{H}om(\tilde{\mathfrak{I}}_{\mathcal{Z}_D}, \tilde{\mathfrak{I}}_{\mathcal{Z}_D})_0.$$

Combined, we have (6.13), and that the last vertical arrow is identical to the (6.14).

Applying [BF97], we have

Proposition 6.5. *The perfect relative obstruction theories of* $\mathfrak{I}^\delta_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ *and of* (6.10) *are compatible with respect to the fiber diagram* (6.8) *(using* (6.9)*). Consequently, we have the identity*

$$[\mathfrak{I}^\delta_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}]^{\mathrm{vir}} = \triangle^!\big([\mathcal{M}_-^\delta]^{\mathrm{vir}} \times [\mathcal{M}_+^\delta]^{\mathrm{vir}}\big). \tag{6.15}$$

We state the cycle version of the degeneration of Donaldson-Thomas invariants.

Theorem 6.6. *Let X/C be a simple degeneration of projective threefolds such that $X_0 = Y_- \cup Y_+$ is a union of two smooth irreducible components. Let $[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} \in A_*\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$ be the virtual class of the good degeneration, and let $\triangle$ be the diagonal morphism in* (6.8)*. Then $i_c^![\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} = [\mathfrak{I}^P_{X_c}]^{\mathrm{vir}}$ for $c \neq 0 \in C$, and*

$$i_0^![\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} = \sum_{\delta \in \Lambda_P^{\mathrm{spl}}} \triangle^!\big([\mathcal{M}_-^\delta]^{\mathrm{vir}} \times [\mathcal{M}_+^\delta]^{\mathrm{vir}}\big). \tag{6.16}$$

Corollary 6.7. *Let the situation be as in Theorem 6.6. Suppose X_c are Calabi-Yau threefolds for $c \neq 0$. Then*

$$\deg\,[\mathfrak{I}^P_{X_c}]^{\mathrm{vir}} = \sum_{\delta \in \Lambda_P^{\mathrm{spl}}} \deg\big(\mathrm{ev}_{-*}[\mathcal{M}_-^\delta]^{\mathrm{vir}} \bullet \mathrm{ev}_{+*}[\mathcal{M}_+^\delta]^{\mathrm{vir}}\big), \tag{6.17}$$

where $\mathrm{ev}_\pm \colon \mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond} = \mathcal{M}^\delta_\pm \to \mathrm{Hilb}_D^{\delta_0}$ is the restriction morphism, and $\bullet$ is the intersection pairing in $A_*\mathrm{Hilb}_D^{\delta_0}$.

Proof. The Theorem follows from Propositions 6.4 and 6.5. The Corollary follows from the Theorem and that deg $i_c^![\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} = \deg i_0^![\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}}$. □

6.3. The degeneration formula

We prove Theorem 1.4 in the Introduction, whose formulation is due to [MNOP06].

Let the situation be as in Theorems 1.4 and 6.6. We define descendant invariants, following [MNOP06]. We continue to denote by

$$\pi : \mathcal{X} \longrightarrow \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}, \quad \pi_X : \mathcal{X} \longrightarrow X, \quad \text{and} \quad \mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_{\mathcal{X}}$$

be the universal family on $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$. Since locally $\mathcal{I}_{\mathcal{Z}}$ admits locally free resolutions of finite length, the Chern character

$$\mathrm{ch}(\mathcal{I}_{\mathcal{Z}}) : A_*\mathcal{X} \longrightarrow A_*\mathcal{X}$$

is well defined.

For any $\gamma \in H^l(X, \mathbb{Z})$, we define

$$\mathrm{ch}_{k+2}(\gamma) : H_*^{BM}\big(\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}, \mathbb{Q}\big) \longrightarrow H^{BM}_{*-2k+2-l}\big(\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}, \mathbb{Q}\big) \tag{6.18}$$

(H_i^{BM} is the Borel-Moore homology) via

$$\mathrm{ch}_{k+2}(\gamma)(\xi) = \pi_*(\mathrm{ch}_{k+2}(\mathcal{I}_{\mathcal{Z}}) \cdot \pi_X^*(\gamma) \cap \pi^*(\xi)),$$

where π^* is the flat pullback.

For cohomology classes $\gamma_i \in H^{l_i}(X, \mathbb{Z})$ of pure degree l_i, we define

$$\left\langle \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i) \right\rangle^P_{\mathfrak{X}} = \left[\prod_{i=1}^r (-1)^{k_i+1} \mathrm{ch}_{k_i+2}(\gamma_i) \cdot \big[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}\big]^{\mathrm{vir}} \right]_2 \in H_2^{BM}\big(\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}, \mathbb{Q}\big),$$

where the term inside the bracket is a homology class of dimension

$$2 \dim \big[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}\big]^{\mathrm{vir}} - \sum_{i=1}^r (2k_i - 2 + l_i),$$

and the $[\cdot]_2$ is taking the dimension two part of the term inside the bracket. This is the family version of the descendent Donaldson-Thomas invariants given in [MNOP06]:

$$\left\langle \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \right\rangle_{X_c}^{P} = \left[\prod_{i=1}^{r} (-1)^{k_i+1} \mathrm{ch}_{k_i+2}(\gamma_i) \cdot \left[\mathfrak{I}_{X_c}^{P}\right]^{\mathrm{vir}} \right]_0 \in H_0^{BM}\left(\mathfrak{I}_{X_c}^{P}, \mathbb{Q}\right).$$

Since P has degree one, we let $P(v) = d \cdot v + n$. We form the partition function of descendent Donaldson-Thomas invariants of X_c

$$Z_d\left(X_c; q \,\middle|\, \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \right) = \sum_{n \in \mathbb{Z}} \deg \left\langle \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \right\rangle_{X_c}^{d \cdot v + n} q^n.$$

Accordingly, for the relative Hilbert schemes $\mathfrak{I}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}^{\delta_\pm,\delta_0}$, we define $\mathrm{ch}_{k+2}(\gamma)$ similarly, and

$$\begin{aligned} \left\langle \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \right\rangle_{\mathfrak{Y}_\pm}^{\delta_\pm} &= \mathrm{ev}_{\pm *} \left(\prod_{i=1}^{r} (-1)^{k_i+1} \mathrm{ch}_{k_i+2}(\gamma_i) \cdot \left[\mathfrak{I}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}^{\delta_\pm,\delta_0}\right]^{\mathrm{vir}} \right) \\ &\in H_*(\mathrm{Hilb}_D^{\delta_0}, \mathbb{Q}). \end{aligned}$$

Let $\beta_1, \ldots, \beta_m$ be a basis of $H^*(D, \mathbb{Q})$. Let $\{C_\eta\}_{|\eta|=k}$ be a Nakajima basis of the cohomology of Hilb_D^k, where η is a cohomology weighted partition w.r.t. β_i. The relative DT-invariants with descendent insertions [MNOP06] are

$$\left\langle \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \,\middle|\, \eta \right\rangle_{\mathfrak{Y}_\pm}^{\delta_\pm} = \left[\prod_{i=1}^{r} (-1)^{k_i+1} \mathrm{ch}_{k_i+2}(\gamma_i) \cap \mathrm{ev}_\pm^*(C_\eta) \cdot \left[\mathfrak{I}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}^{\delta_\pm,\delta_0}\right]^{\mathrm{vir}} \right]_0,$$

which form a partition function

$$Z_{d_\pm,\eta}\left(Y_\pm, D_\pm; q \,\middle|\, \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \right) = \sum_{n \in \mathbb{Z}} \deg \left\langle \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \,\middle|\, \eta \right\rangle_{\mathfrak{Y}_\pm}^{d_\pm \cdot v + n} q^n.$$

Theorem 6.8 (Theorem 1.4). *Fix a basis $\beta_1, \ldots, \beta_m$ of $H^*(D, \mathbb{Q})$. Let γ_i be cohomology classes of X of pure degree l_i. The degeneration formula* *of Donaldson-Thomas invariants has the following form*

$$Z_d\left(X_c;q\,\middle|\,\prod_{i=1}^r\tilde\tau_0(i_c^*\gamma_i)\right)=\sum_{\substack{d_-,d_+;\,\eta\\ d=d_-+d_+}}\frac{(-1)^{|\eta|-l(\eta)}\mathfrak{z}(\eta)}{q^{|\eta|}}$$
$$\cdot Z_{d_-,\eta}\left(Y_-,D_-;q\,\middle|\,\prod_{i=1}^r\tilde\tau_0(i_-^*\gamma_i)\right)$$
$$\cdot Z_{d_+,\eta^\vee}\left(Y_+,D_+;q\,\middle|\,\prod_{i=1}^r\tilde\tau_0(i_+^*\gamma_i)\right)$$

where $i_c\colon X_c\to X$, $i_\pm\colon Y_\pm\to X$ *are the inclusions,* η *are cohomology weighted partitions w.r.t.* β_i, *and* $\mathfrak{z}(\eta)=\prod_i\eta_i|\mathrm{Aut}(\eta)|$.

Proof. Since Gysin maps commute with proper pushforward and flat pullback, we have

$$\deg i_c^!\left\langle\prod_{i=1}^r\tilde\tau_0(\gamma_i)\right\rangle_{\mathfrak{X}}^P=\deg i_0^!\left\langle\prod_{i=1}^r\tilde\tau_0(\gamma_i)\right\rangle_{\mathfrak{X}}^P.$$

By $i_c^![\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}}=[\mathfrak{I}^P_{X_c}]^{\mathrm{vir}}$, the left hand side term equals to $\deg\langle\prod_{i=1}^r\tilde\tau_0(i_c^*\gamma_i)\rangle^P_{X_c}$, which is the Donaldson-Thomas invariants of X_c.

For the other term, we will decompose it into relative invariants using (6.16). Since the universal family $\mathcal{Z}\subset\mathcal{X}$ has codimension at least 2, the cohomology class $-\mathrm{ch}_2(\mathfrak{I}_{\mathcal{Z}})$ is represented by the codimension 2 cycle $[\mathcal{Z}]$, which splits according to (6.11). Applying the operation $\prod_{i=1}^r(-\mathrm{ch}_2(\gamma_i))$ to both sides of (6.16), and using the restriction morphism $\mathrm{ev}_\pm\colon\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}\to\mathrm{Hilb}^{\delta_0}_D$, and

$$\left\langle\prod_{i=1}^r\tilde\tau_0(i_\pm^*\gamma_i)\right\rangle_{\mathfrak{Y}_\pm}^{\delta_\pm}=\mathrm{ev}_{\pm*}\left(\prod_{i=1}^r(-\mathrm{ch}_2(i_\pm^*\gamma_i))\cdot[\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}]^{\mathrm{vir}}\right)\in H_*(\mathrm{Hilb}^{\delta_0}_D,\mathbb{Q}),$$

we obtain

(6.19)
$$\deg i_0^!\left\langle\prod_{i=1}^r\tilde\tau_0(\gamma_i)\right\rangle_{\mathfrak{X}}^P=\sum_{\delta\in\Lambda_P^{\mathrm{spl}}}\deg\left(\left\langle\prod_{i=1}^r\tilde\tau_0(i_-^*\gamma_i)\right\rangle_{\mathfrak{Y}_-}^{\delta_-}\bullet\left\langle\prod_{i=1}^r\tilde\tau_0(i_+^*\gamma_i)\right\rangle_{\mathfrak{Y}_+}^{\delta_+}\right).$$

Let $\beta_1, \ldots, \beta_m$ be a basis of $H^*(D, \mathbb{Q})$, and let η be a cohomology weighted partition with respect to β_i. Following the notation in [MNOP06, Nak99], we denote

$$C_\eta = \frac{1}{\mathfrak{z}(\eta)} P_{\delta_1}[\eta_1] \cdots P_{\delta_s}[\eta_s] \cdot \mathbf{1} \in H^*(\mathrm{Hilb}_D^{|\eta|}, \mathbb{Q})$$

with $\mathfrak{z}(\eta) = \prod_i \eta_i |\mathrm{Aut}(\eta)|$. Then $\{C_\eta\}_{|\eta|=k}$ is the Nakajima basis of the cohomology of Hilb_D^k, and the Kunneth decomposition of the diagonal class $[\triangle] \in H^*(\mathrm{Hilb}_D^k \times \mathrm{Hilb}_D^k, \mathbb{Q})$ takes the form

$$[\triangle] = \sum_{|\eta|=k} (-1)^{k-l(\eta)} \mathfrak{z}(\eta) C_\eta \otimes C_{\eta^\vee}.$$

Since

$$\left\langle \prod_{i=1}^r \tilde{\tau}_0(i_\pm^* \gamma_i) \,\middle|\, \eta \right\rangle_{\mathfrak{Y}_\pm}^{\delta_\pm} = \left[\prod_{i=1}^r (-\mathrm{ch}_2(i_\pm^* \gamma_i)) \cap \mathrm{ev}_\pm^*(C_\eta) \cdot \left[\mathfrak{I}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}^{\delta_\pm, \delta_0}\right]^{\mathrm{vir}} \right]_0$$

is an element in $H_0^{BM}(\mathrm{Hilb}_D^{\delta_0}, \mathbb{Q})$, applying to (6.19), we have

$$\begin{aligned} \deg i_0^! \left\langle \prod_{i=1}^r \tilde{\tau}_0(\gamma_i) \right\rangle_{\mathfrak{X}}^P = \sum_{\delta \in \Lambda_P^{\mathrm{spl}}; |\eta|=\delta_0} & (-1)^{|\eta|-l(\eta)} \mathfrak{z}(\eta) \deg \left\langle \prod_{i=1}^r \tilde{\tau}_0(i_-^* \gamma_i) \,\middle|\, \eta \right\rangle_{\mathfrak{Y}_-}^{\delta_-} \\ & \cdot \deg \left\langle \prod_{i=1}^r \tilde{\tau}_0(i_+^* \gamma_i) \,\middle|\, \eta^\vee \right\rangle_{\mathfrak{Y}_+}^{\delta_+}. \end{aligned}$$

Finally, we form the partition functions of these invariants. Notice that $\delta_- + \delta_+ - \delta_0 = P$, which accounts for the shift of the power of q. This proves Theorem 6.8. □

6.4. Degeneration of stable pair invariants

We fix a simple degeneration $\pi : X \to C$ of projective threefolds with a π-ample H on X; we suppose that $X_0 = Y_- \cup Y_+$ is a union of two smooth irreducible components. For reference, we state the degeneration of PT-invariants, which is proved in [MPT10].

Recall that the coherent systems we considered are homomorphisms $\varphi : \mathcal{O}_X \longrightarrow \mathcal{F}$ so that $\mathcal{F}$ is pure of dimension one and φ has finite cokernel. Let P be a degree one polynomial. Let $\mathfrak{P}_{\mathfrak{X}/\mathfrak{C}}^P$ be the good degeneration of the moduli of coherent systems constructed in this paper. It is a separated and proper Deligne-Mumford stack of finite type over C. We use the relative obstruction theory of $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}^P$ introduced in [PT09] to construct its virtual class $[\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}}$.

Let $\pi : \mathcal{X} \to \mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$ and $\varphi : \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{F}$ be the universal family of $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$, and let $\mathcal{I}^\bullet \in D^b(\mathcal{X})$ be the object corresponds to the complex $[\mathcal{O}_{\mathcal{X}} \to \mathcal{F}]$ with $\mathcal{O}_{\mathcal{X}}$ in degree 0. We denote by $L_{\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}/\mathfrak{C}^P}$ be the truncated relative cotangent complex of $\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}} \to \mathfrak{C}^P$. In [MPT10, Prop 10], using the Atiyah classes a perfect relative obstruction theory is constructed:

$$E^\vee := R\pi_* R\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0[1]^\vee \longrightarrow L_{\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}/\mathfrak{C}^P}.$$

Let $[\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}]^{\mathrm{vir}} \in A_*\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}$ be its associated virtual cycle. In the same paper, for any partition $\delta = (\delta_\pm, \delta_0)$, a perfect relative obstruction theory of $\mathfrak{P}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond} \to \mathfrak{A}^{\delta_\pm,\delta_0}_\diamond$ is also constructed, which gives its virtual class $[\mathfrak{P}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}]^{\mathrm{vir}} \in A_*\mathfrak{P}^{\delta_\pm,\delta_0}_{\mathfrak{Y}_\pm/\mathfrak{A}_\diamond}$.

Let $c \in C$ and $\mathfrak{P}^P_{\mathfrak{X}_c/\mathfrak{C}_c} = \mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}} \times_C c$. Let

$$i^!_c : A_*\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}} \to A_*\mathfrak{P}^P_{\mathfrak{X}_c/\mathfrak{C}_c}$$

be the Gysin map. By Theorem 5.28, we can decompose $\mathfrak{P}^P_{\mathfrak{X}_0/\mathfrak{C}_0}$ as a union of $\mathfrak{P}^\delta_{\mathfrak{X}^\dagger_0/\mathfrak{C}^\dagger_0}$, $\delta \in \Lambda^{\mathrm{spl}}_P$, and obtain the isomorphism (5.42). By going through the argument parallel to the proof of degeneration formula for Hilbert schemes of ideal sheaves, Maulik, Pandharipande and Thomas proved in [MPT10] the degeneration formula of PT stable pair invariants.

Theorem 6.9 (Maulik, Pandharipande and Thomas). *Let X/C be a simple degeneration of projective threefolds such that $X_0 = Y_- \cup Y_+$ is a union of two smooth irreducible components. Then*

$$i^!_c\big[\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}\big]^{\mathrm{vir}} = \big[\mathfrak{P}^P_{X_c}\big]^{\mathrm{vir}} \in A_*\mathfrak{P}^P_{X_c} \quad \textit{for } c \neq 0 \in C,$$

and

$$i^!_0\big[\mathfrak{P}^P_{\mathfrak{X}/\mathfrak{C}}\big]^{\mathrm{vir}} = \sum_{\delta \in \Lambda^{\mathrm{spl}}_P} \triangle^!\Big(\big[\mathfrak{P}^{\delta_-,\delta_0}_{\mathfrak{Y}_-/\mathfrak{A}_\diamond}\big]^{\mathrm{vir}} \times \big[\mathfrak{P}^{\delta_+,\delta_0}_{\mathfrak{Y}_+/\mathfrak{A}_\diamond}\big]^{\mathrm{vir}}\Big),$$

where $\triangle : \mathrm{Hilb}^{\delta_0}_D \to \mathrm{Hilb}^{\delta_0}_D \times \mathrm{Hilb}^{\delta_0}_D$ is the diagonal morphism.

Appendix A. Proof of Lemmas 3.13 and 3.14

Proof of Lemma 3.13. First, because $M_I \otimes_A A_0 \to M_0$ is injective and its image lies in $(M_0)_{I_0}$, $M_I \otimes_A A_0 \to (M_0)_{I_0}$ is injective. We next show that it is surjective.

Since $M_I \otimes_A A_0 \to (M_0)_{I_0}$ is G_{m}-equivariant, it suffices to show that every weight ℓ element in $(M_0)_{I_0}$ can be lifted to a weight ℓ element in $M_I \otimes_A A_0$. Let $v \in (M_0)_{I_0}$ be a weight ℓ element. We first lift v to a weight ℓ element $\bar{v} \in R_0$; we write

$$\bar{v} = \alpha_0 + z_1\alpha_1 + \cdots + z_1^p\alpha_p, \quad \alpha_i \in A[z_2]^{\oplus m}.$$

Let

$$K = \ker\{\varphi : R \longrightarrow M\}, \quad K_0 = \ker\{\varphi \otimes_A A_0 : R_0 \longrightarrow M_0\}.$$

By the definition of $(M_0)_{I_0}$, there is a power z_1^k, $k > 0$, so that $z_1^k\bar{v} \in K_0$. Because M is $\mathbf{k}[t]$-flat, tensoring the exact sequence $0 \to K \to R \to M \to 0$ with A_0, we obtain an exact sequence $0 \to K \otimes_A A_0 \to R_0 \to M_0 \to 0$. Therefore,

$$K \otimes_A A_0 = K_0.$$

We let $w \in K$ be a lift of $z_1^k\bar{v} \in K_0$. We write w in the form

$$w = w_0 + tw_1 + \cdots + t^r w_r, \quad w_i \in R' := B[z_1, z_2]/(z_1z_2)^{\oplus m}.$$

Since M_0 only contains elements of non-negative weights, $\ell \geq 0$. Thus w has weight $\ell + ka$. Since $a > 0$, and since the weights of w_i are $\ell + ka - bi > ka$, we have $w_i = z_i^k w_i'$ for $w_i' \in P'$. For w_0', we can choose it to be $w_0' = \alpha_0 + \cdots z_1^p\alpha_p$. We let

$$w' = w_0' + tw_1' + \cdots + t^r w_r'.$$

Then $\varphi(w') \in M$ is a lift of $v \in (M_0)_{I_0}$.

We claim $\varphi(w') \in M$ is annihilated by z_1^k. This is true because $z_1^k \cdot \varphi(w') = \varphi(z_1^k w') = \varphi(w) = 0$, since $w \in K$. We show that $\varphi(w')$ is also annihilated by a power of z_2. We distinguish two cases. The first is when $\ell > 0$. In this case, the weight of w_i' are $\ell - ib \geq \ell > 0$, thus $z_1 | w_i'$. Hence $z_2\varphi(w') = \varphi(z_2 w') = 0$.

The other case is when $\ell = 0$. In this case, we still have $z_2 w_i' = 0$ for $i > 0$. We claim that for some $h > 0$, $z_2^h\varphi(w_0') = 0$. We pick a z_2^h so that

$z_2^h\bar{v} \in K_0$, (this is possible since $v \in (M_0)_{I_0}$), and then lift $z_2^h\bar{v}$ to a weight 0 element $\tilde{w} \in K$. We write

$$\tilde{w} = \tilde{w}_0 + t\tilde{w}_1 + \cdots + t^s\tilde{w}_s, \quad \tilde{w}_i \in R'.$$

Then $\tilde{w}_{i>0}$ has positive weight in R', thus are annihilated by z_2, and $z_2\tilde{w} = z_2\tilde{w}_0$. Therefore by replacing h by $h+1$, we can assume $\tilde{w}_{i>0} = 0$, and $\tilde{w}_0$ is expressed as an element in $B[z_2]^{\oplus m}$.

Since $\tilde{w}_0$ is a lift of $z_2^h\bar{v} = z_2^h\alpha_0$, and since both are expressed as elements in $A[z_2]^{\oplus m}$, we have $\tilde{w}_0 = z_2^h\alpha_0$. Therefore, since $\tilde{w} = \tilde{w}_0 \in K$,

$$z_2^h\varphi(w') = \varphi(z_2^h w') = \varphi(z_2^h w_0') = \varphi(z_2^h\alpha_0) = \varphi(\tilde{w}_0) = \varphi(\tilde{w}) = 0.$$

This proves that $\varphi(w')$ lies in M_I and is a lift of $v \in (M_0)_{I_0}$. This proves the lemma. $\square$

Propf of Lemma 3.14. Let $\beta \in C_{\text{gen}}$. Then there are $x \in K^-$, t^k and $z_1^h \in A$ such that

$$x = t^k z_1^h \beta \mod (t^{k+1}, z_1^{h+1}).$$

Since the modules involved are G_{m}-equivariant, we can assume that x has weight $ah + bk$. Thus after expressing x as

$$x = t^k z_1^h \beta + t^{k+1}\beta_1 + z_1^{h+1}\beta_2, \quad \beta_1, \beta_2 \in B[z_1, t],$$

and plus the weight consideration, we conclude $\beta_2 = t^{k+1}\beta_2'$ for a $\beta_2' \in B[z_1, t]^{\oplus m}$. Therefore, $z_1 x = t^k(z_1^{h+1}\beta + t\beta_3) \in K$, where $\beta_3 \in B[z_1, t]^{\oplus m}$. Since $K \subset R$, we conclude $z_1^{h+1}\beta + t\beta_3 \in K$. In particular, $z_1^{h+1}\beta \in K_0$ and $\beta \in (K_0^-)_{(z_1)} \cap R_0^-$. This proves $C_{\text{gen}} \subset C_0$.

For the other direction, we let $\gamma \in C_0$. For the same reason, for a positive h and a weight ah, $y \in K_0^-$, $y = z_1^h\gamma_1 + z_1^{h+1}\gamma_2$, $\gamma_i \in A[z_1]^{\oplus m}$. Since $z_1 z_2 = 0$ in B, $y \in K_0$. We let $\tilde{y} \in K$ be a weight ah lifting of y, expressed in the form

$$\tilde{y} = (z_1^h\gamma_1 + z_1^{h+1}\gamma_2) + tf_1 + \cdots + t^q f_q, \quad f_i \in R'.$$

Since $\tilde{y}$ has weight ah, we conclude that $f_i = z_1^{h+1} f_i'$, for some $f_i' \in B[z_1]^{\oplus m}$. Therefore, $\tilde{y} = z_1^h(\gamma + z_1\gamma_3)$, for a $\gamma_3 \in B[z_1, t]^{\oplus m}$, and hence

$$\gamma + z_1\gamma_3 \in (K_{(t)}^-)_{(z_1)} \cap R_{(t)}^-.$$

This implies that γ lies in (3.4), and thus lies in C_0. This proves the Lemma. $\square$

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