Communications in
Analysis and Geometry
Volume 23, Number 4, 717–796, 2015

Harmonic maps of conic surfaces with cone angles less than 2π

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We prove the existence and uniqueness of harmonic maps in degree one homotopy classes of closed, orientable surfaces of positive genus, where the target has non-positive Gauss curvature and conic points with cone angles less than 2π. For a homeomorphism w of such a surface, we prove existence and uniqueness of minimizers in the homotopy class of w relative to the inverse images of the cone points with cone angles less than or equal to π. The latter can be thought of as minimizing maps from *punctured* Riemann surfaces into conic surfaces. We discuss the regularity of these maps near the inverse images of the cone points in detail. For relative minimizers, we relate the gradient of the energy functional with the Hopf differential.

When the genus is zero, we prove the same relative minimization provided there are at least three cone points of cone angle less than or equal to π.

1. Introduction

The study of harmonic maps into singular spaces, initiated in [GS], has reached a refined state; beyond general existence and uniqueness theorems, there are regularity and compactness results in the presence of minimal regularity assumptions on the spaces involved, see, among many others, [KS], [Mese1], [DM], [W2], and [EF].

Particularly detailed results are available in the case of maps of surfaces. In [K], Kuwert studied degree one harmonic maps of closed Riemann surfaces into flat, conic surfaces with cone angles bigger than 2π. He showed that the minimizing maps can be obtained as limits of diffeomorphisms, and that the inverse image under a degree one harmonic map of each point in the singular set is the union of a finite number of vertical arcs of the Hopf differential. Away from this inverse image the map is diffeomorphism onto its image. In [Mese1], Mese proved the same when the target is a metric space with curvature bounded above, in particular when it is a conic surface with cone angles bigger than 2π. In this paper, we study cone angles less than 2π.

We will state our main theorems assuming for simplicity's sake that there is just one cone point. The energy functional is conformally invariant with respect to the domain metric (see §3.1), so we state all results in terms of conformal structures on the domain. First, we have

Theorem 1. *Let Σ be a closed surface of genus > 0, equipped with a conformal structure, $\mathfrak{c}$, and a Riemannian metric G, with a conic point p of cone angle less than 2π and non-positive Gauss curvature away from p. Let $\phi\colon \Sigma \longrightarrow \Sigma$ be a homeomorphism. Then there is a unique map $u\colon \Sigma \longrightarrow \Sigma$ which minimizes energy in the homotopy class of ϕ. This map satisfies*

$$u^{-1}(p) \text{ is a single point}$$

and $u\colon \Sigma - u^{-1}(p) \longrightarrow \Sigma - p$ is a diffeomorphism.

Second, we have a more refined result for surfaces with cone angles less than π, which can be thought of as a theorem about maps from *punctured* Riemann surfaces.

Theorem 2. *Let $\Sigma, \mathfrak{c}$, and G be as in the previous theorem. If the cone angle at p is less than or equal to π, then for each $q \in \Sigma$ and each homeomorphism ϕ of Σ with $\phi(q) = p$, there is a unique map $u\colon \Sigma \longrightarrow \Sigma$ with $u(q) = p$ which* *minimizes energy in the rel. q homotopy class of ϕ (see (2.31) for the definition of relative homotopy class). This map satisfies*

$$u^{-1}(p) = q$$

and $u\colon \Sigma - q \longrightarrow \Sigma - p$ is a diffeomorphism.

See Theorem 3 in Section 2.2 for a precise restatement of these results, including multiple cone points and the genus 0 case.

This problem is motivated by the role of harmonic maps in Teichmüller theory and recent work that extends classical uniformization results to the case of conic metrics on punctured surfaces. The uniformization theorems for cone metrics of McOwen and Troyanov, [Mc1], [Mc2], [Tro], recent work by Schumacher and Trapani, [ST], [ST2], and unpublished work of Mazzeo and Weiss have shown that where hyperbolic metrics are used in standard Teichmüller theory, hyperbolic cone metrics can be used to similar effect in the Teichmüller theory of punctured surfaces. In the unpunctured case, harmonic maps enter the picture in the works of [Tr], [W1], [W2], and [W3], in the creation of various functionals and in the important parametrization of Teichmüller space by holomorphic quadratic differentials. Thus, this paper lays the groundwork for the extension of these results to the punctured case if the uniformizing metrics are conic with cone angles less than or equal to π.

The proofs follow the method of continuity, and accordingly the paper is divided into a portion with a perturbation result and a portion with a convergence result. In both we make frequent use of the fact that minimizers as in both theorems solve a differential equation. Let $\mathcal{M}_{conic}(p,\alpha)$ denote the space of smooth metrics on $\Sigma - p$ with a cone point at p of cone angle $2\pi\alpha$ (see §2.1). The 'tension field' operator τ is a second order, quasi-linear, elliptic partial differential operator, arising as the Euler-Lagrange equation for the energy functional, which takes a triple (u, g, G) of maps and metrics to a vector field over u, denoted by $\tau(u, g, G)$. For $g \in \mathcal{M}_{conic}(q,\alpha)$, $G \in \mathcal{M}_{conic}(p,\alpha)$, in Section 5 we show that a diffeomorphism $u\colon (\Sigma - q, g) \longrightarrow (\Sigma - p, G)$ subject to certain conditions on the behavior near q minimizes energy in its rel. q homotopy class if $\tau(u, g, G) = 0$.

The perturbation result is an application of the Implicit Function Theorem to the tension field operator. Fix $g \in \mathcal{M}_{conic}(q,\alpha)$, $G \in \mathcal{M}_{conic}(p,\alpha)$ and a minimizer u as in Theorem 2. The proofs rest mainly in finding the right space of perturbations of u, call them $\mathcal{P}(u)$, and the right space of perturbations of g, call them $\mathcal{M}^*(g) \subset \mathcal{M}_{conic}(q,\alpha)$, so that τ acting on $\mathcal{P}(u) \times \mathcal{M}^*(g)$ has non-degenerate differential in the $\mathcal{P}(u)$ direction at (u, g).

There are two not-quite-correct points in this last sentence. First, to apply the Implicit Function Theorem, one needs to work with Banach and not Fréchet manifolds like $\mathcal{M}_{conic}(p, \alpha)$; for a precise definition of the spaces we use see Section 3. Second, the map τ actually takes values in a bundle, specifically the bundle $\mathsf{E} \longrightarrow \mathcal{P}(u) \times \mathcal{M}^*(g)$ whose fiber over $(\widetilde{u}, \widetilde{g})$ is (some Banach space of) sections of $\widetilde{u}^* T\Sigma$, so in fact we do not show that τ has surjective differential but rather that τ is transversal to the zero section of E.

If z denotes conformal coordinates near q, the linear operator L can be written $L = |z|^{2\alpha} \widetilde{L}$, where $\widetilde{L}$ falls into a broad class of linear operators known as elliptic b-differential operators, pioneered and elaborated by R. Melrose, and used subsequently in countless settings. For detailed properties of b-operators, see [Me] and [Me-Me]. The main difficulty we encounter is that L is degenerate on a natural space of perturbations. Following the example of previous authors, including [MP], we supplement the domain of τ with 'geometric' perturbations; in our case, we let it act on a space $\mathcal{C}$ of diffeomorphisms of the domain which look like conformal dilations, rotations, and translations near the cone point (§3).

As we will discuss in Section 6.2, the natural domains for L are weighted Banach spaces $r^c \mathcal{X}_b^{k,\gamma}$, which for the moment should be thought of as vector fields vanishing to order r^c near q, with some Hölder regularity naturally adapted to the geometry. In particular L acts from $r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}$ to $r^{1-\epsilon-2\alpha}\mathcal{X}_b^{0,\gamma}$, and is Fredholm for sufficiently small $\epsilon > 0$. Let $\mathcal{K}$ denote its cokernel (see (6.4)). The leading order behavior of a vector field in $\mathcal{K}$ near the inverse image of the cone point is characterized by the homogeneous solutions of a related 'indicial' equation, c.f. Lemma 6.8. A dichotomy between the behavior of elements in $\mathcal{K}$ near cone points of cone angle less than π and cone angle greater than π arises.

Lemma 1.1. *Let $\psi \in \mathcal{K}$, and suppose that G has only one cone point, p, with $u^{-1}(p) = q$. If the cone angle $2\pi\alpha$ is bigger than π, then near q*

$$\psi(z) = w + \frac{\overline{a}}{1-\alpha} |z|^{2(1-\alpha)} + O\left(|z|^{2(1-\alpha)+\delta}\right) \tag{1.1}$$

for some $w, a \in \mathbb{C}$ and $\delta > 0$. If the cone angle is less than π, then near q

$$\psi(z) = \mu z + O\left(|z|^{1+\delta}\right) \tag{1.2}$$

for some $\mu \in \mathbb{C}$ and $\delta > 0$. Here and below, $O(|z|^c)$ denotes a quantity bounded by $C|z|^c$ for some $C > 0$ and $|z|$ sufficiently small.

This lemma is central to the proof of the main theorems, since an accurate appraisal of the cokernel is needed to show that the geometric perturbations (the space $\mathcal{C}$ described above) are sufficient to give a surjective problem.

To prove energy minimization, both in relative and absolute homotopy classes, we use an argument from [CH], in which the authors prove uniqueness for harmonic maps of surfaces with genus bigger than 1. They show that the pullback of the target metric via a harmonic map can be written as a sum of two metrics, one conformal to the domain metric and the other with negative curvature, thereby decomposing the energy functional into a sum of two functionals which the harmonic map jointly minimizes. In Section 5, we adapt this argument to prove uniqueness in the conic setting.

For fixed domain and target structures $\mathfrak{c}$ and G, if the cone point of G has cone angle less than π, the harmonic maps in Theorem 2 from $(\Sigma, \mathfrak{c})$ to (Σ, G) are parametrized by points q and rel. q homotopy classes of diffeomorphisms taking q to p. Denote this space by $\mathcal{H}arm_{\mathfrak{c},G}$. Given any diffeomorphism ϕ of Σ, let $[\phi]_{rel.} = [\phi; \phi^{-1}(p)]$, and denote the corresponding element of $\mathcal{H}arm_{\mathfrak{c},G}$ by $u_{[\phi]_{rel.}}$. One of our main results is a formula for the gradient of the energy functional on $\mathcal{H}arm_{\mathfrak{c},G}$ in terms of the Hopf differential $\Phi(u_{[\phi]_{rel.}})$ of $u_{[\phi]_{rel.}}$. (See Section 5 for a definition of the Hopf differential.) It turns out that $\Phi(u_{[\phi]_{rel.}})$ is holomorphic on $\Sigma - q$ with at most a simple pole at q. Given a path u_t in $\mathcal{H}arm_{\mathfrak{c},G}$ with $u_0 = u_{[\phi]_{rel.}}$, and writing $J := \frac{d}{dt}\big|_{t=0} u_t$, the gradient is given by

$$\left.\frac{d}{dt}\right|_{t=0} E(u_t) = \Re 2\pi i \operatorname{Res}|_q \iota_J \Phi\left(u_{[\phi]_{rel.}}\right), \tag{1.3}$$

where ι is contraction. The one form $\iota_J\Phi(u_{[\phi]_{rel.}})$ is not holomorphic, but admits a residue nonetheless. By Theorem 1 there is a unique choice $[\overline{\phi}]_{rel.}$ such that the corresponding solution $\overline{u}$ is an absolute minimum of energy in the free homotopy class $[\overline{\phi}]$, and we use (1.3) to prove that $\overline{u}$ is the unique element in $\mathcal{H}arm_{\mathfrak{c},G}$ for which $\Phi(\overline{u})$ extends smoothly to all of Σ (§6). In other words, $\overline{u}$ is the unique critical point of $E\colon \mathcal{H}arm_{\mathfrak{c},G} \longrightarrow \mathbb{R}$. We refer to $\overline{u}$ as an *absolute minimizer* to distinguish it from the other *relative minimizers* in $\mathcal{H}arm_{\mathfrak{c},G}$.

As a corollary to (1.3), we prove the following formula for the Hessian of energy at $\overline{u}$. Given a path u_t through $u_0 = \overline{u}$ with derivative $J = \frac{d}{dt}\big|_{t=0} u_t$, we can define the Hopf differential of J by $\Phi(J) = \frac{d}{dt}\big|_{t=0} \Phi(u_t)$. We have

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t) = \Re 2\pi i \operatorname{Res}|_{\overline{q}} \iota_J \Phi(J) \tag{1.4}$$

We use (1.4) to prove the non-degeneracy of the linearized residue map near $\overline{u}$ (§6), and use this to show that in the cone angle less than π case one can perturb in the direction of absolute minimizers as the geometric data varies.

In the closedness portion (Section 7), we adapt standard methods for proving regularity of minimizing maps to the conic setting. First, we prove that the sup norm of the energy density of a minimizing map is controlled by the geometric data (see Proposition 7.4). This uses a standard application of the theorems of DiGiorgi-Nash-Moser and an extension of a Harnack inequality from [He]. It is noteworthy that if a conic metric is chosen on the domain, with cone point at the inverse image of the cone point of a the target via a minimizer from either of the main theorems, then the energy density is bounded from both above and below if and only if the cone angles are equal, and we work with such conformal metrics on the domain in what follows.

Control of the energy density near the (inverse image of) the cone points is insufficient, and to obtain stronger estimates we proceed by contradiction, employing a rescaling argument, and the elliptic regularity of b-differential operators, to produce a minimizing map of the standard round cone. We also classify such maps, and the map produced by rescaling is not among them; thus the desired bounds hold near the cone points (see Proposition 7.9).

Finally, we point out that the fact that the minimizers in Theorem 1 take only a single point to the cone point is consistent with the work of Hardt and Lin on maps into round cones in Euclidean space, see [HL].

This work was completed under the supervision of Rafe Mazzeo for my thesis at Stanford University, and I would like to thank him for many helpful discussions, ideas, and suggestions. I would also like to thank Andras Vasy, Leon Simon and Dean Baskin for their help and encouragement, and Chikako Mese for a helpful email discussion. I also thank the NSF for the financial support I received through Rafe Mazzeo's research grant.

2. Setup and an example

In this section, we give the precise statements of the theorems and outline the method of proof.

Let Σ be a closed surface, $\mathfrak{p} = \{p_1, \dots, p_k\} \subset \Sigma$ a collection of k distinct points, and set

$$\Sigma_{\mathfrak{p}} := \Sigma - \mathfrak{p}. \tag{2.1}$$

Given a smooth metric G on $\Sigma_{\mathfrak{p}}$, let $u\colon \Sigma \longrightarrow \Sigma$ be a continuous map so that $u^{-1}(p)$ is a single point for all $p \in \mathfrak{p}$. Let

$$\mathfrak{p}' = u^{-1}(\mathfrak{p})$$

and assume that

$$u\colon \Sigma_{\mathfrak{p}'} \longrightarrow \Sigma_{\mathfrak{p}} \tag{2.2}$$

is smooth. Let g be a smooth metric on $\Sigma_{\mathfrak{p}'}$. The differential, du, is a section of $T^*\Sigma_{\mathfrak{p}'} \otimes u^*T\Sigma_{\mathfrak{p}}$, which we endow with the metric $g^{-1} \otimes u^*G$. The **Dirichlet energy** of u for the metrics g and G is given by

$$E(u, g, G) = \frac{1}{2}\int_{\Sigma_{\mathfrak{p}'}} \|du\|^2_{g^{-1}\otimes u^*G}\, dVol_g = \frac{1}{2}\int_{\Sigma_{\mathfrak{p}'}} g^{ij} G_{\alpha\beta}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j} dVol_g. \tag{2.3}$$

Maps u in (2.2) which are critical points of the energy with respect to compactly supported perturbations are smooth [Si] and satisfy that the **tension field**

$$\tau(u, g, G)\colon = \operatorname{Tr} \nabla_{(u,g,G)} du = \Delta_g u^\gamma + {}^G\Gamma^\gamma_{\alpha\beta}\left\langle du^\alpha, du^\beta\right\rangle_g = 0 \tag{2.4}$$

where $\nabla_{(u,g,G)}$ is the Levi-Cevita connection on $T^*\Sigma_{\mathfrak{p}'} \otimes u^*T\Sigma_{\mathfrak{p}}$ with metric $g^{-1} \otimes u^*G$. Thus, for a C^∞ map u as in (2.2),

$$\tau(u, g, G) \in \Gamma\left(u^*T\Sigma_{\mathfrak{p}}\right), \tag{2.5}$$

where $\Gamma(B)$ denotes the space of smooth sections of a vector bundle $B \longrightarrow \Sigma$. That is to say, the tension field is a vector field over u, and it is minus the gradient of the energy functional in the following sense [EL].

Lemma 2.1 (First Variation of Energy). *Let (M, g) and $(N, \widetilde{g})$ be smooth Riemannian manifolds, possibly with boundary, and let $u\colon (M, g) \longrightarrow (N, \widetilde{g})$ be a C^2 map. If u_t is a variation of C^2 maps through $u_0 = u$ and $\frac{d}{dt}\big|_{t=0} u_t = \psi$, then*

$$\left.\frac{d}{dt}\right|_{t=0} E(u_t, g, \widetilde{g}) = -\int_M \langle \tau(u, g, \widetilde{g}), \psi\rangle_{u^*\widetilde{g}} dVol_g + \int_{\partial M} \langle u_*\partial_\nu, \psi\rangle_{u^*\widetilde{g}} ds, \tag{2.6}$$

where ∂_ν is the outward pointing normal to ∂M and ds is the area form.

Solutions to (2.4) are called harmonic maps.

Much of what follows can be explained by carefully studying the example of standard cones. For $\alpha < 1$, the standard cone of cone angle $2\pi\alpha$ is the sector in $\mathbb{R}^2$ of angle $2\pi\alpha$ with its boundary rays identified. Thus in polar coordinates $(\widetilde{r}, \widetilde{\theta})$ we can write this as a quotient $\mathbb{R}^+ \times [0, 2\pi\alpha]/\left((\widetilde{r}, 0) \sim (\widetilde{r}, 2\pi\alpha)\right)$, with metric $g_\alpha := d\widetilde{r}^2 + \widetilde{r}^2 d\widetilde{\theta}^2$. Let $\widetilde{z} = \widetilde{r}e^{i\widetilde{\theta}}$, and set

$$\widetilde{z} = \frac{1}{\alpha} z^\alpha \tag{2.7}$$

Then, in terms of z,

$$g_\alpha = |z|^{2(\alpha-1)} |dz|^2 . \tag{2.8}$$

We denote this space by

$$\begin{aligned} C_\alpha &:= (\mathbb{C}, g_\alpha) \\ C_\alpha^* &:= C_\alpha - \{0\} \end{aligned} \tag{2.9}$$

Let $D \subset \mathbb{C}$ be the standard disc of radius one, and let $D^* := D - \{0\}$. We will discuss the Dirichlet problem for harmonic maps from D to C_α. In this case (see (3.5)), the tension field operator for a smooth map $u\colon D^* \longrightarrow C_\alpha^*$ is

$$\tau(u) = u_{z\bar{z}} + \frac{\alpha - 1}{u} u_z u_{\bar{z}}.$$

Therefore the identity map $id(z) = z$ is harmonic. Given a map $\phi\colon \partial D \longrightarrow C_\alpha$ near to $id|_{\partial D}$, we would like to find a harmonic map

$$\begin{aligned} &u\colon D \longrightarrow C_\alpha \\ &\tau(u) = 0 \text{ on } D - u^{-1}(0) \\ &u|_{\partial D} = \phi \end{aligned} \tag{2.10}$$

Let $z = re^{i\theta}$. Initially, we consider τ acting on maps of the form

$$\begin{aligned} u(z) &= z + v(z) \\ v &\in r^{1+\epsilon} C_b^{2,\gamma}(C_\alpha(1)), \end{aligned} \tag{2.11}$$

where, given $v\colon D \longrightarrow \mathbb{C}$,

$$v \in r^c C_b^{2,\gamma}(D) \iff \begin{array}{l} r^{-c}v \text{ has uniformly bounded } C^{2,\gamma} \text{ norm} \\ \text{on balls of uniform size with respect to} \\ \text{the rescaled metric } g_\alpha/r^{2\alpha} = \frac{dr^2}{r^2} + d\theta^2. \end{array} \tag{2.12}$$

We will describe this space more precisely in Section 2.1, but we mention now that in particular $|v(z)| = O(r^{1+\epsilon})$. We refer to the space of u in (2.11) by $\mathcal{B}^{1+\epsilon}$. Let $\mathcal{B}_0^{1+\epsilon} \subset \mathcal{B}^{1+\epsilon}$ be those u with $u|_{\partial D} = id|_{\partial D}$, i.e. those u whose v in (2.11) satisfy $v|_{\partial D} \equiv 0$. Near id we can write $\mathcal{B}^{1+\epsilon}$ as the product of $\mathcal{B}_0^{1+\epsilon} \times C^{2,\gamma}(\mathbb{S}^1;\mathbb{C})$, the latter being the space of $C^{2,\gamma}$ Hölder continuous functions from the circle $\mathbb{S}^1$ into $\mathbb{C}$. To be precise, pick a smooth cutoff function χ with

$$\chi(z) \equiv 0 \text{ for } |z| \leq 1/2 \quad \text{and} \quad \chi(z) \equiv 1 \text{ for } |z| \geq 3/4. \tag{2.13}$$

An open neighborhood of $(id, 0)$ in $\mathcal{B}_0^{1+\epsilon} \times C^{2,\gamma}(\mathbb{S}^1;\mathbb{C})$ can be be identified with an open neighborhood of id in $\mathcal{B}^{1+\epsilon}$ via the map $(u, \phi) \mapsto u + \chi\phi$.

Consider τ acting on a neighborhood of id in $\mathcal{B}_0^{1+\epsilon} \times C^{2,\gamma}(\mathbb{S}^1;\mathbb{C})$. If the derivative of τ at id in the $\mathcal{B}_0^{1+\epsilon}$ were an isomorphism onto its image, then the zero set of τ near id would be a smooth graph over $C^{2,\gamma}(\mathbb{S}^1;\mathbb{C})$ by the Implicit Function Theorem, so for each boundary value sufficiently close to id we would have a unique solution close to id in form (2.11). *This turns out to be false.* The reason for this failure is that form (2.11) is insufficient to encompass the behavior of solutions. Consider the spaces

$$\mathcal{D} = \{M_\lambda(z) = \lambda z : \lambda \in \mathbb{C}\}$$
$$\mathcal{T} = \{T_w(z) = z - w : w \in \mathbb{C}\}$$

and define the two, 2−dimensional spaces using χ from (2.13),

$$\begin{aligned} \mathcal{D}_0 &= \left\{ \widetilde{M}_\lambda = (1-\chi)M_\lambda(z) + \chi \cdot id : |\lambda - 1| < \epsilon \right\} \\ \mathcal{T}_0 &= \left\{ \widetilde{T}_w = (1-\chi)T_w(z) + \chi \cdot id : |w| < \epsilon \right\}, \end{aligned} \tag{2.14}$$

where ϵ is chosen small enough that all these maps are diffeomorphisms of D. In words, $\mathcal{D}_0$ is a space of diffeomorphisms equal to id on ∂D which are conformal dilations and rotations near the cone point, and $\mathcal{T}_0$ is the same but for conformal translations. The dichotomy between cone angles less than π and between π and 2π now enters. It turns out that for $\epsilon > 0$ sufficiently small

$$\begin{aligned} &D_{id}\tau \colon T\left(\mathcal{B}_0^{1+\epsilon} \circ \mathcal{D}_0\right) \longrightarrow r^{-1+\epsilon} C_b^{0,\gamma}(D) \\ &\text{is an isomorphism if } 0 < 2\pi\alpha < \pi \end{aligned} \tag{2.15}$$

while

$$D_{id}\tau\colon T\left(\mathcal{B}_0^{1+\epsilon}\circ\mathcal{D}_0\circ\mathcal{T}_0\right)\longrightarrow r^{-1+\epsilon}C_b^{0,\gamma}(D) \tag{2.16}$$
is an isomorphism if $2\pi > 2\pi\alpha > \pi$.

In the $0 < 2\pi\alpha < \pi$ case, for boundary values ϕ near id in $C^{2,\gamma}$ we can find solutions of the form $u(z) = \lambda z + v(z)$ while in the latter the solutions are in the form $u(z) = \lambda(z-w) + v(z-w)$, i.e. we must move the inverse image of the cone point.

As we will see in Section 3.1, the precomposition of a harmonic map u with a conformal map is harmonic. Define $C_\alpha(1) = C_\alpha \cap \{z : |z| \leq 1\}$. By (2.8), $C_\alpha(1)$ is conformally equivalent to D, so for the moment we think of id not as a map from D to C_α but from $C_\alpha(1)$ to C_α, and applying the inverse of (2.7) to both spaces, think of id as the identity map on the wedge

$$W = \{r \leq 1, 0 \leq \theta \leq 2\pi\alpha\}\,. \tag{2.17}$$

(For simplicity, we have dropped the tildes from the notation.) We can think of a boundary map, ϕ near id as a map from the arc $\{r = 1, 0 \leq \theta \leq 2\pi\alpha\}$ into $\mathbb{C}$, satisfying the condition that

$$\phi(e^{2\pi\alpha i}) = e^{2\pi\alpha i}\phi(1). \tag{2.18}$$

We decompose ϕ in terms of the eigenfunctions of ∂_θ^2 which satisfy (2.18), i.e. we write $\phi(e^{i\theta}) = \sum_{j\in\mathbb{Z}} a_j e^{i(1+\frac{j}{\alpha})\theta}$. Assuming convergence, these are the values on the arc of the harmonic map

$$u(z) = \sum_{j\geq 0} a_j z^{1+\frac{j}{\alpha}} + \sum_{j>0} a_{-j}\overline{z}^{-1+\frac{j}{\alpha}}, \tag{2.19}$$

from the sector into $\mathbb{R}^2$. (In the flat metric $dr^2 + r^2 d\theta^2$ on the sector, the tension field of u is simply Δu, so by the decomposition $\Delta = 4\partial_z\partial_{\overline{z}}$, the sum of a conformal and an anti-conformal function is harmonic.)

Consider the coefficient a_{-1}. If $\alpha > 1/2$ then $-1 + 1/\alpha < 1$, so the term $a_{-1}\overline{z}^{-1+\frac{1}{\alpha}}$ is the leading order term in (2.19) at $z = 0$. The map $\overline{z}^{-1+\frac{1}{\alpha}}$ from the wedge W into $\mathbb{C}$ does not pass to a map of the cone C_α (it wraps around the wrong way), so u as in (2.19) gives a harmonic map of $C_\alpha(1)$ if and only if $a_{-1} = 0$. The Implicit Function Theorem together with (2.16) can be read as saying that, after identifying the wedge W with $C_\alpha(1)$ as in (2.7), composition with a conformal translation $z \mapsto z - z_0$ can be used to produce a harmonic map of $C_\alpha(1)$ with boundary value ϕ close to id.

When $\alpha < 1/2$, any power series as in (2.19) gives a map on the cone, so for any sufficiently regular boundary data near id, there is a harmonic map of D^* with that boundary data. To go deeper, as we show in Section 5, the map u is minimizing among all maps with its boundary values if and only if the residue of the Hopf differential of u is zero, and a simple computation shows that (in the case under consideration,)

$$\operatorname{Res}\Phi(u) = \overline{a_{-1}}\left(-1 + \frac{1}{\alpha}\right).$$

That is, *if $\alpha < 1/2$, a_{-1} is the obstruction to having a minimizer (w.r.t. the boundary values) on all of D, not just D^*.* Thus it is natural, in the case $\alpha < 1/2$, to ask if we can solve the augmented Dirichlet problem

$$\begin{aligned} &u\colon D \longrightarrow C_\alpha \\ &\tau(u) = 0 \text{ on } D - u^{-1}(0) \\ &a_{-1} = 0 \\ &u|_{\partial D} = \phi \end{aligned} \quad . \tag{2.20}$$

In fact we can. By (2.15) there is a graph of solutions to (2.10) over $\mathcal{T}_0 \times C^{2,\gamma}(\mathbb{S}^1;\mathbb{C})$. The solutions lying over $\mathcal{T}_0 \times \{id\}$ solve (2.10) with identity boundary value. Call this space $\mathcal{S}$. In Section 6, we show that the map from $\mathcal{S}$ to a_{-1} has non-degenerate differential. Since it is a map of two dimensional vector spaces, it is an isomorphism, and we get a smooth graph of solutions to (2.20) over $\mathcal{T}_0 \times C^{2,\gamma}(\mathbb{S}^1;\mathbb{C})$, see Figure 1.

2.1. Conic metrics and minimizing maps

Let

$$D(\sigma) = \{z : |z| \leq \sigma\} \subset \mathbb{C}. \tag{2.21}$$

The space $C_b^{k,\gamma}(D(\sigma))$ was defined above as the space of functions with $C^{k,\gamma}$ norm uniformly bounded on balls of uniform size with respect to the rescaled metric $g/(e^{2\mu}|z|^{2\alpha}) = \frac{1}{r^2}dr^2 + d\theta^2$, but here we give an alternative characterization that is easier to work with. Given $f\colon D(R) \longrightarrow \mathbb{C}$, let

$$\|f\|_{C_b^{0,\gamma}(D(R))} = \sup_{0<|z|\leq R} |f| + \sup_{0<|z|,|z'|\leq R} \frac{|f(z) - f(z')|}{|\theta - \theta'|^\gamma + \frac{|r-r'|^\gamma}{|r+r'|^\gamma}}. \tag{2.22}$$

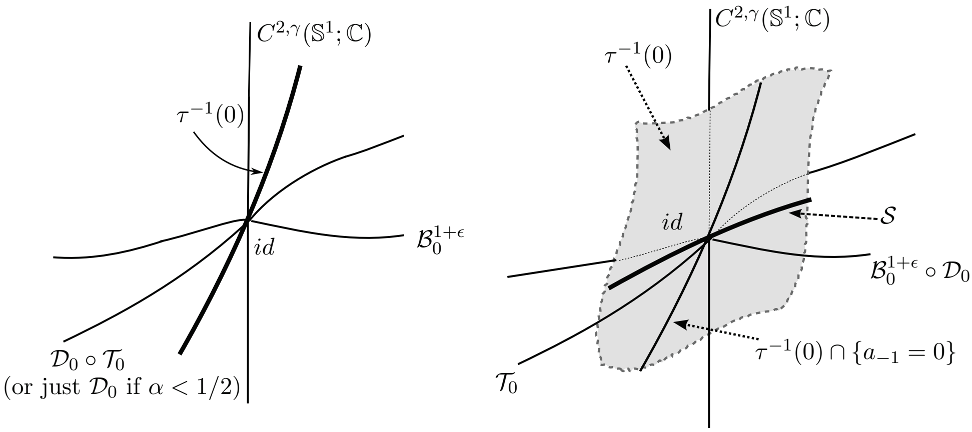

Figure 1: Solutions to the augmented equation (2.20) are found by applying the inverse function theorem to the map from $\mathcal{S}$ to a_{-1}.

(Here, as always, $z = re^{i\theta}$, $z' = r'e^{i\theta'}$.) For the higher regularity spaces, define

$$\|f\|_{C_b^{k,\gamma}(D(R))} = \sum_{i+j\leq k} \left\| (r\partial_r)^i \partial_\theta^j f \right\|_{C_b^{0,\gamma}(D(R))} \tag{2.23}$$

Finally, define the weighted Hölder spaces by

$$\begin{gathered} f \in r^c C_b^{k,\gamma}(D(R)) \iff r^{-c} f \in C_b^{k,\gamma}(D(R)) \\ \|f\|_{r^c C_b^{k,\gamma}(D(R))} = \left\| r^{-c} f \right\|_{C_b^{k,\gamma}(D(R))} \end{gathered} \tag{2.24}$$

The standard cone metric in (2.8) motivates our definition of a conic metric. Given $\alpha_j \in \mathbb{R}_+$ and $\nu_j > 0$, a Riemannian metric G on $\Sigma_{\mathfrak{p}}$ is said to have a conic singularity at $p_j \in \mathfrak{p}$ with cone angle $2\pi\alpha_j$ and type ν_j if there are conformal coordinates z centered at p_j such that

$$\begin{gathered} G = ce^{2\mu_j} |z|^{2(\alpha_j - 1)} |dz|^2 \\ \mu_j = \mu_j(z) \in r^{\nu_j} C_b^{k,\gamma}(D(\sigma)) \\ c > 0. \end{gathered} \tag{2.25}$$

The form (2.25) is invariant under conformal change of coordinates, so our notion of conic surface is well defined. For convenience, we single out those coordinates for which $c = 1$, and we refer to these as *normalized* conformal coordinates. Given $\mathfrak{a} = (\alpha_1, \dots, \alpha_k) \subset (0,1)^k$ and $\nu = (\nu_1, \dots, \nu_k) \in \mathbb{R}^k_+$, we define

$$\mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a}) = \left\{ \begin{array}{l} C^{k,\gamma}_{loc} \text{ metrics on } \Sigma_{\mathfrak{p}} \text{ with cone points } \mathfrak{p} \\ \text{and cone angles } 2\pi\mathfrak{a} \text{ such that in (2.25)} \\ \text{we have } \mu_j \in r^{\nu_j} C^{k,\gamma}_b(D(\sigma)) \text{ for some } \sigma. \end{array} \right\} \tag{2.26}$$

Let κ_G denote the Gauss curvature of $G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$, and near $p_j \in \mathfrak{p}$, write $G = \rho \left|dz\right|^2$ with $\rho = e^{2\mu} \left|z\right|^{2(\alpha-1)}$. Using $\kappa_G = -\frac{1}{\rho}\partial_z \partial_{\overline{z}} \log \rho^2$ and $z\partial_z = \frac{1}{2}\left(r\partial_r - i\partial_\theta\right)$, it follows that

$$k \geq 2 \text{ and } \nu \geq 2\alpha \implies \kappa_G \leq C < \infty, \tag{2.27}$$

and we will always make this assumption below.

We will work with a subspace of the metrics in (2.26).

Definition 2.2. A function $f \colon D(R) \longrightarrow \mathbb{C}$ is **polyhomogeneous** if f is smooth in the interior of D and admits an asymptotic expansion

$$f(r, \theta) \sim \sum_{(s,p) \in \mathcal{E}} r^s \log^p r a_{s,p}(\theta), \tag{2.28}$$

where $\mathcal{E} \subset \mathbb{R} \times \mathbb{N}$ is a discrete set for which each subset $\{s \leq c\} \cap \mathcal{E}$ is finite. In this setup, $\mathcal{E}$ is called the 'index set' of f, and the symbol $\sim$ means that

$$f(r, \theta) - \sum_{(s,p) \in \mathcal{E}, N > s} r^s \log^p r a_{s,p}(\theta) = o(r^N)$$

The metrics of interest are

$$\mathcal{M}^{phg}_{\nu}(\mathfrak{p}, \mathfrak{a}) = \left\{ \begin{array}{l} C^{\infty}_0 \text{ metrics on } \Sigma_{\mathfrak{p}} \text{ with cone points } \mathfrak{p} \\ \text{and cone angles } 2\pi\mathfrak{a} \text{ such that in (2.25)} \\ \text{we have } \mu_j \text{ is polyhomogeneous.} \end{array} \right\} \tag{2.29}$$

We will look for harmonic maps which have specified behavior near the inverse images of the cone points. For easy reference, we state this as

Form 2.3. *We say that $u \colon (\Sigma, g) \longrightarrow (\Sigma, G)$ is in Form 2.3 (with respect to g and G) if*

1) *u is a homeomorphism, and writing $\mathfrak{p}' = u^{-1}(\mathfrak{p})$, $u \colon \Sigma_{\mathfrak{p}'} \longrightarrow \Sigma_{\mathfrak{p}}$ is a $C^{2,\gamma}_{loc}$ diffeomorphism.*

2) For each $p \in \mathfrak{p}$, if z is a centered conformal coordinate around $u^{-1}(p)$ w.r.t. g and w is a centered conformal coordinate around p w.r.t. G, then u is given by

$$w(z) = \lambda z + v(z) \tag{2.30}$$

where $\lambda \in \mathbb{C}^*$ and $v \in r^{1+\epsilon} C_b^{2,\gamma}(D(R))$ for some sufficiently small $\epsilon > 0$.

In words, in normal coordinates near p and $u^{-1}(p)$, u is a dilation composed with a rotation to leading order.

Remark 2.4. Writing $\widetilde{z} = \lambda z$ in (2.30) reduces to the case $\lambda = 1$, but the variable $\widetilde{z}$ is no longer a normalized conformal coordinate for the domain metric g, i.e. the constant c in (2.25) is not equal to 1. If one is willing to scale g by a suitible constant factor, then $\widetilde{z}$ will be a normalized conformal coordinate. In what follows we prefer not to do this scaling and to preserve the λ (except notably in Lemma 6.7 below), because in Section 7, where we have a sequence of domain metrics, the fact that the corresponding sequences of λ's are uniformly bounded is quite subtle.

2.2. Restatement of theorems

Given two maps $u_i : \Sigma \longrightarrow \Sigma$, $i = 1, 2$, and a finite subset $\mathfrak{q} \subset \Sigma$, consider the standard relative homotopy relation

$$u_1 \sim_{\mathfrak{q}} u_2 \iff \begin{array}{l} u_1 \text{ is homotopic to } u_2 \text{ via} \\ F : [0,1] \times \Sigma \longrightarrow \Sigma \\ F_t(q) = u_1(q) \text{ for all } (t,q) \in [0,1] \times \mathfrak{q}. \end{array} \tag{2.31}$$

If $u_1 \sim_{\mathfrak{q}} u_2$, we say that the two maps are homotopic relative to $\mathfrak{q}$ (or simply rel. $\mathfrak{q}$). For fixed u_1, the set of all u_2 satisfying (2.31) is referred to as the rel. $\mathfrak{q}$ homotopy class of u_1, and is denoted by $[u_1; \mathfrak{q}]$. By $u_1 \sim u_2$ we mean that u_1 and u_2 are homotopic with no restrictions on the homotopy.

Define

$$\begin{aligned} \mathfrak{p}_{<\pi} &= \{p_i \in \mathfrak{p} \mid 2\pi\alpha_i < \pi\} \\ \mathfrak{p}_{>\pi} &= \{p_i \in \mathfrak{p} \mid 2\pi\alpha_i > \pi\} \\ \mathfrak{p}_{=\pi} &= \{p_i \in \mathfrak{p} \mid 2\pi\alpha_i = \pi\} \end{aligned} \tag{2.32}$$

and assume that

$$\mathfrak{p}_{=\pi} = \varnothing.$$

We discuss the case $\mathfrak{p}_{=\pi} \neq \varnothing$ in Section 8. Our main theorem is the following.

Theorem 3. *Assume that genus $\Sigma > 0$. For Σ and $\mathfrak{p}$ as above, let $G \in \mathcal{M}^{phg}_{\nu}(\mathfrak{p}, \mathfrak{a})$ have cone angles less than 2π, and $\kappa_G \leq 0$. Let $\mathfrak{c}$ be a conformal class on Σ. Fix a (possibly empty) subset $\mathfrak{q} \subset \mathfrak{p}_{<\pi}$, let $w_0 \colon \Sigma \longrightarrow \Sigma$ be a homeomorphism.*

If $\mathfrak{q} \neq \varnothing$, set $\mathfrak{q}' := w_0^{-1}(\mathfrak{q})$. Then there exists a unique energy minimizing map u in $[w_0; \mathfrak{q}']$. Furthermore,

$$u^{-1}(p_i) \text{ is a single point for all } i = 1, \ldots, k, \tag{2.33}$$

and

$$u \colon \Sigma - u^{-1}(\mathfrak{p}) \longrightarrow \Sigma - \mathfrak{p} \tag{2.34}$$

is a diffeomorphism.

If $\mathfrak{q} = \varnothing$, there exists an energy minimizing map u in $[w_0]$, unique up to precomposition with a conformal automorphism of $(\Sigma, \mathfrak{c})$. These also satisfy (2.33) and (2.34).

If $\Sigma = S^2$ then, assumptions as above, there is a unique energy minimizing map in the rel. $\mathfrak{q}$ homotopy class of w_0 provided $|\mathfrak{q}| \geq 3$. Again the map satisfies (2.33) *and* (2.34).

All of these minimizers are in Form 2.3 with respect to $\mathfrak{c}$ and G.

A simple argument using the isometry invariance of the energy allows us to reduce Theorem 3 to the case in which $\mathfrak{q}' = \mathfrak{q}$ and $w_0 \sim_{\mathfrak{q}} id$. Indeed, given a homeomorphism $w_0 \colon \Sigma \longrightarrow \Sigma$ with $w_0^{-1}(\mathfrak{q}) = \mathfrak{q}'$, let $\widetilde{w}_0$ be a diffeomorphism in $[w_0; \mathfrak{q}']$. (Such a diffeomorphism always exists. When $\mathfrak{q} = \varnothing$, this is the standard fact that every homotopy of surfaces is homotopic to a diffeomorphism [BT, chapter 17]. When $\mathfrak{q} \neq \varnothing$, if F_t, $t \in [0, 1]$ denotes a *free* homotopy between w_0 and a diffeomorphism, then composing with a homotopy through diffeomorphisms which takes $F_t(\mathfrak{q}')$ back to $\mathfrak{q}$ gives the desired relative homotopy.) If $\widetilde{u} \colon (\Sigma, (w_0^{-1})^* g) \longrightarrow (\Sigma, G)$ is the unique minimizer in $[id; \mathfrak{q}]$, then $u := \widetilde{u} \circ w_0^{-1} \colon (\Sigma, g) \longrightarrow (\Sigma, G)$ is the unique minimizer in $[w_0; \mathfrak{q}']$.

Our approach to proving the theorem is to find maps whose tension fields vanish away from $u^{-1}(\mathfrak{p})$. Among diffeomorphisms in Form 2.3 with vanishing tension field, minimizing energy in the sense of Theorem 3 turns out to be equivalent to a condition on the Hopf differential of u, which we discuss now. Given $u \colon \Sigma_{\mathfrak{p}'} \longrightarrow \Sigma_{\mathfrak{p}}$ with $\tau(u, g, G) = 0$ on $\Sigma_{\mathfrak{p}'}$, let $u^* G^{\circ}$

denote the g-trace-free part of u^*G. Among maps that are C^2 away from the cone points

$$\tau(u, g, G) = 0 \implies \delta_g \left(u^* G^\circ\right) = 0, \tag{2.35}$$

where δ_g is the divergence operator for the metric g acting on symmetric $(0,2)-$ tensors. Trace-free, divergence-free tensors are equal to the real parts of holomorphic quadratic differentials w.r.t. the conformal class $[g]$, so in conformal coordinates, we can write

$$u^* G = e\,|dz|^2 + 2\Re\left(\phi(z) dz^2\right) \tag{2.36}$$

where ϕ is holomorphic. Parting slightly with standard notation, e.g. from [W3], we use the symbol Φ to refer to the tensor which in conformal coordinates is expressed $\phi(z)dz^2$; this is called the **Hopf differential** of u. In Section 5 we will show that, writing $\mathfrak{p}'_{<\pi} = u^{-1}(\mathfrak{p}_{<\pi})$, then

$\Phi(u)$ is holomorphic on $\Sigma - \mathfrak{p}'_{<\pi}$ with at most simple poles on $\mathfrak{p}'_{<\pi}$.

Our proof of Theorem 3 then relies on the following lemma, proven in Section 5.

Lemma 2.5. *If genus $\Sigma > 0$, given a harmonic diffeomorphism $u\colon (\Sigma_{\mathfrak{p}'}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, G)$ (not necessarily in Form 2.3,) for any $\mathfrak{q} \subset \mathfrak{p}$, write $\mathfrak{q}' = u^{-1}(\mathfrak{q})$. Then u is energy minimizing in its rel. $\mathfrak{q}'$ homotopy class if and only if the Hopf differential $\Phi(u)$ extends smoothly to all of $\Sigma_{\mathfrak{p}-\mathfrak{q}'}$.*

If $\Sigma = S^2$, the same is true so long as $|\mathfrak{q}| \geq 3$.

Remark 2.6. If $\mathfrak{q} = \varnothing$, this lemma means that u is minimizing in its free homotopy class if and only if $\Phi(u)$ is smooth on all of Σ. We say that such maps are **absolute** minimizers.

Thus a diffeomorphism with vanishing tension field on $\Sigma_{\mathfrak{p}'}$ is minimizing in the sense of Theorem 3 if and only if the residues of its Hopf differential vanish on $\mathfrak{p}'_{<\pi} - \mathfrak{q}$, i.e. when it solves the augmented equation

$$\text{(HME}(\mathfrak{q})) \qquad \begin{aligned} u\colon \Sigma \longrightarrow \Sigma & \quad \text{a homeomorphism} \\ u \sim_{\mathfrak{q}} id & \\ \tau(u, g, G) = 0 & \quad \text{on } \Sigma_{\mathfrak{p}'} \\ \operatorname{Res}|_p \Phi(u) = 0 & \quad \text{for each } p \in \mathfrak{p}'_{<\pi} - \mathfrak{q}. \end{aligned}$$

Definition 2.7. When the subset $\mathfrak{q} \subset \mathfrak{p}$ in Theorem 3 is nonempty, we will call the corresponding minimizing map u a **relative** minimizer. The unique map u' which minimizes energy in the free homotopy class (which exists by taking $\mathfrak{q} = \varnothing$ in Theorem 3) will be called an **absolute** minimizer.

3. The harmonic map operator

In this section we discuss the global analysis of the map τ. We begin by discussing the invariance under conformal change of the domain metric.

3.1. Conformal invariance

Let $g, G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$, and suppose that u is a C^2 map of $\Sigma_{\mathfrak{p}}$. Suppose we have conformal expressions $G = \rho\,|du|^2$ at some $x \in \Sigma_{\mathfrak{p}}$ and $g = \sigma\,|dz|^2$ and near $u^{-1}(q)$. The energy density (the integrand in (2.3)) in conformal coordinates is

$$e(u, g, G)(z) := \frac{1}{2}\,\|du(z)\|^2_{g\otimes u^*G} = \frac{\rho(u(z))}{\sigma(z)}\left(|\partial_z u|^2 + |\partial_{\overline{z}} u|^2\right) \tag{3.1}$$

From this expression it is easy to verify that if

$$\begin{array}{ccccc} (\Sigma_1, g) & \xrightarrow{C} & (\Sigma_1, g) & \xrightarrow{u} & (\Sigma_2, G) \\ z & \longmapsto & w = C(z) & \longmapsto & u(w) \end{array} \tag{3.2}$$

for arbitrary surfaces Σ_i, $i = 1, 2$ and if C is conformal, then

$$e(u \circ C, g, G)(z) = |\partial_z C(z)|^2\, e(u, C^* g, G)(z). \tag{3.3}$$

It follows that

$$E(u \circ C, g, G) = E(u, C^* g, G) = E(u, g, G). \tag{3.4}$$

Now consider the tension field in conformal coordinates [SY]

$$\tau(u, g, G) = \frac{4}{\sigma}\left(u_{z\overline{z}} + \frac{\partial \log \rho}{\partial u} u_z u_{\overline{z}}\right) \tag{3.5}$$

The tension field enjoys a point-wise conformal invariance; in the situation of (3.2), if at a point z_0 we have $\partial_{\overline{z}} C(z_0) = 0$, then

$$\tau(u \circ C, g, G) = \tau(u, C^* g, G) \circ C \tag{3.6}$$

3.2. Global analysis of τ

Given a diffeomorphism $\widetilde{u}_0\colon (\Sigma_{\mathfrak{p}}, g_0) \longrightarrow (\Sigma_{\mathfrak{p}'}, \widetilde{G})$ solving HME($\mathfrak{q}$), we will now discuss spaces of maps and metrics near the triple $\widetilde{u}_0, g_0, \widetilde{G}$. To avoid tedious repetition below, we give a name to our main assumption on the metrics and maps.

Assumption 3.1. *1) $G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$ with $\alpha_j < 1$ for each $2\pi\alpha_j \in \mathfrak{a}$, and $\nu_j > 2\alpha_j$ (see Section 2.1)*

2) g_0 is also in $\mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$.

Note that the identity map $id\colon (\Sigma_{\mathfrak{p}}, g_0) \longrightarrow (\Sigma_{\mathfrak{p}}, \widetilde{u}_0^*\widetilde{G})$ solves HME($\mathfrak{q}$) and is still in Form 3.1. We focus our attention on such a triple (id, g_0, G_0).

Thinking informally for a moment, note that, by (2.5), $\tau(u, g, G)$ lies in a vector space that depends on u. Letting u vary, τ is most naturally viewed as a section of a vector bundle. We will now define precisely the domain of τ and the vector bundle E in which it takes values.

Fix coordinates z_j near each $p_j \in \mathfrak{p}$, conformal with respect to g_0, and let $z_j = r_j e^{i\theta_j}$. Let w_j be conformal coordinates for G near p_j. (We will often omit j from the notation when it is understood that we work with a fixed cone point.) Given any u in Form 2.3 with respect to g_0 and G_0, we define the Banach space $r^c\mathcal{X}_b^{k,\gamma}(u)$ for any $c \in \mathbb{R}^k$ by

$$(3.7) \qquad \psi \in r^c\mathcal{X}_b^{k,\gamma}(u) \iff \begin{cases} \psi \in \Gamma(u^*T\Sigma_{\mathfrak{p}}) \\ \psi \in C^{k,\gamma}_{loc} \text{ away from } u^{-1}(\mathfrak{p}) \\ \psi \in r^{c_j}C_b^{k,\gamma}(D(\sigma)) \text{ near } p_j \in \mathfrak{p}, \end{cases}$$

for some $\sigma > 0$.

Remark 3.2. We will often write $r^{1+\epsilon}\mathcal{X}_b^{k,\gamma}(u)$ for a positive number ϵ, by which we mean $r^c\mathcal{X}_b^{k,\gamma}(u)$ where $c_j = 1 + \epsilon$ for all j. Given $\delta \in \mathbb{R}$, by $c > \delta$ we mean that $c_j > \delta$ for all j.

- Let $\mathcal{B}^{1+\epsilon}(u_0)$ be the space of perturbations of u_0 defined by

$$(3.8) \qquad \mathcal{B}^{1+\epsilon}(u_0) = \left\{ \exp_{u_0} \psi \mid \psi \in r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}(u_0), \|\psi\|_{r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}} < \delta \right\}.$$

  The $\delta > 0$ is a fixed number, small enough that all the maps in $\mathcal{B}^{1+\epsilon}(u_0)$ are diffeomorphisms of $\Sigma_{\mathfrak{p}}$. Note that

$$(3.9) \qquad T_{u_0}\mathcal{B}^{1+\epsilon}(u_0) = r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}(u_0)$$

- We also need automorphisms of Σ, analogous to those in (2.14), that are locally conformal near $\mathfrak{p}$ with respect to g. Scaling the domain metric g_0 by a constant, we may assume that all the conformal coordinates z_j are valid on the unit disc. Let χ be a cutoff function with $\chi(z) \equiv 1$ for $|z| < 1$ and $\chi(z) \equiv 0$ for $|z| > 3/4$. Writing $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^{|\mathfrak{p}|}$, there is an $\epsilon > 0$ so that if $|\theta_j - 1| < \epsilon$ for all j, then the map defined locally by

$$M_\lambda(z_j) = \chi(z_j)\lambda_j z_j + (1 - \chi(z_j))z_j \tag{3.10}$$

is a local diffeomorphism and coincides with the identity when $|z| > 3/4$. Extending M_λ by the identity, we have a $|\mathfrak{p}|$ complex dimensional space,

$$\mathcal{D} := \{M_\lambda : |\lambda_j - 1| < \epsilon \text{ for all } j\} \tag{3.11}$$

satisfying (3.10) near p_j. Similarly, given $\zeta = (\zeta_1, \ldots, \zeta_{\mathfrak{p}}) \in \mathbb{C}^{|\mathfrak{p}|}$, there is an ϵ so that if $|\zeta_j| < \epsilon$ for all j, the maps defined locally by

$$T_\zeta(z_j) = \chi(z_j)(z_j - \zeta_j) + (1 - \chi(z_j))z_j \tag{3.12}$$

is a local diffeomorphism and coincides with *id* outside, $|\zeta_j| > 3/4$. Extending T_ζ by the identity, we have a space

$$\mathcal{T} := \{T_\zeta : |\zeta_j| < \epsilon \text{ for all } j\} \tag{3.13}$$

satisfying (3.12) near p_j. For any subset $\widetilde{\mathfrak{p}} \subset \mathfrak{p}$, define $\mathcal{T}_{\widetilde{\mathfrak{p}}} \subset$ by the condition that that $\zeta_j = 0$ for $p_i \notin \widetilde{\mathfrak{p}}$. This can be done in such a way that

$$\mathcal{T} = \mathcal{T}_{>\pi} \circ \mathcal{T}_{<\pi} \circ \mathcal{T}_{=\pi}. \tag{3.14}$$

where these are, respectively, spaces of local conformal translations near $\mathfrak{p}_{<\pi}$, $\mathfrak{p}_{=\pi}$, and $\mathfrak{p}_{>\pi}$. Finally, set

$$\mathcal{C} := \mathcal{D} \circ \mathcal{T} \tag{3.15}$$

- Finally, given $h_0 \in \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p},\mathfrak{a})$, define a subspace

$$\mathcal{M}^*_{k,\gamma,\nu}(h_0,\mathfrak{p},\mathfrak{a}) \subset \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p},\mathfrak{a})$$

as follows. Let $D_j = \{z_j \leq 1\}$ where now z_j are conformal coordinates for h_0 near p_j.

$$\mathcal{M}^*_{k,\gamma,\nu}(h_0,\mathfrak{p},\mathfrak{a}) = \left\{ h \in \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p},\mathfrak{a}) \;\middle|\; \begin{array}{l} id|_{D_j} \colon (D_j, h_0) \longrightarrow (\Sigma, h) \\ \text{is conformal for all } j \end{array} \right\} \tag{3.16}$$

In words, this means the conformal coordinates for h_0 near $\mathfrak{p}$ **are** conformal coordinates for h. This definition may seem arbitrary, but as we will see in Section 9, it is motivated by the requirement that τ be a continuous map.

To clarify the relationship between $\mathcal{M}^*_{k,\gamma,\nu}(h_0,\mathfrak{p},\mathfrak{a})$ and $\mathcal{M}_{k,\gamma,\nu}(\mathfrak{p},\mathfrak{a})$, we can construct, locally near h_0, a smooth injection from an open ball $\mathcal{U}$ in $\mathcal{M}_{k,\gamma,\nu}(\mathfrak{p},\mathfrak{a})$ into $\mathrm{Diff}_0(\Sigma;\mathfrak{p}) \times \mathcal{M}^*_{k,\gamma,\nu}(h_0,\mathfrak{p},\mathfrak{a})$ by uniformizing locally around $\mathfrak{p}$. To be precise, given any $h \in \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p},\mathfrak{a})$, let $v_h \colon D_j \longrightarrow (D_j, h)$ be the solution to the Riemann mapping problem normalized by the condition that $v_h(0) = 0$ and $v_h(1) = 1$. Let $\chi \colon D_j \longrightarrow \mathbb{R}$ be a smooth cutoff function with $\chi \equiv 1$ near 0 and $\chi \equiv 0$ near ∂D_j. Consider the map

$$\widetilde{v}_h(z) = \begin{cases} \chi(z_j) v_h(z_j) + (1 - \chi(z_j))\, z_j & \text{on } D_j \\ id & \text{elsewhere} \end{cases}$$

Then $\widetilde{v_h}$ is well-defined, and is a diffeomorphism if $\|v_h - id\|_{C^\infty(D)} < \epsilon$. We have

Lemma 3.3. *The map — defined locally near h_0 — given by*

$$\begin{aligned} \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p},\mathfrak{a}) &\longrightarrow \mathrm{Diff}_0(\Sigma;\mathfrak{p}) \times \mathcal{M}^*_{k,\gamma,\nu}(h_0,\mathfrak{p},\mathfrak{a}) \\ h &\longmapsto (\widetilde{v}_h, \widetilde{v}_h^* h) \end{aligned}$$

is an isomorphism onto its image on a ball near h_0.

Proof. This follows from the fact that the solution to the Riemann mapping problem has C^∞ norm controlled by the distance from h to h_0 [TII]. □

Note that, by construction, $id\colon (\Sigma, h_0) \longrightarrow (\Sigma, \widetilde{v}_h^* h)$ is conformal near $\mathfrak{p}$. With these definitions, we consider

$$\begin{aligned}\tau\colon \big(\mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{C}\big) \times \mathcal{M}^*_{2,\gamma,\nu}(g_0, u^{-1}(\mathfrak{p}), \mathfrak{a}) \times \mathcal{M}^*_{2,\gamma,\nu}(G_0, \mathfrak{p}, \mathfrak{a}) &\longrightarrow \mathsf{E} \\ (u, g, G) &\longmapsto \tau(u, g, G),\end{aligned} \tag{3.17}$$

where $\pi\colon \mathsf{E} \longrightarrow \big(\mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{C}\big) \times \mathcal{M}^*_{2,\gamma,\nu}(g_0, u^{-1}(\mathfrak{p}), \mathfrak{a}) \times \mathcal{M}^*_{2,\gamma,\nu}(G_0, \mathfrak{p}, \mathfrak{a})$ is the bundle whose fibers are

$$\pi^{-1}(u \circ C, g, G) = r^{1+\epsilon-2\mathfrak{a}} \mathcal{X}_b^{0,\gamma}(u) \tag{3.18}$$

In Section 9, we will prove

Proposition 3.4. *Let (u_0, g_0, G_0) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. Then the map (3.17) is C^1.*

3.3. Proof of Theorem 3

Let $\mathfrak{c}$ be any conformal structure on Σ and $G \in \mathcal{M}_\nu^{phg}(\mathfrak{p}, \mathfrak{a})$ any metric satisfying the hypotheses of Theorem 3. Given $\mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q} \neq \varnothing$ (resp. $\mathfrak{q} = \varnothing$), we would like to find a map $u\colon (\Sigma, \mathfrak{c}) \longrightarrow (\Sigma, G)$ that minimizes energy in the rel. $\mathfrak{q}$ homotopy class of the identity (resp. the free homotopy class of the identity.) We will do so by varying the conformal structure of the domain, as follows. Let $\mathfrak{c}_0 := [G]$ be the conformal class of G, and let $\mathfrak{c}_t$, $t \in [0, 1]$ be a smooth path of conformal structures from $\mathfrak{c}_0$ to $\mathfrak{c}_1 = \mathfrak{c}$. (That the space of conformal structures is connected follows immediately from the convexity of the space of metrics.) Finally, define

$$\mathcal{H}(\mathfrak{q}) = \left\{ t \in [0,1] \;\middle|\; \begin{array}{l} \text{there is a map } u_t\colon (\Sigma, \mathfrak{c}_t) \longrightarrow (\Sigma, G) \\ \text{so that } (u_t, \mathfrak{c}_t, G) \text{ satisfies (HME}(\mathfrak{q})). \end{array} \right\} \tag{3.19}$$

The following is a consequence of Propositions 4.4 and 7.2

Proposition 3.5. *If genus $\Sigma > 0$, $\mathcal{H}(\mathfrak{q})$ is closed, open, and non-empty. If $\Sigma = S^2$, then the same is true provided $|\mathfrak{q}| \geq 3$.*

Remark 3.6. $\mathcal{H}(\mathfrak{q})$ is non-empty since $\mathfrak{c}_0 = [G]$, so $id\colon (\Sigma, \mathfrak{c}_0) \longrightarrow (\Sigma, G)$ is conformal and thus a global energy minimizers in its homotopy class, [EL].

We can now prove Theorem 3 modulo the proofs of Propositions 3.5 and 5.2.

Proof of Theorem 3. The content of Proposition 5.2 is that solutions to (HME($\mathfrak{q}$)) are minimizing in their rel. $\mathfrak{q}$ (or free if $\mathfrak{q} = \varnothing$) homotopy classes, and that such minimizers are unique in the appropriate sense. Thus it suffices to solve Equation (HME($\mathfrak{q}$)), but Proposition 3.5 implies that a solution always exists. □

4. Openness via non-degeneracy

Our proof that $\mathcal{H}(\mathfrak{q})$ in (3.19) is open relies on Propositions 4.1 and 4.3. We state these now and use them to prove openness.

Let (u_0, g, G) solve (HME($\mathfrak{q}$)) with u_0 in Form 2.3. Let u_t be a C^1 path in $\mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{D} \circ \mathcal{T}_{>\pi}$ and write $\psi := \dot{u}_0$. Define

$$L\psi := \frac{d}{dt}\bigg|_{t=0} \tau(u_t, g, G). \tag{4.1}$$

Assume, as discussed at the beginning of Section 3.2, that $u_0 = id$. By (3.9), the domain of L is $T\left(\mathcal{B}^{1+\epsilon} \circ \mathcal{D} \circ \mathcal{T}_{>\pi}\right) = r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} + T_{id}\mathcal{D} + T_{id}\mathcal{T}_{>\pi}$. We will see that

$$L \colon r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} + T_{id}\mathcal{D} + T_{id}\mathcal{T}_{>\pi} \longrightarrow r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} \tag{4.2}$$

is bounded

To state our non-degeneracy result, we recall that a conformal Killing field for g is a vector field C satisfying $\mathcal{L}_C g = \mu g$ for some function μ, where $\mathcal{L}$ denotes the Lie derivative. This is the derivative of the conformal map equation $F_t^* g = e^{\mu_t} g$ for some family F_t with $F_0 = id$. It is well known that for surfaces the conformal killing fields are exactly the tangent space to the identity component of the conformal group,

$$\mathcal{C}onf_0 = \left\{C \colon (\Sigma, g) \longrightarrow (\Sigma, g) : C^* g = e^{2\mu} g\right\} \tag{4.3}$$

This space contains only the identity map if genus $\Sigma > 1$ and is two or three dimensional if the genus is 1 or 0, respectively. Our first non-degeneracy result is

Proposition 4.1. *Notation as above, if genus $\Sigma \geq 2$ then* (4.2) *is an isomorphism. If genus $\Sigma = 1$ then the kernel* (4.2) *is the space of conformal*

Killing fields and

$$L\left(T\left(\mathcal{B}^{1+\epsilon}\circ\mathcal{D}\circ\mathcal{T}_{>\pi}\right)\right)\oplus\left(T\mathcal{C}on f_0\cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}\right)=r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}. \tag{4.4}$$

If genus $\Sigma=0$ *and* $|\mathfrak{p}_{<\pi}|\geq 3$ *then* (4.2) *is an isomorphism.*

Proposition 4.1 is proven in Section 6.5 below.

To state the second non-degeneracy result, Proposition 4.3, we begin by describing the space of harmonic maps near a given solution to (HME($\mathfrak{q}$)) with fixed geometric data, see (4.10). Proposition 4.1 says that the map (4.2) is *almost* surjective. Though it is not in general surjective, in the cases of interest here, the relevant conclusion of the Implicit Function Theorem is true.

Lemma 4.2. *Let* (u_0,g_0,G) *solve* (HME($\mathfrak{q}$)) *and satisfy Assumption 3.1. If genus* $\Sigma=0$*, assume that* $|\mathfrak{p}_{<\pi}|\geq 3$*. Then there is an open neighborhood* N *of* (u_0,g_0) *in* $\left(\mathcal{B}^{1+\epsilon}(u_0)\circ\mathcal{C}\right)\times\mathcal{M}^*_{2,\gamma,\nu}(g_0,u^{-1}(\mathfrak{p}),\mathfrak{a})$ *such that*

$$\tau^{-1}(0)\cap N \text{ is a smooth Banach manifold.}$$

Excluding the case of genus $\Sigma=1$ *and* $\mathfrak{p}_{<\pi}=\varnothing$*, there is an open set* $\mathcal{U}\subset\mathcal{T}_{<\pi}\times\mathcal{M}^*_{2,\gamma}(g_0,\mathfrak{p},\mathfrak{a})$ *and a map graphing zeros of the tension field* τ*,*

$$\begin{aligned}\mathcal{S}\colon\mathcal{U}&\longrightarrow\mathcal{B}^{1+\epsilon}(u_0)\circ\mathcal{D}\circ\mathcal{T}_{>\pi}\\(T,g)&\longmapsto u \text{ with } \tau(u,g,G)=0,\end{aligned} \tag{4.5}$$

so that $u=\widetilde{u}\circ D\circ T'\circ T$*, where* $T'\in\mathcal{T}_{>\pi}, D\in\mathcal{D},\widetilde{u}\in\mathcal{B}^{1+\epsilon}(u_0)$*.*

If $\Sigma=1$ *and* $\mathfrak{p}_{<\pi}=\varnothing$*, the solutions form a graph over an open subset of* $\mathcal{M}^*_{2,\gamma}(g,\mathfrak{p},\mathfrak{a})\times\mathcal{C}on f_0(g_0)$ *(the identity component of the conformal group for* g_0*.)*

Proof. If genus $\Sigma>1$, then $\mathrm{T}\mathcal{C}on f_0=\varnothing$ and (4.2) is an isomorphism, so the Implicit Function Theorem applies directly.

If genus $\Sigma=0$ and $|\mathfrak{p}_{<\pi}|\geq 3$, then $\mathrm{T}\mathcal{C}on f_0\cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}=\varnothing$ since sections of $r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$ vanish at $\mathfrak{p}_{<\pi}$, and the only conformal Killing field vanishing at three points is identically zero. Thus, again the IFT applies

Now suppose genus $\Sigma=1$. Let $\{C_i\}$ be a basis for $\mathrm{T}\mathcal{C}on f_0\cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$. Note that

$$\mathrm{T}\mathcal{C}on f_0\cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}=\begin{cases}\mathrm{T}\mathcal{C}on f_0 & \text{if } \mathfrak{p}_{<\pi}=\varnothing\\ \{0\} & \text{if } \mathfrak{p}_{<\pi}\neq\varnothing\end{cases} \tag{4.6}$$

(Since conformal Killing fields are nowhere vanishing.) Thus if $\mathfrak{p}_{<\pi} \neq \varnothing$ we can again apply the Implicit Function Theorem.

Finally, assume that $\mathfrak{p}_{<\pi} = \varnothing$ and genus = 1. In this case $\mathcal{T}_{>\pi} = \mathcal{T}$. Consider the quotient bundle $\widetilde{\mathsf{E}}$ whose fiber over (u, g, G) is given by E_u/V_u, where

$$V_u := u_* \mathrm{T}\mathcal{C}onf(g) \subset r^{1-\epsilon-2\mathfrak{a}} \mathcal{X}_b^{2,\gamma}(u)$$

Let $\pi \colon \mathsf{E} \longrightarrow \widetilde{\mathsf{E}}$ be the natural projection. Proposition 4.1 says that the differential of the composition $\pi \circ \tau$ in the direction of $\mathcal{B}^{1+\epsilon} \circ \mathcal{D} \circ \mathcal{T}$ is an isomorphism. Thus the zero set of $\pi \circ \tau$ near (u_0, g, G) is a smooth graph over a neighborhood of $\mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$, so for each $g \in \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$ there is a map $u = \widetilde{u} \circ D \circ T$ with $\mathcal{T}$ such that

$$\tau(u, g, G) \in V_u. \tag{4.7}$$

We will show that (4.7) implies that $\tau(u, g, G) = 0$. Suppose that $\tau(u, g, G) = u_* C$ for some $C \in \mathrm{T}\mathcal{C}onf(g)$. Let $f_t \subset \mathcal{C}onf_0$ be a family with $\frac{d}{dt}\big|_{t=0} f_t = C$. By the conformal invariance of energy, (3.4), we have.

$$\left.\frac{d}{dt}\right|_{t=0} E(u \circ f_t, g, G) = 0 \tag{4.8}$$

On the other hand, we will show using the first variation formula (2.1) and (3.4) that

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} E(u \circ f_t, g, G) &= \int_\Sigma \langle \tau(u, g, G), u_* C \rangle \sqrt{g} dx \\ &= \|u_* C\|^2_{L^2}, \end{aligned} \tag{4.9}$$

with L^2 norm as in (6.29), below. Some care is needed in the proof since in general the boundary term in the first variation (2.1) can be singular. We postpone the rigorous computation to Section 6.4, where several similar computations are done at once, see Lemma 6.11. This implies that $\tau^{-1}(0)$ is a smooth manifold and thus completes the proof. □

Fix $\mathfrak{q} \subset \mathfrak{p}_{<\pi}$, and let $\mathcal{T}_{\mathfrak{p}_{<\pi} - \mathfrak{q}} := \{T_w : w_i = 0 \text{ for } p_i \in \mathfrak{p}_{<\pi} - \mathfrak{q}\}$. Given the identification of $\mathcal{T}_{\mathfrak{p}_{<\pi} - \mathfrak{q}}$ with a ball $U \subset \mathbb{C}^{|\mathfrak{p}_{<\pi} - \mathfrak{q}|}$ around the origin, if U is sufficiently small we can define a $2\,|\mathfrak{p}_{<\pi} - \mathfrak{q}|$-dimensional manifold of harmonic maps fixing $\mathfrak{q}$ by

$$\mathcal{H}arm_{\mathfrak{q}} = \{u_w := \mathcal{S}(T_w, g_0) : w \in U\}\,, \tag{4.10}$$

We identify the tangent space $T_{u_0}\mathcal{H}arm_{\mathfrak{q}}$ with $\mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|}$ by mapping $w \in \mathbb{C}^{\mathfrak{p}_{<\pi}-\mathfrak{q}}$ to the Jacobi field

$$J_w := \left.\frac{d}{dt}\right|_{t=0} u_{tw}. \tag{4.11}$$

Consider the residue map,

$$\begin{aligned} \operatorname{Res}_{u_w^{-1}(\mathfrak{p}_{<\pi}-\mathfrak{q})} \colon \mathcal{H}arm_{\mathfrak{q}} &\longrightarrow \mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|} \\ u_w &\longmapsto \operatorname{Res}|_{u_w^{-1}(\mathfrak{p}_{<\pi}-\mathfrak{q})}\, \Phi(u_w), \end{aligned}$$

where $\Phi(u_w)$ is the Hopf differential defined in (2.35)–(2.36). Differentiating Res at u_0 gives

$$\begin{aligned} D\operatorname{Res}_{\mathfrak{p}_{<\pi}-\mathfrak{q}} \colon T_{u_0}\mathcal{H}arm_{\mathfrak{q}}. &\longrightarrow \mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|} \\ J_w &\longmapsto \operatorname{Res}(\Phi(J_w)). \end{aligned} \tag{4.12}$$

(The J_w also have Hopf differentials, defined by $\Phi(J_w) = \frac{d}{dt}\big|_{t=0}\Phi(u_{tw})$.) We will prove the following

Proposition 4.3. *For any Jacobi field J_w as above, if $\operatorname{Res}(\Phi(J_w)) = 0 \in \mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|}$ then $J_w \in TConf_0$, i.e. J_w is a conformal Killing field.*

Proposition 4.12 is proven in Section 6.6 below.

Assuming for the moment that Propositions 4.1 and 4.3 are true, we can now prove

Proposition 4.4. *$\mathcal{H}(\mathfrak{q})$ in (3.19) is open.*

Proof. Given $t_0 \in \mathcal{H}(\mathfrak{q})$, let $g_0 \in \mathfrak{c}_{t_0}$ be a metric satisfying Assumption 3.1. (See Section 3.3 for definitions.) We now use Lemma 3.3; there is a small $\delta > 0$ and a path of diffeomorphisms $\widetilde{v}_t$, each isotopic to the identity, for $t \in (t_0 - \delta, t_0 + \delta)$, such that the pullback conformal structures $\widetilde{v}_t^*\mathfrak{c}_t$ have the property that $id\colon (\Sigma, \mathfrak{c}_{t_0}) \longrightarrow (\Sigma, \widetilde{v}_t^*\mathfrak{c}_t)$ is conformal near $\mathfrak{p}$. Let $\widetilde{g}_t \in \widetilde{v}_t^*\mathfrak{c}_t$ be any family of metrics that equal the standard conic metric of cone angle α_j on the conformal ball D_j near p_j. This can be done uniformly for t near t_0 since all the conformal structures $\widetilde{v}_t^* \mathfrak{c}_t$ are equal. Thus $\widetilde{g}_t \in \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$. We claim that for t near t_0 there exists a unique solution $\widetilde{u}_t : (\Sigma, \widetilde{g}_t) \longrightarrow (\Sigma, G)$ to (HME($\mathfrak{q}$)). Assuming this for the moment, the proof is finished, since $\left(\widetilde{v}_t^{-1}\right)^* \widetilde{g}_t \in \mathfrak{c}_t$, and

$$u_t := \widetilde{u}_t \circ \widetilde{v}_t^{-1} \colon (\Sigma, \left(\widetilde{v}_t^{-1}\right)^* \widetilde{g}_t) \longrightarrow (\Sigma, G) \tag{4.13}$$

solves (HME($\mathfrak{q}$)) and the zeroes of the Hopf differential of $\widetilde{u}_t$ are the same as those of u_t. (See Section 5, where this last fact is made obvious.) Thus the proposition is proven modulo the existence of $\widetilde{u}_t$.

Consider the manifold $\mathcal{H}arm_{\mathfrak{q}}$ of harmonic maps from $(\Sigma, \widetilde{g}_{t_0})$ to (Σ, G) that fix $\mathfrak{q}$. If $\mathfrak{q} = \mathfrak{p}_{<\pi}$, the existence of the $\widetilde{u}_t$ is implied by Lemma 4.2.

Assume $\mathfrak{q} \neq \mathfrak{p}_{<\pi}$. In all the remaining cases under consideration except for genus 1 and $\mathfrak{q} = \varnothing$, Proposition 4.3 implies that $D\,\mathrm{Res}_{\mathfrak{p}_{<\pi}-\mathfrak{q}} \colon T_{u_0}\mathcal{H}arm_{\mathfrak{q}} \longrightarrow \mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|}$ is injective (and thus an isomorphism since the two spaces have the same dimension.) This is simply because in all these cases there are no conformal Killing fields in $T_{u_0}\mathcal{H}arm_{\mathfrak{q}}$. (If genus > 1 there are no non-trivial conformal Killing fields. For genus 1, vectors in $T_{u_0}\mathcal{H}arm_{\mathfrak{q}}$ fix $\mathfrak{q}$ are only conformal Killing if they are identically zero.) The Implicit Function Theorem then implies that the set $\left\{\tau^{-1}(0)\right\} \cap \left\{\mathrm{Res}^{-1}_{\mathfrak{p}_{<\pi}-\mathfrak{q}}(0)\right\}$ is locally a graph over $\mathcal{T}_{\mathfrak{q}} \times \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$, and therefore there exists a $\widetilde{u}_t$ as above.

Finally, suppose that genus $= 1$ and $\mathfrak{q} = \varnothing$. We have already coverd the case $\mathfrak{p}_{<\pi} = \mathfrak{q}$, so assume that $\mathfrak{p}_{<\pi} \neq \varnothing$. Lift to the universal cover of Σ, which we can take to be $\mathbb{C}$, and let z denote the coordinate there. By integrating around a fundamental domain, we claim that, writing $\phi_w dz^2 = \Phi(J_w)$ then

$$\sum_{p_i \in \mathfrak{p}_{<\pi}} \mathrm{Res}|_{p_i} \phi_w = 0. \tag{4.14}$$

This follows immediately from the fact that the deck transformations are $z \mapsto z + z_0$ for some z_0, so ϕ_w is actually a periodic function with respect to the deck group. Define a subset $V \subset \mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|}$ by $V = \mathrm{span}\langle(1, \dots, 1), (i, \dots, i)\rangle$. Thus by (4.14), $V^\perp = D\,\mathrm{Res}(T\mathcal{H}arm_{\mathfrak{q}})$, where the orthocomplement is taken with respect to the standard hermitian inner product on $\mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|}$. Letting $X \subset \mathcal{H}arm_{\mathfrak{q}}$ be any subspace whose tangent space is complimentary to $\mathrm{T}\mathcal{C}onf_0$, by Proposition 4.3 and the fact that $\mathrm{T}\mathcal{C}onf_0$ has one complex dimension,

$$D\,\mathrm{Res}\colon T\mathcal{H}arm'_{\mathfrak{q}} \longrightarrow V^\perp$$

is an isomorphism, and again the existence of the $\widetilde{u}_t$ follows from the Implicit Function Theorem. $\square$

5. Uniqueness and convexity

The main result of this section is Proposition 5.2, which states that if (u, g, G) is a solution to (HME($\mathfrak{q}$)) in Form 2.3 satisfying Assumption 3.1, then, up to conformal automorphism, u is uniquely energy minimizing in its rel. $\mathfrak{q}$ homotopy class.

Let (u, g, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. In conformal coordinates, (3.1) yields

$$u^*G = e(u)\sigma\,|dz|^2 + 2\Re\rho(u)u_z\overline{u}_z dz^2. \tag{5.1}$$

where $g = \sigma\,|dz|^2$. The Hopf differential (see (2.36)) thus satisfies

$$\Phi(u) := \phi(z)dz^2 = \rho(u)u_z\overline{u}_z dz^2. \tag{5.2}$$

It follows directly (see Section 9 of [S]) that for $z_0 \in \Sigma_{\mathfrak{p}}$

$$\begin{aligned} \tau(u, g, G)(z_0) = 0 &\implies \partial_{\overline{z}}\phi(z_0) = 0 \\ \partial_{\overline{z}}\phi(z_0) = 0 \quad \& \quad J(u)(z_0) \neq 0 &\implies \tau(u, g, G)(z_0) = 0, \end{aligned} \tag{5.3}$$

where $J(u)(z_0)$ is the Jacobian determinant of u. Thus among local diffeomorphisms the vanishing of the tension field is equivalent to the holomorphicity of the (locally defined) function ϕ. By Form 2.3, near $p \in \mathfrak{p}$ we have $u(z) = \lambda z + v(z)$ with $\lambda \in \mathbb{C}^*$ and $v(z) \in r^{1+\epsilon}C_b^{2,\gamma}$ for some $\epsilon > 0$. By the definition of $C_b^{2,\gamma}$ and $\partial_z = \frac{1}{2z}(r\partial_r - i\partial_\theta)$,

$$\begin{aligned} \phi(z) &= \left(|\lambda z|^{2(\alpha-1)} + o(|z|^{2(\alpha-1)})\right)(\lambda + o(1))\,O(|z|^\epsilon) \\ &= O(|z|^{-2+2\alpha+\epsilon}). \end{aligned} \tag{5.4}$$

Since $-2 + 2\alpha + \epsilon > -2$, the function ϕ has at worst a simple pole at $z = 0$. If $\alpha \geq 1/2$ then $-2 + 2\alpha + \epsilon > -1$, so ϕ extends to a holomorphic function over p. Thus we have proven

Lemma 5.1. *Let $\Phi(u)$ be the Hopf differential of a solution (u, g, G) to (HME($\mathfrak{q}$)) in Form 2.3, with $G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$.*

$\Phi(u)$ is holomorphic on $\Sigma - \mathfrak{p}_{<\pi}$ with at worst simple poles on $\mathfrak{p}_{<\pi}$.

We will prove

Proposition 5.2. *Let (u, g, G) solve (HME($\mathfrak{q}$)) with u in Form 2.3, and assume that* $\operatorname{Res}\Phi(u)|_{\mathfrak{q}} \neq 0$ *and that* $\Phi(u)$ *is holomorphic on* $\Sigma_{\mathfrak{q}}$.

If $\mathfrak{q} \neq \varnothing$, then for any $w\colon \Sigma \to \Sigma$ with $w \sim_{\mathfrak{q}} u$ (see (2.31); in particular $w|_{\mathfrak{q}} = u|_{\mathfrak{q}}$) we have

$$E(u, g, G) \leq E(w, g, G)$$

with equality if and only if $u = w$.

If $\mathfrak{q} = \varnothing$ and $w\colon \Sigma \longrightarrow \Sigma$ satisfies $w \sim u$, then

$$E(u, g, G) \leq E(w, g, G)$$

with equality if and only if $u = w \circ C$ for $C \in \mathcal{C}onf_0$. In particular, if genus $\Sigma > 1$, equality holds if and only if $u = w$.

Proof. Assume $\mathfrak{q} = \varnothing$. We use a trick from [CH] to reduce to the smooth case.

First assume that the genus of $\Sigma > 1$. Let $\omega(z)\,|dz|^2$ be the unique constant curvature -1 metric in the conformal class of $[g]$, see [TII]. By Section 5, for any $\epsilon > 0$ in local coordinates we can write

$$\begin{aligned} u^*G &= e(u)\sigma dz d\overline{z} + 2\Re\phi(z)dz^2 \\ &= \underbrace{\left(e(u)\sigma - \left(\epsilon\omega^2 + |\phi|^2\right)^{1/2}\right)|dz|^2}_{:=H_1} \\ &\quad + \underbrace{\left(\epsilon\omega^2 + |\phi|^2\right)^{1/2}|dz|^2 + 2\Re\phi(z)dz^2}_{:=H_2} \end{aligned} \tag{5.5}$$

For ϵ sufficiently small, H_1 is positive definite on $\Sigma_{\mathfrak{p}}$; this follows from the fact that for u^*G to be positive definite we must have that $e(u)\sigma(z) > |\phi(z)|$. As for H_2, if $\mathfrak{q} = \varnothing$, ϕ extends smoothly to all Σ, so H_2 is a smooth metric. It is slightly more involved (See Appendix B of [CH]) to verify that the Gauss curvature of H_2 satisfies

$$\kappa_{H_2} < 0. \tag{5.6}$$

Since u is a local diffeomorphism by Form 2.3), (5.3) implies that

$$\begin{aligned} &id\colon (\Sigma, g) \longrightarrow (\Sigma, H_1) \text{ is conformal} \\ &id\colon (\Sigma, g) \longrightarrow (\Sigma, H_2) \text{ is harmonic} \end{aligned}$$

From Equation (2.3), for any $w\colon \Sigma \longrightarrow \Sigma$ we have

$$E(w, g, G) = E(w, g, H_1) + E(w, g, H_2), \tag{5.7}$$

Unique minimization now follows from (5.7), the fact that degree one conformal maps are energy minimizing, and fact that a harmonic diffeomorphism into a negatively curved surface is uniquely energy minimizing in its homotopy class (see e.g. [Tr], [Har]).

If the genus of Σ is 1, then lifting to the universal cover $\mathbb{C}$, we obtain a harmonic map $\widetilde{u}\colon (\mathbb{C}, \pi^* g) \longrightarrow (\mathbb{C}, \widetilde{\pi}^* G)$, where π and $\widetilde{\pi}$ are conformal covering maps with respect to the standard conformal structure on $\mathbb{C}$. The metric $|dz|^2$ descends to $\Sigma_{\mathfrak{p}}$ and is in the unique ray of flat metrics in the conformal class of g. Here $\Phi(\widetilde{u}) = \widetilde{\phi}(\widetilde{z}) d\widetilde{z}^2$ is defined globally on $\mathbb{C}$ and $\widetilde{\phi}$ is entire and periodic with respect to the deck transformations, hence bounded, hence constant. Write $\Phi(\widetilde{u}) = a d\widetilde{z}^2$. For sufficiently small $\epsilon > 0$, we decompose

$$\widetilde{u}^*(\pi^* G) = \underbrace{(e(\widetilde{u})\sigma - \epsilon) |d\widetilde{z}|^2}_{:=\widetilde{K}_1} + \underbrace{\epsilon |d\widetilde{z}|^2 + 2\Re\left(a d\widetilde{z}^2\right)}_{:=\widetilde{K}_2} \tag{5.8}$$

Since $e(\widetilde{u})\sigma > 2|a|$ and $(e(\widetilde{u})\sigma)(\widetilde{z})$ is periodic, there is an ϵ so that the $\widetilde{K}_i$ are positive definite. If $K_i := \pi_* \widetilde{K}_i$, then $id\colon (\Sigma, g) \longrightarrow (\Sigma, K_i)$ is harmonic for both $i = 1, 2$. We now argue as above, invoking both the minimality of conformal maps, and the minimality up to conformal automorphisms for harmonic maps of flat surfaces. This completes the $\mathfrak{q} = \varnothing$ case.

Finally, if genus $= 0$, it is standard that $\Phi(u) \equiv 0$, and thus u is conformal away from $\mathfrak{q}$, hence globally conformal. Since conformal maps are energy minimizing this case is complete.

This completes the $\mathfrak{q} = \varnothing$ case. To relate the $\mathfrak{q} = \varnothing$ case to the $\mathfrak{q} \neq \varnothing$ case, we will use the following

Lemma 5.3. *Given a closed Riemann surface $R = (\Sigma, \mathfrak{c})$ (here $\mathfrak{c}$ is the conformal structure) with genus > 0 and any finite subset $\mathfrak{q} \subset \Sigma$, there is a finite sheeted conformal covering space $\pi\colon S \longrightarrow R$ which has non-trivial branch points exactly on $f^{-1}(\mathfrak{q})$. Furthermore, genus $S >$ genus Σ.*

If $R = S^2$, the above is true provided $|\mathfrak{q}| \geq 3$.

This is a well-known fact from the theory of Riemann surfaces. We include a sketch of a proof here for the convenience of the reader. First assume genus $\Sigma > 0$. For each $q \in \mathfrak{q}$, there is a branched cover $Y_q \longrightarrow R$ ramified above q, constructed as follows. The fundamental group of $R - \{q\}$

is a free group with $2g$ generators, where g is the genus. Let Z_q be any normal covering space of $R - q$ that is not a covering of R, i.e. let Z_q correspond to a normal subgroup of $\pi_1(R - \{q\})$ that is not the pullback of a normal subgroup in $\pi_1(R)$ via the induced map. Let Y_q be the unique closure of Z_q. Finally, let $\widetilde{S}$ be the composite of the Z_q, i.e. the covering of $R - \mathfrak{q}$ corresponding to the intersection of the groups corresponding to the Z_q. Then $\widetilde{S}$ is normal and has a unique closure S with a branched covering of R that factors through each of the $Y_q \longrightarrow R$. The deck transformations of $\widetilde{S}$ extend to S and by normality act transitively on the fibers, and therefore S is ramified exactly above $\mathfrak{q}$. If $\Sigma = S^2$ and $|\mathfrak{q}| \geq 3$, let $\mathfrak{q}_0 \subset \mathfrak{q}$ have exactly three points. There is a branched cover of Σ by a torus branched over $\mathfrak{q}_0$. Applying the previous argument to the torus and branching over all the lifts of cone points to the torus gives the result. [Ya]

Assuming $\mathfrak{q} \neq \varnothing$, let S be a covering map branched exactly over $\mathfrak{q}$. (By our assumption that $|\mathfrak{q}| \geq 3$ in case $\Sigma = S^2$, such a cover exists.) and lift u to a map $\widetilde{u}$ to obtain the commutative diagram

(5.9)
$$\begin{array}{ccc} (S, \pi^* g) & \xrightarrow{\widetilde{u}} & (S, \pi^* G) \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ (\Sigma_{\mathfrak{p}}, g) & \xrightarrow{u} & (\Sigma_{\mathfrak{p}}, G). \end{array}$$

In particular, $\widetilde{u}^* \pi^* G = \pi^* u^* G$, so the Hopf differential of $\widetilde{u}$ is the pullback via the conformal map π of the Hopf differential of u. Pick any $p \in \mathfrak{q}$, and let $\phi(z)dz^2$ be a local expression of the Hopf differential of u in a conformal neighborhood centered at p. Given $q \in \pi^{-1}(p)$, since q is a non-trivial branch point we can choose conformal coordinates $\widetilde{z}$ near q so that the map π is given by $\widetilde{z}^k = z$ for some $k \in \mathbb{N}$, $k > 1$. By Lemma 5.1

$$\phi(z)dz^2 = \left(\frac{a}{z} + h(z)\right) dz^2$$

where h is holomorphic. Pulling back to $\widetilde{z}$ gives

$$\left(\frac{a}{z} + h(z)\right) dz^2 = k^2 \left(a\widetilde{z}^{k-2} + \widetilde{z}^{2k-2} h(\widetilde{z}^k)\right) d\widetilde{z}^2,$$

so $\Phi(\widetilde{u})$ is holomorphic on all of S, and therefore $\widetilde{u}$ solves (HME($\mathfrak{q}$)) with respect to the pullback metrics. Note that by Lemma 5.3, the genus of S is at least 2, so we are in the right situation to apply the preceding argument. Finally, suppose that $w \sim_{\mathfrak{q}} u$, and thus there exists a commutative diagram as in (5.9) with u and $\widetilde{u}$ replaced by w and $\widetilde{w}$. Then $\widetilde{u} \sim \widetilde{w}$ and

$$\begin{aligned}(\text{ \# of sheets }) E(u, g, G) &= E(\widetilde{u}, \pi^* g, \pi^* G) \\ &\leq E(\widetilde{w}, \pi^* g, \pi^* G) \\ &= (\text{ \# of sheets }) E(w, g, G)\end{aligned}$$

with equality if and only if $\widetilde{u} = \widetilde{w}$, i.e. $u = w$. □

6. Linear Analysis

6.1. The linearization

We now explicitly compute the derivative of τ at u_0 in the $\mathcal{B}^{1+\epsilon}(u_0)$ direction. That is, given a solution (u_0, g_0, G) to (HME($\mathfrak{q}$)) with u_0 in Form 2.3 and $g_0, G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$, and a path $u_t \in \mathcal{B}^{1+\epsilon}(u_0)$ through u_0 with $\frac{d}{dt}\big|_{t=0} u_t = \psi$, we compute

$$L\psi := \frac{d}{dt}\bigg|_{t=0} \tau(u_t, g_0, G).$$

For any $u \in \mathcal{B}^{1+\epsilon}(u_0)$, near $p \in \mathfrak{p}$, by abuse of notation, we write

$$\begin{aligned} u_0 &= \lambda z + v \\ u &= u_0 + \widetilde{v}, \\ v, \widetilde{v} &\in r^{1+\epsilon} C_b^{2,\gamma}, \end{aligned}$$

that is, we think of u_0 and u as complex valued functions, and write the metrics as $g_0 = \sigma |dz|^2, G = \rho |du|^2$. From (3.5), we have

$$\begin{aligned} \left(\frac{\sigma}{4}\right) \tau(u_0 + \widetilde{v}, g_0, G) &= \partial_z \partial_{\overline{z}} (u_0 + \widetilde{v}) + \frac{\partial \log \rho(u)}{\partial u} \partial_z (u_0 + \widetilde{v}) \partial_{\overline{z}} (u_0 + \widetilde{v}) \\ &= \frac{\sigma}{4} \tau(u_0, g_0, G) + \partial_z \partial_{\overline{z}} \widetilde{v} \\ &\quad + \left(\frac{\partial \log \rho(u)}{\partial u} - \frac{\partial \log \rho(u_0)}{\partial u} \right) \partial_z u_0 \partial_{\overline{z}} u_0 \\ &\quad + \frac{\partial \log \rho(u)}{\partial u} \left(\partial_z u_0 \partial_{\overline{z}} \widetilde{v} + \partial_z \widetilde{v} \partial_{\overline{z}} u_0 + \partial_z \widetilde{v} \partial_{\overline{z}} \widetilde{v} \right). \end{aligned} \tag{6.1}$$

Using $\tau(u_0, g_0, G) = 0$, we have

$$\begin{aligned}(6.2)\qquad \frac{\sigma}{4}L\psi &= \frac{d}{dt}\left(\frac{\sigma}{4}\tau(u_t, g_0, G)\right)\Big|_{t=0} \\ &= \partial_z\partial_{\overline{z}}\psi + \frac{\partial \log \rho(u)}{\partial u}\left(\partial_z u_0 \partial_{\overline{z}}\psi + \partial_{\overline{z}} u_0 \partial_z \psi\right) + A\psi\end{aligned}$$

where, writing $\rho = |u|^{2(\alpha-1)} e^{2\mu} |du|^2$, $\mu \in r^\nu C_b^{2,\gamma}$ by definition and

$$\begin{aligned}(6.3)\qquad \frac{\sigma}{4}A\psi &:= \partial_z u_0 \partial_{\overline{z}} u_0 \cdot \frac{d}{dt}\Big|_{t=0} \frac{\partial \log \rho(u_t)}{\partial u} \\ &= 2\partial_z u_0 \partial_{\overline{z}} u_0 \left(\frac{\partial^2 \mu}{\partial u \partial \overline{u}}\overline{\psi} + \frac{\partial^2 \mu}{\partial u^2}\psi\right) + \partial_z u_0 \partial_{\overline{z}} u_0 \frac{\alpha - 1}{u_0^2}\psi.\end{aligned}$$

As a preliminary to the analysis below, we can now see that L is a bounded map of weighted Hölder spaces (see (2.24)):

$$(6.4)\qquad L\colon r^c \mathcal{X}_b^{2,\gamma} \longrightarrow r^{c-2\alpha}\mathcal{X}_b^{0,\gamma}$$

In fact, if we define the operator $\widetilde{L} := \frac{\sigma|z|^2}{4}L$ then

$$(6.5)\qquad \widetilde{L}\psi = I(\widetilde{L})\psi + E(\psi)$$

where

$$(6.6)\qquad I(\widetilde{L})\psi = (z\partial_z)\left(\overline{z}\partial_{\overline{z}}\right)\psi + (\alpha - 1)\,\overline{z}\partial_{\overline{z}}\psi$$

and E is defined (locally near p) by this equation. It follows immediately that

$$(6.7)\qquad I(\widetilde{L})\colon r^c C_b^{2,\gamma}(D) \longrightarrow r^c C_b^{0,\gamma}(D)$$

Furthermore, using $|z|\left|\frac{\partial \mu}{\partial u}\right| + |\partial_z v| + |\partial_{\overline{z}} v| = O\left(|z|^\epsilon\right)$ we see that if $\psi \in r^c C_b^{2,\gamma}$ near $p \in \mathfrak{p}$ then

$$(6.8)\qquad E(\psi) \in r^{c+\epsilon} C_b^{0,\gamma}(D).$$

The boundedness of (6.4) follows from (6.7) and (6.8).

For the arguments in Section 7 we must also compute the mapping properties of the locally defined operator

$$Q\widetilde{v} = \tau(u_0 + \widetilde{v}, g_0, G) - L(\widetilde{v}) \tag{6.9}$$

From (6.1)–(6.3), we have

$$\frac{\sigma}{4}\left(\tau(u_0 + \widetilde{v}, g_0, G) - L(\widetilde{v})\right) = \left(\frac{\partial \log \rho(u)}{\partial u} - \frac{\partial \log \rho(u_0)}{\partial u}\right)\partial_z u_0 \partial_{\overline{z}} u_0 + \frac{\partial \log \rho(u)}{\partial u}\partial_z \widetilde{v}\partial_{\overline{z}}\widetilde{v} - A\widetilde{v}$$

From this formula and the fact that $\widetilde{v} \in r^{1+\epsilon}C_b^{2,\gamma}$, $\mu \in r^{\epsilon}C_b^{1,\gamma}$, and $\partial_{\overline{z}} u \in r^{\epsilon}C_b^{2,\gamma}$ we see that

$$\|Q(\widetilde{v})\|_{r^{1+2\epsilon-2\alpha}C_b^{1,\gamma}} < C\,\|\widetilde{v}\|_{r^{1+\epsilon}C_b^{2,\gamma}} \tag{6.10}$$

holds for $\epsilon > 0$ small, $k \in \mathbb{N}$ and some $\gamma \in (0,1)$.

6.2. The b-calculus package for L

This section links the study of L to a large body of work on b-differential operators. For more detailed definitions and proofs of what follows we refer the reader to [Me]. Fixing conformal coordinates z_i near each $p_i \in \mathfrak{p}$, we make a smooth, positive function $r\colon \Sigma_{\mathfrak{p}} \longrightarrow \mathbb{C}$ that is equal to $|z_i|$ in a neighborhood of each p_i. There is a smooth manifold with boundary $[\Sigma;\mathfrak{p}]$ constructed via radial blowup and a smooth map $\beta\colon [\Sigma;\mathfrak{p}] \longrightarrow \Sigma$ which is a diffeomorphism from the interior of $[\Sigma;\mathfrak{p}]$ onto $\Sigma_{\mathfrak{p}}$, with $\beta^{-1}(\mathfrak{p}) = \partial[\Sigma;\mathfrak{p}] \simeq \cup_{i=1}^{k} S^1$, on which the map $r \circ \beta$ is smooth up to the boundary. Smooth functions on $[\Sigma,\mathfrak{p}]$ are pullbacks via β of those functions on Σ that are polyhomogeneous with only integer powers in r. Finally, let

$$\mathcal{V}_b = \begin{array}{l}\text{smooth vector fields on } [\Sigma;\mathfrak{p}] \\ \text{which are tangent to the boundary}\end{array} \tag{6.11}$$

The Lie algebra $\mathcal{V}_b$ generates a filtered algebra of differential operators called b-differential operators.

For a more concrete definition, $V \in \mathcal{V}$ if and only if

$$V = ar\partial_r + b\partial_\theta \quad \text{for } a, b \in C^\infty([\Sigma,\mathfrak{p}]). \tag{6.12}$$

let B_i, $i = 1, 2$ be any two vector bundles over $[\Sigma; \mathfrak{p}]$. Then P is a differential b-operator of order N on sections of B if it admits a local expression

$$P = \sum_{i+j \leq N} a_{i,j} \, (r\partial_r)^i \, \partial_\theta^j \quad \text{where } a_I \in C^\infty \left([\Sigma; \mathfrak{p}]; End(B_1; B_2)\right) \tag{6.13}$$

near the boundary of $[\Sigma; \mathfrak{p}]$. We will call P **b-elliptic** if for all (r, θ) and $(\xi, \eta) \in \mathbb{R}^2 - \{(0, 0)\}$,

$$\sum_{i+j \leq N} a_{i,j}(r, \theta) \xi^i \eta^j \quad \text{is invertible.}$$

Let (u, g, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. In (6.5), the operator $\widetilde{L}$ can be defined globally by extending the local functions σ to positive smooth functions on Σ. We immediately see that $\widetilde{L}$ is an elliptic b-operator, and that E is a b-operator which, in local coordinates as in (6.13), has coefficients a_I tending to zero at a polynomial rate. In particular, $\widetilde{L}\colon r^c \mathcal{X}_b^{k,\gamma} \longrightarrow r^c \mathcal{X}_b^{k-2,\gamma}$ is continuous for any $c \in \mathbb{R}$ and $k \in \mathbb{N}$. It follows that

$$L\colon r^{c-2\mathfrak{a}} \mathcal{X}_b^{k,\gamma} \longrightarrow r^c \mathcal{X}_b^{k-2,\gamma}, \tag{6.14}$$

is continuous for any $c \in \mathbb{R}$ and $k \in \mathbb{N}$.

We will now state the properties of elliptic b-differential operators to be used below. To begin we define the set $\Lambda \subset \mathbb{C}$ of indicial roots of $\widetilde{L}$. Given $p \in \mathfrak{p}$, let Λ_p consist of all $\zeta \in \mathbb{C}$ such that for some function $a = a(\theta)\colon S^1 \longrightarrow \mathbb{C}$

$$\widetilde{L} r^\zeta a(\theta) = o(r^\zeta). \tag{6.15}$$

The total set of indicial roots is

$$\Lambda = \{z \in \mathbb{C}^n \colon z_i \in \Lambda_{p_i} \text{ for some } i\}. \tag{6.16}$$

(We will compute Λ in (6.42) below.) By [Me, Theorem 4.26], there is a discrete set $\widetilde{\Lambda} \supset \Lambda$ such that for all $c \notin \Re\widetilde{\Lambda}$, (6.14) is Fredholm, meaning its range is closed and its kernel and cokernel are finite dimensional.

The same is true for b-Sobolev spaces, which are defined as follows. Let $d\mu$ be a smooth, nowhere-vanishing density on $\Sigma_\mathfrak{p}$ so that near $\mathfrak{p}$

$$d\mu = \frac{dr d\theta}{r}$$

Let $L^2_b(T^*\Sigma, d\mu)$ be the completion of $r^\infty \mathcal{X}^\infty_b$ (smooth vector fields vanishing to infinite order at $\mathfrak{p}$) with respect to the norm

$$\|\psi\|^2_{L^2_b(d\mu)} = \int_\Sigma \|\psi\|^2_{u^*G}\, d\mu$$

As in the Hölder case, we write $\psi \in r^c L^2_b(T^*\Sigma, d\mu)$ if and only if $r^{-c}\psi \in L^2_b(T^*\Sigma, d\mu)$. Also, $r^c L^2_b(T^*\Sigma, d\mu)$ is a Hilbert space with inner product

$$\langle \psi, \psi' \rangle_{r^c L^2_b(d\mu)} = \langle r^{-c}\psi, r^{-c}\psi' \rangle_{L^2_b(d\mu)} \tag{6.17}$$

We define the weighted b-Sobolev spaces by

$$\psi \in r^c H^k_b(T^*\Sigma, d\mu) \iff \begin{array}{l} \text{for all } k\text{-tuples } V_1, \dots, V_k \in \mathcal{V}_b, \\ \nabla_{V_1} \cdots \nabla_{V_k}\psi \in r^c L^2_b(T^*\Sigma, d\mu) \end{array} \tag{6.18}$$

It follows that

$$L\colon r^c H^k_b(T^*\Sigma, d\mu) \longrightarrow r^{c-2\mathfrak{a}} H^{k-2}_b(T^*\Sigma, d\mu), \tag{6.19}$$

is bounded. In fact, for $c \notin \widetilde{\Lambda}$ as above, (6.19) is also Fredholm [Me-Me].

It follows that for each such $c \notin \widetilde{\Lambda}$, there is a generalized inverse $\mathcal{G}_c : r^{c-2\mathfrak{a}} H^{k-2}_b(T^*\Sigma, d\mu) \longrightarrow r^c H^k_b(T^*\Sigma, d\mu)$, defined by the equations

$$\begin{aligned} \mathcal{G}_c \circ L &= I - \pi_{ker} \\ L \circ \mathcal{G}_c &= I - \pi_{coker}, \end{aligned} \tag{6.20}$$

where π_{ker} is the $r^c H^k_b(T^*\Sigma, d\mu)$ orthogonal projection onto the kernel of L, and π_{coker} is the $r^{c-2\mathfrak{a}} H^{k-2}_b(T^*\Sigma, d\mu)$ orthogonal projections onto orthogonal compliment of the range of L

See [M], Section 3 for more on the relationship between b-Hölder and b-Sobolev spaces and for the proof of the following lemma.

Lemma 6.1. *For any $c \notin \Lambda$,*

$$\begin{aligned} &Ker\left(L\colon r^c H^2_b(T^*\Sigma, d\mu) \longrightarrow r^{c-2\mathfrak{a}} H^0_b(T^*\Sigma, d\mu)\right) \\ &= Ker\left(L : r^c \mathcal{X}^{2,\gamma}_b \longrightarrow r^{c-2\mathfrak{a}} \mathcal{X}^{0,\gamma}_b\right), \end{aligned} \tag{6.21}$$

and

$$L(r^c \mathcal{X}^{2,\gamma}_b) \oplus W = r^{c-2\mathfrak{a}} \mathcal{X}^{0,\gamma}_b, \tag{6.22}$$

where

$$W = Coker\left(L\colon r^c H_b^2(T^*\Sigma, d\mu) \longrightarrow r^{c-2\mathfrak{a}} H_b^0(T^*\Sigma, d\mu)\right) := \left(L\left(r^c H_b^2(T^*\Sigma, d\mu)\right)\right)^{\perp}. \tag{6.23}$$

Here the orthocomplement is computed with respect to the $r^{c-2\mathfrak{a}} H_b^0(T^*\Sigma, d\mu)$ *inner product.*

If in addition to the above assumptions we assume both that u_0 is polyhomogeneous (See (2.28) above) and that $G \in \mathcal{M}_\nu^{phg}(\mathfrak{p}, \mathfrak{a})$, then approximate solutions to the equation $L\psi = 0$ admit partial expansions. To be precise, from Corollary 4.19 in [M], if $\psi \in r^c \mathcal{X}_b^{k,\gamma}$ and $L\psi \in r^{c+\delta-2\mathfrak{a}} \mathcal{X}_b^{k-2,\gamma}$ for $\delta > 0$, i.e. if $L\psi$ vanishes faster than the generically expected rate of $r^{c-2\mathfrak{a}}$ from (6.4), then ψ decomposes as

$$\begin{aligned} \psi &= \psi_1 + \psi_2 \quad \text{where} \\ \psi_1 &\in r^{c+\delta} \mathcal{X}_b \\ \psi_2(z) &= \sum_{(s,p)\in\mathcal{E}, c+\delta>s>c} a_{s,p}(\theta) r^s \log^p r, \end{aligned} \tag{6.24}$$

for some discrete $\mathcal{E} \subset \mathbb{C} \times \mathbb{N}$ which intersects $\{\Re z < C\}$ at a finite number of points and functions $a_{s,p} \in C^\infty(\mathbb{S}^1)$. In particular, we have

Lemma 6.2. *Assume that* u_0 *is polyhomogeneous and that* $G \in \mathcal{M}_\nu^{phg}(\mathfrak{p}, \mathfrak{a})$. *Then solutions to* $L\psi = 0$ *are polyhomogeneous. (See Definition* (2.28).*)*

This follows from the mapping properties of the generalized inverse $\mathcal{G}_c$ in (6.20), proved e.g. in [M]; if $\mathcal{A}_{phg}^{\mathcal{E}}$ denotes the space of polyhomogeneous functions with index set $\mathcal{E}$, then, given $c' > c$,

$$\mathcal{G}_c\colon r^{c'-2\mathfrak{a}} \mathcal{X}_b^{0,\gamma} \longrightarrow r^c \mathcal{X}_b^{2,\gamma} + \left(\mathcal{A}_{phg}^{\mathcal{E}} \cap r^{c'} \mathcal{X}_b^{2,\gamma}\right) \tag{6.25}$$

We can also use (6.25) to prove that if u_0 is polyhomogeneous, $G \in \mathcal{M}_\nu^{phg}(\mathfrak{p},\mathfrak{a})$, and $u \in \mathcal{B}^{1+\epsilon}(u_0)$ is a solution to (HME($\mathfrak{q}$)), then u is polyhomogeneous. Recall from (6.9) that $\tau(u, g, G) = Lv + Q(v)$ where $u = \lambda z + v$. Since u is harmonic, we have $L(v) = -Q(v)$. We now want to let our parametrix $\mathcal{G}_c$, with $c_i = 1 + \epsilon$ for all i, act on both sides of this equation and to use the fact that L is injective on $r^{1+\epsilon} \mathcal{X}_b^{2,\gamma}$, as we show below, in particular $\mathcal{G}_c L = I$.

This yields

$$v = -\mathcal{G}Q(v) \tag{6.26}$$

Working locally, $v \in r^{1+\epsilon}C_b^{2,\gamma}$. From (6.10) we have $Q(v) \in r^{1+2\epsilon-2\alpha}C_b^{1,\gamma}$, so by (6.25), $v = -\mathcal{G}Q(v) = v_1 + v_2$ where $v_1 \in r^{1+2\epsilon}C_b^{1,\gamma}$ and $v_2 \in r^{1+\epsilon}\mathcal{A}_{phg}^{\mathcal{E}}$. The full expansion follows from induction; assuming that $v = v_{1,k} + v_{2,k}$, where $v_{1,k} \in \mathcal{X}_{k+1,\gamma}^{1+k\epsilon}$ and $v_{2,k} \in \mathcal{A}_{phg}^{\mathcal{E}}$, a trivial computation shows that $Q(v) = w_1 + w_2$ where $w_1 \in r^{1-2\mathfrak{a}+(k+1)\epsilon}C_b^{0,\gamma}$ and $w_2 \in r^{1-2\mathfrak{a}+\epsilon}\mathcal{A}_{phg}^{\mathcal{E}}$. Applying the same logic to (6.26), we have proved the following

Proposition 6.3. *Let u_0 be polyhomogeneous and assume that $G \in \mathcal{M}_\nu^{phg}(\mathfrak{p},\mathfrak{a})$. Then any solution u to* (HME($\mathfrak{q}$)) *in $\mathcal{B}^{1+\epsilon}(u_0)$ is polyhomogeneous.*

6.3. The cokernel of $L\colon r^{1-\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$

We continue to let (u_0, g_0, G) denote a solution to (HME($\mathfrak{q}$)) in Form 2.3 satisfying Assumption 3.1, and to let L denote the linearization of τ at u_0.

An important step in the proof of Proposition 4.1 is an accurate identification of 'the' cokernel of the map

$$L\colon r^{1-\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} \tag{6.27}$$

(note the '$-\epsilon$.') In this section we will prove the following lemma.

Lemma 6.4.

$$r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} = L\left(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}\right) \oplus \mathcal{K}$$

where

$$\boxed{\mathcal{K} := \mathrm{Ker}(L\colon r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b \longrightarrow r^{1+\epsilon-4\mathfrak{a}}\mathcal{X}_b),} \tag{6.28}$$

and this decomposition is L^2 orthogonal with respect to the inner product in (6.29)*.*

Given two vector fields $\psi, \psi' \in \Gamma(u_0^* T\Sigma_{\mathfrak{p}})$ which are smooth and vanish to infinite order near $\mathfrak{p}$, we define the geometric L^2 pairing by

$$\langle \psi, \psi' \rangle_{L^2} := \int_\Sigma \langle \psi, \psi' \rangle_{u_0^*(G)} \, dVol_g = \Re \int_\Sigma \psi\overline{\phi}\rho(u_0)\sigma(z)\,|dz|^2 \tag{6.29}$$

It is straightforward to check using $g, G \in \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$ and $u_0 - \lambda z \in r^{1+\epsilon} C_b^{2,\gamma}$ near each cone point, that if $\psi \in r^c \mathcal{X}_b^{0,\gamma}(u_0)$ and $\psi' \in r^{c'} \mathcal{X}_b^{0,\gamma}(u_0)$, then

$$c_i + c_i' > 2 - 4\alpha_i \quad \text{for each } p_i \in \mathfrak{p} \implies \langle \psi, \psi' \rangle_{L^2} < \infty \tag{6.30}$$

It is standard (see e.g. [EL]), that the linearization of τ in (4.1) is symmetric with respect to this inner product and appears in the formula of the Hessian of the energy functional near a harmonic map. For reference, we state this here as

Lemma 6.5 (Second Variation of Energy). *Let (M, h) and $(N, \widetilde{h})$ be smooth Riemannian manifolds, possibly with boundary, and let $u_0 : (M, h) \longrightarrow (N, \widetilde{h})$ be a C^2 map satisfying $\tau(u_0, h, \widetilde{h}) = 0$. If u_t is a variation of C^2 maps through $u_0 = 0$ with $\frac{d}{dt}\big|_{t=0} u_t = \psi$, and L is the linearization of τ at u_0 (see (4.1)), then*

$$L\psi = \nabla^* \nabla \psi + \operatorname{tr}_h R^{\widetilde{h}}(du, \psi) du,$$

where ∇ is the natural connection on $u_0^*(TN)$ induced by $\widetilde{h}$, and $R^{\widetilde{h}}$ is its curvature tensor. If ∂_ν denotes the outward pointing normal to ∂M, then

$$\begin{aligned} &\frac{d^2}{dt^2} E(u_t)\Big|_{t=0} \\ &= \int_M \left(\|\nabla \psi\|^2_{u^*\widetilde{h}} - \operatorname{tr}_h R^{\widetilde{h}}(du, \psi, \psi, du) \right) dVol_h \\ &\quad + \int_{\partial M} \langle \nabla_{u_* \partial_\nu} \psi, \psi \rangle_{u^* \widetilde{h}} ds \end{aligned} \tag{6.31}$$

$$= - \langle L\psi, \psi \rangle_{L^2} + \int_{\partial M} \left(\langle \nabla_\psi \psi, u_* \partial_\nu \rangle_{u^*\widetilde{h}} + \langle \nabla_{u_*\partial_\nu} \psi, \psi \rangle_{u^*\widetilde{h}} \right) ds \tag{6.32}$$

Note that the boundary term in the last line can be expressed in terms of the Lie derivative

$$\begin{aligned} &\int_{\partial M} \left(\langle \nabla_\psi \psi, u_* \partial_\nu \rangle_{u^*\widetilde{h}} + \langle \nabla_{u_*\partial_\nu} \psi, \psi \rangle_{u^*\widetilde{h}} \right) ds \\ &= \int_{\partial M} \mathcal{L}_{u_*\psi} \widetilde{h}(u_* \psi, u_* \partial_\nu) ds. \end{aligned} \tag{6.33}$$

Furthermore, L is symmetric with respect to the L^2 inner product; as we will see, if $\psi \in r^c \mathcal{X}_b^{2,\gamma}$ and $\psi' \in r^{c'} \mathcal{X}_b^{2,\gamma}$, then

$$\langle L\psi, \psi' \rangle_{L^2} = \langle \psi, L\psi' \rangle_{L^2} \quad \text{if } c_i + c_i' > 2(1 - \alpha_i). \tag{6.34}$$

If we take the target (N, h) in (6.31) to be (Σ, G) and the domain to be $(\Sigma - \cup_{p_i \in \mathfrak{p}} D_i(\epsilon), g_0)$ where $D_i(\epsilon)$ is the conformal ball around p_i with $|z_i| \leq r_i < \epsilon$, and if $\psi \in r^c \mathcal{X}_b^{2,\gamma}(u_0)$, then the integrand in (6.33) satisfies $\mathcal{L}_{u_* \psi} \widetilde{h}(u_* \psi, u_* \partial_\nu) ds = O(r^{2c_i + 2(\alpha_i - 1) - 1}) dr d\theta$. Thus,

$$\text{(6.35)} \quad \text{for } u_t \in \mathcal{B}^{1-\mathfrak{a}+\epsilon}, \ \left.\frac{d}{dt}\right|_{t=0} u_t = \psi, \quad \left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, G) = -\langle L\psi, \psi \rangle_{L^2}$$

To see that (6.34) holds, note that using (6.2) and the subsequent estimates

$$\text{(6.36)} \quad \begin{aligned} &\langle L\psi, \psi' \rangle_{L^2} - \langle \psi, L\psi' \rangle_{L^2} \\ &= \sum_i \int_0^{2\pi} (-(r\partial_r \psi)\overline{\psi}\rho + \psi r \partial_r(\overline{\psi}\rho) - 2(\alpha_i - 1)\psi\overline{\psi})(1 + O(r^\epsilon)) d\theta \end{aligned}$$

The proof of (6.35) is similar.

To prove Lemma 6.4, we will need a version of (6.34) for b-Sobolev spaces. Below, given a constant $\delta \in \mathbb{R}$ and $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$, we will write $c + \delta$ for the weight $(c_1 + \delta, \ldots, c_k + \delta)$

Lemma 6.6. *Again, let (u_0, g_0, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. If $c \in \mathbb{R}^k$ and if, notation as in* (6.18)*, $\psi \in r^{c-\mathfrak{a}+1} H_b^k(T^*\Sigma, d\mu)$, $\psi' \in r^{-c-\mathfrak{a}+1} H_b^{k-2}(T^*\Sigma, d\mu)$, we have*

$$\text{(6.37)} \qquad \big\langle L\psi, \psi' \big\rangle_{L^2} = \big\langle \psi, L\psi' \big\rangle_{L^2}$$

Proof. Equation (6.37) holds for $\psi, \psi' \in r^\infty \mathcal{X}_b^{2,\gamma}$ by (6.34), and both sides of (6.37) are continuous with respect to the stated norms. □

We can now prove Lemma 6.4.

Proof of Lemma 6.4. There exist positive functions f and g such that for $\psi \in r^{c-\mathfrak{a}+1} H_b^2(T^*\Sigma, d\mu)$ and $\widetilde{\psi} \in r^{-c-\mathfrak{a}+1} H_b^2(T^*\Sigma, d\mu)$

$$\text{(6.38)} \qquad \begin{aligned} \langle L\psi, f\widetilde{\psi} \rangle_{r^{c-3\mathfrak{a}+1} L^2(d\mu)} &= \langle L\psi, \widetilde{\psi} \rangle_{L^2} = \langle \psi, L\widetilde{\psi} \rangle_{L^2} \\ &= \langle g\psi, L\widetilde{\psi} \rangle_{r^{-c-3\mathfrak{a}+1} L^2(d\mu)}, \end{aligned}$$

where $f \sim r^{2(c_i - \alpha_i)}$, $g \sim r^{-2(c_i - \alpha_i)}$ near $p_i \in \mathfrak{p}$. Note that the middle equality in (6.38) is a consequence of Lemma 6.6. This implies that

$$\text{(6.39)} \qquad \begin{aligned} &(L(r^{c-\mathfrak{a}+1} H_b^2(T^*\Sigma, d\mu))^\perp \\ &= f \operatorname{Ker}(L \colon r^{-c-\mathfrak{a}+1} H_b^2(T^*\Sigma, d\mu) \longrightarrow r^{-c-3\mathfrak{a}+1} L^2(T^*\Sigma, d\mu)) \end{aligned}$$

where the orthogonal compliment on the left is taken with respect to the $r^{c-3\mathfrak{a}+1}L^2(T^*\Sigma, d\mu))$ inner product. In (6.39) we take $c - \mathfrak{a} + 1 = 1 - \epsilon$, i.e. $c = \mathfrak{a} - \epsilon$. Thus, using Lemma 6.1

$$
\begin{aligned}
(6.40) \qquad & (L(r^{1-\epsilon}H^2_b(T^*\Sigma, d\mu))^{\perp} \\
&= f\,\mathrm{Ker}(L\colon r^{1+\epsilon-2\mathfrak{a}}H^2_b(T^*\Sigma, d\mu) \longrightarrow r^{1+\epsilon-3\mathfrak{a}}L^2(T^*\Sigma, d\mu)) \\
&= f\,\mathrm{Ker}(L\colon r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}^{2,\gamma}_b \longrightarrow r^{1+\epsilon-3\mathfrak{a}}\mathcal{X}^{0,\gamma}_b)
\end{aligned}
$$

This shows that the space $\mathcal{K}$ in (6.28) has the correct dimension. Since $\mathcal{K} \subset r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}^{2,\gamma}_b$, it is complimentary to $L(r^{1-\epsilon}H^2_b(T^*\Sigma, d\mu))$ is and only if they have trivial intersection. Given $\psi' \in \mathcal{K}$ and $\psi \in r^{1-\epsilon}H^2_b(T^*\Sigma, d\mu$, again by Lemma we have 6.6 $\langle L\psi, \psi'\rangle_{L^2} = \langle \psi, L\psi'\rangle_{L^2} = 0$. This shows both that the intersection is empty and that the compliment is orthgonal, so Lemma 6.4 is true. □

We now compute the indicial roots of L defined in (6.15) above. It is sufficient to find all r-homogeneous solutions to

$$
(6.41) \qquad I(\widetilde{L})\widetilde{\psi} = 0,
$$

with $I(\widetilde{L})$ defined in (6.6). Fixing $p \in \mathfrak{p}$, solutions to (6.41) take the form $r^s e^{ij\theta}$, and

$$
(6.42) \qquad I(\widetilde{L})r^s e^{ij\theta} = e^{ij\theta}r^s\left(s^2 + 2\,(\alpha - 1)\,s - j^2 - 2\,(\alpha - 1)\,j\right).
$$

$s \in \{j, 2 - 2\alpha - j\}$. Setting

$$
(6.43) \qquad \Lambda_p = \bigcup_{j\in\mathbb{Z}} \{j, 2 - 2\alpha - j\}
$$

As in (6.16), the set of indicial roots of L is the union $\Lambda = \cup_{\mathfrak{p}}\Lambda_p$.

We now prove precise asymptotics for solutions to (HME($\mathfrak{q}$)).

Lemma 6.7 (Leading order asymptotics). *Notation as above, near a cone point p of cone angle $2\pi\alpha > \pi$, we have $u = \lambda z + v$ where $v \in r^{3-2\alpha}C^{2,\gamma}_b \cap \mathcal{A}^{\mathcal{E}}_{phg}$. Moreover,*

$$
(6.44) \qquad u = \lambda z + c r^{2-2\alpha}\overline{z} + o(|z|^{3-2\alpha}).
$$

Here and below $o(|z|^c)$ denotes a quantity such that $\limsup_{|z|\to 0}\left||z|^{-c}\,o(|z|^c)\right| = 0$.

Proof. By Proposition 6.3, $u \in \mathcal{A}^{\mathcal{E}}_{phg}$. By (6.1), the leading order term of v, call it v_0, satisfies $I(\widetilde{L})v_0 = 0$. Since $v_0 \in r^{1+\epsilon}C^{2,\gamma}_b$, this implies that it in fact vanishes to the first order indicial root bigger than $1+\epsilon$. By (6.43) and (6.42), this root is $3-2\alpha$ and in fact $v_0 = cr^{2-2\alpha}\overline{z}$ for some $c \in \mathbb{C}$. □

We now describe the behavior of the elements in $\mathcal{K}$ (see (6.28)) near the conic set.

Lemma 6.8. *Let $\psi \in \mathcal{K}$ (see (6.28)), let $p_i \in \mathfrak{p}$, and let z be conformal coordinates near p_i. If $2\pi\alpha_i < \pi$, then*

$$\psi(z) = \mu_i z + O(|z|^{1+\delta}) \tag{6.45}$$

for some $\mu_i \in C$, $\delta > 0$.

If $2\pi\alpha_i > \pi$, and the solution $u_0(z) = z + v$ (which we may assume by rescaling the domain metric, see Remark 2.4), then

$$\psi(z) = w_i + \frac{\overline{a_i}}{1-\alpha}|z|^{2(1-\alpha)} + O\left(|z|^{2(1-\alpha)+\delta}\right) \tag{6.46}$$

for some $w_i, a_i \in \mathbb{C}$, $\delta > 0$.

Proof. Fix $\psi\in\mathcal{K}$. By Lemma 6.2, ψ is polyhomogeneous. Fix $p\in\mathfrak{p}$. By (6.6), the lowest order homogeneous term in the expansion of ψ at p, which we can write as $f(r,\theta) = r^\delta a(\theta)$ for $a\colon S^1 \longrightarrow \mathbb{C}$, must solve $I(\widetilde{L})f = 0$ for $I(\widetilde{L})$ in (6.6), and so δ must be an indicial root, i.e. $\delta \in \Lambda_p$ (see (6.43)). If $p \in \mathfrak{p}_{<\pi}$, then the smallest such indicial root is 1. The eigenvector is λz, so the lemma is proven in this case. If $p \in \mathfrak{p}_{>\pi}$, the smallest such indicial root is 0, and the eigenvector for this indicial root is a constant complex number w_i. To get the second term in (6.46), we consider the Hopfy differential of ψ defined by the linearization of (5.2). Precisely, suppose $\frac{d}{dt}\big|_{t=0} u_t = \psi$, and define

$$\Phi(\psi) := \left(\left(\frac{\partial\rho}{\partial u}\psi + \frac{\partial\rho}{\partial\overline{u}}\overline{\psi}\right)\partial_z u_0 \partial_z \overline{u}_0 + \rho(u_0)\left(\partial_z\psi\partial_z\overline{u}_0 + \partial_z\overline{\psi}\partial_z u_0\right)\right)dz^2 \tag{6.47}$$

It follows from (6.2) that $\Phi(\psi)$ is holomorphic on $\Sigma_{\mathfrak{p}}$. Near $p \in \mathfrak{p}_{>\pi}$, since $\psi = w + \psi_0$ for $\psi_0 \in r^c C^{2,\gamma}_b$ for some $c > 0$ and $v \in r^{3-2\alpha}C^{2,\gamma}_b$ (Lemma 6.7),

$\Phi(\psi)$ has at most a simple pole. Suppose that

$$\text{Res}|_p \Phi(\psi) = a \tag{6.48}$$

We claim that the only term in (6.47) that contributes to this residue is $\rho(u_0)\partial_z\overline{\psi}\partial_z u_0$. To see this, using $\rho = e^{2\mu_j} |u|^{2(\alpha_j-1)} |du|^2$ note that

$$\begin{aligned} \frac{\partial \rho}{\partial u} &= (\alpha-1)\overline{\lambda} |z|^{2\alpha-4} \overline{z} + o(|z|^{2\alpha-3}) \quad \text{and} \\ \frac{\partial \rho}{\partial \overline{z}} &= (\alpha-1)\lambda |z|^{2\alpha-4} z + o(|z|^{2\alpha-3}), \end{aligned} \tag{6.49}$$

and that by (6.44),

$$\partial_z u = \lambda \quad \text{and} \quad \partial_z \overline{u} = |z|^{2-2\alpha} (2-\alpha). \tag{6.50}$$

Combining (6.49) and (6.50) gives $\left(\frac{\partial\rho}{\partial u}\psi + \frac{\partial\rho}{\partial \overline{u}}\overline{\psi}\right)\partial_z u_0 \partial_z \overline{u}_0 = c' |z|^{-2} (\overline{\lambda}\overline{z}w - \lambda z\overline{w}) + o(|z|^{-1})$, which implies

$$\lim_{\epsilon\to 0} \int_{|z|=1} \left(\frac{\partial\rho}{\partial u}\psi + \frac{\partial\rho}{\partial \overline{u}}\overline{\psi}\right) \partial_z u_0 \partial_z \overline{u}_0 dz = 0. \tag{6.51}$$

Using 6.7, the other term is term is $\rho(u_0)\partial_z\psi\partial_z\overline{u}_0 = o(|z|^{-1})$, and therefore does not contribute to the residue.

Finally, let $|z|^\delta b(\theta)$ be the lowest order homogeneous term in $\psi_0 = \psi - w$. Then the leading order part of $\rho(u_0)\partial_z\overline{\psi}\partial_z u_0$ is $|z|^{2(\alpha-1)} \partial_z \overline{|z|^\delta b(\theta)}$, which implies that

$$|z|^\delta b(\theta) = \frac{\overline{a}}{1-\alpha} |z|^{2(1-\alpha)}. \qquad \square$$

Remark 6.9. Note that, as a consequence of the proof, u_0 is a solution to (HME($\mathfrak{q}$)) in Form 2.3 for which $\lambda = 1$ (see Remark d 2.4), then $\text{Res}|_{p_i}\Phi(\psi) = a_i$ with a_i as in (6.46).

Writing $\mathfrak{p}_{>\pi} = \{q_1, \dots, q_{|\mathfrak{p}_{>\pi}|}\}$, the map

$$\begin{aligned} \text{Res}\colon \mathcal{K} &\longrightarrow \mathbb{C}^{|\mathfrak{p}_{>\pi}|} \\ \psi &\longmapsto \text{Res}\,\Phi(\psi) = (\text{Res}|_{q_1}\Phi(\psi), \dots, \text{Res}|_{q_{|\mathfrak{p}_{>\pi}|}}\Phi(\psi)) \end{aligned}$$

is obviously linear. We define a basis of $\mathcal{K}$, $K_{a^1}, \dots, K_{a^{m_1}}, C_1, \dots, C_{m_2}$ with $a^j \in \mathbb{C}^{|\mathfrak{p}_{>\pi}|}$, so that $C_1, \dots, C_{m_2}$ is a basis of $\ker(\text{Res}\colon \mathcal{K} \longrightarrow \mathbb{C}^{|\mathfrak{p}_{>\pi}|})$, i.e.

$$\text{Res}\, C_j = 0 \in \mathbb{C}^{|\mathfrak{p}_{>\pi}|} \quad \text{and} \quad \text{Res}\, K_{a^j} = a^j \in \mathbb{C}^{|\mathfrak{p}_{>\pi}|}. \tag{6.52}$$

It follows that the a^j are linearly independent.

6.4. The C_j are conformal Killing fields

We will prove that the C_j are conformal Killing fields. The most important step in the proof is that they are zeros of the Hessian of the energy functional, and we begin by proving this.

Lemma 6.10. *Let (u_0, g_0, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. Let $C \in \mathcal{K}$ have* $\operatorname{Res} C = 0 \in \mathbb{C}^n$. *By Lemma 6.8, we can find $u_t \in \mathcal{B}^c \circ \mathcal{D} \circ \mathcal{T}_{>\pi}$, $\epsilon > 0$ so that $\frac{d}{dt}\big|_{t=0} u_t = C$, where*

$$\begin{aligned} c_i &> 1 \quad \text{for } p_i \in \mathfrak{p}_{<\pi} \\ c_i &> 2 - 2\alpha_i \quad \text{for } p_i \in \mathfrak{p}_{>\pi}. \end{aligned} \tag{6.53}$$

Then

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g_0, G) = 0 \tag{6.54}$$

The proof hinges on a nice cancellation of boundary terms related to the conformal invariance of energy. We begin by proving Equation (4.9) above, which illustrates this phenomenon in a simpler setting.

Lemma 6.11. *Equation (4.9) holds.*

Proof. We are given a $u \in \mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{D} \circ \mathcal{T}_{>\pi}$, and we assume that $\mathfrak{p}_{<\pi} = \varnothing$ and genus $\Sigma = 1$. In particular, near each $q \in u^{-1}(\mathfrak{p})$ we can choose conformal coordinates so that $u \sim \lambda z$. Let $C \in \mathrm{T}\mathcal{C}on f_0$. Choose $f_t \in \mathcal{C}on f_0$ with $\frac{d}{dt}\big|_{t=0} f_t = C$ and consider $E(u \circ f_t, g, G)$. By (4.8) above,

$$\left.\frac{d}{dt}\right|_{t=0} E(u \circ f_t, g, G) = 0.$$

By lifting to the universal cover we can assume that

$$f_t(z) = z - tw$$

for some fixed w. Let

$$D_w(r) = \{z : |z - w| < r\}, \tag{6.55}$$

be the conformal disc (not necessarily a geodesic ball), so $f_t(D_{tw}(r)) = D_0(r)$. In particular, for $\delta > 0$ sufficiently small, $f_t(\Sigma - D_{tw}(\delta)) = \Sigma - D_0(\delta)$.

For the moment we drop the geometric data from the notation and, given a subset $A \subset \Sigma$, let

$$E(w, A) = \int_A e(w, g, G)\sqrt{g}dx$$

By the conformal invariance of the energy functional, for all t,

$$E(u \circ f_t, \Sigma - D_{tw}(\delta)) = E(u, \Sigma - D_0(\delta)) \quad \text{and}$$
$$E(u \circ f_t, D_{tw}(\delta)) = E(u, D_0(\delta)).$$

Thus the functions $E(u \circ f_t, \Sigma)$, $E(u \circ f_t, \Sigma - D_{tw}(\delta))$, and $E(u \circ f_t, D_{tw}(\delta))$ are all constant in t. In particular

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} E(u \circ f_t, \Sigma) \\ &= \frac{d}{dt}\bigg|_{t=0} E(u \circ f_t, \Sigma - D_{tw}(\delta)) + \frac{d}{dt}\bigg|_{t=0} E(u \circ f_t, D_{tw}(\delta)) \\ &= \frac{d}{dt}\bigg|_{t=0} E(u \circ f_t, \Sigma - D_{tw}(\delta)) \end{aligned}$$

We can also evaluate this last expression using the first variation formula (2.1) and the chain rule to get the expression

$$\begin{aligned} &\frac{d}{dt}\bigg|_{t=0} E(u \circ f_t, \Sigma - D_{tw}(\delta)) \\ &= -\int_{\Sigma - D_0(\delta)} \langle \tau(u, g, G), u_* C\rangle_{u^*G} dVol_g + \int_{\partial D_0(\delta)} \langle u_*\partial_\nu, u_*C\rangle_{u^*G} ds \\ &\quad + \frac{d}{dt}\bigg|_{t=0} \left(\int_{\Sigma - D_{tw}(\delta)} e(u, g, G)\sqrt{g}dx \right) \end{aligned} \tag{6.56}$$

Here ∂_ν is the outward pointing normal from $\Sigma - D_0(\delta)$. The last integral satisfies

$$\begin{aligned} &\frac{d}{dt}\bigg|_{t=0} \left(\int_{\Sigma - D_{tw}(\delta)} e(u, g, G)\sqrt{g}dx \right) \\ &= -\int_{\partial D_{tw}(\delta)} e(u, g, G)\langle \partial_\nu, u_* C\rangle_g ds \end{aligned} \tag{6.57}$$

To compare this integral to the second term in (6.56), we use the decomposition (5.1), $u^*G = e(u)g + u^*G^\circ$.

$$\int_{\partial D_0(\delta)} \langle u_* \partial_\nu, u_* C\rangle_{u^*G} ds = \int_{\partial D_0(\delta)} e(u, g, G)\langle \partial_\nu, u_* C\rangle_g ds + \int_{\partial D_0(\delta)} \langle \partial_\nu, u_* C\rangle_{u^*G^\circ} ds$$

By (6.57), the first term on the right exactly cancels the last term in (6.56). For the second term, note that although u^*G° is not holomorphic, the bound from (5.4) still holds, so

$$\int_{\partial D_0(\delta)} \langle \partial_\nu, u_* C\rangle_{u^*G^\circ} ds = \int_0^{2\pi} O\left(r^{2\alpha-2+\epsilon}\right) r d\theta, \tag{6.58}$$

which goes to zero since $\alpha > 1/2$. Looking back at (6.56), we have proven that

$$\lim_{\delta\to 0}\left\{\left(\left.\frac{d}{dt}\right|_{t=0} E(u\circ f_t, \Sigma - D_{tw}(\delta))\right) + \int_{\Sigma - D_0(\delta)} \langle \tau(u, g, G), u_* C\rangle_{u^*G} dVol_g\right\} = 0$$

This proves (4.9) □

Remark 6.12. Note that this last proof works if $\alpha > 1/2$ is replaced by $\alpha \geq 1/2$ since (6.58) still holds. This will be important when we deal with that case in Section 8.

A similar cancellation of boundary terms will lead to the proof of Lemma 6.10. Here we take two derivatives, so the relevant boundary terms look slightly different. To illustrate this, let $g = \sigma\left|dz\right|^2$ be a conformal metric on $\mathbb{C}$ with finite area, and let $T_t = z - tw$, and $\left.\frac{d}{dt}\right|_{t=0} T_t = C(= w)$. Notation as above

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(T_t, \mathbb{C} - D_{tw}(\delta)) = 0$$

As above, a direct computation of the second derivative using (6.5) and the chain rule will produce boundary terms which must cancel one another. If we let $e_t = e(T_t, \left|dz\right|^2, g)$, and let ∂_{ν_t} be the outward pointing normal to $\mathbb{C} - D_{tw}(\delta)$, then a simple computation using (6.32)–(6.33) and the product rule shows that

$$\begin{aligned}(6.59)\qquad \left.\frac{d^2}{dt^2}\right|_{t=0} E(T_t, \mathbb{C} - D_{tw}(\delta)) &= \left.\frac{d^2}{dt^2}\right|_{t=0} E(T_t, \mathbb{C} - D_{tw}(\delta)) \\ = \int_{\partial D_0(\delta)} \mathcal{L}_{\dot{T}_0} g \left(\dot{T}_0, \partial_\nu\right) ds &- 2 \int_{\partial D_0(\delta)} \dot{e}_0 \langle \dot{T}_0, \partial_\nu \rangle_g ds \\ - \int_{\partial D_0(\delta)} e_0 \left.\frac{d}{dt}\right|_{t=0} &\langle \dot{T}_{-t}, \partial_{\nu_{-t}} \rangle_{g(T_{-t})} ds.\end{aligned}$$

Thus the expression on the right *must be equal to zero.*

Proof of Lemma 6.10. Assume that $u_0 = id$. By (3.1) and conformal invariance, we may replace g by $g/e(u_0)$ and assume that

$$e(u_0) \equiv 1. \qquad (6.60)$$

We arrange it so that

$$\begin{aligned}&u_t = \widetilde{u}_t \circ T_t \\ &\widetilde{u}_t \in r^{2(1-\alpha)} C_b^{2,\gamma} \quad \text{near } p \in \mathfrak{p}_{>\pi} \\ &T_t \in \mathcal{T}_{>\pi},\end{aligned}$$

with c as in (6.53), so $T_t(z_i) = z_i - tw_i$ near $p_i \in \mathfrak{p}_{>\pi}$. Define

$$D_{i,t}(\delta) = D_{tw_i}(\delta) \text{ in conformal coordinates } z_i \text{ near } p_i\ ,$$

where $D_{tw_i}(\delta)$ is the conformal disc defined in (6.55). We can then write

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E\left(u_t, \Sigma\right) = \underbrace{\left.\frac{d^2}{dt^2}\right|_{t=0} E\left(u_t, \Sigma - \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_{i,t}(\delta)\right)}_{:=A(\delta)} + \underbrace{\left.\frac{d^2}{dt^2}\right|_{t=0} E\left(u_t, \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_{i,t}(\delta)\right)}_{:=B(\delta)}.$$

For the term $B(\delta)$ we can use conformal invariance

$$E\left(u_t, \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_t(\delta)\right) = E\left(\widetilde{u}_t, \bigcup_{p_i \in \mathfrak{p}_{>\pi}} T_t\left(D_t(\delta)\right)\right)$$
$$= E\left(\widetilde{u}_t, \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_0(\delta)\right).$$

Using the computations in (6.34)–(6.37), since $\widetilde{u}_t \in r^{2-2\alpha_i+\epsilon} C_b^{2,\gamma}$ near $p_i \in \mathfrak{p}_{>\pi}$, $B(\delta) \to 0$ as $\delta \to 0$. Note that for fixed t and δ the integrals $E\big(u_t, \Sigma - \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_{i,t}(\delta)\big)$ are improper since we did not delete balls around the points $p \in \mathfrak{p}_{<\pi}$, but again the boundary contributions from deleted discs here do not contribute to $A(\delta)$. Let

$$\Sigma(\delta) := \Sigma - \cup D_{i,t}(\delta). \tag{6.61}$$

We use the same reasoning as in (6.59) to deduce that

$$A(\delta) = \int_{\Sigma(\delta)} \mathcal{L}_C G(C, \partial_\nu) ds - 2 \int_{\Sigma(\delta)} \dot{e}_0 \langle \dot{T}_0, \partial_\nu \rangle_g ds$$
$$- \int_{\Sigma(\delta)} e_0 \left.\frac{d}{dt}\right|_{t=0} \langle \dot{u}_{-t}, \partial_{\nu_{-t}} \rangle_{g(T_{-t})} ds,$$

where we have once again set $e_t = e(u_t)$. At this point we use that C is a Jacobi field, so in conformal coordinates $g = \sigma \left|dz^2\right|$, writing $\Phi(C) = \phi(z)dz^2$ for the the Hopf differential of C,

$$\mathcal{L}_C G = \dot{e}_0 g + 2\Re\phi(z)dz^2.$$

Since $\operatorname{Res} C = 0$, $\phi(z)dz^2$ is smooth on all of Σ. Plugging in and separating everything from the Hopf differential bit, we have (using (6.60))

$$\begin{aligned} A(\delta) = &\sum_{p \in \mathfrak{p}_{>\pi}} \int_0^{2\pi} 2\left(\Re\phi(z)dz^2\right)(C, \partial_r)\, \delta d\theta \\ &+ \sum_{p \in \mathfrak{p}_{>\pi}} \int_0^{2\pi} \bigg(\dot{e}_0 \langle C, \partial_r \rangle_g - 2\dot{e}_0 \langle \dot{T}_0, \partial_r \rangle_g \\ &\qquad - \left.\frac{d}{dt}\right|_{t=0} \langle \dot{u}_{-t}, \partial_{r_{-t}} \rangle_{g(T_{-t})} \bigg) \delta d\theta \end{aligned} \tag{6.62}$$

Since ϕ is bounded, the first term on the right hand side goes to zero as $\delta \to 0$.

The point is that up to terms vanishing with δ the bottom line consists exactly of the cancelling terms from (6.59), proving Lemma 6.10. To show this, we start with the rightmost term. Since $\ddot{T} \equiv 0$ near $p \in \mathfrak{p}_{>\pi}$ and $\ddot{\widetilde{u}}_0 \in r^{2-2\mathfrak{a}+\epsilon}\mathcal{X}_b^{2,\gamma}$, we see that

$$\begin{aligned}
&\int_{\partial\Sigma(\delta)} e_0 \left.\frac{d}{dt}\right|_{t=0} \langle \dot{u}_{-t}, \partial_{\nu_{-t}}\rangle_{g(T_{-t})} ds \\
&= \sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \left(\langle \dot{T}_t + \dot{\widetilde{u}}_t, \partial_{r_{-t}}\rangle_{g(T_{-t})} \right) \delta d\theta \\
&= \sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \langle \dot{T}_t, \partial_{r_{-t}}\rangle_{g(T_{-t})} \delta d\theta \\
&\quad + \sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \left(\langle \dot{\widetilde{u}}_t, \partial_{r_{-t}}\rangle_{euc} \sigma(T_{-t}) \right) \delta d\theta \\
&= \sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \langle \dot{T}_t, \partial_{r_{-t}}\rangle_{g(T_{-t})} \delta d\theta + \sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} O(\delta^{2-2\alpha+\epsilon}) \sigma(z) \delta d\theta
\end{aligned}$$

but in the last term, since $\sigma = O(\delta^{2(\alpha-1)})$, the integrand is $O(\delta^{1+\epsilon}) d\theta$ so

$$\begin{aligned}
&\int_{\partial\Sigma(\delta)} e_0 \left.\frac{d}{dt}\right|_{t=0} \langle \dot{u}_{-t}, \partial_{\nu_{-t}}\rangle_{g(T_{-t})} ds \\
&= \sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \langle \dot{T}_t, \partial_{r_{-t}}\rangle_{g(T_{-t})} \delta d\theta + O(\delta^{1+\epsilon})
\end{aligned} \tag{6.63}$$

Thus as $\delta \to 0$ this approaches the final term in (6.59). Note that in the end $\ddot{\widetilde{u}}_0$ disappears completely. Next comes the middle term in (6.62). Using that $T_0 = id$, $\widetilde{u}_0 = id$, we have

$$\dot{e}_0 = \left.\frac{d}{dt}\right|_{t=0} e(T_t) + \left.\frac{d}{dt}\right|_{t=0} e(\widetilde{u}_t),$$

and by (3.1), $\left.\frac{d}{dt}\right|_{t=0} e(\widetilde{u}_t) = O(\delta^{1+\epsilon-2\alpha})$. Using this and $\ddot{T}_0 \equiv 0$ near $\mathfrak{p}$, we see that

$$\sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} -2\dot{e}_0 \langle \dot{T}_0, \partial_r\rangle_g \delta d\theta = \sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} e(T_t) \langle \dot{T}_0, \partial_\nu\rangle_g + O(\delta^\epsilon),$$

And this indeed approaches the middle term of (6.59). That the first term in (6.62) limits to the first term in (6.59) follows in the same way. This completes the proof □

We can now prove

Corollary 6.13. *The C_j are conformal Killing fields.*

This is an immediate corollary of Lemma 6.10 and the following.

Lemma 6.14. *The Hessian of the energy $E(\cdot, g_0, G)$, considered as a bilinear form on $T(\mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{D}) + span\{C_i\}$ is positive semi-definite and vanishes only on the conformal Killing fields.*

Proof. Assume that $u_0 = id$. For genus $\Sigma > 1$, $\mathcal{C}onf_0 = \{id\}$. From the decomposition (5.7) and our work above, it follows that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, G) = \left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, H_1) + \left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, H_2).$$

Note that this is a non-trivial statement, since it is by no means clear that, for example, the function $t \longmapsto E(u_t, g, H_1)$ is twice differentiable; but it indeed is, by exactly the same computations we just preformed. The fact that conformal maps are global minimizers of energy now implies that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, H_1) \geq 0,$$

so since the left hand side of our first equation is zero by Lemma 6.10, we arrive at

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, H_2) = 0.$$

But in fact $E(\cdot, g, H_2)$ if positive definite, as we now show. Cutting out conformal discs $D_j(\epsilon)$ as above, near $p \in \mathfrak{p}_{>\pi}$ the boundary term in (6.31) is

$$\begin{aligned}\lim_{\epsilon \to 0} \int_{\partial D_j(\epsilon)} \langle \nabla_{u_* \partial_\nu} C, C \rangle_{u^* H_2} ds &= \lim_{\epsilon \to 0} \int_0^{2\pi} \partial_r \langle C, C \rangle_{u^* H_2} \epsilon d\theta \\ &= \lim_{\epsilon \to 0} \int_0^{2\pi} \partial_r \left| w + O(r^{2-2\alpha_j}) \right| \epsilon d\theta \\ &= 0.\end{aligned}$$

Near $p \in \mathfrak{p}_{<\pi}$ the same computation shows that the contribution is also zero. We now see from (6.31) that

$$\int_M \left(\langle \nabla C, \nabla C \rangle_{u^* H_2} + \mathrm{tr}_g \, R^{H_2}(du, C, C, du) \right) dVol_g = 0,$$

which immediately implies that $C = 0$ by negative curvature. (In particular C is conformal Killing.)

Now assume the genus of Σ is 1. As above we lift to the universal cover and use (5.8). As in the previous paragraph we conclude that

$$\int_M \left(\langle \nabla C, \nabla C\rangle_{u^*K_2} + \operatorname{tr}_g R^{K_2}(du, C, C, du)\right) dVol_g = 0,$$

but now we can only have flatness and thus can only conclude that

$$^{K_2}\nabla C = 0. \tag{6.64}$$

The lift of K_2 to the universal cover, $\widetilde{K}_2$, written with respect to the global coordinates $\widetilde{z}$ on $\mathbb{C}$, is a constant coefficient metric. Hence (6.64) means $C \equiv v$ for some constant vector $v \in \mathbb{C}$, which, as desired, is conformal Killing. The first statement follows exactly as in the genus $\Sigma > 1$ case. □

6.5. Proof of Proposition 4.1

Recall from Lemma 6.4 that the Fredholm map

$$L\colon r^{1-\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$$

satisfies

$$L(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}) \oplus \mathcal{K} = r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}, \tag{6.65}$$

and the sum is L^2 orthogonal. Thus

$$\Big(L(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}) \oplus \mathcal{K}\Big) \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}. \tag{6.66}$$

Since $L\colon r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$ is also Fredholm for ϵ small, $L\big(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}\big) \subset L(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}) \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$ is a finite index inclusion, and we can find

$$W \subset L(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}) \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} \tag{6.67}$$

so that

$$\left(L\left(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}\right) + W\right) \oplus \mathcal{K} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{2,\gamma}. \tag{6.68}$$

We now use (6.25) and the subsequent paragraph to show that

$$\psi \in W \implies \begin{cases} \psi \in r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} \\ \psi = L\psi' \quad \text{for } \psi' \in r^{1-\epsilon}\mathcal{X}_b^{0,\gamma}, \end{cases} \tag{6.69}$$

so from (6.24) we have $\psi' = \psi_1 + \psi_2$, where

$$\psi_1 = \sum_{(s,p)\in\Lambda\cap[1-\epsilon,1+\epsilon]} r^s \log^p(r) a_{s,p}(\theta),$$

and $\psi_2 \in r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}$. Looking at the eigenvectors in (6.42) shows that $\psi_1(z) = \lambda z$ for some $\lambda \in \mathbb{C}$. Thus $\psi' \in T_{id}\mathcal{D} + r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}$, which implies that $L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}) + W = L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} + T_{id}\mathcal{D})$. By (6.68)

$$L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} + T_{id}\mathcal{D}) \oplus \mathcal{K} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}, \tag{6.70}$$

where $\mathcal{K} \perp L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} + T_{id}\mathcal{D})$. Let

$$\pi_{\mathcal{K}} = \text{projection onto } \mathcal{K} \text{ in (6.70)}.$$

Now we add $T_{id}\mathcal{T}_{>\pi}$ to the domain of L. Let $\psi \in T_{id}\mathcal{T}_{>\pi}$ corresponding to $w \in \mathbb{C}^n$, so near $p_i \in \mathfrak{p}_{>\pi}$ we have $\psi \equiv w_i$ Since near p_i we have $Lw_i \equiv 0$, we know that

$$L\psi \in r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}. \tag{6.71}$$

Using the basis for $\mathcal{K}$ in (6.52), (6.71) implies that we can write $\pi_{\mathcal{K}}L\psi = \sum_{j=1}^N \langle L\psi, K_{a^i}\rangle_{L^2} K_{a^i} + \sum_{k=1}^M \langle L\psi, C_k\rangle_{L^2} C_k$, for the L^2 inner product in (6.29). Using (6.36) for any $\widetilde{\psi} \in \mathcal{K}$, $\langle L\psi, \widetilde{\psi}\rangle_{L^2} = -4\pi\Re\sum_{p_i\in\mathfrak{p}_{>\pi}} w_i \operatorname{Res}|_{p_i}\Phi(\widetilde{\psi})$. This immediately implies that $\pi_{\mathcal{K}}L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \oplus T_{id}\mathcal{D} \oplus T_{id}\mathcal{T}_{>\pi}) = \operatorname{span}\langle K_{a^i}\rangle$, and thus we have shown that

$$L\left(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \oplus T_{id}\mathcal{D} \oplus T_{id}\mathcal{T}_{>\pi}\right) \oplus \mathrm{T}\mathcal{C}onf_0 \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma},$$

which is what we wanted.

6.6. Proof of Proposition 4.3

For the proof of Proposition 4.3 we will need a formula for the second variation of energy in the direction of an arbitrary $J_w \in T_{u_0}\mathcal{H}arm_{\mathfrak{q}}$. First, we compute the first variation near a solution (HME($\mathfrak{q}$)).

Proposition 6.15. *Let* (u_0, g_0, G) *solve (HME*$(\mathfrak{q})$*) and satisfy Assumption 3.1. With notation as in the previous section, for any* $w \in \mathbb{C}^{|\mathfrak{q}|}$ *we have*

$$\frac{d}{dt}\bigg|_{t=0} E(u_{tw}, g_0, G) = \Re\left(2\pi i \sum_{p_i \in \mathfrak{p}_{<\pi}} \mathrm{Res}|_{p_i} \left(\iota_{J_w}\Phi(u_0)\right)\right), \tag{6.72}$$

and if $\Phi(u_0) = \phi_{u_0} dz^2$

$$\mathrm{Res}|_{p_i} \left(\iota_{J_w}\Phi\right) = w_i \,\mathrm{Res}|_{p_i}\phi_{u_0}. \tag{6.73}$$

Corollary 6.16. *Let* (u_0, g_0, G) *be a solution to (HME*$(\mathfrak{q})$*). Then* u_0 *minimizes* $E(\cdot, g_0, G)$ *in its free homotopy class if and only if it solves (HME*$(\mathfrak{q})$*) with* $\mathfrak{q} = \varnothing$*, i.e. if and only if* $\Phi(u_0)$ *is holomorphic on* (Σ, g_0)*.*

Proof. That $\mathrm{Res}\,\Phi(u_0) = 0$ implies the minimizing property is the content of Proposition 5.2.

For the other direction, if (u_0, g_0, G) solves (HME$(\mathfrak{q})$) and satisfies Assumption 3.1, then

$$\mathrm{Res}\,\Phi(u) \neq 0 \implies u_0 \text{ is not energy minimizing}$$

since by (6.72), if $w \in \mathbb{C}^{|\mathfrak{q}|}$ has $\Re 2\pi i \sum_{p_i \in \mathfrak{p}_{<\pi}} w_i \,\mathrm{Res}|_{p_i}\phi_{u_0} \neq 0$ (which is easy to arrange), then

$$\frac{d}{dt}\bigg|_{t=0} E(u_{tw}, g_0, G) \neq 0,$$

contradicting minimality. □

Remark 6.17. The one form $\iota_{J_w}\Phi(u_0)$ is not holomorphic (since J_w is not), but we will show that it still has a residue, meaning a limit

$$\mathrm{Res}\,\iota_{J_w}\Phi := \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|z|=e} \iota_{J_w}\Phi(z).$$

By our assumption that $u_0 = id$ and decomposition (2.36), in conformal coordinates the metric G is expressed by

$$G = \sigma \,|dz|^2 + 2\Re\phi(z) dz^2. \tag{6.74}$$

Below, these coordinates are used on both the domain and the target.

Proof of Proposition 6.15. The proof is similar to the proof of Lemma 6.10. As always assume $u_0 = id$. We can write $u_{tw} = \widetilde{u}_{tw} \circ T_{tw}$ where $\widetilde{u}_{tw} \in \mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{D} \circ \mathcal{T}_{>\pi}$ and $T_{tw} \in \mathcal{T}_{\mathfrak{q}}$, and $\frac{d}{dt}\big|_{t=0} u_{tw} = J_w$ (see(4.11)). Then

$$\frac{d}{dt}\bigg|_{t=0} E\left(u_{tw}, \Sigma\right) = \underbrace{\frac{d}{dt}\bigg|_{t=0} E\left(u_{tw}, \Sigma - \bigcup_{i=1}^{|p|} D_{tw}(\delta)\right)}_{:=A(\delta)} + \underbrace{\frac{d}{dt}\bigg|_{t=0} E\left(u_{tw}, \bigcup_{i=1}^{|p|} D_{tw}(\delta)\right)}_{:=B(\delta)}.$$

As in Lemma 6.10 $B(\delta) \to 0$ as $\delta \to 0$. Thus we arrive at

$$\frac{d}{dt}\bigg|_{t=0} E\left(u_{tw}, \Sigma\right) = \lim_{\delta \to 0} A(\delta).$$

By the chain rule,

(6.75)
$$A(\delta) = \underbrace{\frac{d}{dt}\bigg|_{t=0} E\left(u_{tw}, \Sigma - \bigcup_{\mathfrak{p}} D_0(\delta)\right)}_{:=A_1(\delta)} + \underbrace{\frac{d}{dt}\bigg|_{t=0} E\left(u_0, \Sigma - \bigcup_{\mathfrak{p}} D_{tw}(\delta)\right)}_{:=A_2(\delta)}$$

As we will see shortly, the term $A_2(\delta)$ is not in general bounded as $\delta \to 0$, but we will show that $A_1(\delta)$ decomposes into a sum of two terms, $A_1(\delta) = A_1^1(\delta) + A_1^2(\delta)$ where $A_1^1(\delta) \sim -A_2(\delta)$ (i.e. it cancels the singularity), and $A_1^2(\delta)$ converges to the expression in (6.72). A_1 is an integral over a smooth manifold with boundary so by the first variation formula (2.6), and in the last line using decomposition (6.74), if $\Sigma(\delta)$ is as in (6.61), we have

$$\begin{aligned} A_1 &= \int_{\partial\Sigma(\delta)} \langle J_w, \partial_\nu \rangle_G ds \\ &= \underbrace{-\sum_{p_i \in \mathfrak{p}_{<\pi}} \int_0^{2\pi} g\left(J_w, \partial_r\right) \delta d\theta}_{:=A_1^1} - \underbrace{\sum_{p_i \in \mathfrak{p}_{<\pi}} \int_0^{2\pi} \Re\Phi\left(J_w, \partial_r\right) \delta d\theta}_{A_1^2}. \end{aligned}$$

It thus remains only to show: 1) $\lim_{\delta\to 0} A_1^2 = \Re\left(2\pi i \operatorname{Res}\left(\iota_{J_w}\Phi\right)\right)$, 2) (6.73) holds, and 3) $\lim_{\delta\to 0}\left|A_1^1 + A_2\right| = 0$.

We prove numbers 1 and 2 together:

$$\int_{\partial D_0(\delta)} \Re\Phi\left(J_w, \partial_r\right) \delta d\theta = \int_{\partial D_0(\delta)} \Re\phi(z) dz^2 \left(J_w, \partial_r\right) \delta d\theta.$$

Since

$$dz^2\left(J_w, \partial_r\right) = J_w dz\left(\partial_r\right) = J_w \frac{z}{|z|},$$

we have

$$\int_0^{2\pi} \Re\Phi\left(J_w, \partial_r\right) \delta d\theta = \Re \int_0^{2\pi} J_w \phi(z) dz, \tag{6.76}$$

and by definition

$$\int_0^{2\pi} \Re\Phi\left(J_w, \partial_r\right) \delta d\theta = \Re \int_0^{2\pi} \iota_{J_w}\Phi(z). \tag{6.77}$$

So

$$\begin{aligned} \sum_{p_i \in \mathfrak{p}_{<\pi}} \lim_{\delta\to 0} \int_0^{2\pi} J_w \phi(z) dz &= \sum_{p_i \in \mathfrak{p}_{<\pi}} \lim_{\delta\to 0} \int_0^{2\pi} \left(-w_i + O(|z|)\right) \phi(z) dz \\ &= -2\pi i \sum_{p_i \in \mathfrak{p}_{<\pi}} w_i \operatorname{Res}|_{p_i} \phi_{u_0}. \end{aligned} \tag{6.78}$$

Putting (6.78) together with (6.77) gives us what we wanted.

For number 3, note that from the expression $g = e^{2\mu} |z|^{2(\alpha-1)}$ we have

$$A_1^1(\delta) = \int_0^{2\pi} \langle -w_i + O(|z|), \partial_r\rangle_g r d\theta = \int_0^{2\pi} \langle -w_i, \partial_r\rangle_g r d\theta + O(\delta^{2\alpha})$$

Using (6.57) and that fact that near $p \in \mathfrak{p}$, $T_{tw}^{-1}(z) = z + tw$ parametrizes the boundary of $\partial D_{tw}(\delta)$ we have

$$A_2 = \sum_{i=1}^{|p|} \int_0^{2\pi} \langle \dot{T}_0^{-1}, \partial_r\rangle_g r d\theta = \sum_{i=1}^{|p|} \int_0^{2\pi} \langle w_i, \partial_r\rangle_g ds + O(\delta^{2\alpha}).$$

Thus $\left|A_1^1 + A_2\right| = O(\delta^{2\alpha})$, and the proof is finished. □

Now suppose that u_0 solves (HME($\mathfrak{q}$)). Thus u_0 is an absolute minimum of $E(\cdot, g, G)$. By differentiating again, we get the following as a corollary to Proposition 6.15.

Corollary 6.18. *Suppose that* (u_0, g, G) *solves* (HME($\mathfrak{q}$)) *and satisfies Assumption 3.1. Then*

$$\frac{d^2}{dt^2}\bigg|_{t=0} \widetilde{E}(u_{tw}) = \Re\left(2\pi i \sum_{p_i \in \mathfrak{p}_{<\pi}} \mathrm{Res}|_{p_i}\left(\iota_{J_w}\Phi(J_w)\right)\right) = \Re\left(2\pi i \sum_{p_i \in \mathfrak{p}_{<\pi}} w_i\,\mathrm{Res}|_{p_i}\phi(J_w)\right) \tag{6.79}$$

where $\Phi(J_w)dz^2 = \frac{d}{dt}\big|_{t=0}\phi_{u_{tw}}dz^2$

We now conclude the proof Proposition 4.3 using Proposition 6.15

Proof of Proposition 4.3. We proceed by interpreting the formula for the Hessian in Corollary 6.18 in light of the characterization of the Hessian of the energy functional at a solution to (HME($\mathfrak{q}$)) in Corollary 6.13. By Corollary 6.13, the Hessian of the energy functional is positive definite on any compliment of the conformal Killing fields, while by Corollary 6.18, if $\mathrm{Res}\, J_w = 0$, J_w is a zero of the Hessian, thus a conformal Killing field. □

7. Closedness: limits of harmonic diffeomorphisms

In this section we prove that $\mathcal{H}(\mathfrak{q})$ in (3.19) is closed. The following is a consequence of Corollary 7.10 below.

Theorem 7.1. *Let* (u_k, g_k, G_k) *be a sequence of solutions to (HME(*$\mathfrak{q}$*)) where* $g_k \to g_0$ *with* $g_k \in \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$*,* $G_k \to G_0$ *with* $G_k \in \mathcal{M}^*_{2,\gamma,\nu}(G_0, \mathfrak{p}, \mathfrak{a})$*, and all the* u_k *in Form 2.3. Assume that the scalar curvature* $\kappa_{G_k} \leq 0$ *and that each* u_k *has non-vanishing Jacobian away from* $u_k^{-1}(\mathfrak{p})$*. Then the* u_k *converge to a map* u_0 *so that* (u_0, g_0, G_0) *solves (HME(*$\mathfrak{q}$*)) in Form 2.3. The convergence is* $C^{2,\gamma}_{loc}$ *away from* $u_0^{-1}(\mathfrak{p})$*. For a precise description of the convergence near the cone points, see Corollary 7.10 below.*

To be precise about the convergence of the G_k, in conformal coordinates near $p \in \mathfrak{p}$ we have

$$G_k = c_k e^{2\mu_k}|w|^{2(\alpha-1)}|dw|^2, \quad \text{and} \quad G_0 = c_0 e^{2\mu_0}|w|^{2(\alpha-1)}|dw|^2.$$

By $G_k \to G_0$, we mean that for some $\sigma > 0$,

$$\begin{aligned} &\mu_k \to \mu \text{ in } r^\nu C^{2,\gamma}_b(D(\sigma)) \\ &c_k \to c_0 \end{aligned} \tag{7.1}$$

(The b-Hölder spaces are defined in (2.22)–(2.24)). Away from the cone points $G_k \to G$ in $C^{2,\gamma}_{loc}$. We can (and do) reduce to the case $c_k = c_0 = 1$ by replacing G_k by G_k/c_k and G_0 by G_0/c_0. Note that (7.1) easily implies that near each cone point the scalar curvature functions κ_{G_k} converge to κ_{G_0} in $C^{0,\gamma}_b$ (see (2.27)); in particular the is a constant $c > 0$ independent of k for which

$$\kappa_{G_k} \geq c. \tag{7.2}$$

For the g_k, we make the stronger assumption that near $u^{-1}(\mathfrak{p})$ the metrics look like the standard round conic metric g_α (see (2.8)). To do this uniformly, we need the uniform bound on the modulus of continuity obtained in the next section; the precise statement of this assumption is in Section 7.1. In the end the theorem is true as stated (i.e. without this stronger assumption), since we will change the domain metric in a bounded way and in its conformal class.

We refer the reader to the introduction for an outline of the subsequent arguments. Before we prove Theorem 7.1, we use it to prove

Proposition 7.2. *$\mathcal{H}(\mathfrak{q})$ in (3.19) is closed.*

Proof. Let $t_k \in \mathcal{H}(\mathfrak{q})$ be a sequence such that $t_k \to t_0$, and let $\mathfrak{c}_k$ be the corresponding conformal structures, so $\mathfrak{c}_k \to \mathfrak{c}_0$. As in the proof of Proposition 4.4, we uniformize locally, i.e. we choose diffeomorphisms v_k so that $id\colon (\Sigma, \mathfrak{c}_0) \longrightarrow (\Sigma, v_k^*\mathfrak{c}_k)$ is conformal near $\mathfrak{p}$ in such a way that $v_k \to id$ in C^∞. By assumption, there is a rel. $\mathfrak{q}$ minimizer $u_k\colon (\Sigma, \mathfrak{c}_k) \longrightarrow (\Sigma, G)$. Let g_k be metrics in $\mathfrak{c}_k$ and g_0 be a metric in $\mathfrak{c}_0$ that are conic near near $u_k^{-1}(\mathfrak{p})$ with cone angles $\mathfrak{a}$. Then $u_k \circ v_k\colon (\Sigma, v_k^* g_k) \longrightarrow (\Sigma, G)$ are also rel. $\mathfrak{q}$ minimizers, but now $v_k^* g_k \to g_0$ in $\mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$, and thus Theorem 7.1 applies. The limiting map u_0 is the minimizer we desire, so $\mathcal{H}(\mathfrak{q})$ is closed. □

7.1. Energy bounds, uniform continuity, energy density, and the Jacobian.

The u_k in Theorem 7.1 are in fact a rel. $\mathfrak{q}$ energy minimizing sequence for the metric G_0, meaning that

$$\limsup_{k\to\infty} E(u_k, g_k, G_0) = \inf_{u \sim_{rel. \mathfrak{q}} id} E(u, g_k, G_0). \tag{7.3}$$

This follows from Proposition 5.2, since for any $u \sim_{rel.\mathfrak{q}} id$

$$\limsup_{k\to\infty} E(u_k, g_0, G_0) = \limsup_{k\to\infty} E(u_k, g_k, G_k) \\ \leq \lim_k E(u, g_k, G_k) = E(u, g_0, G_0).$$

Assuming that genus $\Sigma > 0$, by the Courant-Lebesgue Lemma, the u_k are an equicontinuous sequence. Thus they subconverge. Let

$$R := u_0^{-1}(\Sigma_{\mathfrak{p}}).$$

It is standard that the u_k converge uniformly in $C^{2,\gamma}_{loc}$ with Jacobian uniformaly bounded below on compact subsets of R, [Tr], [J]. For $\Sigma = S^2$, lifting as in (5.9) to a branched cover and applying the above arguments gives the same results. To summarize, we have

Lemma 7.3. *The u_k converge in $C^0(\Sigma) \cap C^{2,\gamma}_{loc}(R)$ to a map u_0. On each compact subset of R the Jacobian of u_0 is bounded below by a positive constant.*

Fix $p \in \mathfrak{p}$, and let

$$q_k = u_k^{-1}(p),$$

we can pass to a subsequence so that

$$q_k \to q_0,$$

for some q_0. Let $S \subset \Sigma$ be any set containing $u_0^{-1}(p)$ so that $S \cap u_0^{-1}(\mathfrak{p} - \{p\}) = 0$ and S is diffeomorphic to a disc, and choose conformal maps

$$(7.4) \qquad \begin{aligned} F_k \colon D &\longrightarrow S \\ 0 &\longmapsto q_k \end{aligned}$$

so that $F_k \to F_0$ in C^∞. Finally, define

$$(7.5) \qquad w_k := u_k \circ F_k.$$

Thus we have a sequence of harmonic maps $w_k \colon D \longrightarrow (\Sigma, G_k)$ with $w_k(0) = p$. By uniform continuity, we may choose a single conformal coordinate chart containing $w_k(D) = u_k \circ F_k(D)$ for all k. By abuse of notation, we denote these coordinates by w. Our goal is to prove uniform estimates for the w_k. Specifically, we wish to control their energy densities and Jacobians.

Consider the maps

$$w_k\colon (D, g_\alpha) \longrightarrow (\Sigma, G_k), \tag{7.6}$$

where $2\pi\alpha$ is the cone angle at p and again $g_\alpha = |z|^{2(\alpha-1)}\,|dz|^2$.

Proposition 7.4. *The maps (7.6) have uniformly bounded energy density (see (3.1)), i.e. for some $C > 0$*

$$e_k(z) = e(w_k, g_\alpha, G_k)(z) < C \text{ for } |z| \leq 1/2. \tag{7.7}$$

At $z = 0$ we have the lower bound

$$0 < c \leq \lim_{z\to 0} e_k(z, g_\alpha, G_k) \tag{7.8}$$

for some uniform c.

Remark 7.5. The uniform bounds on the energy density e_k from above and below holds only if the domain is given the type of cone metric it is given in Theorem 7.1. This may seem enigmatic, since the question of energy minimization is not influenced by the metric on the domain, only its conformal class. Indeed, the conic geometry of the domain, though naturally suited to the analysis, is not essential; it merely provides the cleanest mode of exposition.

Before we begin the proof, recall that by Form 2.3,

$$w_k(z) = \lambda_k z + v_k(z) \quad \text{with} \quad v_k \in r^{1+\epsilon} C_b^{2,\gamma}. \tag{7.9}$$

Proof of Proposition 7.4. For any harmonic map $w\colon (D, \sigma\,|dz|^2) \longrightarrow (D, \rho\,|dw|^2)$, as in [W3], let

$$h(z) = \frac{\rho(w(z))}{\sigma(z)}\,|\partial_z w|^2, \quad \ell(z) = \frac{\rho(w(z))}{\sigma(z)}\,|\partial_{\overline{z}} w|^2, \tag{7.10}$$

so the energy density and the Jacobian satisfy, respectively

$$e = h + \ell \quad \& \quad J = h - \ell. \tag{7.11}$$

The proposition will follow from analysis of the following inequality and identity, which are standard and can be found for example in [SY], where they appear as equations (1.19) and (1.17), respectively.

$$\begin{aligned} \Delta e(u) &\geq -2\kappa_\rho J + 2\kappa_\sigma e(u) \\ \Delta \log h &= -2\kappa_\rho (h - \ell) + \kappa_\sigma . \end{aligned} \tag{7.12}$$

Here κ_ρ and κ_σ are the scalar curvature functions for the range and domain, respectively, and Δ is the Laplacian for $\sigma \left|dz\right|^2$. The second equation holds only when $h(z) \neq 0$, and of course both equations make sense only when σ and ρ are sufficiently regular. For the w_k in (7.6), the equations simplify as follows: 1) $\sigma(z) \left|dz\right|^2 = g_\alpha$ has $\kappa_\sigma \equiv 0$ away from $z = 0$, 2) $\kappa_{\rho_k} \leq 0, J \geq 0$ by the assumptions of Theorem (7.1), and 3) $\kappa_{\rho_k} > -C$ by (7.2). Therefore, we restrict our attention to the inequalities

$$\Delta e_k \geq 0 \tag{7.13}$$

$$\Delta \log h_k \leq C h_k . \tag{7.14}$$

To prove (7.7), we use (7.13) as follows. Since $\Delta = \left|z\right|^{2(1-\alpha)} \Delta_0$ where $\Delta_0 = 4\partial_z \partial_{\overline{z}}$ is the euclidean Laplacian, we also have

$$\Delta_0 e_k \geq 0, \tag{7.15}$$

away from $z = 0$. We claim that in fact each e_k is a subsolution to (7.15) on all of D. To see this, write $e_k = h_k + \ell_k$ as in (7.11). Using (7.9), we have

$$\begin{aligned} h_k(z) &= \frac{\rho_k(w_k(z))}{\left|z\right|^{2(\alpha-1)}} \left|\partial_z w_k(z)\right|^2 \\ &= \lambda_k^{2(\alpha-1)} \left|1 + \left|z\right|^{2(1-\alpha)} v_k(z)\right|^{2(\alpha-1)} e^{2\mu_k(w_k(z))} \left|\lambda_k + \partial_z v_k(z)\right|^2 \\ &= \lambda_k^{2\alpha} \left|1 + \left|z\right|^{2(1-\alpha)} v_k(z)\right|^{2(\alpha-1)} e^{2\mu_k(w_k(z))} \left|1 + \partial_z v_k(z)/\lambda_k\right|^2 , \end{aligned}$$

so since $v_k \in r^{1+\epsilon} C_b^{2,\gamma}$ and $\mu_k \in r^\nu C_b^{2,\gamma}$, we have

$$h_k - \lambda_k^{2\alpha} \in r^\epsilon C_b^{1,\gamma} \tag{7.16}$$

for some $\epsilon > 0$. Similarly, using $\partial_{\overline{z}} z = 0$,

$$\ell_k \in r^\epsilon C_b^{1,\gamma}. \tag{7.17}$$

Therefore

$$e_k = \lambda_k^{2\alpha} + f_k(z) \quad \text{for } f_k \in r^\epsilon C_b^{1,\gamma}(D). \tag{7.18}$$

In particular, $r\partial_r e_k \to 0$ as $r \to 0$. Therefore, for any non-negative function $\zeta \in C_c^\infty(D)$.

$$(7.19)\quad \int_D \nabla e_k \cdot \nabla \zeta dxdy = -\lim_{\epsilon\to 0} \int_{D-D(\epsilon)} (\Delta e_k)\zeta dxdy - \int_{r=\epsilon} (\partial_r e_k)\zeta r d\theta \le 0$$

so the e_k are indeed subsolutions. By (7.18), each e_k is a bounded function, so the standard theory of subsolutions to elliptic linear equations [Mo, Section 5] implies that for some $C > 0$,

$$\sup_{z\in D(1/2)} e_k(z) \le C \int_D e_k dxdy.$$

The right hand side is controlled by the energy,

$$\int_D e_k dxdy \le \int_D e_k \, |z|^{2(\alpha-1)} \, dxdy = E(w_k, D, G_k),$$

and this establishes (7.7).

It remains to prove (7.8). From (7.16) and (7.17), we now see that

$$(7.20)\quad \lim_{z\to 0} e_k(z) = \lim_{z\to 0} h_k(z) = \lambda_k^{2\alpha},$$

Thus, to prove (7.8), it is equivalent to prove that $\lambda_k \ge c > 0$ for some c independent of k. To do so, we use (7.14). Dropping the k's for the moment, by (7.16) and the fact that the logarithm is smooth and vanishes simply at 1, we have

$$(7.21)\quad \log h - \log|\lambda|^{2\alpha} \in r^\epsilon C_b^{2,\gamma}.$$

We will now apply the assumption from Theorem 7.1 regarding the Jacobian, specifically that $J = h - \ell > 0$ on compact sets away from 0, and thus by continuity and (7.20) we may choose a δ (depending on h) satisfying

$$0 < \delta \le \frac{1}{2} \inf_{z\in D} h(z).$$

Thus h/δ is bounded from below by 2. We also need control from above. We already know by (7.7) that $h + \ell = e < c$ for some $c > 0$, so $\sup_{z\in D} h(z) < c$

for some constant depending only on the energy. Using this and the elementary bound $h \leq \frac{c}{\log(c'/\delta)} \log\left(\frac{h}{\delta}\right)$ we conclude from (7.14) that

$$\Delta \log(h/\delta) \leq \frac{c}{\log(c'/\delta)} \log\left(\frac{h}{\delta}\right). \tag{7.22}$$

Note that $\log h/\delta \geq \ln 2 > 0$, so this inequality looks promising for an application of the Harnack inequality. We use the following explicit inequality, inspired by Lemma 6 of [He].

Remark 7.6. We prove the following lemma under more general assumptions than currently necessary so that it may be applied in Section 8 where we deal with the case $\mathfrak{p}_{=\pi} \neq \varnothing$.

Lemma 7.7 (Harnack Inequality). *Let Δ denote the Laplacian on the standard cone (D, g_α). Let $f\colon D \longrightarrow \mathbb{R}$, $f \in C^2(\overline{D} - \{0\})$, $f > 0$, and assume that for some $\sigma \in \mathbb{R}$,*

$$\left(\Delta - \sigma^2\right) f \leq 0 \quad \text{on } D - \{0\}.$$

Furthermore, assume that

$$f = a + b(\theta) + v(r, \theta) \quad \text{for } v \in r^\epsilon C_b^{2,\gamma}(D), \tag{7.23}$$

for $a \in \mathbb{C}$ and $b \in C^\infty(S^1)$. Then if $\sigma < \epsilon$, there is a constant $c > 0$ such that

$$\liminf_{z \to 0} f = a + \inf b \geq e^{-c\sigma^2} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta. \tag{7.24}$$

Before proving the lemma, we conclude the proof of (7.8) (and thus of Proposition 7.4) by applying the lemma to (7.22) as follows. By (7.21), the lemma applies to (7.22) with $f = \log h/\delta$, $\sigma^2 = c/\log(c'/\delta)$, and $b \equiv 0$. We will choose $\eta > 0$ small so that if

$$\delta_k = \min\left\{\eta, \frac{1}{2} \inf_{z \in D - 0} h_k(z)\right\} \tag{7.25}$$

then the hypotheses of the lemma are satisfied. Thus by the lemma

$$\lim_{z\to 0}\log(h_k(z)/\delta_k) \geq e^{\frac{-c}{\log(c'/\delta_k)}}\frac{1}{2\pi}\int_0^{2\pi}\log(h_k/\delta_k)d\theta$$
$$\lim_{z\to 0}\log h_k(z) \geq e^{\frac{-c}{\log(c'/\delta_k)}}\left(\int_0^{2\pi}\log h_k\ d\theta\right) - \left(e^{\frac{-c}{\log(c'/\delta_k)}} - 1\right)\log\delta_k.$$

Since $\left(\exp(\frac{-c}{\log(c'/\delta_k)}) - 1\right)\log\delta_k \geq c'' > 0$ for some c'' independent of δ_k and $\exp(\frac{-c}{\log(c'/\delta_k)})$ is bounded above, for some $c > 0$, we have

$$\lim_{z\to 0}\log h_k(z) = |\lambda_k|^{2\alpha} \geq -c\left|\int_0^{2\pi}\log h_k(e^{i\theta})\ d\theta\right| + c \tag{7.26}$$

By Lemma 7.3, the Jacobians, J_k, and thus the h_k, are uniformly bounded below on ∂D. Thus (7.26) gives a uniform lower bound for $h_k(0)$ from below, which finishes the proof of Proposition 7.4 modulo the proof of Lemma 7.7. □

Proof of Lemma 7.7. To prove the lemma we will work in normal polar coordinates. Let $\phi = \theta, \rho = r^\alpha/\alpha$. In these coordinates $g_\alpha = d\rho^2 + \alpha^2\rho^2 d\phi^2$ and $\Delta = \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\alpha^2\rho^2}\partial_\phi^2$. The fact that $f - (a + b(\theta)) \in r^\epsilon C_b^{2,\gamma}$ implies that $f - (a + b(\theta)) \in \rho^{\epsilon/\alpha}C_b^{2,\gamma}$. The exact vanishing rate is irrelevant, and we refer to it henceforth as ϵ. The proof proceeds by comparing f to the solution $\widetilde{f}$ to the equation

$$\begin{aligned}(\Delta - \sigma^2)\widetilde{f} &= 0\\ \widetilde{f}|_{\partial D} &= f|_{\partial D}.\end{aligned} \tag{7.27}$$

In fact, we have (noting that ∂D now occurs at $\rho = 1/\alpha$, the explicit solution

$$\widetilde{f}(\rho,\phi) = \sum_{n\in\mathbb{Z}}\frac{I_{n/\alpha}(\sigma r)}{I_{n/\alpha}(\sigma/\alpha)}a_n e^{in\phi} \tag{7.28}$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi}e^{-in\phi}f(1/\alpha,\phi)d\phi$, and the $I_{n/\alpha}$ are the standard modified Bessel functions [AS] and we have written f as a function of ρ,ϕ. In particular, by [AS, Equation 9.6.10]

$$\widetilde{f}(z) - \widetilde{f}(0) \in r^{1/\alpha}C_b^{2,\gamma} \tag{7.29}$$

It follows that $F = f - \widetilde{f}$ satisfies

$$\begin{aligned} \left(\Delta - \sigma^2\right) F \leq 0 \quad & \text{away from } z = 0 \\ F|_{\partial D} = 0 \quad & \text{where } \partial D = \{\rho = 1/\alpha\}. \end{aligned} \tag{7.30}$$

We will show that

$$F(\rho, \phi) \geq 0. \tag{7.31}$$

Assuming this for the moment, we have in particular that

$$\liminf_{z \to 0} f(z) \geq \widetilde{f}(0) = \frac{a_0}{I_0(\sigma/\alpha)} = \frac{1}{I_0(\sigma/\alpha)2\pi} \int_0^{2\pi} e^{-in\phi} f(1/\alpha, \phi) d\phi. \tag{7.32}$$

But by [AS, Equation 9.6.10], $I_0(\sigma/\alpha) \leq e^{-c\sigma^2}$ for some some $c > 0$, which implies (7.24).

It remains to show that $F \geq 0$. Switch back to conformal coordinates, $z = re^{i\theta}$. By assumption (7.23), $F(z) = F(r, \theta)$ is continuous on $[0, 1]_r \times S^1_\theta$, thus attains a minimum. If that minimum is on $r = 1$, we are done by (7.30). If it is in the interior and away from $z = 0$, say at z_0, then (7.30) implies that $0 \leq \Delta F \leq \sigma^2 F$, so since $\sigma^2 \geq 0$, $F(z_0) \geq 0$. Finally, assume that F attains its minimum on $r = 0$, and define

$$-m := \inf_{r=0} F(r, \theta) < 0.$$

Consider the function

$$F_\mu(z) = F(z) - \mu r^\nu, \text{ where } \mu > 0, \nu \geq \sigma. \tag{7.33}$$

Since $(\Delta - \sigma)r^\nu = r^\nu \left(r^{-2\alpha}\nu^2 - \sigma\right) \geq 0$, F_μ also satisfies $\left(\Delta - \sigma^2\right) F_\mu(z) \leq 0$. Since $F_\mu|_{\partial D} \equiv -\mu$ and $\liminf_{z \to 0} F_\mu(z) = -m$, if $-m \leq -\mu$ then $F_\mu = F_\mu(r, \theta)$ has a (negative) minimum away from ∂D. On the other hand, given any (r, θ) with $r > 0$,

$$F_\mu(0, \theta) \geq F_\mu(r, \theta) \iff -\mu \leq \frac{F(0, \theta) - F(r, \theta)}{r^\nu}.$$

Thus if we can find $\mu > 0$ so that for some $r > 0, \theta, \nu$

$$-m \leq -\mu \leq \frac{F(0, \theta) - F(r, \theta)}{r^\nu}, \tag{7.34}$$

then F_μ will have a negative interior minimum. By the assumptions of the lemma and (7.29) we have that $F(0,\theta) - F(r,\theta) \in r^\epsilon C_b^{2,\gamma}$, so

$$\nu < \epsilon \implies \frac{|F(0,\theta) - F(r,\theta)|}{r^\nu} \to 0 \text{ as } r \to 0.$$

Thus, (7.34) can be obtained as long as

$$\sigma < \epsilon. \tag{7.35}$$

□

Now we prove a classification lemma for harmonic maps of the standard cone C_α defined in (2.8) and (2.9) above.

Lemma 7.8. *Let $u\colon C_\alpha \longrightarrow C_\alpha$ be a smooth harmonic map (i.e. a solution to (HME($\mathfrak{q}$))) in Form 2.3, with uniformly bounded energy density, that is the uniform limit of a sequence of orientation-preserving homeomorphisms fixing the cone point. Then $u(z) = \lambda z$ for some $\lambda \in \mathbb{C} - \{0\}$.*

Proof. Let e be the energy density function of u (7.11), and let h, ℓ, and J be the functions defined in (7.10). We will use a differential inequality for ℓ, similar to those used in the proof of Proposition 7.4. Namely, (1.18) from [SY] gives

$$\Delta\ell \geq 2\kappa_\rho J\ell + 2\kappa_\sigma\ell,$$

where κ_ρ and κ_σ are the scalar curvatures on the domain and target, respectively, and the inequality holds only away from $z = 0$. Since C_α is flat away from the cone point, this gives $\Delta\ell(z) \geq 0$ when $z \neq 0$. The assumption that the energy density of u is uniformly bounded implies ℓ is uniformly bounded ($\ell < e$), and on the other hand, as in (7.19), ℓ is a subsolution on all of $\mathbb{C}$. Thus ℓ is identically constant. (There are no non-constant, bounded entire subsolutions.) Since $\ell \in r^\epsilon C_b^{2,\gamma}$, $\lim_{z\to 0} \ell = 0$, and thus $\ell \equiv 0$, i.e. $\partial_{\bar{z}} u = 0$ when $z \neq 0$. Since u bounded near 0, it is in fact entire. The fact that u is the uniform limit of homeomorphisms and takes Form 2.3 implies that it is $1-1$ on the inverse image of an open ball around 0. Since the only entire holomorphic function with this behavior that fixes 0 is $u(z) = \lambda z$ for $\lambda \in \mathbb{C}^*$, the proof is complete. □

7.2. Uniform $C_b^{2,\gamma}$ bounds near $\mathfrak{p}$ and the preservation of Form 2.3

We will now make precise the sense in which the w_k in (7.6) are uniformly bounded near $\mathfrak{p}$. We will need the well-known Rellich lemma for b-Hölder spaces; given $0 < \gamma' < \gamma < 1$, $c' < c$, and non-negative integers $k' \leq k$ the containment

$$r^c C_b^{k,\gamma}(D(R)) \subset r^{c'} C_b^{k',\gamma'}(D(R)) \tag{7.36}$$

is compact. (Here $D(R)$ is as in (2.21).) Given a smooth function $f \colon \mathbb{C} \longrightarrow \mathbb{R}$, define

$$\|f\|_{c,k,\gamma,R} := \|f\|_{r^c C_b^{k,\gamma}(D(R))}. \tag{7.37}$$

(See (2.22)–(2.23).) And for any map $w \colon D(R) \longrightarrow D(R)$ with

$$w(z) = \lambda z + v(z) \quad \text{and} \quad v \in r^c C_b^{2,\gamma}(D(R)), \tag{7.38}$$

let

$$[w]_{c,k,\gamma,R} := \|v\|_{c,k,\gamma,R} \tag{7.39}$$

Having established the uniform energy density bound, we will now prove

Proposition 7.9. *For the w_k in (7.6), there exist uniform constants $\sigma, C, \epsilon > 0$ such that*

$$[w_k]_{1+\epsilon,2,\gamma,\sigma} < C. \tag{7.40}$$

This proposition implies the following.

Corollary 7.10. *With F_k as in* (7.4) *and $w_k = \lambda_k z + v_k$ as in* (7.5)*, some subsequence of the w_k converges to a map $w_0 = \lambda_0 z + v_0$, in the sense that*

$$\begin{aligned} &\lambda_k \to \lambda_0 \\ &v_k \to v_0 \ \textit{in} \ r^{1+\epsilon} C_b^{2,\gamma}, \end{aligned} \tag{7.41}$$

for some $\epsilon, \gamma > 0$. Since the F_k in $u_k \circ F_k = w_k$ converge in C^∞ to some univalent conformal map F_0, the limit in $u_k \to u_0$ is of Form 2.3.

Proof of Corollary 7.10. The first line of (7.41) follows from Proposition 7.4 and (7.20). The second line of (7.41) follows from Proposition 7.9 and the fact that the containment (7.36) is compact. □

Before we prove Proposition 7.9, we discuss scaling properties of the norm in (7.37). Let $R > 0$, $\sigma > 0$, and let f be a function defined on $D(R)$. Define

$$f_\sigma(z) = f(\sigma z).$$

From (2.22)–(2.24),

$$\|f_\sigma\|_{c,k,\gamma,R/\sigma} = \sigma^c \|f\|_{c,k,\gamma,R} \tag{7.42}$$

For $w\colon D \longrightarrow D$ as in (7.38), $\frac{1}{\tau} w_\sigma(z)$ makes sense locally and equals $\frac{\lambda\sigma}{\tau} z + \frac{v_\sigma(z)}{\tau}$. By (7.42) we have

Lemma 7.11. *If $w(z) = \lambda z + v(z)$, then*

$$\left[\frac{1}{\sigma} w_\tau\right]_{c,k,\gamma,R/\tau} = \frac{\tau^c}{\sigma} [w]_{c,k,\gamma,R}$$

We will apply this lemma directly to the w_k in (7.6). For these maps $G_k = e^{2\mu_k} |w|^{2(\alpha-1)} |dw|^2$, and the map $\frac{1}{\sigma}(w_k)_\tau(z)$ is an expression in normalized conformal coordinates (see (2.25)) of the map

$$w_k\colon (D, g_\alpha/\tau^{2\alpha}) \longrightarrow (\Sigma, G_k/\sigma^{2\alpha}), \tag{7.43}$$

meaning simply that if we write z for z/τ and w for w/σ, then

$$g_\alpha/\tau^{2\alpha} = |z|^{2(\alpha-1)} |dz|^2 \quad \text{and} \quad G_k/\sigma^{2\alpha} = e^{2\mu_k(\sigma w)} |w|^{2(\alpha-1)} |dw|^2.$$

Before we begin the proof of Proposition 7.9, we recall that by (7.8) we know that $\lambda_k \to \lambda_0$ for some $\lambda_0 > 0$, and, by setting $G_k = G_k/\lambda_k^{2\alpha}$, we may assume without loss of generality that

$$\lambda_k \equiv 1$$

so that the normalized conformal coordinate expression is $w_k = z + v_k(z)$.

Proof of Proposition 7.9. We proceed by contradiction. Supposing Proposition 7.9 false, we will produce a sequence $\sigma_k \to 0$ so that the scaled maps

$$\frac{1}{\sigma_k} w_{k,\sigma_k},$$

converge to a harmonic map $w_\infty\colon C_\alpha \longrightarrow C_\alpha$ satisfying the assumptions but not the conclusion of Lemma 7.8. This contradiction proves the lemma.

If (7.40) does not hold then for all $\sigma, C > 0$ there is a k such that

$$[w_k]_{1+\epsilon,2,\gamma,\sigma} > C. \tag{7.44}$$

Thus, for every $l \in \mathbb{N}$ there is a k_l such that $[w_{k_l}]_{1+\epsilon,2,\gamma,1/l} > l^\epsilon$, and passing to a subsequence, we assume that $[w_k]_{1+\epsilon,2,\gamma,1/k} > k^\epsilon$. Since the the semi-norm in (7.39) is monotone increasing in R and tends to 0 as $R \to 0$, for each k there is a number $\sigma_k < 1/k$ such that $[w_k]_{1+\epsilon,2,\gamma,\sigma_k} = \sigma_k^{-\epsilon}$. Thus

$$\left[\frac{1}{\sigma_k} w_{k,\sigma_k}\right]_{1+\epsilon,2,\gamma,1} = 1. \tag{7.45}$$

By the remarks immediately preceding the proof, this map, which is the normalized coordinate expression of the map

$$w_k \colon (D, g/\sigma_k^{2\alpha}) \longrightarrow (\Sigma, G_k/\sigma_k^{2\alpha}) \tag{7.46}$$

is harmonic and in normalized conformal coordinates satisfies $\frac{1}{\sigma_k} w_{k,\sigma_k}(z) = z + \frac{1}{\sigma_k} v_{k,\sigma_k}(z)$. Define $\widetilde{v}_k := \frac{1}{\sigma_k} v_{k,\sigma_k}(z)$. Since $e(w_k, g/\sigma_k^2, G_k/\sigma_k^2) = e(w_k, g, G_k)$, the uniform bound (7.7) holds for the maps in (7.46), and thus they converge on compact subsets of $(D(1/\sigma_k), g_\alpha)$ to a map $w_\infty \colon C_\alpha \longrightarrow C_\alpha$. The following will finish the proof of Proposition 7.9 since it contradicts Lemma 7.8.

Claim 7.12. *The map w_∞ is in Form 2.3, with*

$$w_\infty(z) = z + v_\infty(z) \tag{7.47}$$

and $\widetilde{v}_k \to v_\infty \in r^{1+\epsilon} C_b^{2,\gamma}(D)$, for $\epsilon', \gamma' > 0$ sufficiently small. Furthermore

$$v_\infty(z) \not\equiv 0 \tag{7.48}$$

To prove the claim, set

$$\widetilde{G}_k = G_k/\sigma_k^{2\alpha}$$

Using the elliptic theory of b-differential operators from Section 6.2, we will prove that, for $\epsilon' < \epsilon$ as above and $0 < \gamma' < \gamma < 1$ we have the inequality

$$\|\widetilde{v}_k\|_{1+\epsilon,2,\gamma,1} \leq C \, \|\widetilde{v}_k\|_{1+\epsilon',2,\gamma',2} \, . \tag{7.49}$$

Note the shift in regularity and the fact that the norm on the right is on a ball of larger radius than the norm on the left. The left hand side of (7.49) is bounded from below by (7.45), so the right hand side is also bounded from below. By the compact containment (7.36), the v_{k,σ_k} converge strongly in the norm on the left, thus they converge to a non-zero function, i.e. (7.48) holds.

Thus it remains to prove the estimate (7.49). To do so, we apply Taylor's theorem to the Harmonic map operator τ around the map $id\colon (D, g_\alpha) \longrightarrow (D, \widetilde{G}_k)$. Let $r^c \dot{C}_b^{k,\gamma}(C_\alpha)$ denote the set of maps $v \in r^c C_b^{k,\gamma}(D(2))$ which vanish on $\partial D(2)$. By Section 6.1 we have

$$\begin{aligned} \tau(w_k, g_\alpha, \widetilde{G}_k) &= \tau(id, g_\alpha, \widetilde{G}_k) + L_k \widetilde{v}_k + Q_k(\widetilde{v}_k) \\ L_k \widetilde{v}_k &= -Q_k(\widetilde{v}_k) \end{aligned}$$

All the materlial in Section 6.2 applies to

$$L_k\colon r^{1+\epsilon} \dot{C}_b^{j,\gamma}(D(2)) \longrightarrow r^{1+\epsilon-2\mathfrak{a}} C_b^{j-2,\gamma}(D(2)). \tag{7.50}$$

In particular, it is Fredholm for any γ, j, and ϵ small, and it has an generalized inverse $\mathcal{G}_k$ which satisfies the mapping properties analogous to (6.25). By Lemma 6.6, (7.50) is injective, so $\mathcal{G}_k L_k = I$. From (6.10), for $\epsilon' < \epsilon$ as above, we have $\|Q_k(\widetilde{v}_k)\|_{1+2\epsilon'-2\alpha,j,\gamma,2} \leq C \|\widetilde{v}_k\|_{1+\epsilon',j+1,\gamma,2}$. Let $\chi(r)$ be a cut-off function that is 1 on D and supported in $D(2)$. Then we have

$$\mathcal{G}_k \chi Q_k(\widetilde{v}_k) = -\mathcal{G}_k \chi L_k(\widetilde{v}_k) = -\mathcal{G}_k [L_k, \chi] \widetilde{v}_k - \chi \widetilde{v}_k,$$

so $\chi \widetilde{v}_k = \mathcal{G}_k \chi Q_k(\widetilde{v}_k) - \mathcal{G}_k [L_k, \chi] \widetilde{v}_k$. Since $[L_k, \chi]$ is the zero operator near the cone point, it maps $r^{1+\epsilon} C_b^{j,\gamma}(D(2))$ to $r^N C_b^{j-1,\gamma}(D(2))$ for any $N > 0$. Tracing through all of the boundedness properties above, we get

$$\begin{aligned} \|\widetilde{v}_k\|_{1+2\epsilon',2,\gamma,1} &\leq c \|\chi \widetilde{v}_k\|_{1+2\epsilon',2,\gamma,2} \\ &\leq c \left(\|\mathcal{G}_k \chi Q_k(\widetilde{v}_k)\|_{1+2\epsilon',2,\gamma,2} + \|\mathcal{G}_k [L_k, \chi] \widetilde{v}_k\|_{1+2\epsilon',2,\gamma,2} \right) \\ &\leq c \|\widetilde{v}_k\|_{1+\epsilon',1,\gamma,2}, \end{aligned}$$

for a constant $c > 0$ whose value varies from line to line. But ϵ' is an arbitrary positive number less than ϵ, so regardless of the ϵ present in (7.45), we can choose $\epsilon > \epsilon' > \epsilon/2$ and (7.49) is proven.

This completes the proof of Claim 7.12 and thus the proof of Proposition 7.9. □

8. Cone angle π

We now discuss the case $\mathfrak{p}_{=\pi} \neq \varnothing$.

Let $id = u_0 \colon (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, G)$ be energy minimizing and fix $p \in \mathfrak{p}_{=\pi}$. Let $\phi_1 \colon D \longrightarrow (\Sigma_{\mathfrak{p}}, g)$ and $\phi_2 \colon D \longrightarrow (\Sigma_{\mathfrak{p}}, G)$, so that the ϕ are conformal and $\phi_i(0) = p$. Pick conformal coordinates z and w on the domain and target, respectively. The double cover $f \colon D \longrightarrow D$ with $f(z) = z^2$ can be used to pull back G to a metric $\overline{G} \colon = f^* \phi_2^* G$ on D with (not necessarily smooth) cone angle 2π. The map $w = \phi_2 \circ u_0 \circ \phi_1^{-1}$ lifts to a harmonic map

$$\begin{array}{ccc} D & \xrightarrow{\widetilde{w}} & (D, \overline{G}) \\ \downarrow f & & \downarrow f \\ D & \xrightarrow{w} & (D, \phi_2^* G) \end{array} \tag{8.1}$$

Since $\overline{G}$ has cone angle 2π, by Section 2.1, in conformal coordinates v we can write

$$\overline{G} = e^{2\mu} \left| d\widetilde{w} \right|^2 .$$

The reason we treat this case separately is that the form of these harmonic maps near $\mathfrak{p}_{=\pi}$ is different than Form 2.3. We have

Form 8.1 ($\mathfrak{p}_{=\pi} \neq \varnothing$). *We say that $u \colon (\Sigma, g) \longrightarrow (\Sigma, G)$ is in Form 2.3 (with respect to g and G) if*

1) *u is a homeomorphism, and writing $\mathfrak{p}' = u^{-1}(\mathfrak{p})$, $u \colon \Sigma_{\mathfrak{p}'} \longrightarrow \Sigma_{\mathfrak{p}}$ is a $C^{2,\gamma}_{loc}$ diffeomorphism.*
2) *For each $p \in \mathfrak{p} - \mathfrak{p}_{=\pi}$, if z is a centered conformal coordinate around $u^{-1}(p)$ w.r.t. g and w is a centered conformal coordinate around p w.r.t. G, then u is given by $w(z) = \lambda z + v(z)$, where $\lambda \in \mathbb{C}^*$ and $v \in r^{1+\epsilon} C^{2,\gamma}_b(D(R))$ for some sufficiently small $\epsilon > 0$.*
3) *Near $p \in \mathfrak{p}_{=\pi}$, if w is defined as in (8.1), then*

$$\widetilde{w}(\widetilde{z}) = a\widetilde{z} + b\overline{\widetilde{z}} + v(\widetilde{z})$$

where $v \in r^{1+\epsilon} C^{2,\gamma}_b(D(R))$, for some sufficiently small $\epsilon > 0$, $a, b, \in \mathbb{C}$, and

$$|a| > |b| . \tag{8.2}$$

It is easy to check that Lemma 5.1 holds, i.e. that harmonic maps in Form 8.1 have Hopf differentials that are holomorphic with at worst simple poles at p. In fact, since $\Phi(\widetilde{w})$ is invariant under the deck transformation, we can just compute $\Phi(\widetilde{w})$ and use $\Phi(w) = f_*\Phi(\widetilde{w})$.

$$\Phi(\widetilde{w}) = e^{2\mu} v_{\widetilde{z}} \overline{v}_{\widetilde{z}} = (a\overline{b} + O(|\widetilde{z}|))d\widetilde{z}^2.$$

By holomorphicity and the invariance of Φ under the deck transformation,

$$\Phi(\widetilde{w}) = (a\overline{b} + g(\widetilde{z}^2))d\widetilde{z}^2,$$

where g is a holomorphic function with $g(0) = 0$. Thus

$$\Phi(w) = f_*\Phi(\widetilde{w}) = \frac{1}{4}\left(\frac{a\overline{b}}{z} + g(z)\right) dz^2.$$

Thus, we have shown more than Lemma 5.1, namely,

Lemma 8.2. *Suppose* $u\colon (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, G)$ *is harmonic and is in Form 8.1 with* $a \neq 0$*, then* u *is actually in Form 2.3, i.e.* $b = 0$*, if and only if* $\Phi(u)$ *extends holomorphically over* $\mathfrak{p}_{=\pi}$*.*

It follows that *all of the results of Section 5 hold*, since the only way Form 2.3 we used in this section was in proving Lemma 5.1.

Given a harmonic u_0 in Form 8.1, the space $\mathcal{B}^{1+\epsilon}(u_0)$ is defined so that $u \in \mathcal{B}^{1+\epsilon}(u_0)$ if and only if near $q \in \mathfrak{p}$, $u - u_0 \in r^{1+\epsilon} C_b^{2,\gamma}$. For $p \in \mathfrak{p}_{=\pi}$, writing the lift of w_0 as $\widetilde{w}_0(\widetilde{z}) = a_0\widetilde{z} + b_0\overline{\widetilde{z}} + v_0(\widetilde{z})$, we see that

$$u \in \mathcal{B}^{1+\epsilon}(u_0) \implies \widetilde{w}(z) = a_0\widetilde{z} + b_0\overline{\widetilde{z}} + v(\widetilde{z}) \quad \text{for some } v \in r^{1+\epsilon} C_b^{2,\gamma}$$

where w is the localized lift of u from (8.1). As above, we allow the tension field operator τ to act on a space of geometric perturbations. Near $p \in \mathfrak{p}_{=\pi}$ they can be described simply; given $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, consider the (locally defined) map

$$\widetilde{w}(\widetilde{z}) = (a_0 + \lambda_1)\widetilde{z} + (b_0 + \lambda_2)\overline{\widetilde{z}} + v(\widetilde{z}). \tag{8.3}$$

Following the above arguments we let $\mathcal{D}'$ be a space of automorphisms which look like (3.10) near $q \in \mathfrak{p} - \mathfrak{p}_{=\pi}$ and which lift to look like (8.3) near each $p \in \mathfrak{p}_{=\pi}$.

8.1. $\mathcal{H}(\mathfrak{q})$ is open ($\mathfrak{p}_{=\pi} \neq \varnothing$)

We state and sketch the proof of the main lemma

Lemma 8.3. *The tension field operator* τ *acting on* $\mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{D}' \circ \mathcal{T}_{>\pi}$ *is* C^1*. Its linearization* $L = D_u \tau$ *is bounded as a map*

$$L \colon r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \oplus T_{id}\mathcal{D}' \oplus T_{id}\mathcal{T}_{>\pi} \longrightarrow r^{1+\epsilon-2\alpha}\mathcal{X}_b^{0,\gamma},$$

and is transverse to $(T\mathcal{C}onf_0 \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma})^{\perp}$.

By Remark 6.12, the lemma implies the openness statement exactly as it did in the case $\mathfrak{p}_{=\pi} = \varnothing$. The map

$$L \colon r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1+\epsilon-2\alpha}\mathcal{X}_b^{0,\gamma} \tag{8.4}$$

is Fredholm for small ϵ, as is $L \colon r^{1-\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1-\epsilon-2\alpha}\mathcal{X}_b^{2,\gamma}$. The latter map has cokernel $\mathcal{K} = \operatorname{Ker} L|_{r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{2,\gamma}}$. The cokernel of (8.4) can again be written as $\widetilde{W} \oplus \mathcal{K}$ where $\widetilde{W}$ consists of vectors $\psi \in r^{1-\epsilon}\mathcal{X}$ with $L\psi \in r^{1+\epsilon-2\alpha}\mathcal{X}$, and again such vectors have expansions determined by the indicial roots. Near $p \in \mathfrak{p}_{=\pi}$, using the lifts at the beginning of this section makes the asymptotics easy to calculate. Note that

$$H(u, g, G) = (\phi_1)_* f_* H(\widetilde{w}, D, \overline{G}),$$

where $\widetilde{z}$ is defined by (8.1), which immediately implies (initially on $\mathcal{X}^{\infty}$), that near $p \in \mathfrak{p}_{=\pi}$, $L_{u_0,g,G} = (\phi_1)_* f_* D_{\widetilde{w}} H|_{\widetilde{w}_0,D,\overline{G}}$. If $\overline{L} := D_{\widetilde{z}} H|_{\widetilde{w}_0,D,\overline{G}}$, then setting $\overline{\overline{L}} = (|\widetilde{z}|^2/4)\overline{L}$ we have

$$\overline{\overline{L}} = (\widetilde{z}\partial_{\widetilde{z}})\left(\overline{\widetilde{z}}\partial_{\overline{\widetilde{z}}}\right) + E(\widetilde{z}).$$

Any $\psi \in r^{1+\epsilon-2\alpha}\mathcal{X}$ is in $r^{\epsilon}C_b^{2,\gamma}$ near $p \in \mathfrak{p}_{=\pi}$ since $\alpha_p = 1/2$. It is now easy to see that a solution $L\psi = 0$ has lift

$$\overline{\psi}(\widetilde{z}) = \lambda_1 \widetilde{z} + \lambda_2 \overline{\widetilde{z}} + \overline{\psi}' \text{ where } \overline{\psi}' \in r^{1+\epsilon}C_b^{2,\gamma} \tag{8.5}$$

near such a p, and this shows that

$$L\left(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \oplus \mathcal{D}'\right) \oplus \mathcal{K} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}.$$

In analogy with Lemma 6.8, we have that, near $p \in \mathfrak{p}_{=\pi}$, elements of $\mathcal{K}$ look like (8.5). All of the material in Section 6 follows after replacing $\mathcal{D}$ by $\mathcal{D}'$.

8.2. $\mathcal{H}(\mathfrak{q})$ is closed ($\mathfrak{p}_{=\pi} \neq \varnothing$).

The proof again requires only minor modifications. The main difference is the following; consider a sequence of maps u_k and converging metrics metrics $G_k \to G_0$ and $g_k \to g_0$ as in the statement of Theorem 7.1. In the same way as in the $\mathfrak{p}_{=\pi} = \varnothing$ case, we reduce to the local analysis of the u_k near a cone point $p \in \mathfrak{p}_{=\pi}$. By analogy with the treatment of the λ_k in the $\mathfrak{p}_{=\pi} = \varnothing$ case, we want want to show that the condition (8.2) persists in the limit, and for this we need some uniform control of the a_k, b_k. In fact, we claim that

$$|a_k| \leq c \quad \text{and} \quad |a_k| - |b_k| \geq c > 0,$$

for some uniform constant c. As in the previous case, the a_k and b_k are relate to the energy density e_k and the function h_k defined in (7.10). There exist $f_1, f_2 \in r^{\epsilon} C_b^{2,\gamma}$ such that

$$\begin{aligned} h_k(z) &\sim |a_k|^{2\alpha} \left| 1 + \frac{b_k}{a_k} e^{-2i\theta} \right|^{2(\alpha-1)} + f_2 \\ e_k(z) &\sim |a_k|^{2(\alpha-1)} \left| 1 + \frac{b_k}{a_k} e^{-2i\theta} \right|^{2(\alpha-1)} \left(|a_k|^2 + |b_k|^2 \right) + f_1 \end{aligned} \tag{8.6}$$

If follows that the e_k are uniformly bounded, since they are still subsolutions. In the second line, choosing θ so that $\frac{b_k}{a_k} e^{-2i\theta} = \left| \frac{b_k}{a_k} \right| < 1$ and using the uniform bound $e_k < c$ gives $|a_k| < c$. We apply Lemma 7.7 to the $\log h_k/\delta_k$ for δ_k defined as in (7.25), noting that the hypotheses are satisfied by (8.6). As above, this leads to the lower bound

$$\inf_{z \in D - 0} h_k(z) \geq c > 0,$$

so choosing θ such that $1 + \frac{b_k}{a_k} e^{-2i\theta} = 1 - \left| \frac{b_k}{a_k} \right|$ we get that $|a_k| - |b_k| \geq c > 0$

The rest of the argument proceeds as in the $\mathfrak{p}_{=\pi} = \varnothing$ case, with a_k playing the role of λ_k. Assuming the same type of blow-up near $p \in \mathfrak{p}_{=\pi}$, and rescaling in the exact same way, on the local double cover a harmonic map of $w_\infty \colon \mathbb{C} \longrightarrow \mathbb{C}$ results with $w_\infty = a_\infty z + b_\infty \overline{z} + v_\infty$ with $v_\infty \in r^{1+\epsilon} C_b^{2,\gamma}$ not identically zero. This is a contradiction by the following argument from [D], which we outline briefly; an orientation preserving harmonic mapping map of $\mathbb{C}$ can be written, globally, as a sum $f + \overline{g}$ where f and g are holomorphic. The ratio $\partial_z g / \partial_z f$ is bounded by the orientation preserving property and is clearly holomorphic, hence constant. Integrating proves the statement.

9. H is continuously differentiable

Finally, we discuss that the map (3.17) in detail. To study its properties we trivialize the bundle $\mathsf{E} \longrightarrow \mathcal{B}^{1+\epsilon}_{2,\gamma}(u_0) \circ \mathcal{C} \times \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a}) \times \mathcal{M}^*_{2,\gamma,\nu}(G_0, \mathfrak{p}, \mathfrak{a})$ (see (3.17)–(3.18)), and use the trivializing map to define the topology of E. We define a map

$$\Xi \colon \mathsf{E} \longrightarrow \mathcal{X}^{1+\epsilon-2\mathfrak{a}}_{0,\gamma}(u_0) \tag{9.1}$$

as follows. Let $((u \circ C, g, G), \psi) \in \mathsf{E}$ where $\psi \in \mathcal{X}^{1+\epsilon-2\mathfrak{a}}_{0,\gamma}(u \circ C)$. By the definition of $u \in \mathcal{B}^{1+\epsilon}(u_0)$, there is a unique $\widetilde{\psi} \in \mathcal{X}^{1+\epsilon}(u_0)$ so that $u = \exp_{u_0}(\widetilde{\psi})$. Assuming for the moment that $C = id$, let $\Xi((u, g, G), \psi) = \Xi_u(\psi)$ where

$$\Xi_u(\psi) := \begin{array}{l} \text{parallel translation of } \psi \text{ along} \\ \gamma_t := \exp_{u_0}(t\widetilde{\psi}) \text{ from } t = 1 \text{ to } t = 0. \end{array} \tag{9.2}$$

In general, motivated by pointwise conformal invariance of τ (see (3.6)), define $\Xi((u \circ C, g, G), \psi) = \Xi_u(\psi \circ C^{-1})$. Obviously, Ξ is an isomorphism on each fiber, and we endow E with the pullback topology induced by Σ. Thus a section σ of E is C^1 if and only if $\Xi \circ \sigma$ is C^1. The purpose of this section is to prove Proposition 3.4, which states that if g_0 and G_0 satisfy Assumption 3.1, then the map 9.1 is C^1.

We reduce the proposition to a computation in local coordinates. Near $p \in \mathfrak{p}$, we have

$$\begin{aligned} H(u \circ C, g, G) &= \Xi\left(\tau(u, C^*g, G)\right) \\ &= \Xi_u\left(\tau^i(u, C^*g, G)\partial_i\right) \\ &= \tau^i(u, C^*g, G)(\Xi_u)_i^{\ j}\partial_j, \end{aligned}$$

where $(\Xi_u)_i^{\ j}$ is the local coordinate expression for parallel translation of ∂_α along the path in (9.2).

Since (u_0, g_0, G_0) solves (HME($\mathfrak{q}$)) we have $\tau(u_0, g_0, G_0) = 0$, and since G and G_0 have the same the same conformal coordinates near $p \in \mathfrak{p} - \mathfrak{p}_{=\pi}$, by the computations of Section 6.1,

$$\tau^i(u, C^*g, G) \in \frac{4}{\sigma} r^{\epsilon-1} C^{0,\gamma}_b = r^{1+\epsilon-2\mathfrak{a}} C^{0,g}_b \tag{9.3}$$

Near $p \in \mathfrak{p}_{=\pi}$, recall that $C = D'_{\lambda_1,\lambda_2} \circ T_w$ where $T_w(z) = z - w$ and $D'_{\lambda_1,\lambda_2}(\widetilde{z}) = \lambda_1\widetilde{z} + \lambda_2\overline{\widetilde{z}}$, for $\widetilde{z}^2 = z$ and $\widetilde{u}^2 = u$ coordinates on the local double cover.

Again we have, locally

$$\begin{aligned} H(u \circ D_{\lambda_1,\lambda_2} \circ T_w, g, G) &= \Xi\left(\tau(u \circ D_{\lambda_1,\lambda_2}, T_w^* g, G)\right) \\ &= \Xi_u\left(\tau^i(u \circ D_{\lambda_1,\lambda_2}, T_w^* g, G)\partial_i\right) \\ &= \tau^i(u \circ D_{\lambda_1,\lambda_2}, T_w^* g, G)(\Xi_u)_i^{\ j}\partial_j, \end{aligned}$$

The local computation of τ can now be done in the lifted coordinates $\widetilde{z}$, where $\widetilde{u} = a\widetilde{z} + b\overline{\widetilde{z}} + v$ for $v \in r^{1+\epsilon}C_b^{2,\gamma}$ coordinates. The pulled back tension field is

$$\frac{4}{\sigma}\left(\widetilde{u}_{\widetilde{z}\overline{\widetilde{z}}} + \frac{\partial \log \rho}{\partial \widetilde{u}}((a+\lambda_1) + \widetilde{u}_{\widetilde{z}})((b+\lambda_2) + \widetilde{u}_{\overline{\widetilde{z}}})\right)$$

So by $\frac{\partial \log \rho(\widetilde{u}_0)}{\partial \widetilde{u}} = 2\frac{\partial \widetilde{\mu}}{\partial \widetilde{u}}$, we have $\tau^i \in r^\epsilon C_b^{0,\gamma}$, which is (9.3) in this context.

As for the expression $(\Xi_u)_i^{\ j}\partial_j$, a simple exercise in ODEs shows that (if we assume $u - z \in r^{1+\epsilon}C_b^{2,\gamma}$), then $(\Xi_u)_i^{\ j}\partial_j - \partial_i \in r^{1+\epsilon}C_b^{0,\gamma}$. Thus we have established

Lemma 9.1. *If (u_0, g_0, G_0) satisfy (HME($\mathfrak{q}$)) and u_0 is in Form 2.3, then*

$$\begin{aligned} \tau\colon \mathcal{B}^{1+\epsilon}_{2,\gamma}(u_0) \circ \mathcal{D} \times \mathcal{M}^*_{2,\gamma,\nu}(G_0, \mathfrak{p}, \mathfrak{a}) &\longrightarrow \mathsf{E} \\ (u \circ C, g, G) &\longrightarrow \tau(u \circ C, g, G) \end{aligned} \tag{9.4}$$

is C^1 near u_0.

Index of Notation

$\mathfrak{a}$	a point in $(0,1)^k$, page 730.
$\mathcal{B}^{1+\epsilon}(u_0)$	$r^{1+\epsilon}C_b^{2,\gamma}$ perturbations of u_0, page 736.
$r^c C_b^{2,\gamma}$	weighted b-Hölder spaces, page 729.
C_α	the standard flat cone of cone angle $2\pi\alpha$, page 725.
g_α	metric on the standard flat cone, page 725.
$\mathcal{D}$	local conformal dilations, page 736.
$D(R)$	the disc in $\mathbb{C}$ of radius R, page 730.
$E(u, g, G)$	The energy of a map $u\colon (\Sigma, g) \longrightarrow (\Sigma, G)$, page 724.
E	bundle in which τ takes values., page 738.
$\Gamma(B)$	the sections of a bundle B., page 725.
$\mathcal{C}on f_0$	The space of conformal automorphisms of $((\Sigma, g)$ that are homotopic to the identity., page 739.

$\mathcal{H}arm_{\mathfrak{q}}$	harmonic maps with fixed geometric data fixing $\mathfrak{q}$, page 742.
κ_h	scalar curvature of the metric h, page 730.
$\mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$	is the space of conic metrics with cone points at $\mathfrak{p}$ of cone angles $2\pi\mathfrak{a}$, page 730.
$\mathcal{M}^*_{k,\gamma,\nu}(h_0, \mathfrak{p}, \mathfrak{a})$	metrics locally conformal to h_0, page 737.
$\mathcal{M}^{phg}_{\nu}(\mathfrak{p}, \mathfrak{a})$	polyhomogeneous conic metrics, page 730.
$\Phi(u)$	the Hopf differential, page 733.
$\mathfrak{p}_{<\pi}, \mathfrak{p}_{>\pi}, \mathfrak{p}_{=\pi}$	points with various angle specifications, page 731.
$\mathfrak{q}$	the cone points in whose relative homotopy class we minimize, page 732.
Σ	a closed, orientable, smooth surface, page 724.
$\mathfrak{p}$	a finite subset of Σ, page 724.
$\Sigma_{\mathfrak{p}}$	$\Sigma - \mathfrak{p}$, page 724.
$\mathcal{T}$	local conformal translations, page 736.
$\mathcal{T}_{\widetilde{\mathfrak{p}}}$	local conformal translations near $\widetilde{\mathfrak{p}}$, page 736.
$\tau(u, g, G)$	the tension field of $u : (\Sigma, g) \longrightarrow (\Sigma, G)$, page 725.
$r^{1+\epsilon}\mathcal{X}^{k,\gamma}_b(u)$	$r^{1+\epsilon}C^{2,\gamma}_b$ vector fields over u, page 735.

References

[AS] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* National Bureau of Standards Applied Mathematics Series, Volume 55, U.S. Government Printing Office, Washington, D.C., (1964).

[BT] R. Bott and L. Tu, *Differential forms in algebraic topology.* Graduate Texts in Mathematics, **82**, Springer-Verlag, New York (1982).

[C] Personal Communication with Brian Conrad.

[CH] J. M. Coron and F. Helein, *Harmonic diffeomorphisms, minimizing harmonic maps, and rotational symmetry.* Composito Mathematica, **69**, 175–228, (1989).

[DM] G. Daskalopoulos and C. Mese, *Harmonic maps from a simplicial complex and geometric rigidity.* J. Differential Geometry, **79**, 277–334, (2008).

[DM2] G. Daskalopoulos and C. Mese, *Harmonic Maps Between Singular Spaces I.* Communications in Analysis and Geometry, to appear.

[D] P. Duren, *Harmonic Mappings in the Plane*. Cambridge tracts in mathematics, **156**, Cambridge University Press, (2004).

[EE] C. J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*. J. Differential Geometry, **3**, 19–43, (1969).

[EF] J. Eells and B. Fuglede, *Harmonic Maps Between Riemannian Polyhedra*. Cambridge Tracts in Mathematics, **142**, Cambridge University Press, Cambridge, (2001).

[EL] J. Eells and L. Lemaire, *Report on harmonic maps*. Bull. London Math. Soc., **10**, 1–68, (1978).

[ES] J. Eells and J. H. Sampson, *Harmonic Maps into Riemannian Manifolds*. American Journal of Mathematics, **86**, 109–160, (1964).

[FT] A. E. Fischer and A. J. Tromba, *A new proof that Teichmüller space is a cell*. Trans. Amer. Math. Soc., **303**, 257–262, (1987).

[GR] M. Gerstenhaber and H. E. Rauch, *On extremal quasi-conformal mappings. I, II*. Proc. Nat. Acad. Sci. U.S.A., **40**, 808–812, 991–994, (1954).

[GS] M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*. IHES Publications Mathématiques, **76**, 165–246, (1992).

[Ham] R. Hamilton, *Harmonic Maps of Manifolds with Boundary*. Lecture Notes in Math., **471**, Springer-Verlag, (1967).

[HL] R. Hardt and F. Lin, *Harmonic maps into round cones and singularities of nematic liquid crystals*. Mathematische Zeitschrift, **213**, 575–593, (1993).

[Har] P. Hartman, *On homotopic harmonic maps*. Canad. J. Math., **19**, 673–687, (1967).

[He] E. Heinz, *On certain nonlinear elliptic differential equations and univalent mappings*. J. D'Anal. Math., **5**, 197–272, (1956/57).

[J] J. Jost, *Harmonic Maps Between Surfaces*. Lecture Notes in Math., **1062**, Springer-Verlag, (1984).

[JS] J. Jost and R. Schoen, *On the existence of harmonic diffeomorphisms between surfaces.* Invent. math., **66**, 353–359, (1982).

[KS] N. Korevaar and R. Schoen, *Sobolev spaces and harmonic maps for metric space targets.* Communications in Analysis and Geometry, **1**, 561–659, (1993).

[K] E. Kuwert, *Harmonic maps between flat surfaces with conical singularities.* Mathematische Zeitschrift, **221**, 421–436, (1996).

[L] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings.* Bull. Amer. Math. Soc., **42**, 689–692, (1936).

[M] R. Mazzeo, *Elliptic theory of differential edge operators.* Communications in Partial Differential Equations, **16**, 1616–1664, (1991).

[MP] R. Mazzeo and F. Pacard, *Constant mean curvature surfaces with Delaunay ends.* Comm. Anal. Geom., **9** (1), 169–237, (2001).

[Mc1] R. McOwen, *Prescribed curvature and singularities of conformal metrics on Riemann surfaces.* J. Math. Anal. Appl., **177** (1), 287–298, (1993).

[Mc2] R. McOwen, *Point singularities and conformal metrics on Riemann surfaces.* Proc. Amer. Math. Soc., **103** (1), 222–224, (1988).

[Me] R. B. Melrose, *The Atiyah-Patodi-Singer index theorem.* A. K. Peters, Ltd., Boston, Mass., (1993).

[Me-Me] R. B. Melrose and G. Mendoza, *Elliptic pseudodifferential operators of totally characteristic type.* MSRI preprint, (1983).

[Mese1] C. Mese, *Harmonic maps into surfaces with an upper curvature ound in the sense of Alexandrov.* Mathematische Zeitschrift, **242**, 633–661, (2002).

[Mese2] C. Mese, *Harmonic Maps Between Surfaces and Teichmüller Spaces.* Amer. J. Math., **124**, 451–481, (2002).

[Mo] Charles B. Morrey Jr., *Multiple Integrals in the Calculus of Variations.* Classiscs in Mathematics, Springer-Verlag, Berlin, (2008).

[R] T. Radó. *Aufgabe 41.* Jahresber. Deutsch Math. Verein, **35**, 49, (1926).

[S] J. H. Sampson, *Some properties and applications of harmonic mappings.* Ann. Sci. École Norm. Sup., **4**, 211–228, (1978).

[SY] R. Schoen and S. Yau, *On Univalent Harmonic Maps Between Surfaces.* Inventiones math., **44**, 265–278, (1978).

[ST] G. Schumacher and S. Trapani, *Variation of cone metrics on Riemann surfaces.* J. Math. Anal. Appl., **311** (1), 218–230, (2005).

[ST2] G. Schumacher and S. Trapani, *Weil-Petersson Geometry for families of hyperbolic conical Riemann surfaces.* `arXiv:0809.0058 [math.CV]`.

[Se] T. Serbinowski, *Harmonic maps into metric spaces with curvature bounded above.* Ph.D. thesis, University of Utah, (1995).

[Si] L. Simon, *Theorems on the Regularity and Singularity of Energy Minimizing Maps.* Birkhäuser, Basel, (1996).

[TII] M. E. Taylor, *Partial Differential Equations II: Qualitative Studies of Linear Equations.* Springer-Verlag, New York, (1996).

[TIII] M. E. Taylor, *Partial Differential Equations III: Nonlinear Equations.* Springer-Verlag, New York, (1997).

[Th] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces.* Bull. Amer. Math. Soc. (N.S.), **19** (2), 417–431, (1988).

[Tr] A. Tromba, *Teichmüller Theorey in Riemannian Geometry.* Birkhäuser, Lectures in Mathematics, ETH Zürich, (1992).

[Tro] M. Troyanov, *Prescribing curvature on compact surfaces with conical singularities.* Trans. Amer. Math. Soc., **324** (2), 793–821, (1991).

[W1] M. Wolf, *Harmonic maps from surfaces to R-trees.* Math. Z., **218** (4), 577–593, (1995).

[W2] M. Wolf, *High energy degeneration of harmonic maps between surfaces and rays in Teichmüller space.* Topology, **30**, 517–540, (1991).

[W3] M. Wolf, *The Teichmüller Theory of Harmonic Maps.* J. Differential Geometry, **29**, 449–479, (1989).

[Y] S. Yamada, *Weil-Peterson convexity of the energy functional on classical and universal Teichmüller spaces.* J. Differential Geometry, **51** (1), 35–96, (1999).

[Ya] K. Yano, *On Belyi functions on elliptic curves of degree* 6. Master thesis, Sophia University, (2006).

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Received March 20, 2012