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A uniqueness theorem for gluing calibrated submanifolds

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‘Gluing’ is a technique of constructing solutions to non-linear (elliptic) partial differential equations such as Yang-Mills equations, minimal surface equations and Einstein equations. Calibrated submanifolds are a certain class of minimal surfaces, and there are various examples of them constructed by the gluing technique. We have existence theorems in that sense, but there seems to have been no uniqueness theory for higher-dimensional ones such as special Lagrangian submanifolds, which we discuss in the present paper.

1. Introduction

As we have mentioned above ‘gluing’ is a technique of constructing solutions to non-linear (elliptic) partial differential equations such as Yang-Mills equations, minimal surface equations and Einstein equations. Solutions constructed by the gluing technique are usually parametrized by small $s > 0$ and tending to something *singular* as $s \to +0$; for example Taubes [15] constructed a one-parameter family of Yang-Mills ASD (anti-self-dual) instantons A_s with curvature tending to a δ-function as $s \to +0$. There are many other examples of Yang-Mills instantons, minimal surfaces and Einstein metrics constructed by the gluing technique, including calibrated submanifolds (which are a certain class of minimal surfaces); for instance various authors [2, 8–10] constucted various kinds of special Lagrangian submanifolds (which are a higher-dimensional example of calibrated submanifolds).

What we shall study in the present paper is a *uniqueness* problem: given a singular solution and a family of (non-singular) solutions parametrized by $s > 0$ and tending to the singular one as $s \to +0$ then need they be re-constructed by the gluing technique?

The answer is ‘yes’ in the situation of Taubes: all ASD instantons with curvature close to a δ-function may be re-constructed by the method of Taubes, which was proved by Donaldson [3]. Something similar holds for pseudo-holomorphic curves in symplectic manifolds, and they give a key step to the definition of Donaldson invariants (which 'count' ASD instantons), Gromov-Witten invariants (which 'count' pseudo-holomorphic curves) and Floer homologies in Yang-Mills gauge theory or in symplectic geometry.

There seems to have been no such kind of uniqueness results proved for calibrated submanifolds of higher dimension; pseudo-holomorphic curves are calibrated submanifolds of dimension 2 and by 'higher' we mean $\geqslant 3$.

We shall now recall an outline of the proof of Donaldson. Let A_s be an instanton whose curvature is close to a δ-function supported at a point x in a manifold X (of dimension 4 and supposed to be compact). There are mainly three things to do:

(i) One first proves that A_s tends to the trivial instanton over each compact subset of $X \setminus \{x\}$ and that there exists $\epsilon_s > 0$ such that if A_s is re-scaled by ϵ_s^{-1} about x then the re-scaled instanton $\epsilon_s^{-1} A_s$ will tend to an instanton B over $T_x X \cong \mathbb{R}^4$ decaying to the trivial instanton at infinity.

(ii) There is a well-known classification result for such instantons, which implies that B is a basic instanton used by Taubes (which is unique up to re-scaling). Thus A_s will be close to the trivial instanton on $M \setminus U$ for some neighbourhood U of x in X and to the re-scaled instanton $\epsilon_s B$ on a smaller neighbourhood U_s of x in U, but we have not seen yet the behaviour of A_s in $U \setminus U_s$.

(iii) We may suppose that U and U_s are open balls about x in X. Since B decays at infinity in $\mathbb{R}^4$ it follows that $\epsilon_s B$ is close to the trivial instanton near the boundary of U_s and so A_s is close to the trivial instanton near the two boundaries of the annulus $U \setminus U_s$. The final step is to prove that A_s is close to the trivial instanton over the whole annulus $U \setminus U_s$, which will readily imply that A_s is gauge-equivalent to one of the instantons constructed by the gluing technique.

We wish to recover the steps (i)–(iii) for calibrated submanifolds in place of instantons, but it seems too difficult to do in general. We shall therefore focus upon a situation of Joyce [8]. Let L be a compact special Lagrangian submanifold with isolated conical singularities in his sense. Joyce proved that if there are local models of desingularizing the tangent cones to L then one can glue them to L to get compact special Lagrangian submanifolds (without singularities) parametrized by $s > 0$ and tending to L as $s \to +0$ (in the sense of geometric measure theory).

We wish to prove that all compact special Lagrangian submanifolds tending to L may be re-constructed by the method of Joyce. For that purpose it suffices to prove analogues to (i)–(iii) above, and in the present paper we shall prove the analogue to (iii). It seems difficult to prove analogues to (i) and (ii) in full generality, but is doable in some interesting situations, which we do in a sequel to the present paper [7] (the analogue to (i) holds for special Lagrangian Jacobi-integrable smooth cones and the analogue to (ii) hold for stable T^2-cones, which are automatically special Lagrangian Jacobi-integrable).

The analogue to (iii) may be stated as follows:

Theorem 1.1. *Let $B(\rho)$ be the open ball of radius $\rho > 0$ about $\mathbf{0}$ in $\mathbb{R}^n$. Let $s \in (0, 1)$, let ϕ be a calibration of degree m on $\mathbb{R}^n$, and let CX be a ϕ-calibrated smooth cone in $\mathbb{R}^n$ with $X \equiv CX \cap S^{n-1}$ being a compact submanifold of S^{n-1}. Let M be a properly-embedded ϕ-calibrated submanifold of $B(1) \setminus \overline{B(s)}$ with ∂M being a smooth hypersurface of $\overline{M}$ contained in $\partial B(1) \cup \partial B(s)$. Let $\partial M \cap \partial B(1)$ and $\partial M \cap \partial B(s)$ be C^1-close to $CX \cap \partial B(1)$ and $CX \cap \partial B(s)$ respectively. Then M is C^1-close to $CX \cap B(1) \setminus \overline{B(s)}$.*

Remark 1.2. This holds for general calibrated submanifolds which need not be special Lagrangian. The cone CX plays the rôle of the trivial instanton in the step (iii) above, and M plays the rôle of A_s.

Remark 1.3. We do not suppose anything particular about the behaviour of M away from ∂M, but the conclusion of Theorem 1.1 implies that M is diffeomorphic to $CX \cap B(1) \setminus \overline{B(s)}$.

Remark 1.4. The statement above will be refined in Theorem 2.2 below. We have supposed so far that the metric is flat and the calibration is constant, but shall deal with more general metrics and calibrations in Theorem 2.2; it will be necessary for the situation of gluing special Lagrangian submanifolds in Calabi-Yau manifolds where the metric need not be flat and the calibration need not be constant. In Theorem 2.2 we shall also say how close M is to CX.

The proof of Theorem 1.1 may be sketched as follows. Donaldson used a method of Uhlenbeck [16] in the step (iii) above, and we shall use a method of Simon [13, 14] for the proof of Theorem 1.1; Uhlenbeck proved a removable singularity theorem for Yang-Mills instantons in dimension 4 and Simon proved the uniqueness of multiplicity 1 smooth tangent cones to minimal surfaces. Let M be a minimal surface of dimension m in $\mathbb{R}^n$ with an isolated singularity at $\mathbf{0}$. It is well-known that $\mathrm{area}\big(M \cap B(\rho)\big)/\rho^m$ is a monotone non-decreasing function in ρ which plays a central rôle in the proof of Simon. In the situation of Theorem 1.1 however we have to work in *annuli* instead of balls. We shall therefore make the following version of monotonicity formula.

Let r be the radius function on $\mathbb{R}^n$. For each compact $(m-1)$-dimensional submanifold Σ of S^{n-1} let

$$F(\Sigma) \equiv \int_\Sigma r^{1-m} \frac{\partial}{\partial r} \lrcorner \phi. \tag{1.1}$$

Let M be as in Theorem 1.1. Then $F\big(M \cap \partial B(\rho)\big)$ will be a monotone non-decreasing in ρ (we note that for ρ generic $M \cap \partial B(\rho)$ is a submanifold of $\partial B(\rho) \cong S^{n-1}$ which makes $F\big(M \cap \partial B(\rho)\big)$ well-defined almost everywhere in ρ); the functional F is a higher-dimensional analogue of Hofer's functional for pseudo-holomorphic curves in symplectizations of contact manifolds [6, pp. 534–539]. We also note that F is similar to the Chern-Simons functional in the step (iii) above.

Morally speaking $M \cap \partial B(\rho)$ behaves like a gradient flow of F where ρ may be regarded as 'time'. Theorem 1.1 assumes that the flow starts and ends near X, and concludes that the flow stays near X for all time. It will follow from Simon's estimates including a version of Łojasiewicz inequality [13, Lemma 1, p. 542].

The remainder of the paper will be organized as follows:

- In §2 we state the refined version of Theorem 1.1.
- In §3 we prove the monotonicity formula for F_ρ. We shall have error terms in general if the metric is non-flat or the calibration is non-constant.
- In §4 we show how to use Simon's estimates [13, 14] including a version of Łojasiewicz inequality [13, Lemma 1, p. 542].
- In §5 we complete the proof of Theorem 1.1.

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2. Statement of main result

We begin with a review of calibrated geometry [5]. Let W be a Riemannian manifold. An m-form ϕ on W is said to be of comass $\leqslant 1$ if $\phi(v_1, \ldots, v_m) \leqslant 1$ for every orthonormal vector fields $v_1, \ldots, v_m$ on W. A closed m-form of comass $\leqslant 1$ on W is called a calibration of degree m on W. Let ϕ be a calibration of degree m on W. Let M be an oriented submanifold of W. We call M a ϕ-submanifold of W if $\phi|_M$ is the volume form of M. By a theorem of Harvey and Lawson [5], ϕ-submanifolds of W are minimal submanifolds of W.

We shall set up the notation which we use in the statement of Theorem 2.2 below. Let g' be the Euclidean metric on $\mathbb{R}^n$, i.e.,

$$g' = dy^1 \otimes dy^1 + \cdots + dy^n \otimes dy^n$$

in the coordinates $(y^1, \ldots, y^n)$ on $\mathbb{R}^n$. Let ϕ' be a calibration of degree m on $(\mathbb{R}^n, g')$. Suppose ϕ' is parallel, i.e.,

$$\phi' = \phi'_{i_1 \ldots i_m} dy^{i_1} \wedge \cdots \wedge dy^{i_m}$$

for some $\phi'_{i_1 \ldots i_m} \in \mathbb{R}$. Let r be the radial coordinate $|\bullet|$ on $(\mathbb{R}^n \setminus \{0\}, g')$. Set

$$\psi' = \left(\partial_r \lrcorner \phi'\right)|_{S^{n-1}}, \tag{2.1}$$

where ∂_r is the vector field $\partial/\partial r$, $\lrcorner$ is the interior product of vector fields with differential forms, and S^{n-1} is the unit sphere of $(\mathbb{R}^n, g')$. For every orthonormal vector fields $v_1, \ldots, v_{m-1}$ on S^{n-1}, we have

$$\psi'(v_1, \ldots, v_{m-1}) = \phi'(\partial_r, v_1, \ldots, v_{m-1}) \leqslant 1 \tag{2.2}$$

since $\partial_r, v_1, \ldots, v_{m-1}$ are orthonormal. Therefore, ψ' is an $(m-1)$-form of comass $\leqslant 1$ on S^{n-1}. Let X be a oriented submanifold of S^{n-1}. We call X a ψ'-submanifold if $\psi'|_X$ is the volume form of X.

Proposition 2.1. *ψ'-submanifolds of S^{n-1} are minimal submanifolds of S^{n-1}.*

Proof. Let X be a ψ'-submanifold of S^{n-1}. Set

$$CX = \{rx \in \mathbb{R}^n \mid r \in (0,\infty), x \in X\}.$$

Then, by (2.2), CX is a ϕ'-submanifold of $(\mathbb{R}^n, g')$. Therefore, CX is a minimal submanifold of $(\mathbb{R}^n, g')$. Therefore, X is a minimal submanifold of S^{n-1}. □

Let I be an open interval of $(0,\infty)$, and X a submanifold of S^{n-1}. We embed $I \times S^{n-1}$ into $\mathbb{R}^n$ by $(r,y) \mapsto ry$. Let ν be a normal vector field on $I \times X$ in $(I \times S^{n-1}, g')$. Set

$$\|\nu\|_{C^0_{\mathrm{cyl}}} = \sup_{I\times X} |\nu|/r, \quad \|\nu\|_{C^1_{\mathrm{cyl}}} = \sup_{I\times X} \big(|\nu|/r + |D\nu|\big),$$

where $D\nu$ is the covariant derivative of ν. These are induced by the cylindrical metric g'/r^2 on $(0,\infty) \times S^{n-1}$. Set

$$G_{\mathrm{cyl}}(\nu) = \left\{ \frac{r}{\sqrt{r^2 + |\nu(rx)|^2}} \big(rx + \nu(rx)\big) \,\middle|\, r \in I, x \in X \right\}.$$

We are ready now to refine the statement of Theorem 1.1:

Theorem 2.2. *Let $B(\rho)$ be the ball of radius ρ about $\mathbf{0}$ in $(\mathbb{R}^n, g')$. Let ϕ' be a parallel calibration of degree m on the Euclidean space $(\mathbb{R}^n, g')$, and ψ' the $(m-1)$-form (2.1) on the unit sphere S^{n-1} of $(\mathbb{R}^n, g')$. Let X be a compact ψ'-submanifold of S^{n-1}. Let $0 < l < 1$. Then, there exist $\epsilon_0, C_0, c_0 > 0$ depending only on l, m, n, X, ϕ' such that if:*

(A0) $0 < \epsilon < \epsilon_0$;

(A1) $0 < a_0 < b_0 < a_1 < b_1$, $a_0/b_0 = a_1/b_1 = l$;

(A2) *g is a Riemannian metric on $B(b_1)$ with*

$$\|g - g'\|_{C^1(B(b_1))} \leqslant \epsilon, \quad \|g - g'\|_{C^2(B(b_1))} \leqslant 1$$

with respect to g';

(A3) *ϕ is a calibration on $(B(b_1), g)$ with*

$$\left(1 + \log\frac{b_1}{a_0}\right) \sup_{B(b_1)} |\phi - \phi'| \leqslant \epsilon,$$

where $|\bullet|$ is with respect to g';

(A4) *M is a closed subset of $(a_0, b_1) \times S^{n-1}$, and M is a ϕ-submanifold with respect to g;*

(A5) *there exists a normal vector field ν_i on $(a_i, b_i) \times X$ in $((a_i, b_i) \times S^{n-1}, g'/r^2)$, where $i = 0, 1$, such that*

$$M \cap ((a_i, b_i) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu_i) \quad \textit{with} \quad \|\nu_i\|_{C^1_{\mathrm{cyl}}} \leqslant \epsilon,$$

then there exists a normal vector field ν on $(a_0, b_1) \times X$ in $((a_0, b_1) \times S^{n-1}, g'/r^2)$ such that

$$M = G_{\mathrm{cyl}}(\nu) \quad \textit{with} \quad \|\nu\|_{C^1_{\mathrm{cyl}}} \leqslant C_0 \epsilon^{c_0}. \tag{2.3}$$

Remark 2.3. One sufficient condition for (A3) to hold is that we have $\phi|_{\mathbf{0}} = \phi'$ and $a_0 = s^\alpha$, $b_1 = s$ for some $s > 0$ small enough and $\alpha \in (0, 1)$ independent of s; if so we have

$$\left(1 + \log \frac{b_1}{a_0}\right) \sup_{B(b_1)} |\phi - \phi'| = (1 + (1 - \alpha) \log s) O(s)$$

which tends to 0 as $s \to +0$.

3. A monotonicity formula

In this section we prove a monotonicity formula for calibrated submanifolds of annuli; see Proposition 3.4. This is a higher-dimensional analogue of an energy estimate of Hofer [6, pp. 534–539] for pseudo-holomorphic curves in symplectizations of contact manifolds.

Let g be a Riemannian metric on $\mathbb{R}^n$, and ϕ a calibration of degree m on $(\mathbb{R}^n, g)$.

Proposition 3.1. *Let M be a ϕ-submanifold of $(\mathbb{R}^n, g)$. If ν is a normal vector field on M in $(\mathbb{R}^n, g)$, then we have*

$$(\nu \lrcorner \phi)|_M = 0.$$

Proof. It suffices to prove that for every point $p \in M$ and orthonormal vectors $v_1, \ldots, v_{m-1} \in T_p M$, we have

$$\phi_p(\nu_p, v_1, \ldots, v_{m-1}) = 0. \tag{3.1}$$

Choose $v \in T_pM$ so that $\phi_p(v, v_1, \ldots, v_{m-1}) = 1$. Consider

$$t \mapsto \phi_p((\sin t)\nu_p + (\cos t)v, v_1, \ldots, v_{m-1}).$$

By the definition of calibration, this attains maximum 1 at $t = 0$. Differentiating it at $t = 0$, we have (3.1). $\square$

Let g' be the Euclidean metric on $\mathbb{R}^n$. Let r be the radial coordinate on the Euclidean space $(\mathbb{R}^n, g')$, and ∂_r the vector field ∂/∂_r. In the same way as Harvey and Lawson [5, Lemma 5.11, II.5], we shall prove the following

Proposition 3.2. *Let M be a ϕ-submanifold of $(\mathbb{R}^n, g)$. Then, we have*

$$\langle \overrightarrow{TM}, \partial_r \lrcorner dr \wedge \phi \rangle = |\mathrm{pr}_{TM^\perp} \partial_r|^2, \tag{3.2}$$

where $\langle \bullet, \bullet \rangle$ is the canonical pairing of poly-vector fields and differential forms, $\overrightarrow{TM}$ is the m-vector field on M dual to $\phi|_M$, $r = |\bullet|$ is with respect to the Euclidean metric g', and $\mathrm{pr}_{TM^\perp}$ is the projection of $\mathbb{R}^n$ onto the normal bundle of M in $(\mathbb{R}^n, g)$.

Proof. By Proposition 3.1, we have

$$\langle \nu \wedge \overrightarrow{TM}, dr \wedge \phi \rangle = \langle \nu, dr \rangle \langle \overrightarrow{TM}, \phi \rangle, \quad \text{where} \quad \nu = \mathrm{pr}_{TM^\perp} \partial_r.$$

This proves (3.2). $\square$

Set

$$\psi = \frac{m}{r^m} \int_0^r (\partial_r \lrcorner \phi) dr. \tag{3.3}$$

Proposition 3.3. *ψ is an $(m-1)$-form on $\mathbb{R}^n \setminus \{0\}$ such that*

$$\phi = d\left(\frac{r^m}{m}\psi\right). \tag{3.4}$$

Proof. Set $\chi = \partial_r \lrcorner \phi$, and $\omega = \partial_r \lrcorner dr \wedge \phi$. Then, we have

$$\phi = dr \wedge \chi + \omega. \tag{3.5}$$

Since $\partial_r \lrcorner \chi = \partial_r \lrcorner \omega = 0$, we may regard χ and ω as smooth families of differential forms on S^{n-1}. By the definition of calibration, $d\phi = 0$. Therefore, we have

$$d_{S^{n-1}}\chi = \partial_r \omega, \tag{3.6}$$

where $d_{S^{n-1}}$ is the exterior differentiation on S^{n-1}. By (3.5) and (3.6), we have

$$\phi = d\left(\int_0^r \chi dr\right) = d\left(\int_0^r (\partial_r \lrcorner \phi) dr\right).$$

By (3.3), this proves (3.4). □

Let ϕ' a parallel calibration of degree m on the Euclidean space $(\mathbb{R}^n, g')$, Set

$$\psi' = r^{1-m}\partial_r \lrcorner \phi'. \tag{3.7}$$

Then, (3.3) holds with ϕ', ψ' in place of ϕ, ψ respectively.

We shall prove a monotonicity formula with an error term. When $\phi = \phi'$, it has no error term.

Proposition 3.4. *There exists $C_{m,n} > 0$ depending only on m, n such that*

$$\left| m^{-1} d\psi - r^{-m}\partial_r \lrcorner dr \wedge \phi \right|_{\mathrm{cyl}} \leqslant C_{m,n} \sup |\phi - \phi'|, \tag{3.8}$$

where $|\bullet|_{\mathrm{cyl}}$ is with respect to the metric g'/r^2.

Proof. By (3.4) and (3.7), we have

$$\begin{aligned} & m^{-1} d\psi - r^{-m}\partial_r \lrcorner dr \wedge \phi \\ &= dr/r \wedge (r^{1-m}\partial_r \lrcorner \phi - r^{1-m}\partial_r \lrcorner \phi' + \psi' - \psi). \end{aligned} \tag{3.9}$$

By (3.3) and (3.7), we have

$$\begin{aligned} |r^{1-m}\partial_r \lrcorner \phi - r^{1-m}\partial_r \lrcorner \phi'|_{\mathrm{cyl}} &\leqslant c \sup |\phi - \phi'|, \\ |\psi - \psi'|_{\mathrm{cyl}} &\leqslant c \sup |\phi - \phi'| \end{aligned}$$

for some $c > 0$ depending only on m, n. Therefore, by (3.9), we have (3.8). □

We shall prove a proposition which we use in the proof of Lemma 3.6 below. We also use it in the key step to proof of the main result of this paper.

Proposition 3.5. *Let M be a ϕ-submanifold of $(\mathbb{R}^n, g)$, and suppose M is a closed subset of $(a,b) \times S^{n-1}$, where $(a,b) \times S^{n-1}$ is embedded into $\mathbb{R}^n$ by $(r,y) \mapsto ry$. There exist $\epsilon_{m,n}, C'_{m,n} > 0$ depending only on m, n such that if*

$$\left(1 + m \log \frac{b}{a}\right) \sup_{(a,b)\times S^{n-1}} |\phi - \phi'| \leqslant \epsilon_{m,n}, \quad \sup_{(a,b)\times S^{n-1}} |g - g'| \leqslant 1, \tag{3.10}$$

then we have

$$\begin{aligned} \mathrm{Vol}(M, g/r^2) \leqslant{}& C'_{m,n} \log \frac{b}{a} \limsup_{r\to b} \left| \int_{M\cap\{r\}\times S^{n-1}} \psi \right| \\ &+ C'_{m,n} \left(1 + m \log \frac{b}{a}\right) \int_M |\mathrm{pr}_{TM^\perp} \partial_r|^2 \,\mathrm{dVol}(M, g/r^2). \end{aligned} \tag{3.11}$$

Proof. By (3.4), we have

$$\mathrm{Vol}(M, g/r^2) = \int_M \phi/r^m = \int_M (dr/r) \wedge \psi + m^{-1} \int_M d\psi.$$

By (3.8), we have

$$m^{-1} \int_M d\psi \leqslant \int_M |\mathrm{pr}_{TM^\perp} \partial_r|^2 \,\mathrm{dVol}(M, g/r^2) + C_{m,n} \sup |\phi - \phi'| \,\mathrm{Vol}(M, g'/r^2).$$

By (3.8) and (3.2), we have

$$\begin{aligned} \int_M (dr/r) \wedge \psi &\leqslant \log \frac{b}{a} \limsup_{r\to b} \left| \int_{M\cap\{r\}\times S^{n-1}} \psi + \int_{M\cap([a,r]\times S^{n-1})} d\psi \right| \\ &\leqslant m \log \frac{b}{a} \limsup_{r\to b} \left| \int_{M\cap\{r\}\times S^{n-1}} m^{-1}\psi \right| \\ &\quad + \int_M |\mathrm{pr}_{TM^\perp} \partial_r|^2 \,\mathrm{dVol}(M, g'/r^2) \\ &\quad + m C_{m,n} \log \frac{b}{a} \sup |\phi - \phi'| \,\mathrm{Vol}(M, g'/r^2). \end{aligned}$$

Thus, we have

$$\begin{aligned}\mathrm{Vol}(M, g/r^2) \leqslant m \log \frac{b}{a} \limsup_{r \to b} \left| \int_{M \cap \{r\} \times S^{n-1}} \psi \right| \\ + \left(1 + m \log \frac{b}{a}\right) \int_M |\mathrm{pr}_{TM^\perp} \partial_r|^2 \,\mathrm{dVol}(M, g/r^2) \\ + C_{m,n} \left(1 + m \log \frac{b}{a}\right) \sup |\phi - \phi'| \,\mathrm{Vol}(M, g'/r^2).\end{aligned}$$

By (3.10), we have

$$C_{m,n} \left(1 + m \log \frac{b}{a}\right) \sup |\phi - \phi'| \,\mathrm{Vol}(M, g'/r^2) \leqslant (1/2)\,\mathrm{Vol}(M, g/r^2).$$

Thus, we have (3.11). □

We shall prove a lemma which we use in the key step to the proof of the main result of this paper. It is similar to a lemma of Simon [13, Lemma 3, p. 561]. We however use the monotonicity formula for ϕ-submanifolds of annuli.

Lemma 3.6. *Let ϕ' be a parallel calibration of degree m on the Euclidean space $(\mathbb{R}^n, g')$, and let ψ' be as in (3.7). Let X be a compact ψ'-submanifold of S^{n-1}. Let $\epsilon > 0$, and $0 < \lambda < \lambda'' < \lambda' < 1$. Then, there exists $\delta > 0$ such that if:*

- (P1) *g is a Riemannian metric on $B(1)$ with $\|g - g'\|_{C^1(B(1))} \leqslant \delta$, where $\| \bullet \|_{C^1}$ is with respect to g', and $B(1)$ is the unit ball of $(\mathbb{R}^n, g')$;*
- (P2) *ϕ is a calibration on $(B(1), g)$ with $\sup_{B(1)} |\phi - \phi'| \leqslant \delta$, where $| \bullet |$ is with respect to g';*
- (P3) *M is a ϕ-submanifold of $(\mathbb{R}^n, g)$, and M is a closed subset of $(\lambda, 1) \times S^{n-1}$, where $(\lambda, 1) \times S^{n-1}$ is embedded into $\mathbb{R}^n$ by $(r, y) \mapsto ry$;*
- (P4) *there exists a normal vector field ν on $(\lambda', 1) \times X$ in $((\lambda', 1) \times S^{n-1}, g'/r^2)$ such that*

  $$M \cap ((\lambda', 1) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu) \quad \text{with} \quad \|\nu\|_{C^1_{\mathrm{cyl}}} \leqslant \delta$$

  in the notation of Section 2;
- (P5) *$\int_M |\mathrm{pr}_{TM^\perp} \partial_r|^2 \,\mathrm{dVol}(M, g/r^2) \leqslant \delta$,*

then there exists a normal vector field ν' on $(\lambda,1)\times S^{n-1}$ in $((\lambda,1)\times S^{n-1}, g'/r^2)$ such that

$$M = G_{\mathrm{cyl}}(\nu') \quad \textit{with} \quad \|\nu'|_{(\lambda'',\lambda')\times S^{n-1}}\|_{C^{1,1/2}_{\mathrm{cyl}}} \leqslant \epsilon,$$

where $C^{1,1/2}_{\mathrm{cyl}}$ is the Hölder space with respect to the metric g'/r^2 on $(\lambda,1)\times S^{n-1}$.

Proof. Suppose there does not exist such δ. Then, for every $j = 2,3,4,\ldots$, there exist g_j, ϕ_j, M_j such that (P1), (P2), (P3), (P4) and (P5) hold with $\delta = 1/j$, and the following holds:

(P6) there does not exist any normal vector field ν'_j on $(\lambda'',\lambda')\times X$ in $((\lambda'',\lambda')\times S^{n-1}, g'/r^2)$ such that

$$M_j = G_{\mathrm{cyl}}(\nu'_j) \quad \text{with} \quad \|\nu'_j\|_{C^{1,1/2}_{\mathrm{cyl}}} \leqslant \epsilon.$$

By (P1), (P2) and (P3), we may apply Proposition 3.5. Therefore, by (3.11), (P4) and (P5), we have

$$\sup_{j=2,3,4,\ldots} \mathrm{Vol}(M_j, g_j/r^2) < \infty. \tag{3.12}$$

Therefore, by (P1), we have

$$\sup_{j=2,3,4,\ldots} \mathrm{Vol}(M_j, g') < \infty.$$

By (P1) and (P3), we have

$$\lim_{j\to\infty} \big(\text{the mean curvature of } M_j \text{ in } ((\lambda,1)\times S^{n-1}, g')\big) = 0 \tag{3.13}$$

in the C^0-topology. Thus, by Allard's compactness theorem [1, Theorem 5.6], there exists a subsequence M_{j_k} converging as varifolds to some rectifiable varifold M_∞ in $((\lambda,1)\times S^{n-1}, g')$.

Let $\|M_\infty\|$ be the Radon measure on $((\lambda,1)\times S^{n-1}, g')$ induced by M_∞. We shall prove

$$a^m \|M_\infty\|(a^{-1}E) = \|M_\infty\|(E) \tag{3.14}$$

for every $a > 0, E \subset (\lambda, 1) \times S^{n-1}$ with $aE \subset (\lambda, 1) \times S^{n-1}$. It suffices to prove

$$\frac{d}{da} a^m \int_{(\lambda,1)\times S^{n-1}} f(ar) h d\|M_\infty\| = 0 \tag{3.15}$$

for every smooth functions $h : S^{n-1} \to [0, \infty)$ and $f : (\lambda, 1) \to [0, \infty)$ with $a(\mathrm{supp} f) \subset (\lambda, 1)$. By (3.8), (P2), (P5) and (3.12), we have

$$\lim_{j\to\infty} \int_{M_j} d\psi_j \to 0,$$

where ψ_j is as in (3.3) with ϕ_j in place of ϕ. Therefore, by (P3) and (3.4), we have

$$\begin{aligned}
\text{the left-hand side of (3.15)} &= \frac{d}{da} \lim_{k\to\infty} a^m \int_{M_{j_k}} f(ar) h d\left(\frac{r^m}{m}\psi_{j_k}\right) \\
&= \lim_{k\to\infty} \int_{M_{j_k}} \frac{d}{da}((ar)^m f(ar)) \frac{dr}{r} \wedge h\psi_{j_k} \\
&= \lim_{k\to\infty} \int_{M_{j_k}} a^{-1} \frac{d}{dr}((ar)^m f(ar)) dr \wedge h\psi_{j_k} \\
&= \lim_{k\to\infty} - \int_{M_{j_k}} a^{-1}(ar)^m f(ar) dh \wedge \psi_{j_k}.
\end{aligned}$$

Therefore, by (3.7), (P2) and (3.12), we have

$$\text{the left-hand side of (3.15)} = \lim_{k\to\infty} - \int_{M_{j_k}} a^{-1}(ar)^m f(ar) r^{1-m} \partial_r \lrcorner (dh \wedge \phi_{j_k}).$$

By Proposition 3.1, we have

$$\int_{M_{j_k}} r^{-m} \partial_r \lrcorner (dh \wedge \phi_{j_k}) = \int_{M_{j_k}} \langle \mathrm{pr}_{TM_{j_k}^\perp} \partial_r, dh \rangle \, \mathrm{dVol}(M_{j_k}, g_{j_k}/r^2).$$

This converges to 0 by (P5) and (3.12). Thus, we have (3.15). This proves (3.14).

By (P4), the restriction of M_∞ to $(\lambda', 1) \times S^{n-1}$ is equal to $(\lambda', 1) \times X$ as varifolds in $((\lambda', 1) \times S^{n-1}, g')$. Therefore, by (3.14), we have

$$M_\infty = (\lambda, 1) \times X \text{ as varifolds in } ((\lambda, 1) \times S^{n-1}, g').$$

Therefore, M_{j_k} converges to $(\lambda, 1) \times X$ as varifolds in $((\lambda, 1) \times S^{n-1}, g')$. Therefore, by (3.13) and Allard's regularity theorem [1, Theorem 8.19], M_{j_k}

converges to $(\lambda, 1) \times X$ in the local $C^{1,1/2}$-topology in $(\lambda, 1) \times S^{n-1}$. This contradicts (P6), which completes the proof of Lemma 3.6. $\square$

4. Simon's estimates

In this section we show how to use Simon's estimates [13, 14] including a version of Łojasiewicz inequality [13, Lemma 1, p. 542].

Let X be a compact smooth Riemannian manifold, V a smooth real vector bundle on X with a fibre metric and a metric connection. Let C_x^∞ be the space of smooth sections of $V \to X$. Let $E : C_x^\infty \to \mathbb{R}$ satisfy

$$Ev = \int_X F(x, v, D_x v) dx \tag{4.1}$$

for every $v \in C_x^\infty$, where $D_x v$ is the covariant derivative of v, and $F = F(x, v, p)$ is a $\mathbb{R}$-valued smooth function of $x \in X$, $v \in V|_x$, $p \in T_x^* X \otimes V|_x$. Suppose F satisfies the following conditions:

(C1) $(v, p) \mapsto F(x, v, p)$ is a real-analytic function on the vector space $V|_x \oplus (T_x^* X \otimes V|_x)$ for every $x \in X$;

(C2) there exists $c > 0$ such that for every $x \in X, \xi \in T_x^* X, v \in V|_x$,

$$\frac{d^2}{dh^2} F(x, 0, h^2 \xi \otimes v)\Big|_{h=0} > c|\xi|^2 |v|^2.$$

By (C1), one can use the Łojasiewicz estimate [11]. This is important in the proof of a result of Simon; for the statement, see Proposition 4.1 below. (C2) is called the Legendre-Hadamard condition. Let $-\operatorname{grad} E : C_x^\infty \to C_x^\infty$ be the Euler-Lagrange operator of E, i.e.,

$$\big(\operatorname{grad} E(v), v'\big)_{L_x^2} = \frac{d}{dh} E(v + hv')\Big|_{h=0}$$

for every $v, v' \in C_x^\infty$, where

$$(v'', v')_{L_x^2} = \int_X \big(v''(x), v'(x)\big) dx; \tag{4.2}$$

here $\big(v''(x), v'(x)\big)$ is the inner product on the fibre $V|_x$ at $x \in X$. Suppose

$$\operatorname{grad} E(0) = 0, \quad \text{where} \quad 0 \in C_x^\infty. \tag{4.3}$$

Let $t_0 < t_\infty$. Let $C_{t,x}^\infty(t_0, t_\infty)$ be the space of all smooth sections $u = u(t, x)$ with $u(t, x) \in V|_x$ for every $(t, x) \in (t_0, t_\infty) \times X$. Let $C_{t,x}^{k,\mu}(t_0, t_\infty)$ be the

Hölder spaces with respect to the product metric on $(t_0, t_\infty) \times X$. Set $u(t) = u(t, \bullet) \in C^\infty_x$ for every $u = u(t,x) \in C^\infty_{t,x}(t_0, t_\infty)$.

We shall state a result of Simon which we use in the proof of Lemma 4.3 below.

Proposition 4.1 (Simon [13, Lemma 1, p. 542]). *There exist $\delta_0, \theta > 0$ depending only on X, V, E such that if $t_0 < t_3 < t_4 < t_\infty, u \in C^\infty_{t,x}(t_0, t_\infty)$, $\delta > 0$ and if*

$$\begin{aligned} &\|u\|_{C^{2,1/2}_{t,x}(t_3,t_4)} \leqslant \delta_0, \\ &\sup_{t\in[t_3,t_4]} \big(E(0) - E\big(u(t)\big)\big) \leqslant \delta, \\ &\|\partial_t u(t) + \operatorname{grad} E\big(u(t)\big)\|_{L^2_x} \leqslant (3/4)\|\partial_t u(t)\|_{L^2_x} \quad \textit{for every } t \in [t_3, t_4], \end{aligned} \tag{4.4}$$

then we have

$$\int_{t_3}^{t_4} \|\partial_t u(t)\|_{L^2_x} dt \leqslant (4/\theta)\big(\big|E\big(u(t_3)\big) - E(0)\big|^\theta + \delta^\theta\big).$$

Here, $\| \bullet \|_{L^2_x}$ is with respect to (4.2).

Consider $u = u(t,x) \in C^\infty_{t,x}(t_0, t_\infty)$ satisfying

$$\partial_t^2 u - \partial_t u - \operatorname{grad} E(u) + R(u, \partial_t u, \partial_t^2 u) = f \tag{4.5}$$

as in Simon [13], where $f \in C^\infty_{t,x}(t_0, t_\infty)$ satisfies

$$\|\partial_t^k f(t)\|_{C^2_x} \leqslant C_f e^{-2(t-t_0)} \quad \text{for every } t \in (t_0, t_\infty), k = 0, 1, 2 \tag{4.6}$$

for some $C_f > 0$, and $R : C^\infty_x \times C^\infty_x \times C^\infty_x \to C^\infty_x$ satisfies

$$\begin{aligned} R(v, v^{(1)}, v^{(2)}) = {} & A(x, v, D_x v, v^{(1)}) \cdot D_x^2 v \otimes v^{(1)} \\ & + \sum_{(k,l)=(0,1),(1,1),(0,2)} B_{kl}(x, v, D_x v, v^{(1)}) \cdot D_x^l v^{(k)} \end{aligned} \tag{4.7}$$

for every $v, v^{(1)}, v^{(2)} \in C^\infty_x$, where $A = A(x, v, p, q)$, $B_{kl} = B_{kl}(x, v, p, q)$ are smooth functions of $x \in X$, $v \in V|_x$, $p \in T^*_x X \otimes V|_x$, $q \in V|_x$ with $A(x, v, p, q) \in \operatorname{Hom}(\bigotimes^2 T^*_x X \otimes V|_x \otimes V|_x, V|_x)$, $B_{kl}(x, v, p, q) \in \operatorname{Hom}(\bigotimes^l T^*_x X \otimes V|_x, V|_x)$ and $B_{kl}(x, 0, 0, 0) = 0$ for every $x \in X$, $(k, l) = (1, 0), (1, 1), (2, 0)$. Then, for every $C_2' > 0$, there exists $\delta_4 = \delta_4(X, V, E, R, C_2') > 0$ such that if

$\|u\|_{C^{1,1/2}_{t,x}(t_0,t_\infty)} \leqslant \delta_4$, then we have

$$(4.8)\qquad |R(u(t), \partial_t u(t), \partial_t^2 u(t))| \leqslant C_2'(|\partial_t u(t)| + |D_x \partial_t u(t)| + |\partial_t^2 u(t)|).$$

Let $H : C^\infty_x \to C^\infty_x$ be the linearized operator of grad E at $0 \in C^\infty_x$. Then (4.5) is of the form

$$(4.9)\qquad \partial_t^2 u - \partial_t u - Hu = \sum_{0 \leqslant k+l \leqslant 2} a_{kl}(x, u, D_x u, \partial_t u) \cdot D_x^l \partial_t^k u + f,$$

where $a_{kl} = a_{kl}(x, u, p, q)$ are smooth functions of $x \in X$, $v \in V|_x$, $p \in T^*_x X \otimes V|_x$, $q \in V|_x$ with $a_{kl}(x, v, p, q) \in \mathrm{Hom}(\bigotimes^l T^*_x X \otimes V|_x, V|_x)$, $a_{kl}(x, 0, 0, 0) = 0$ for every $x \in X$, $0 \leqslant k + l \leqslant 2$. Therefore, there exists $\delta_2 = \delta_2(X, V, E, R) > 0$ such that if $u \in C^\infty_{t,x}(t_0, t_\infty)$ with $\|u\|_{C^{1,1/2}_{t,x}(t_0,t_\infty)} \leqslant \delta_2$, then we have

$$(4.10)\qquad \max_{0 \leqslant k+l \leqslant 2} \|a_{kl}(x, u, D_x u, \partial_t u)\|_{C^{0,1/2}_{t,x}(t_0,t_\infty)} \leqslant \delta_1,$$

where $\delta_1 = \delta_1(X, V, E) > 0$ is given below. By the Legendre-Hadamard condition (C2), $\partial_t^2 - \partial_t - H$ is elliptic on $C^\infty_{t,x}(t_0, t_\infty)$. Therefore, there exists $\delta_1 = \delta_1(X, V, E) > 0$ such that if $T > 0$, if $w, g \in C^\infty_{t,x}(-T/3, T/3)$ and if

$$(4.11)\qquad \partial_t^2 w - \partial_t w - Hw = \sum_{0 \leqslant k+l \leqslant 2} b_{kl}(t, x) \cdot D_x^l \partial_t^k w + g$$

with $\max_{0 \leqslant k+l \leqslant 2} \|b_{kl}\|_{C^{0,1/2}_{t,x}(-T/3,T/3)} \leqslant \delta_1$, then we have

$$(4.12)\qquad \|w\|_{C^{2,1/2}_{t,x}(-T/5,T/5)} \leqslant C_1 \|w\|_{L^2_{t,x}(-T/4,T/4)} + C_1 \|g\|_{C^{0,1/2}_{t,x}(-T/4,T/4)}$$

for some $C_1 = C_1(X, V, E; T) > 0$; here $L^2_{t,x}(t', t'')$ is with respect to the product metric on $(t', t'') \times X$. (4.12) is a Schauder estimate for elliptic systems; see Douglis-Nirenberg [4] and Morrey [12].

We shall state a proposition which we use in the proof of Lemma 4.3 below. One can prove it in the same way as a result of Simon; see [13, Lemma 2, p. 549] or [14, Lemma 3.3, Part II].

Proposition 4.2. *There exist $h, T_3, \delta_3 > 0$ depending only on X, V, E such that if $T > T_3$, if $w, g \in C^\infty_{t,x}(0, 3T)$ satisfy (4.11) with $\|b_{kl}\|_{C^0_{t,x}(0,3T)} \leqslant \delta_3$,* *and if*

$$\|g\|_{L^2_{t,x}(0,3T)} \leqslant {\delta_3}^{1/3}\|w\|_{L^2_{t,x}(T,2T)} \quad \text{with} \quad \|w\|_{L^2_{t,x}(0,3T)} < \infty,$$

then we have

$$\begin{aligned}
&\|w\|_{L^2_{t,x}(2T,3T)} \leqslant e^{-hT}\|w\|_{L^2_{t,x}(T,2T)} \Longrightarrow \|w\|_{L^2_{t,x}(T,2T)} \leqslant e^{-hT}\|w\|_{L^2_{t,x}(0,T)},\\
&\|w\|_{L^2_{t,x}(T,2T)} \geqslant e^{hT}\|w\|_{L^2_{t,x}(0,T)} \Longrightarrow \|w\|_{L^2_{t,x}(2T,3T)} \geqslant e^{hT}\|w\|_{L^2_{t,x}(T,2T)},\\
&\|w\|_{L^2_{t,x}(T,2T)} \geqslant e^{-hT}\|w\|_{L^2_{t,x}(0,T)} \text{ and } \|w\|_{L^2_{t,x}(2T,3T)} \leqslant e^{hT}\|w\|_{L^2_{t,x}(T,2T)}\\
&\Longrightarrow \|w(t)\|_{L^2_x} \leqslant (3/2)\|w(t')\|_{L^2_x} \text{ for every } t, t' \in (T, 2T)\\
&\text{and } \|\partial_t w(t)\|_{L^2_x} \leqslant (1/2)\|w(t)\|_{L^2_x} \text{ for every } t \in (T, 2T).
\end{aligned}$$

We shall prove a lemma which we use in the key step to the main result of this paper. It is similar to a result of Simon [13, Theorem 1, p. 534]. Simon's result is an a-priori estimate on $(0,\infty)\times X$. We however consider $(t_0, t_\infty)\times X$ with (t_0, t_∞) bounded. We prove the lemma for completeness.

Lemma 4.3. *Let X, V, E, R be as above. Let $t_0 < t_\infty$, and $f \in C^\infty_{t,x}(t_0, t_\infty)$ with (4.6) for some $C_f > 0$. Then, there exist $\theta, \delta_*, C_* > 0$ depending only on X, V, E, R, C_f such that if $t_* \in (t_0, t_\infty)$, if $u \in C^\infty_{t,x}(t_0, t_*)$ satisfies (4.5) and if*

$$\|u\|_{C^{1,1/2}_{t,x}(t_0,t_*)} \leqslant \delta_*, \tag{4.13}$$

$$\limsup_{t\to t_0} \|u(t)\|_{L^2_x} \leqslant \delta, \tag{4.14}$$

$$\sup_{t\in(t_0,t_*)} \big(E(0) - E\big(u(t)\big)\big) \leqslant \delta, \tag{4.15}$$

$$\|\partial_t u\|_{L^2_{t,x}(t_0,t_*)} \leqslant \sqrt{\delta} \tag{4.16}$$

for some $0 < \delta < \min\{1, \delta_*\}$, then we have

$$\sup_{t\in(t_0,t_*)} \|u(t)\|_{L^2_x} < C_*\delta^\theta. \tag{4.17}$$

Proof. By (4.14), it suffices to prove

$$\int_{t_0}^{t_*} \|\partial_t u(t)\|_{L^2_x} dt < C_*\delta^\theta. \tag{4.18}$$

By the Schwartz inequality and (4.16), for every $(t', t'') \subset (t_0, t_*)$, we have

$$\int_{t'}^{t''} \|\partial_t u(t)\|_{L^2_x} dt \leqslant \sqrt{t''-t'}\|\partial_t u\|_{L^2_{t,x}(t',t'')} \leqslant \sqrt{(t''-t')\delta}. \tag{4.19}$$

Let $T > 0$ be a sufficiently large constant; in the proof of Lemma 4.3 a constant means a real number depending only on X, V, E, R, C_f. If $t_* - t_0 < 8T$, then by (4.19), we have (4.18); we may therefore assume $t_* - t_0 \geqslant 8T$. Choose $t_1, t_6 \in (t_0, t_*)$ so that $T \leqslant t_1 - t_0 \leqslant 2T$, $T \leqslant t_* - t_6 \leqslant 2T$ and $t_6 - t_1 = jT$ for some integer $j \geqslant 4$. Then, by (4.19), we have

$$\int_{t_0}^{t_1} \|\partial_t u(t)\|_{L^2_{t,x}} \leqslant \sqrt{T\delta}, \quad \int_{t_6}^{t_*} \|\partial_t u(t)\|_{L^2_{t,x}} \leqslant \sqrt{T\delta}. \tag{4.20}$$

By (4.13), u satisfies (4.9) with (4.10). Therefore, u satisfies the Schauder estimate (4.12). Therefore, by (4.13) and (4.6), we have

$$\|u\|_{C^{2,1/2}_{t,x}(t_1,t_6)} \leqslant C_1'\delta_* + C_1'' e^{-2T}$$

for some constants $C_1', C_1'' > 0$. We may therefore assume that

$$\|u\|_{C^{2,1/2}_{t,x}(t_1,t_6)} \text{ is sufficiently small.} \tag{4.21}$$

Differentiating (4.5) with respect to t and using (4.21), we have:

$$\begin{aligned} &w = \partial_t u, g = \partial_t f \text{ satisfy (4.11)} \\ &\text{with } \|b_{kl}\|_{C^{0,1/2}_{t,x}(t_1,t_6)} \text{ sufficiently small.} \end{aligned} \tag{4.22}$$

We may therefore apply Proposition 4.2 to $\partial_t u$ repeatedly on (t_1, t_6) since $t_6 - t_1 \geqslant 4T$ is assumed to be sufficiently large. Therefore, there exist constants $h, \delta_3, c_3 > 0$ and integers i_1, i_2 with $1 \leqslant i_1 \leqslant i_2 \leqslant j-1$ such that: if $1 < i_1$, then we have

$$\begin{aligned} &\text{either } \|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \leqslant e^{-hT} \|\partial_t u\|_{L^2_{t,x}(t_1+(i-1)T, t_1+iT)} \\ &\text{or } {\delta_3}^{1/3} \|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \leqslant \|C_f e^{-2(t-t_0)}\|_{L^2_{t,x}(t_1+(i-1)T, \infty)} \end{aligned} \tag{4.23}$$

for every $i \in \{1, \dots, i_1 - 1\}$; if $i_1 < i_2$, then we have

$$c_3 e^{-2(t-t_0)} \leqslant \|\partial_t u(t)\|_{L^2_x} \leqslant (3/2) \|\partial_t u(t')\|_{L^2_x} \tag{4.24}$$

for every $t, t' \in (t_1 + i_1 T, t_1 + i_2 T)$ with $|t' - t| \leqslant T$ and we have

$$\|\partial_t^2 u(t)\|_{L^2_x} \leqslant (1/2) \|\partial_t u(t)\|_{L^2_x} \tag{4.25}$$

for every $t \in (t_1 + i_1 T, t_1 + i_2 T)$; if $i_2 < j - 1$, then we have

$$\|\partial_t u\|_{L^2_{t,x}(t_1+(i-1)T, t_1+iT)} \leqslant e^{-hT} \|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \tag{4.26}$$

for every $i \in \{i_2+1, \ldots, j-1\}$. Set $t_5 = t_1 + i_2 T$. Then, by (4.26) and (4.19), we have

$$\begin{aligned} \int_{t_5}^{t_6} \|\partial_t u(t)\|_{L^2_x} dt &\leqslant \sum_{i=i_2}^{j} \sqrt{T} \|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \\ &\leqslant \sqrt{T}(1-e^{-hT})^{-1}\sqrt{\delta}. \end{aligned} \tag{4.27}$$

In a similar way, by (4.23), there exists a constant $C_{T,h} > 0$ such that

$$\int_{t_1}^{t_2} \|\partial_t u(t)\|_{L^2_x} dt \leqslant C_{T,h}\sqrt{\delta}. \tag{4.28}$$

If $i_1 = i_2$, then by (4.20) and (4.28), we have (4.18); we may therefore assume $i_1 < i_2$. Set $t_3 = t_2 + T/3$, $t_4 = t_5 - T/3$. Then, by (4.19), we have

$$\int_{t_2}^{t_3} \|\partial_t u(t)\|_{L^2_x} dt \leqslant \sqrt{(T/3)\delta}, \ \int_{t_4}^{t_5} \|\partial_t u(t)\|_{L^2_x} dt \leqslant \sqrt{(T/3)\delta}, \tag{4.29}$$

$$\int_{t_2}^{t_3+T/4} \|\partial_t u(t)\|_{L^2_x} dt \leqslant \sqrt{(7T/12)\delta}. \tag{4.30}$$

By (4.22), we may apply the Schauder estimate (4.12) to $w = \partial_t u, g = \partial_t f$. Therefore, by (4.6) and (4.24), there exists a constant $C_2 > 0$ such that for every $t \in [t_3, t_4]$, we have

$$\|D_x \partial_t u(t)\|_{L^2_x} \leqslant C_2 \|\partial_t u(t)\|_{L^2_x}. \tag{4.31}$$

By (4.21), u satisfies (4.5) with R satisfying (4.8). Therefore, by (4.25) and (4.31), for every $t \in [t_3, t_4]$, we have

$$\|\partial_t u(t) + \operatorname{grad} E\big(u(t)\big)\|_{L^2_x} = \|\partial_t^2 u(t) + R\|_{L^2_x} \leqslant (3/4)\|\partial_t u(t)\|_{L^2_x}.$$

Therefore, by (4.21) and (4.15), we have (4.4). Therefore, by Proposition 4.1, we have

$$\int_{t_3}^{t_4} \|\partial_t u(t)\|_{L^2_x} dt \leqslant (4/\theta)\left(\Big|E\big(u(t_3)\big) - E(0)\Big|^{\theta} + \delta^{\theta}\right) \tag{4.32}$$

for some constant $\theta > 0$. Since E satisfies (4.1) with (4.3) and u satisfies the Schauder estimate (4.12), there exist constants $C_3', C_3 > 0$ such that

$$\begin{aligned}(4.33)\quad \big|E\big(u(t_3)\big) - E(0)\big| &\leqslant C_3' \|u(t_3)\|^2_{C^1_x} \\ &\leqslant C_3 \left(\sup_{t \in (t_3 - T/4, t_3 + T/4)} \|u(t)\|_{L^2_x} + e^{-2(t_3 - t_0)} \right)^2 .\end{aligned}$$

By (4.14), (4.20), (4.28) and (4.30), there exists a constant $C_4 > 0$ such that

$$\sup_{t \in (t_3 - T/4, t_3 + T/4)} \|u(t)\|_{L^2_x} \leqslant \limsup_{t \to t_0} \|u(t)\|_{L^2_x} + \int_{t_0}^{t_3 + T/4} \|\partial_t u(t)\|_{L^2_x} dt \leqslant C_4 \sqrt{\delta}.$$

By (4.24) and (4.19), there exists a constant $C_5 > 0$ such that

$$e^{-2(t_3 - t_0)} \leqslant C_5 \sqrt{\delta}.$$

Thus, (4.33) is bounded by $C_6 \delta^\theta$ for some constant $C_6 > 0$. Therefore, (4.32) is bounded by $C_7 \delta^\theta$ for some constant $C_7 > 0$. Therefore, by (4.20), (4.27), (4.28) and (4.29), we have (4.18). This completes the proof of Lemma 4.3. □

5. Completion of the proof

We are ready now to complete the proof of Theorem 2.2:

Proof of Theorem 2.2. Let ϕ' be a parallel calibration of degree m on the Euclidean space $(\mathbb{R}^n, g')$, and let ψ' be as in (2.1) in Section 2, or equivalently as in (3.7) in Section 3. Let X be a compact ψ'-submanifold of S^{n-1}. Let $0 < l < 1$. Suppose:

(S0) $\epsilon > 0$ is sufficiently small;

(S1) $0 < a_0 < b_0 < a_1 < b_1$, $a_0/b_0 = a_1/b_1 = l$;

(S2) g is a Riemannian metric on $B(b_1)$ with

$$\|g - g'\|_{C^1} \leqslant \epsilon, \quad \|g - g'\|_{C^2} \leqslant 1$$

with respect to g';

(S3) ϕ is a calibration of degree m on $(B(b_1), g)$ with

$$\left(1 + \log \frac{b_1}{a_0}\right) \sup_{B(b_1)} |\phi - \phi'| \leqslant \epsilon$$

where $|\bullet|$ is with respect to g';

(S4) M is a ϕ-submanifold of $(\mathbb{R}^n, g)$, and M is a closed subset of $(a_0, b_1) \times S^{n-1}$, where $(a_0, b_1) \times S^{n-1}$ is embedded into $\mathbb{R}^n$ by $(r, y) \mapsto ry$;

(S5) there exists a normal vector field ν_i on $(a_i, b_i) \times X$ in $((a_i, b_i) \times S^{n-1}, g')$, where $i = 0, 1$, such that

$$M \cap ((a_i, b_i) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu_i) \quad \text{with} \quad \|\nu_i\|_{C^1_{\mathrm{cyl}}} \leqslant \epsilon$$

in the notation of Section 2.

Let ψ be as in (3.3) in Section 3. Then, by (S3) and (S5), we have

$$\sup_{r \in (a_0, b_0) \cup (a_1, b_1)} \left| \int_{M \cap \{r\} \times S^{n-1}} \psi - \mathrm{Vol}(X) \right| \leqslant C\epsilon \tag{5.1}$$

for some constant $C > 0$; in the proof of Theorem 2.2 a constant means a real number depending only on l, m, n, X and ϕ'.

By Proposition 3.4, the Stokes Theorem and (5.1), we have

$$\begin{aligned} &\int_M |\mathrm{pr}_{TM^\perp} \partial_r|^2 \, \mathrm{dVol}(M, g/r^2) \\ &\leqslant (2C/m)\epsilon + C_{m,n} \sup |\phi - \phi'| \, \mathrm{Vol}(M, g'/r^2). \end{aligned} \tag{5.2}$$

Therefore, by Proposition 3.5 and (5.1), we have

$$\begin{aligned} \mathrm{Vol}(M, g'/r^2) \leqslant{}& C' \log \frac{b_1}{a_0} \\ &+ C' \left(1 + m \log \frac{b_1}{a_0}\right) \left(\epsilon + \sup |\phi - \phi'| \, \mathrm{Vol}(M, g'/r^2)\right) \end{aligned}$$

for some constant $C' > 0$. Therefore, by (S3), we have

$$\mathrm{Vol}(M, g'/r^2) \leqslant C'' \log \frac{b_1}{a_0} \tag{5.3}$$

for some constant $C'' > 0$. Therefore, by (5.2), we have

$$\int_M |\mathrm{pr}_{TM^\perp}\partial_r|^2 \,\mathrm{dVol}(M, g/r^2) \leqslant C'''\epsilon \tag{5.4}$$

for some constant $C''' > 0$. Choose a constant $\epsilon_* > 0$ so that if I is an open interval of $(0, \infty)$, and if ν is a normal vector field on $I \times X$ in $(I \times S^{n-1}, g')$ with $\|\nu\|_{C^0_{\mathrm{cyl}}} \leqslant \epsilon_*$, then $G_{\mathrm{cyl}}(\nu)$ is contained in a tubular neighbourhood of $I \times X$ in $(I \times S^{n-1}, g')$. Here, $G_{\mathrm{cyl}}(\nu)$ is as in Section 2. If ϵ_* is sufficiently small, then we have

$$|r\partial_r(\nu/r)|^2 \leqslant 2|\mathrm{pr}_{TM^\perp}\partial_r|^2,$$

as in [13, (7.13), p. 561] or [14, 3.2, Part I]. Therefore, by (5.4), we have

$$\int_{M\cap(I\times S^{n-1})} |r\partial_r(\nu/r)|^2 \,\mathrm{dVol}(M, g/r^2) \leqslant 2C'''\epsilon. \tag{5.5}$$

Choose $0 < \lambda < \lambda'' < \lambda' < 1$ so that $l\lambda' < \lambda < \lambda'' < l < \lambda'$. By (5.4), we may apply Lemma 3.6 to $M \cap ((\lambda b_1, b_1)) \times S^{n-1}$. Therefore, there exists a normal vector field ν on $(\lambda b_1, b_1) \times X$ in $((\lambda b_1, b_1) \times S^{n-1}, g'/r^2)$ such that

$$M \cap ((\lambda b_1, b_1) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu) \quad \text{with} \quad \|\nu|_{(\lambda'' b_1, \lambda' b_1)\times X}\|_{C^{1,1/2}_{\mathrm{cyl}}} \leqslant \epsilon_*. \tag{5.6}$$

Let S_* be the set of all $b_* \in [\lambda' a_0/\lambda, a_1)$ such that there exists a normal vector field ν on $(b_*, a_1) \times X$ in $((b_*, a_1) \times S^{n-1}, g'/r^2)$ such that

$$M \cap ((b_*, b_1) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu) \quad \text{with} \quad \|\nu|_{(b_*, a_1)\times X}\|_{C^{1,1/2}_{\mathrm{cyl}}} \leqslant \epsilon_*. \tag{5.7}$$

S_* is non-empty since $\lambda'' b_1 \in S_*$ by (5.6).

Proposition 5.1. *Suppose $b_* \in S_* \cap [\lambda a_0/\lambda', b_1)$, and let ν be as in* (5.7). *Then, there exist constants $c_{10}, C_{10} > 0$ such that*

$$\|\nu|_{(b_*, b_1)\times X}\|_{C^1_{\mathrm{cyl}}} \leqslant C_{10}\epsilon^{c_{10}}.$$

Proof. By Proposition 2.1, X is a minimal submanifold of S^{n-1}. By (S4), M is a minimal submanifold of $((a_0, b_1) \times S^{n-1}, g)$ with $\|g - g'\|_{C^2(B(b_1))} \leqslant 1$ as in (S2). Set

$$u(t, x) = e^{t/m}\nu(e^{-t/m}x), \quad t_0 = -m \log b_1, \quad t_* = -m \log b_*.$$

Then, by a result of Simon [14, Remark 3.3, Part I], u satisfies (4.5) for some E, R, f depending only on m, n, X. We shall apply Lemma 4.3 to u.

By (5.7), we have

$$\|\nu|_{(b_*,a_1)\times X}\|_{C^{1,1/2}_{\mathrm{cyl}}} = \|u\|_{C^{1,1/2}_{t,x}(-m\log a_1,t_*)} \leqslant \epsilon_*. \tag{5.8}$$

Therefore, we have (4.13). By (S5), we have (4.14). By (5.5), (5.8) and (S2), there exists a constant $C_{11}>0$ such that

$$\|\partial_t u\|^2_{L^2_{t,x}(t_0,t_*)} = \int_{(b_*,b_1)\times X} |r\partial_r(\nu/r)|^2 dr/r\,\mathrm{dVol}(X) \leqslant C_{11}\epsilon.$$

Therefore, we have (4.16). It suffices therefore to prove (4.15). In a way similar to (5.1), by (5.7), we have

$$\begin{aligned}&\sup_{b\in(b_*,b_1)} \mathrm{Vol}(X) - \mathrm{Vol}\left(M\cap\{b\}\times S^{n-1}, g'/r^2\right)\\ \leqslant\; &\sup_{b\in(b_*,b_1),b'\in(a_1,b_1)} \int_{M\cap\{b'\}\times S^{n-1}}\psi - \int_{M\cap\{b\}\times S^{n-1}}\psi + C_{12}\epsilon\end{aligned}$$

for some constant $C_{12}>0$. By Proposition 3.4, (5.4), (5.3) and (S3), we have

$$\sup_{b\in(b_*,b_1),b'\in(a_1,b_1)} \int_{M\cap\{b'\}\times S^{n-1}}\psi - \int_{M\cap\{b\}\times S^{n-1}}\psi \leqslant C_{13}\epsilon$$

for some constant $C_{13}>0$. Thus, there exists a constant $C_{14}>0$ such that

$$\sup_{b\in(b_*,b_1)} \mathrm{Vol}(X) - \mathrm{Vol}\left(M\cap\{b\}\times S^{n-1}, g'/r^2\right) \leqslant C_{14}\epsilon.$$

Therefore, we have (4.15). We may now apply Lemma 4.3 to u. Therefore, as in (4.17), we have

$$\sup_{t\in(t_0,t_*)} \|u\|_{L^2_x} \leqslant C_{15}\epsilon^{c_{15}}$$

for some constants $c_{15}, C_{15}>0$. Therefore, by interpolation and (5.8), we have

$$\|\nu|_{(b_*,a_1)\times X}\|_{C^1_{\mathrm{cyl}}} = \|u\|_{C^1_{t,x}(-m\log a_1,t_*)} \leqslant C_{10}\epsilon^{c_{10}}$$

for some constants $C_{10}, c_{10}>0$. By (S5), this proves Proposition 5.1. ☐

Suppose $b_*\in S_*$. Then, by Proposition 5.1, we may apply Lemma 3.6 to $M\cap((\lambda b_*/\lambda', b_*/\lambda')\times S^{n-1})$. Therefore, $\lambda'' b_*/\lambda'\in S_*$. Therefore, b_* is an interior point in S_*. S_* is thus an open subset of $[\lambda' a_0/\lambda, a_1)$.

By definition, S_* is a closed subset of $[\lambda' a_0/\lambda, a_1)$. S_* is thus a non-empty open closed subset of $[\lambda' a_0/\lambda, a_1)$. Therefore, $S_* = [\lambda' a_0/\lambda, a_1)$. Therefore, $\lambda' a_0/\lambda \in S_*$. Therefore, by (5.7) and Proposition 5.1, we have

$$M \cap ((\lambda' a_0/\lambda, b_1) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu) \quad \text{with} \quad \|\nu|_{(\lambda' a_0/\lambda, b_1)\times X}\|_{C^1_{\mathrm{cyl}}} \leqslant C_{10}\epsilon^{c_{10}}.$$

Therefore, by (S5), we have (2.3). This completes the proof of Theorem 2.2. □

References

[1] W. K. Allard, *On the First Variation of a Varifold*. Annals of Mathematics, second series, Vol. **95**, No. 3 (May, 1972), 417–491.

[2] A. Butscher, *Regularizing a singular special Lagrangian variety*. Communications in Analysis and Geometry, Vol. **12**, No. 4, (2004), 733–791.

[3] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*. J. Differential Geom., **18** (1983), 279–315.

[4] A. Douglis and L. Nirenberg, *Interior Estimates for Elliptic Systems of Partial Differential Equations*. Communications on Pure and Applied Mathematics, **8** (1955), 503–538.

[5] R. Harvey and H. B. Lawson, *Calibrated geometries*. Acta Mathematica, **148** (1982), 47–157.

[6] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*. Invent. math., **114** (1993), 515–563.

[7] Y. Imagi, *Surjectivity of a gluing for stable T^2-cones in special Lagrangian geometry*. Preprint, `arXiv:1102.4309`.

[8] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities V. Survey and applications*. Journal of Differential Geometry, **63** (2003), 279–347.

[9] D. A. Lee, *Connected Sums of Special Lagrangian Submanifolds*. Communications in Analysis and Geometry, Vol. **12**, No. 3, (2004), 553–579.

[10] Y. Lee, *Embedded Special Lagrangian Submanifolds in Calabi–Yau Manifolds*. Communications in Analysis and Geometry, Vol. **11**, No. 3, (2003), 391–423.

[11] S. Łojasiewicz, *Ensembles Semi-analytiques.* Institut des Hautes Etudes Scientifiques, 1965.

[12] C. B. Morrey, *Second Order Elliptic Systems of Differential Equations.* In: Contributions to the Theory of Partial Differential Equations (Edited by L. Bers, S. Bochner and F. John), Annals of Mathematics Studies, **33**, 101–159, Princeton University Press, 1954.

[13] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems.* Annals of Mathematics, second Series, **118**, No.3, (1983), 529–571.

[14] L. Simon, *Isolated singularities of extrema of geometric variational problems.* In: Harmonic Mappings and Minimal Immersions (Edited by E. Giusti), Montecanini, Italy 1984, Springer.

[15] C. H. Taubes, *Self-dual Yang-Mills connections on non-self-dual* 4-*manifolds.* J. Differential Geom., **17** (1982), 139–170.

[16] K. K. Uhlenbeck, *Removable singularities in Yang-Mills fields.* Commun. Math. Phys., **83** (1982), 11–29.

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