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Complete spacelike hypersurfaces in the de Sitter space with prescribed Gauss map

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We study the geometry of complete spacelike hypersurfaces with constant scalar curvature immersed into the de Sitter space. In this setting, we show that such a hypersurface must be totally umbilical, provided that its Gauss map has some suitable behavior. Our approach is based on the use of an appropriated generalized maximum principle, which can be seen as a sort of extension to complete Riemannian manifolds of the classical Hopf's maximum principle.

1. Introduction

The interest in the study of spacelike hypersurfaces immersed in spacetimes is motivated by their nice Bernstein-type properties. For instance, it was proved by Calabi [6] (for $n \leq 4$) and by Cheng and Yau [11] (for all n) that a complete maximal spacelike hypersurface in the Minkowski space $\mathbb{L}^{n+1}$ is totally geodesic. In [22], Nishikawa obtained similar results for others Lorentzian manifolds. In particular, he proved that a complete maximal spacelike hypersurface in de Sitter space $\mathbb{S}_1^{n+1}$ is totally geodesic (that is, its second fundamental form vanishes identically).

However, in the case that the mean curvature is a positive constant, Treibergs [33] surprisingly showed that there are many entire solutions of the corresponding constant mean curvature equation in $\mathbb{L}^{n+1}$ and, moreover, he classified them through their projective boundary values at infinity.

In [16], Goddard conjectured that the complete spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ with constant mean curvature H must be totally umbilical.

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Ramanathan proved Goddard's conjecture in [29] for $\mathbb{S}_1^3$ and $0 \leq H \leq 1$. Moreover, if $H > 1$ he showed that the conjecture is false as can be seen from an example due to Dajczer and Nomizu in [14]. In [2], Akutagawa proved that Goddard's conjecture is true when $n = 2$ and $H^2 \leq 1$ or when $n \geq 3$ and $H^2 < \frac{4(n-1)}{n^2}$. He also constructed complete spacelike rotation surfaces in $\mathbb{S}_1^3$ with constant H satisfying $H > 1$ and which are not totally umbilical. Later, Montiel [21] solved Goddard's problem in the compact case proving that the only closed spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature are the totally umbilical hypersurfaces.

Another Goddard-like problem is to study hypersurfaces immersed in a Lorentz space with constant scalar curvature. In this setting, Zheng [39] proved that a compact spacelike hypersurface in $\mathbb{S}_1^{n+1}$ with constant normalized scalar curvature R, $R < 1$ and nonnegative sectional curvatures is totally umbilical. Later, Cheng and Ishikawa [10] showed that Zheng's result is also true without additional assumptions on the sectional curvatures of the hypersurface. When Σ^n is a complete spacelike hypersurface immersed in $\mathbb{S}_1^{n+1}$, $n \geq 3$, with bounded mean curvature H and constant normalized scalar curvature R, as an application of the generalized maximum principle of Omori [26], Camargo, Chaves and Sousa [7] extended a technique developed by Cheng and Yau in [12] to show that if $\frac{n-2}{n} \leq R \leq 1$, then Σ^n is totally umbilical. Such result corresponds to a partially affirmative answer to a question posed by Li in [19]. Moreover, they also proved that if $R \leq 1$ and the square of the length of the second fundamental form S of Σ^n satisfies $\sup_\Sigma S < 2\sqrt{n-1}$, then Σ^n is totally umbilical.

On the other hand, it is well known that the geometry of the Gauss map of spacelike hypersurfaces immersed into a Lorentzian space form can impose several restrictions in its own geometry. For instance, Choi and Treibergs [13] interpreted properties of constant mean curvature spacelike hypersurfaces of $\mathbb{L}^{n+1}$ in terms of the corresponding Gauss map, which is a harmonic map to the hyperbolic space $\mathbb{H}^n$. Given an arbitrary closed set in the ideal boundary at infinity of $\mathbb{H}^n$, they showed that there exists a complete entire constant mean curvature spacelike hypersurface whose Gauss map is a diffeomorphism onto the interior of the hyperbolic space convex hull of the set. Also working in this context, Aiyama in [1] and Xin in [35], simultaneous and independently, used the generalized maximum principle of Omori-Yau [26, 37] in order to characterize the spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces in $\mathbb{L}^{n+1}$ having the image of its Gauss map contained in a geodesic ball of $\mathbb{H}^n$ (see also [28] for a weaker first version of this result given by B. Palmer). Afterwards, Xin and Ye [36] improved such previous results showing that if the image of the

Gauss map of a complete constant mean curvature spacelike hypersurface of $\mathbb{L}^{n+1}$ lies in a horoball of $\mathbb{H}^n$, then it must be a hyperplane. More recently, the third author [15] obtained another extension of the Aiyama-Xin theorem concerning complete spacelike hypersurfaces immersed with bounded mean curvature in $\mathbb{L}^{n+1}$.

When the ambient space is the de Sitter space, Aledo and Alías [3] have studied complete constant mean curvature spacelike hypersurfaces immersed in $\mathbb{S}_1^{n+1}$. In this setting, they applied the generalized maximum principle of Omori-Yau to show that the spacelike geodesic round spheres are the only complete constant mean curvature hypersurfaces in $\mathbb{S}_1^{n+1}$ having the image of its hyperbolic Gauss map contained in a geodesic ball of $\mathbb{H}^{n+1}$. In the recent paper [5], the first and third authors applied a suitable Simons-type formula due to Montiel [21] in order to show that a complete spacelike hypersurface immersed with constant mean curvature in $\mathbb{S}_1^{n+1}$, and whose image of its Gauss map lies in a totally umbilical hypersurface of $\mathbb{H}^{n+1}$, must be totally umbilical.

Here, motivated by the works above described, we deal with complete spacelike hypersurfaces with constant normalized scalar curvature R immersed into the de Sitter space $\mathbb{S}_1^{n+1}$. First, when $R < 1$, we show that such a hypersurface with bounded mean curvature must be a totally umbilical hypersurface of $\mathbb{S}_1^{n+1}$, provided that its hyperbolic Gauss map has some suitable behavior (cf. Theorem 4.1). Our approach is based on the use of an appropriated generalized maximum principle, which can be seen as a sort of extension to complete Riemannian manifolds of the classical Hopf's maximum principle (see Lemma 3.3).

Afterwards, in the case that $R = 1$, we use suitable Reilly type formulas for the height and angle functions (for more details, see Section 2), in order to prove that a complete simply connected spacelike hypersurface whose image of the hyperbolic Gauss map is contained in a totally umbilical hypersurface of the hyperbolic space $\mathbb{H}^{n+1}$, must be a totally geodesic sphere of $\mathbb{S}_1^{n+1}$ (cf. Theorem 4.3). We observe that such result can be regarded as a sort of Lorentzian version of a classical theorem due to Nomizu and Smyth [23], which establishes that the geodesic spheres are the only complete hypersurfaces of the Euclidean sphere $\mathbb{S}^{n+1}$ such that their image by the Gauss map lies in a totally umbilical hypersurface of $\mathbb{S}^{n+1}$.

2. Preliminaries

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric

$$\langle v, w \rangle = \sum_{j=1}^{n+1} v_j w_j - v_{n+2} w_{n+2}.$$

The $(n+1)$-dimensional de Sitter space is given as the following hyperquadric of $\mathbb{L}^{n+2}$:

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2}; \langle p, p\rangle = 1\}.$$

Throughout this paper, we will deal with connected and oriented isometrically immersed spacelike hypersurfaces $\psi : \Sigma^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ (that is, the induced metric via ψ is a Riemannian metric on Σ^n). The commutative ring of smooth real functions on Σ^n will be denoted by $\mathcal{D}(\Sigma)$. We recall that a unit normal timelike vector field N of Σ^n can be regarded as a map $N : \Sigma^n \to \mathbb{H}^{n+1}$, where $\mathbb{H}^{n+1}$ stands for the $(n+1)$-dimensional hyperbolic space, that is,

$$\mathbb{H}^{n+1} = \{p \in \mathbb{L}^{n+2}; \langle p\, ,\, p\rangle = -1\, ;\, p_{n+2} \geq 1\}.$$

In this setting, N is called the *hyperbolic Gauss map* of Σ^n.

Let us denote by $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the Weingarten endomorphism of Σ^n with respect to the normal vector field N. If ∇^0, $\overline{\nabla}$ and ∇ stand for the Levi-Civita connections in $\mathbb{L}^{n+2}$, $\mathbb{S}_1^{n+1}$ and Σ^n, respectively, then the Gauss and Weingarten formulas for such hypersurfaces are given by

$$\nabla^0{}_X Y = \nabla_X Y - \langle AX, Y\rangle N - \langle X, Y\rangle \psi$$

and

$$AX = -\overline{\nabla}_X N = -\nabla^0{}_X N,$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$.

Now, we recall that, as in [27], the curvature tensor R of Σ^n is given by

$$R(X, Y)Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y]Z,$$

where [,] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$. A well known fact is that the curvature tensor R of a spacelike hypersurface Σ^n can be described in terms of its Weingarten operator A by the so-called Gauss equation, given by

$$R(X,Y)Z = -\langle Y,Z\rangle X + \langle X,Z\rangle Y - \langle AX,Z\rangle AY + \langle AY,Z\rangle AX, \tag{2.1}$$

for every tangent vector fields $X,Y,Z \in \mathfrak{X}(\Sigma)$. On the other hand, the Codazzi equation of Σ^n is given by

$$(\nabla_Y A)X = (\nabla_X A)Y, \tag{2.2}$$

where $\nabla_X A$ denotes the covariant derivative of A (cf. [27], Chapter 4).

Associated to the Weingarten operator A of Σ^n one has n smooth functions $S_r : \Sigma^n \to \mathbb{R}$ defined by

$$\det(tI - A) = \sum_{k=0}^{n}(-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by construction. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n), \tag{2.3}$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the $r-$th elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$. An immediate consequence of such definition is the following equality

$$|A|^2 + 2S_2 = S_1^2.$$

Furthermore, if R denotes the scalar curvature of Σ^n, follows from Gauss' Equation (2.1) that

$$n(n-1)(1-R) = 2S_2. \tag{2.4}$$

Thus, from previous equality, we conclude that S_2 is a constant function on Σ^n if, and only if, the scalar curvature R of Σ^n is constant.

In what follows, we will work with the so-called *Newton transformation* $P : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which is defined by

$$P = -S_1 I + A, \tag{2.5}$$

where I is the identity operator. Naturally associated with the Newton transformation P, we have the second order linear differential operator

$L_1 : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$ given by

$$L_1 f = \mathrm{tr}(P \circ \nabla^2 f). \tag{2.6}$$

Here $\nabla^2 f : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator metrically equivalent to the hessian of f, and it is given by

$$\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle,$$

for all $X, Y \in \mathfrak{X}(\Sigma)$. It is also important to note that such operator is divergence type provided that the ambient space is a space form. This fact was proved by Rosenberg in [31] and it reads as follows

$$L_1 f = \mathrm{div}(P\,\nabla f).$$

For a fixed arbitrary vector $a \in \mathbb{L}^{n+2}$, let us consider the *height* and *angle* functions, which are defined, respectively, by $l_a = \langle \psi, a\rangle$ and $f_a = \langle N, a\rangle$. A direct computation allows us to conclude that the gradient of such functions are given by $\nabla l_a = a^\top$ and $\nabla f_a = -A(a^\top)$, where $a^\top$ is the orthogonal projection of a onto the tangent bundle $T\Sigma$, that is,

$$a^\top = a + f_a N - \, l_a \psi. \tag{2.7}$$

Taking into account that $\nabla^0 a = 0$ and using Gauss and Weingarten formulas, it is not difficult to verify that

$$\nabla_X \nabla l_a = -f_a A X - l_a X, \tag{2.8}$$

for all $X \in \mathfrak{X}(\Sigma)$. Now, we use (2.8) jointly with Codazzi Equation (2.2) to deduce

$$\nabla_X \nabla f_a = f_a A^2 X + l_a A X - (\nabla_{a^\top} A)(X), \tag{2.9}$$

for all $X \in \mathfrak{X}(\Sigma)$. Moreover, based on the classical paper [30] of Reilly, it is possible to obtain the following identities related with the action of L_1 on these functions:

$$L_1 l_a = 2S_2 f_a + (n-1) S_1 l_a \tag{2.10}$$

and

$$L_1 f_a = (3S_3 - S_1 S_2) f_a - 2S_2 l_a + \langle \nabla S_2, a^\top\rangle. \tag{2.11}$$

where S_1, S_2 and S_3 are the first three elementary symmetric functions of the principal curvatures of Σ^n which was defined in (2.3).

For what follows, it is also convenient to consider the *traceless operator* associated to the second fundamental form, $\Phi : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which is given by

$$\Phi(X) = AX + HX,$$

for all $X \in \mathfrak{X}(\Sigma)$, where $H = -(1/n)\mathrm{trace}(A) = -(1/n)S_1$ stands for the mean curvature of Σ^n.

At this point, we observe that the choice of the sign in our definition of H is motivated by the fact that in that case the mean curvature vector is given by $\overrightarrow{H} = HN$. Hence, $H(p) > 0$ at a point $p \in \Sigma^n$ if, and only if, $\overrightarrow{H}(p)$ is in the same time-orientation as $N(p)$ (in the sense that $\langle \overrightarrow{H}, N\rangle_p < 0$).

We can check that the Hilbert-Schmidt norm of Φ (that is, $|\Phi|^2 = \mathrm{tr}(\Phi^*\Phi)$, where Φ^* stands for the adjoint of Φ) satisfies

$$|\Phi|^2 = \frac{1}{2n}\sum_{i,j=1}^{n}(\lambda_i - \lambda_j)^2.$$

Consequently, Σ^n is totally umbilical hypersurface if, and only if, $|\Phi|^2 = 0$.

3. Key lemmas

This section is devoted to present the main analytical tools which will be used to prove our results. The first one is a classical algebraic result due to Okumura [25], and completed with the equality case proved in [4] by Alencar and do Carmo.

Lemma 3.1. *Let $\mu_1, \dots, \mu_n$ be real numbers such that $\sum_{i=1}^{n} \mu_i = 0$ and $\sum_{i=1}^{n} \mu_i^2 = \beta^2$, where $\beta \geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_{i=1}^{n}\mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{3.1}$$

and equality holds if, and only if, either at least $(n-1)$ of the numbers μ_i are equal.

The second one corresponds to a suitable characterization of totally umbilical hypersurfaces in a semi-Riemannian space form due to D-S. Kim et al. [18], which can be regarded as a converse of a theorem due to R. Sharma and K. Duggal in [32].

Lemma 3.2. *Let Σ^n be a connected semi-Riemannian hypersurface immersion in a semi-Riemannian space form $\mathbb{M}_c^{n+1}$. Suppose that $\mathbb{M}_c^{n+1}$ carries a conformal vector field V whose tangential component $V^\top$ on Σ^n becomes a conformal vector field. Then, one of the following holds:*

(a) *Σ^n is a totally umbilical hypersurface;*

(b) *the restriction of V to Σ^n reduces to a tangent vector field on Σ^n.*

In order to present our next auxiliary lemma, we quote the following version of Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold Σ^n due to Yau [38]: if $\omega \in \Omega^{n-1}(\Sigma)$ is an integrable $(n-1)$-differential form on Σ^n, then there exists a sequence B_i of domains on Σ^n such that $B_i \subset B_{i+1}$, $\Sigma^n = \bigcup_{i\geq 1} B_i$ and

$$\lim_{i\to+\infty} \int_{B_i} d\omega = 0.$$

Now, suppose Σ^n is oriented by the volume element $d\Sigma$, and let $\mathcal{L}^1(\Sigma)$ be the space of Lebesgue integrable functions on Σ^n. In this setting, taking $\omega = \iota_X d\Sigma$ the contraction of $d\Sigma$ in the direction of a smooth vector field X on Σ^n, Caminha [9] obtained the following consequence of Yau's result (cf. Proposition 2.1 of [9]; see also the Theorem due to Karp in [17]):

Lemma 3.3. *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold Σ^n, such that* $\mathrm{div}X$ *does not change sign on Σ^n. If $|X| \in \mathcal{L}^1(\Sigma)$, then* $\mathrm{div}X = 0$.

4. Umbilicity of spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$

In order to establish our first theorem, we will describe some particular regions of the hyperbolic space $\mathbb{H}^{n+1}$. For this, we will consider $\mathbb{H}^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$:

$$\mathbb{H}^{n+1} = \{p \in \mathbb{L}^{n+2}; \langle p, p\rangle = -1, p_{n+2} \geq 1\}.$$

In this setting, we recall that $\mathbb{H}^{n+1}$ admits a foliation by means of totally umbilical hypersurfaces

$$L_\tau = \{p \in \mathbb{H}^{n+1}; \langle p, a\rangle = \tau\},$$

where $a \in \mathbb{L}^{n+2}$ is a fixed vector and $\tau^2 + \langle a, a\rangle > 0$ (see, for instance, [20]).

In particular, when a is a nonzero null vector, we have that such hypersurfaces L_τ are exactly the horospheres of $\mathbb{H}^{n+1}$. In this case, fixed a $\tau \in \mathbb{R}$, we will refer to the *interior domain* enclosed by L_τ the set

$$\{p \in \mathbb{H}^{n+1}; \langle p, a\rangle < \tau\},$$

while the *exterior domain* enclosed by L_τ is given by

$$\{p \in \mathbb{H}^{n+1}; \langle p, a\rangle > \tau\}.$$

On the other hand, when a is a spacelike vector, the level set

$$L_0 = \{p \in \mathbb{H}^{n+1}; \langle p, a\rangle = 0\}$$

defines a totally geodesic hypersphere in $\mathbb{H}^{n+1}$. So, in analogy to the context of the Euclidean sphere $\mathbb{S}^{n+1}$, we will refer to such hypersphere as the *equator* of $\mathbb{H}^{n+1}$ determined by a. This equator divides $\mathbb{H}^{n+1}$ into two connected components, which (proceeding with our analogy between $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$) will be called *hemispheres* of $\mathbb{H}^{n+1}$ determined by a.

Now, we are in position to present our first result whose proof is based on a nice computation which appears, in a slightly different form, in [34].

Theorem 4.1. *Let $\psi : \Sigma^n \to \mathbb{S}_1^{n+1}$ be a complete spacelike hypersurface, $n \geq 3$, with bounded mean curvature H and constant normalized scalar curvature $R < 1$. Suppose that, for some nonzero vector $a \in \mathbb{L}^{n+2}$, we have $|a^\top| \in \mathcal{L}^1(\Sigma)$. If one of the following conditions is satisfied:*

- (i) *a is timelike;*
- (ii) *a is null and the image of the hyperbolic Gauss map of Σ^n is contained in the closure of a domain enclosed by a horosphere of $\mathbb{H}^{n+1}$ determined by a;*
- (iii) *a is spacelike and the image of the hyperbolic Gauss map of Σ^n is contained in the closure of a hemisphere of $\mathbb{H}^{n+1}$ determined by a,*

then Σ^n is a totally umbilical hypersurface of $\mathbb{S}_1^{n+1}$.

Proof. Let $p \in \Sigma^n$ and $\{e_1, \dots, e_n\}$ be a local orthonormal frame on p, geodesic at p and diagonalizing A at p, with $Ae_k = \lambda_k e_k$ for all $1 \leq k \leq n$.

Then, we have

$$\operatorname{div}(A(\nabla f_a)) = \sum_{i=1}^{n} \langle \nabla_{e_i} A(\nabla f_a), e_i \rangle. \tag{4.1}$$

From Codazzi Equation (2.2), we can rewrite Equation (4.1) in the following way

$$\operatorname{div}(A(\nabla f_a)) = \operatorname{tr}\big(\nabla_{\nabla f_a} A\big) + \sum_{i=1}^{n} \nabla^2 f_a(e_i, Ae_i). \tag{4.2}$$

Now, we observe that

$$\operatorname{tr}\big(\nabla_{\nabla f_a} A\big) = \operatorname{div}(-nH\nabla f_a) + nH\Delta f_a.$$

Thus, using the above expression, from Equations (2.9) and (4.2) we obtain

$$\begin{aligned} \operatorname{div}(A(\nabla f_a) + nH\nabla f_a) &= nH|A|^2 f_a - n^2H^2 l_a + n^2Ha^\top(H) \\ &\quad + \sum_{i=1}^{n} \nabla^2 f_a(e_i, Ae_i). \end{aligned} \tag{4.3}$$

On the other hand, taking into account that $|A|^2 - n^2H^2$ is constant and that $(\nabla_{e_i} e_j)(p) = 0$ for all $i, j \in \{1, \dots, n\}$, with a straightforward computation we get

$$\sum_{i=1}^{n} \nabla^2 f_a(e_i, Ae_i) = -n^2Ha^\top(H) + f_a \sum_{i=1}^{n} \lambda_i^3 + |A|^2 l_a.$$

Moreover, it is not difficult to verify that

$$\sum_{i=1}^{n} \lambda_i^3 = -nH^3 - 3H|\Phi|^2 + \sum_{i=1}^{n} (\lambda_i + H)^3. \tag{4.4}$$

Hence, Equation (4.3) becomes

$$\begin{aligned} &\operatorname{div}\Big(A(\nabla f_a) + nH\nabla f_a + \frac{1}{n}(|A|^2 - n^2H^2)\nabla l_a\Big) \\ &= \Big((n-2)H|\Phi|^2 + \sum_{i=1}^{n} (\lambda_i + H)^3\Big) f_a. \end{aligned} \tag{4.5}$$

We also observe that, from Equation (2.4), our restriction on the normalized scalar curvature R implies that the mean curvature H does not vanish on Σ^n. Consequently, we can choose an orientation for Σ^n in such a way that $H > 0$ and, hence, we deduce that

$$H > \frac{|\Phi|}{\sqrt{n(n-1)}}.$$

Now, since

$$\sum_{i=1}^{n}(\lambda_i + H) = 0 \quad \text{and} \quad \sum_{i=1}^{n}(\lambda_i + H)^2 = |\Phi|^2,$$

it follows from Lemma 3.1 that

$$\sum_{i=1}^{n}(\lambda_i + H)^3 \geq -\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3.$$

Thus, the previous inequalities allow us to conclude that

$$(n-2)H|\Phi|^2 + \sum_{i=1}^{n}(\lambda_i + H)^3 \geq (n-2)|\Phi|^2\left(H - \frac{|\Phi|}{\sqrt{n(n-1)}}\right) \geq 0. \tag{4.6}$$

At this point, we note that, when a is a spacelike vector, our hypothesis on the hyperbolic Gauss map of Σ^n means that there exists a nonzero vector $a \in \mathbb{L}^{n+2}$ such that the corresponding angle function $f_a = \langle N, a\rangle$ does not change sign on Σ^n and, when a is a null vector, such hypothesis implies that f_a has strict sign on Σ^n. Furthermore, when a is a timelike vector, the simple fact that Σ^n is a spacelike hypersurface implies that f_a always has strict sign on it.

Consequently, from Equations (4.5) and (4.6), we conclude that

$$\operatorname{div}\big(A(\nabla f_a) + nH\nabla f_a + \frac{1}{n}(|A|^2 - n^2H^2)\nabla l_a\big)$$

does not change sign on Σ^n when a is a spacelike vector and has strict sign on Σ^n when a is either a null or timelike vector. Moreover, we observe that

$$\begin{aligned} &|A(\nabla f_a) + nH\nabla f_a + \frac{1}{n}(|A|^2 - n^2H^2)\nabla l_a| \\ &\leq \big(2|A|^2 + nH|A| + n^2H^2\big)|a^\top|. \end{aligned} \tag{4.7}$$

Thus, since we are supposing that $a^\top$ has norm integrable on Σ^n, from (4.7) we have that the same occurs with the vector field $X = A(\nabla f_a) + nH\nabla f_a + \frac{1}{n}(|A|^2 - n^2H^2)\nabla l_a$. Then, we can apply Lemma 3.3 in order to infer that the divergence of X vanishes identically on Σ^n.

Hence, from (4.5) and (4.6) we get $|\Phi|^2 f_a^2 = 0$ on Σ^n. This allows us to conclude that $|\Phi|^2 = 0$ on Σ^n when a is either a timelike or null vector and, therefore, Σ^n must be totally umbilical.

When a is a spacelike vector, the equality $|\Phi|^2 f_a^2 = 0$ implies that the corresponding height function l_a satisfies

$$\nabla^2 l_a = \frac{1}{n}(\Delta\, l_a) g,$$

where g stands for the Riemannian metric of Σ^n. Consequently, $\nabla l_a = a^\top$ is a conformal vector field on Σ^n. On the other hand, taking into account once more that $|a^\top| \in \mathcal{L}^1(\Sigma)$, a can not be a tangent vector to Σ^n. Therefore, from Lemma 3.2 we conclude that Σ^n is a totally umbilical hypersurface of $\mathbb{S}_1^{n+1}$. $\square$

Remark 4.2. We point out that, in Theorem 4.1, the integrability on the spacelike hypersurface of the norm of the tangential component of the fixed vector $a \in \mathbb{L}^{n+2}$ is a necessary hypothesis. In fact, it is not difficult to verify that the hyperbolic cylinder

$$\mathcal{C}^n(\rho) = \mathbb{S}^1(\rho) \times \mathbb{H}^{n-1}(\sqrt{\rho^2 - 1}) \hookrightarrow \mathbb{S}_1^{n+1}$$

has the following hyperbolic Gauss map

$$N(p) = -\frac{1}{\rho\sqrt{\rho^2 - 1}}(\nu(p) - \rho^2 p),$$

where $\nu : \mathcal{C}^n(\rho) \to \mathbb{L}^{n+2}$ is given by $\nu(p) = (p_1, p_2, 0, \ldots, 0)$, and that the Weingarten operator A of $\mathcal{C}^n(\rho)$ with respect to N has the following principal curvatures

$$\lambda_1 = \frac{\sqrt{\rho^2 - 1}}{\rho} \quad \text{and} \quad \lambda_2 = \cdots = \lambda_n = \frac{\rho}{\sqrt{\rho^2 - 1}}.$$

On the other hand, such a hyperbolic cylinder $\mathcal{C}^n(\rho)$ is an example of complete spacelike isoparametric hypersurface of $\mathbb{S}_1^{n+1}$. So, let us denote by H the mean curvature of $\mathcal{C}^n(\rho)$. Thus, we have from the above identities that

$$|A|^2 = \frac{\rho^2 - 1}{\rho^2} + (n-1)\frac{\rho^2}{\rho^2 - 1} \quad \text{and} \quad H = \frac{1 - n\rho^2}{n\rho\sqrt{\rho^2 - 1}}.$$

Now, after a straightforward computation, we note that, for all $n \geq 2$,

$$n(n-1)(1-R) = n^2H^2 - |A|^2 = \frac{n-1}{\rho^2-1}(n\rho^2-2) > 0.$$

Therefore, the scalar curvature of $\mathcal{C}^n(\rho)$ satisfies $R < 1$ and, when $a \in \mathbb{L}^{n+2}$ is a timelike vector, this allows us to conclude that $|a^\top|$ can not be integrable on $\mathcal{C}^n(\rho)$.

Now, in order to obtain a sort of Lorentzian version of a classical theorem due to Nomizu and Smyth [23] (which was mentioned in the end of Section 1), we will explore a natural duality between the foliations of the de Sitter and hyperbolic spaces through totally umbilical hypersurfaces. Such duality follows from the fact that the totally umbilical hypersurfaces of $\mathbb{S}_1^{n+1}$ can be realized in the Lorentz-Minkowski model in the following way

$$\mathcal{L}_\varrho = \{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle = \varrho\},$$

where $a \in \mathbb{L}^{n+2}$ is a fixed vector, and $\varrho^2 > \langle a, a\rangle$ (cf. [21], Example 1).

Furthermore, from a straightforward computation, we see that the hyperbolic Gauss map of such a totally umbilical hypersurface is given by

$$N(p) = \frac{1}{\sqrt{\varrho^2 - \langle a, a\rangle}}(a - \varrho p) \in \mathbb{H}^{n+1}.$$

Consequently, from the previous expression we obtain that the corresponding angle function f_a satisfies

$$f_a = -\sqrt{\varrho^2 - \langle a, a\rangle} = \text{constant}.$$

Hence, from the description of totally umbilical spacelike hypersurfaces of the de Sitter space $\mathbb{S}_1^{n+1}$ due to Montiel in [21], we conclude that:

(i) if a is a unit spacelike vector, then $N(\mathcal{L}_\varrho)$ is isometric to an n-dimensional hypersphere of $\mathbb{H}^{n+1}$;

(ii) if a is a nonzero null vector, then $N(\mathcal{L}_\varrho)$ is isometric to a horosphere of $\mathbb{H}^{n+1}$;

(iii) if a is a unit timelike vector, then $N(\mathcal{L}_\varrho)$ is isometric to an n-dimensional sphere of $\mathbb{H}^{n+1}$.

The previous digression allows us to state and prove our second result.

Theorem 4.3. *Let $\psi : \Sigma^n \to \mathbb{S}_1^{n+1}$ be a complete spacelike hypersurface, $n \geq 3$, with constant normalized scalar curvature $R = 1$. If the image of the hyperbolic Gauss map of Σ^n is contained in a totally umbilical hypersurface of $\mathbb{H}^{n+1}$, then Σ^n is a totally geodesic round sphere of $\mathbb{S}_1^{n+1}$.*

Proof. Let us denote by A the Weingarten operator of Σ^n associated to the unit normal vector field N and $\Phi = A + HI$ its associated traceless operator. It is not difficult to verify that $\operatorname{tr}(A^3) = S_1^3 - 3S_1S_2 + 3S_3$, where S_1, S_2 and S_3 are the first three elementary symmetric functions of Σ^n. Thus, a straightforward computation allows us to conclude that

$$3S_3 - S_1S_2 = nH|\Phi|^2 + n(n-1)H^3 - 3H|\Phi|^2 + \operatorname{tr}(\Phi^3). \tag{4.8}$$

On the other hand, taking into account our hypothesis under the image of the hyperbolic Gauss map N of Σ^n, we guarantee that there exists a vector $a \in \mathbb{L}^{n+2}$ such that the angle function f_a satisfies $f_a = \langle N, a\rangle = \tau$ on Σ^n, for some $\tau \in \mathbb{R}$ with $\tau^2 + \langle a, a\rangle > 0$. Consequently, if a is a timelike or nonzero null vector, then $\tau \neq 0$. Now, from formula (2.11) we have

$$(3S_3 - S_1S_2)\tau - 2S_2 l_a + \langle \nabla S_2, a^\top\rangle = 0.$$

But, since $R = 1$, from Gauss equation we get $S_2 = 0$. Hence, from the above expression, we can apply Proposition 1 of [8] to conclude that

$$S_2 = S_3 = \cdots = S_n = 0.$$

We have from (4.8) that

$$\operatorname{tr}(\Phi^3) = (2-n)H|\Phi|^2.$$

Therefore, from the above expression we get

$$|\operatorname{tr}(\Phi^3)| = \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3,$$

and it follows from Lemma 3.1 that at least $n-1$ of the eigenvalues of Φ are equal and, hence, at least $n-1$ of the eigenvalues of the Weingarten operator A of Σ^n are equal. Let us denote by

$$\lambda_1 = \mu \quad \text{and} \quad \lambda_2 = \cdots = \lambda_n = \nu,$$

such eigenvalues and by $e_1, \ldots, e_n$ the corresponding eigenvectors. Since $S_n = 0$, immediately it follows that $\mu\nu = 0$ on Σ^n. Now, using once more that $S_2 = 0$, we obtain $\nu = 0$.

Now, denote by σ_{ij} the 2-dimensional subspace of $T_p\Sigma$ generated by e_i e e_j. Thus, from Gauss Equation (2.1) we get

$$K(\sigma_{ij}) = 1 - \lambda_i\lambda_j,$$

where K stands for the sectional curvature of Σ^n. Thus, using that $\lambda_i\lambda_j = 0$, we obtain from previous expression that $K(\sigma_{ij}) = 1$. Hence, Σ^n has constant sectional curvature equals to 1. Now, we consider the universal covering $\widetilde{\Sigma}^n$ of Σ^n endowed with its standard covering metric. Thus, since $\widetilde{\Sigma}^n$ is a simply connected hypersurface of constant sectional curvature 1, we have from Cartan's classification theorem that $\widetilde{\Sigma}^n$ is isometric to the unit Euclidean sphere $\mathbb{S}^n$ and, consequently, the same holds for Σ^n. Therefore, since $S_2 = 0$, we have that Σ^n must be a totally geodesic sphere of $\mathbb{S}_1^{n+1}$.

In the case that $\tau = 0$, from (2.8) it follows that the hessian of the corresponding height function l_a satisfies

$$\nabla^2 l_a = -l_a g,$$

where g stands for the Riemannian metric of Σ^n. Taking into account once more the description of the totally umbilical hypersurfaces of $\mathbb{S}_1^{n+1}$ given by Montiel in [21], we observe that the height function can not be constant in this case, since a is a spacelike vector. Therefore, from the classical Obata's theorem (cf. [24], Theorem A) we also conclude that Σ^n is isometric to a totally geodesic round sphere of $\mathbb{S}_1^{n+1}$. $\square$

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