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Asymptotic Hodge theory of vector bundles

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We introduce several families of filtrations on the space of vector bundles over a smooth projective variety. These filtrations are defined using the large k asymptotics of the kernel of the Dolbeault Dirac operator on a bundle twisted by the kth power of an ample line bundle. The filtrations measure the failure of the bundle to admit a holomorphic structure. We study compatibility under the Chern isomorphism of these filtrations with the Hodge filtration on cohomology.

1. Introduction

Let M be a compact complex manifold of complex dimension m. Let $\mathrm{Vect}(M)$ denote the isomorphism classes of complex vector bundles over M, and let $\mathrm{Vect}(M, r)$ denote the subset of isomorphism classes of bundles of rank r. Given $E \in \mathrm{Vect}(M)$ equipped with a connection A with curvature F_A, the *Chern character* is defined to be

$$\mathrm{ch}(E) := \left[\mathrm{tr}\,\exp\left(\frac{iF_A}{2\pi}\right)\right] \in H^*(M, \mathbb{Q}).$$

The Chern character extends to a surjective map

$$\mathrm{ch}\colon \mathrm{Vect}(M) \otimes \mathbb{Q} \twoheadrightarrow H^{\mathrm{even}}(M, \mathbb{Q}).$$

When M is Kähler, the Hodge decomposition,

$$H^d(M, \mathbb{C}) = \oplus_{p+q=d} H^{p,q}(M)$$

and the Hodge filtration,

$$S^p_H H^d := \oplus_{j \geq p} H^{j,d-j}(M),$$

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are powerful tools in complex geometry. In this note, we introduce and analyze for projective varieties, M, a natural filtration $S_V^{\cdot}$ on $\mathrm{Vect}(M)$, which is analogous to the Hodge filtration on $H^d(M,\mathbb{C})$. Heuristically, the filtration measures degree of failure to admit a holomorphic structure. In fact, S_V^0 comprises bundles admitting holomorphic structures.

Let L be an ample holomorphic line bundle over a smooth projective variety M of complex dimension m. Then L admits a metric, h, whose induced Chern connection has curvature F^L which satisfies $F^L(v,\bar{v}) > 0$, for all nonzero holomorphic tangent vectors v. We call such a metric *admissible.* An admissible metric on an ample line bundle determines a Kähler form for M, by defining the Kähler form ω to be $\omega = iF^L$. We call a choice of Kähler structure on M induced by an ample line bundle with an admissible metric a *polarization* of M. We call (L,h), (or L when h is understood) a *polarizing line bundle.*

Let E be a complex vector bundle of rank r, with connection A. We do not assume that E is holomorphic. The connection A on E and the Chern connection on a polarizing L induce connections $A(k)$ on $E\otimes L^k$. Let $\bar\partial_{A(k)}$ denote the associated $\bar\partial$-operator. Define

$$D_{A(k)} = \sqrt{2}(\bar\partial_{A(k)} + \bar\partial^*_{A(k)}).$$

For k sufficiently large, the dimension of the kernel of $D_{A(k)}$ is the index of $D^{\text{even}}_{A(k)}$, the restriction of $D_{A(k)}$ to $E\otimes L^k$ valued even forms. Let $s \in \ker(D_{A(k)})$. Let s^j denotes the $(0,j)$ component of s, and write

$$s = s^0 + s^2 + \cdots + s^{2\lfloor \frac{m}{2}\rfloor},$$

where $\lfloor x\rfloor$ denotes the integer part of x. When E is holomorphic, the same estimates which imply that $\ker(D^{\text{odd}}_{A(k)}) = 0$ for k large imply that $s = s^0$. Generically, for E nonholomorphic, one does not expect to find any $s \in \ker(D_{A(k)})$ satisfying $s = s^0$. This leads us to the first definition of our filtration.

Definition 1.1. Let (L,h) be a polarization of M. We say $E \in S^q_{V,L,h}\mathrm{Vect}(M)$ if E admits a connection A so that for all k sufficiently large, $s \in \ker(D_{A(k)})$ implies $s^{2j} = 0, \forall j > q$. We call A an $S^q_{V,L,h}$*-compatible* connection, or simply an S^q_V-compatible connection if we do not wish to specify the polarization data. We say $E \in S^q_V\mathrm{Vect}(M)$ if $E \in S^q_{V,L,h}\mathrm{Vect}(M)$ for *some* choice of polarization (L,h). We say E is of *Hodge type* q if $E \in S^q_V\mathrm{Vect}(M) \setminus S^{q-1}_V\mathrm{Vect}(M)$. We say $E \in IS^q_V\mathrm{Vect}(M)$ if E admits a connection A which is $S^q_{V,L,h}$ compatible for *every* choice of polarization (L,h).

For all q it is easy to construct examples of bundles of Hodge type $q \leq \frac{m}{2}$ on complex m manifolds. A bundle is of Hodge type 0 if and only if it admits a holomorphic structure. For $0 < q < \frac{m}{2}$, simply consider two projective varieties M_1 and M_2 equipped with bundles E_1 and E_2. Assume E_1 is holomorphic. Then the Hodge type of $E_1 \times E_2$ is the Hodge type of E_2, by a separation of variables computation. If $\dim_{\mathbb{C}} M_2 = 2q$, and $H^{0,2q}(M_2) \neq 0$ then there exists E_2 of type q on M_2, thus yielding bundles of type q on $M_1 \times M_2$. In general, the conditions defining $S^{\cdot}_V$ appear to be very difficult to establish, and may be too rigid for many applications. Consequently, we introduce quantized versions, $S^{\cdot}_{V,p}$ and $IS^{\cdot}_{V,p}$, of the filtrations, which are easier to treat in many applications. The quantized filtrations satisfy $S^{\cdot}_V \subset S^{\cdot}_{V,j+1} \subset S^{\cdot}_{V,j}$, $j = 1, 2, \ldots$, and similarly for $IS^{\cdot}_{V,*}$. In order to motivate these new filtrations, we need two preliminary results.

Let Π denote the unitary projection onto $\ker(D_{A(k)})$. Let P_j denote the projection onto $E \otimes L^k$ valued $(0,j)$-forms. Then (see [16, Theorem 4.1.1] or Proposition 3.11 below)

$$\mathrm{Tr}\Pi = \frac{k^m}{2^m \pi^m} \mathrm{Vol}(M) \mathrm{rk}(E) + O(k^{m-1}), \tag{1.1}$$

and (see Proposition 4.1 and Proposition 4.3)

$$\mathrm{Tr} P_{2j} \Pi = \frac{k^{m-2j}}{2^{m-2j} \pi^m (j!)^2} \|(F_A^{0,2})^{\wedge j}\|^2_{L_2} + O(k^{m-2j-1}). \tag{1.2}$$

Definition 1.2. We say $E \in S^q_{V,p,L,h}\mathrm{Vect}(M)$ if E admits a connection A satisfying $\mathrm{Tr} P_{2q+2} \Pi = O(k^{m-2q-2-p})$, for all k sufficiently large. We call A an $S^q_{V,p,L,h}$-*compatible* connection, or simply $S^q_{V,p}$-compatible if we do not specify the polarization. We say $E \in S^q_{V,p}$ if $E \in S^q_{V,p,L,h}$ for *some* choice of polarization (L,h). We say $E \in IS^q_{V,p}$ if E admits a connection A which is $S^q_{V,p,L,h}$ compatible for *every* choice of polarization (L,h). We call such a connection $IS^q_{V,p}$ compatible. We say $E \in MS^q_{V,p}$ if $E \in S^q_{V,p,L,h}\mathrm{Vect}(M)$ for *every* choice of polarization (L,h) (but with A possibly depending on the polarization).

For $q = 0$, the filtrations all agree. This is an immediate consequence of (1.2), which shows that an $S^0_{V,1}$ compatible connection satisfies $F_A^{0,2} = 0$, and thus defines a holomorphic structure on E. For $q > 0$, (1.2) also implies

$S^q_{V,1} = IS^q_{V,1}$; in fact an $S^q_{V,1}$ compatible connection is also $IS^q_{V,1}$ compatible. In general an $S^q_{V,p}$ compatible connection need not be $IS^q_{V,p}$ compatible for $p > 1$.

Question 1.3. Do these filtrations eventually stabilize? In other words, is there some $N(q)$ so that for $p_1, p_2 \geq N(q)$, $S^q_{V,p_1} = S^q_{V,p_2}$?

For $q = 0$, the filtrations all stabilize at $N(0) = 1$ on holomorphic bundles. The stabilization question may therefore be thought of as an extended integrability condition.

In order to support our claim that $S^{\cdot}_V$ and its quantum extensions are analogous to the Hodge filtration on cohomology, we consider the compatibility of these filtrations under the Chern isomorphism. For line bundles, it follows from (1.2) that the Chern character is compatible with the filtrations.

Theorem 1.4.

$$\mathrm{ch}_p(S^q_V \mathrm{Vect}(M,1)) \subset (S^{p-q}_H \cap \bar{S}^{p-q}_H) H^{2p}(M,\mathbb{Q}).$$

We conjecture that this theorem extends to arbitrary rank.

Conjecture 1.5.

$$\mathrm{ch}_p(S^q_V \mathrm{Vect}(M)) \subset (S^{p-q}_H \cap \bar{S}^{p-q}_H) H^{2p}(M,\mathbb{Q}).$$

The conjecture is true for IS^1_V for restricted p.

Theorem 1.6 (See Proposition 6.2). *For $p < 7$,*

$$\mathrm{ch}_p(IS^1_V \mathrm{Vect}(M)) \subset (S^{p-1}_H \cap \bar{S}^{p-1}_H) H^{2p}(M,\mathbb{Q}).$$

For $q > 1$, we have filtration compatibility for p in a restricted range.

Theorem 1.7 (See Corollary 4.6).

$$\mathrm{ch}_p(S^q_V \mathrm{Vect}(M)) \subset (S^{p-q}_H \cap \bar{S}^{p-q}_H) H^{2p}(M,\mathbb{Q}), \ \ \forall p < q+3.$$

It is generally easier to prove results about the quantized versions of our filtration than for $S^{\cdot}_V$ directly. In fact, the preceding results follow from computations of the implications of inclusion in $S^{\cdot}_{V,j}$ and $IS^{\cdot}_{V,j}$, $j = 1, 2$ and 3. It is also easier to establish functorial properties of the quantized filtrations. For example, we have the following theorem, which is an immediate consequence of Equation (1.2).

Theorem 1.8. *If M and N are two smooth projective varieties, and $f: M \to N$ is holomorphic, then $E \in S^q_{V,1}\mathrm{Vect}(N)$ implies $f^*E \in S^q_{V,1}\mathrm{Vect}(M)$.*

We have required M to be projective in this discussion because the definitions of our filtrations required an ample holomorphic line bundle L. This is analogous to defining operations on cohomology only through the intermediary of harmonic forms. A metric free definition is essential for applications. Perhaps we should view the role of the polarization as providing 'enough' *global* solutions to $D_{A(k)}s = 0$. More generally, we might define a filtration by requiring there to be 'enough' local solutions to $(\bar\partial_A + \bar\partial_A^*)s = 0$, with degree $s \le 2q$, but this still requires a metric and seems unnatural. A completely metric free condition similar to $E \in IS^q_V$ is the following : there exists a local frame $\{s_a\}_a$ for E so that $\bar\partial_A^{2q+1}s_a = 0$ for all a.

Question 1.9. If $E \in IS^q_V\mathrm{Vect}(M)$, does there exist a local frame $\{s_a\}_a$ for E so that $\bar\partial_A^{2q+1}s_a = 0$ for all a?

For $q = 0$, the answer is, of course, yes.

The study of the large k asymptotics of $\ker(D_{A(k)})$ and of the Bergman kernel for $E \otimes L^k$ has been very fruitful when E is holomorphic (for example, Berman [2], Berman–Berndtsson–Sjöstrand [3], Bismut [4], Bouche [5], Catlin [6], Dai–Liu–Ma [7], Demailly [8, 9], Donaldson [10], Getzler [11], Keller [13], Liu–Lu [14], Ma–Marinescu [16], Tian [18], L. Wang [19], X. Wang [20], Zelditch [21] and many others. See [16] for an extensive bibliography.) The nonholomorphic case has also been treated by many authors in many contexts, for example, see Ma–Marinescu [15] and [16]. Rather than specializing the extensive general results of [16] to our situation, we instead quickly rederive the asymptotic expansion of the Bergman kernel, using familiar constructions from index theory.

We were led to the structures examined here when investigating whether a similar analysis for nonholomorphic bundles might be used to improve our understanding of which bundles fail to admit a holomorphic structure. When we drop the assumption that E is holomorphic, the natural analog of the Bergman kernel is the L_2 projection Π. See [16] for an extensive treatment of the Bergman kernel in both the holomorphic and nonholomorphic cases. In the holomorphic case, Π is an endomorphism of sections of $E \otimes L^k$; in the nonholomorphic case, Π is an endomorphism of $E \otimes L^k$ valued *forms*. In particular, it defines maps Π^q_0 from sections to $E \otimes L^k$ valued $(0, 2q)$ forms. For $q > 0$, these maps lie deeper in the asymptotics of the Bergman kernel than have previously been computed, but are actually quite computable.

The above theorems all follow from computations of these asymptotics. It seems likely that Conjecture 1.5 can be proved to hold in a wider range by an extension of the computation of asymptotics which we have thus far undertaken.

Unfortunately, the fact that the $q = 0$ filtration stabilizes at $N(0) = 1$ limits the immediate application of this approach to discovering new obstructions to the existence of holomorphic structures. Nonetheless, we hope that these computations may be useful in other contexts.

Plan of the paper

In Section 2, we show that approximating the projection Π onto $\ker(D_{A(k)})$ by $e^{-tD^2_{A(k)}}$, with $t \geq k^{-1/2}$, leads to $O(e^{-\sqrt{k}/2})$ errors in Hilbert–Schmidt and trace class norms.

In Section 3, we construct q_t, an approximation to the Schwartz kernel of $e^{-tD^2_{A(k)}}$. We first gather elementary results on geodesic coordinates in Subsection 3.1, examine the interaction between the complex structure and normal coordinates in Subsection 3.2, and develop basic results about parallel transport in Subsection 3.3. After these geometric preliminaries, we give an inductive construction of q_t in Subsection 3.4. The inductive construction requires an approximate inverse of an operator L. In Subsection 3.5, we construct this approximate inverse and then estimate the magnitude of the summands of q_t. We then use these estimates to show that for k large, the operator $Q_{k^{-1/2}}$ with Schwartz kernel $q_{k^{-1/2}}$ built after N steps in the inductive procedure approximates Π with error $O(k^{-\frac{m-N-1}{2}})$, where m is the complex dimension. We finish this section by computing $\mathrm{Tr}\Pi$ to leading order.

In Section 4, we derive refined asymptotics for the summands Π^{2b}_{2a} of Π which map E-valued $(0,2a)$-forms to E-valued $(0,2b)$-forms. We then use these asymptotics to translate the filtration conditions $S^q_{V,j}$ $j = 1,2$ to constraints on the location of $\mathrm{ch}_p(E)$ in the Hodge diamond and on the curvature of E; for example, we show that these filtration conditions imply $(F^{0,2}_A)^{q+1} = 0$.

In Section 5, we begin investigating the additional constraints placed on $IS^q_{V,p}$ compatible connections by computing the metric variation of the constraints imposed by $S^q_{V,p}$. We find new curvature constraints such as $F^{0,2}_A \wedge i_Z F^{0,2}_A = 0$ for all vector fields Z, if our connection is $IS^1_{V,p}$ compatible.

In Section 6, we derive further asymptotic results by analyzing the fine structure of an operator H arising in the inductive construction, and deduce Theorem 1.6. We then show that $IS^1_{V,3}$ compatibility implies $F_A^{0,2}$ takes values in a commutative nilpotent subalgebra of $\mathrm{ad}(E)$.

In an appendix, we show how to derive pointwise bounds from the Hilbert–Schmidt bounds obtained in the main part of the paper.

2. Approximating Π

We retain the notation from the introduction. Fix a polarization (L, h) for M. Let ω denote the corresponding Kähler form. We may then write

$$F_{A(k)} = -ik\omega + F_A. \tag{2.1}$$

When we wish to emphasize the bundle rather than the connection, we will write F^E for F_A.

Let J denote the complex structure operator. Let $\{Z_j\}_{j=1}^m$ be a local frame for the holomorphic tangent bundle, and $\{w^j\}_{j=1}^m$ a dual coframe. We will be dealing with numerous curvature operators. Let R denote the curvature 2 form induced by the Levi–Civita connection on the exterior algebra bundle. Set $F_{\bar{j}l} := F_{A(k)}(\bar{Z}_j, Z_l) + R(\bar{Z}_j, Z_l)$, and $\hat{F}_{\bar{j}l} := F_A(\bar{Z}_j, Z_l) + R(\bar{Z}_j, Z_l)$. It is convenient to let $e(w)$ denote exterior multiplication on the left by the differential form w, and by $e^*(w)$, the adjoint operation. With this notation, a standard Bochner–Kodaira–Nakano computation gives

$$D^2_{A(k)} = (\nabla^{0,1})^*\nabla^{0,1} - 2e(\bar{w}^j)e^*(\bar{w}^l)F_{\bar{j}l} + 2e(F_A^{0,2}) + 2e^*(F_A^{0,2}). \tag{2.2}$$

Expanding the second term on the right, we have on $(0, q)$ forms

$$-2e(\bar{w}^j)e^*(\bar{w}^l)F_{\bar{j}l} = 2kq - 2e(\bar{w}^j)e^*(\bar{w}^l)\hat{F}_{\bar{j}l}. \tag{2.3}$$

From (2.2) and (2.3), we see that on forms of odd degree, there exists $C_A > 0$ independent of k such that

$$\langle D^2_{A(k)}f, f\rangle_{L_2} \geq (2k - C_A)\|f\|^2. \tag{2.4}$$

The nonzero spectrum of $D^2_{A(k)}$ on even forms is the same as the nonzero spectrum of $D^2_{A(k)}$ on odd forms. Hence the spectrum of $D^2_{A(k)}$ on $E \otimes L^k$-valued even forms is contained in $\{0\} \cup [2k - C_A, \infty)$. The spectral gap of the Laplace operator associated with high tensor powers of a line bundle was observed in [12, Theorem 1].

The large spectral gap implies that for k large and $Tk \gg 1$, $e^{-TD^2_{A(k)}}$ is a good approximation to Π in various operator norms, including the Trace, Hilbert–Schmidt, and supremum norms[1]. For the convenience of the reader, we recall a few elementary features of the Trace and Hilbert–Schmidt norms, which we denote $\|\cdot\|_{Tr}$ and $\|\cdot\|_{HS}$ respectively. For an operator B with singular values λ_j,

$$\|B\|_{Tr} = \sum_j \lambda_j, \quad \text{and} \quad \|B\|_{HS} = \left(\sum_j \lambda_j^2\right)^{\frac{1}{2}}.$$

If B is given by integrating against a kernel $b(x,y)$, then

$$\|B\|^2_{HS} = \int \mathrm{tr} b^*(y,x)b(y,x)dydx. \tag{2.5}$$

For bounded operators A and C,

$$\|ABC\|_{Tr} \leq \|A\|_{sup}\|B\|_{Tr}\|C\|_{sup}, \tag{2.6}$$

$$\|AB\|_{Tr} \leq \|A\|_{HS}\|B\|_{HS}, \tag{2.7}$$

and

$$\|ABC\|_{HS} \leq \|A\|_{sup}\|B\|_{HS}\|C\|_{sup}. \tag{2.8}$$

The spectral gap and Equations (2.6) and (2.7) imply for k large and $Tk \gg 1$ that

$$\|\Pi - e^{-TD^2_{A(k)}}\|_{Tr} \leq \frac{1}{k}\|D^2_{A(k)}e^{-TD^2_{A(k)}}\|_{Tr} \leq e^{-\frac{T}{2}k}\|e^{-\frac{T}{2}D^2_{A(k)}}\|_{Tr}, \tag{2.9}$$

and

$$\|\Pi - e^{-TD^2_{A(k)}}\|_{HS} \leq \frac{1}{k}\|D^2_{A(k)}e^{-TD^2_{A(k)}}\|_{HS} \leq e^{-\frac{T}{2}k}\|e^{-\frac{T}{2}D^2_{A(k)}}\|_{HS}. \tag{2.10}$$

[1]Our methods also yield pointwise results for the kernels. Puchol and Zhu have a derivation [17] of the asymptotic expansion making these pointwise estimates explicit. For the convenience of the reader, we include the details of our pointwise estimates in an appendix, although they are not used in this paper.

It follows from Corollary 3.8 and Equation 3.35 that for $\frac{1}{4} \leq s \leq 1$, $\exists c > 0$ independent of k large such that $\|e^{-sk^{-1/2}D^2_{A(k)}}\|_{HS} \leq ck^{\frac{m}{2}}$, which immediately implies that $\|e^{-2sk^{-1/2}D^2_{A(k)}}\|_{Tr} \leq c^2k^m$. Hence, the error in approximating $A\Pi C$ by $Ae^{-k^{-1/2}D^2_{A(k)}}C$, for A and C bounded is

$$O\left(\|A\|_{sup}\|B\|_{sup}k^m e^{-\frac{k^{1/2}}{2}}\right)$$

in the Hilbert–Schmidt, Trace, and supremum norm.

Equations (2.9) and (2.10) reduce estimates of the errors introduced by replacing Π by $e^{-k^{-1/2}D^2_{A(k)}}$ to estimates of heat kernels, and the problem of approximating Π reduces to the familiar problem of approximating heat kernels. Before embarking on the approximations, we first recall the standard estimates for the errors associated to heat kernel approximations.

Let k_t denote the heat kernel, meaning the Schwartz kernel for $e^{-tD^2_{A(k)}}$, and let q_t denote an approximate kernel. Let Q_t denote the operator corresponding to the kernel q_t. Set

$$\epsilon_s := \left(\frac{\partial}{\partial s} + D^2_{A(k)}\right) q_s.$$

We write the difference of the two kernels as

$$q_t - k_t = \int_0^t k_{t-s}\epsilon_s ds. \tag{2.11}$$

Then

$$\|Q_t - e^{-tD^2_{A(k)}}\|_{HS} \leq \int_0^t \|\epsilon_s\|_{HS} ds, \tag{2.12}$$

since $\|e^{-tD^2_{A(k)}}\|_{sup} \leq 1$. In the next section we construct q_t with ϵ_t small.

3. Approximate heat kernel

3.1. Geodesic coordinates

In this subsection and the next, for the convenience of the reader we gather elementary results on geodesic coordinates on Riemannian and Kähler manifolds. Let x be geodesic normal coordinates centered at y, defined in some neighborhood B_y of y. Let r denote distance from y, and let $\frac{\partial}{\partial r}$ denote the radial vector field. In addition to the geodesic coordinate frame, it is useful to work with an orthonormal frame, $\{e_j\}_j$ satisfying $\nabla_{\frac{\partial}{\partial r}} e_j = 0$, and at y, $e_j = \frac{\partial}{\partial x^j}$. Define the operator

$$\Phi(X) := \nabla_X r\frac{\partial}{\partial r} - X. \tag{3.1}$$

It is easy to expand Φ recursively.

$$\begin{aligned}
r\nabla_{\frac{\partial}{\partial r}}\Phi(e_j) &= \nabla_{r\frac{\partial}{\partial r}}\nabla_{e_j} r\frac{\partial}{\partial r} \\
&= R\left(r\frac{\partial}{\partial r}, e_j\right) r\frac{\partial}{\partial r} + \nabla_{e_j} r\frac{\partial}{\partial r} + \nabla_{[r\frac{\partial}{\partial r}, e_j]} r\frac{\partial}{\partial r} \\
&= R\left(r\frac{\partial}{\partial r}, e_j\right) r\frac{\partial}{\partial r} + e_j + \Phi(e_j) - \nabla_{e_j+\Phi(e_j)} r\frac{\partial}{\partial r} \\
&= R\left(r\frac{\partial}{\partial r}, e_j\right) r\frac{\partial}{\partial r} - \Phi(e_j) - \Phi(\Phi(e_j)).
\end{aligned}$$

Hence, using the radially constant frame to define the integrals, we have

$$\Phi(e_j) = \frac{1}{r}\int_0^r s^2 R\left(\frac{\partial}{\partial r}, e_j\right)\frac{\partial}{\partial r} ds - \frac{1}{r}\int_0^r \Phi(\Phi(e_j)) ds. \tag{3.2}$$

In particular,

$$\Phi(X) = \frac{r^2}{3}R(y)\left(\frac{\partial}{\partial r}, X\right)\frac{\partial}{\partial r} + \frac{r^3}{4}\left(\nabla_{\frac{\partial}{\partial r}} R\right)\left(\frac{\partial}{\partial r}, X\right)\frac{\partial}{\partial r} + O(r^4\nabla). \tag{3.3}$$

Here we have Taylor expanded R in a radially covariant constant frame. We write $R(y)$ for the radial parallel translation of R from y.

We may now compare our coordinate frame to the radially constant frame. (See for example [1, Prop 1.28]). Compute

$$\begin{aligned}
r\frac{\partial}{\partial r}(e_j x^p) &= e_j x^p + \left[r\frac{\partial}{\partial r}, e_j\right] x^p = -\Phi(e_j)x^p \\
&= -\left[\frac{r^2}{3}R(y)\left(\frac{\partial}{\partial r}, e_j, \frac{\partial}{\partial r}, e_m\right)\right. \\
&\quad \left. + \frac{r^3}{4}\left(\nabla_{\frac{\partial}{\partial r}} R\right)\left(\frac{\partial}{\partial r}, e_j, \frac{\partial}{\partial r}, e_m\right)\right] e_m(x^p) + O(r^4).
\end{aligned}$$

Hence

$$e_j x^p = \delta_j^p + \frac{1}{6}R(y)\left(r\frac{\partial}{\partial r}, e_j, e_p, r\frac{\partial}{\partial r}\right) + \frac{1}{12}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(r\frac{\partial}{\partial r}, e_j, e_p, r\frac{\partial}{\partial r}\right) + O(r^4),$$

and

$$e_j = \frac{\partial}{\partial x^j} + \frac{1}{6}R(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^p}, r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial x^p} + \frac{1}{12}(\nabla_{r\frac{\partial}{\partial r}}R)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^p}, r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial x^p} + O(r^4\nabla). \tag{3.4}$$

We write $O(r^4\nabla)$ above, instead of $O(r^4)$, to denote a vector field of magnitude $O(r^4)$. Inverting gives

$$\frac{\partial}{\partial x^j} = e_j - \frac{1}{6}R(y)\left(r\frac{\partial}{\partial r}, e_j, e_p, r\frac{\partial}{\partial r}\right)e_p - \frac{1}{12}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(r\frac{\partial}{\partial r}, e_j, e_p, r\frac{\partial}{\partial r}\right)e_p + O(r^4\nabla). \tag{3.5}$$

The expansion for the metric in geodesic coordinates follows immediately from Equation (3.5) :

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R(y)\left(\frac{\partial}{\partial x^i}, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j}\right) - \frac{1}{6}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(\frac{\partial}{\partial x^j}, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^i}\right) + O(r^4), \tag{3.6}$$

(3.7)

$$\begin{aligned}
\Gamma_{ij}^{\mu}(x) &= \frac{1}{3}R(y)\left(\frac{\partial}{\partial x^i}, r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^j}\right) + \frac{1}{3}R(y)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\mu}, r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j}\right) \\
&- \frac{1}{12}(\nabla_{e_j}R)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^i}, r\frac{\partial}{\partial r}\right) - \frac{1}{12}(\nabla_{e_i}R)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^j}, r\frac{\partial}{\partial r}\right) \\
&+ \frac{1}{12}\left(\nabla_{\frac{\partial}{\partial x^\mu}}R\right)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}, r\frac{\partial}{\partial r}\right) + \frac{1}{6}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}, r\frac{\partial}{\partial r}\right) \\
&+ \frac{1}{6}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\mu}\right) + O(r^3),
\end{aligned}$$

$$dv(x) = \left(1 - \frac{1}{6}Ric(y)(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}) + O(r^3)\right)dx^1 \wedge \cdots \wedge dx^{2m}. \tag{3.8}$$

From these expansions, we also compute

$$\begin{aligned}\Delta r^2 &= d^*d\sum_j (x^j-y^j)^2 = -2\sum_{i,j} e^*(dx^i)\nabla_{\frac{\partial}{\partial x^i}}\big((x^j-y^j)dx^j\big)\\ &= -2\sum_j e^*(dx^j)dx^j - 2\sum_{i,j}(x^j-y^j)e^*(dx^i)\nabla_{\frac{\partial}{\partial x^i}}(dx^j)\\ &= -2\sum_j g^{jj} + 2\sum_{i,j}(x^j-y^j)g^{il}\Gamma^j_{il}\\ &= -4m + \frac{2}{3}Ric\left(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}\right) + O(r^3).\end{aligned}$$

In particular,

$$4m + \Delta r^2 = O(r^2). \tag{3.9}$$

3.2. Complex structure in normal coordinates

Let J denote the complex structure operator. It is also convenient to define $J_0 \in C^\infty(B_y, \mathrm{End}(TM))$, with $J_0^2 = -1$, as follows. Choose the geodesic coordinates to satisfy at y, $J\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^{j+m}}$, for $1 \le j \le m$. Define J_0 to extend this relation to all of B_y:

$$J_0\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^{j+m}}, \qquad \text{for } 1 \le j \le m.$$

We introduce J_0-complex coordinates, $z^j = x^j - y^j + i(x^{j+m} - y^{j+m})$, $1 \le j \le m$. Then

$$r\frac{\partial}{\partial r} = z^l\frac{\partial}{\partial z^l} + \bar z^l\frac{\partial}{\partial \bar z^l}.$$

Let $\{f_a := \frac{1}{2}(e_a - ie_{a+m})\}_{a=1}^m$ be a radially covariant constant frame of the holomorphic tangent bundle, with dual coframe $\{\eta^a\}_{a=1}^m$. Let $f_{\bar a} = \bar f_a$, and $\eta^{\bar a} = \bar\eta^a$. Then from Equation (3.4) we have

$$\begin{aligned}f_a = \frac{\partial}{\partial z^a} &+ \frac{1}{3}R(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^p}, r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial \bar z^p}\\ &+ \frac{1}{3}R(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar z^p}, r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial z^p}\\ &+ \frac{1}{6}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^p}, r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial \bar z^p}\\ &+ \frac{1}{6}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar z^p}, r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial z^p} + O(r^4\nabla),\end{aligned} \tag{3.10}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial z^a} = f_a &- \frac{1}{3}R(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^p}, r\frac{\partial}{\partial r}\right) f_{\bar{p}} \\
&- \frac{1}{3}R(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^p}, r\frac{\partial}{\partial r}\right) f_p \\
&- \frac{1}{6}\left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^p}, r\frac{\partial}{\partial r}\right) f_{\bar{p}} \\
&- \frac{1}{6}\left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^p}, r\frac{\partial}{\partial r}\right) f_p + O(r^4\nabla).
\end{aligned}
\tag{3.11}
$$

It is convenient to express $\bar{\partial}$ in a mixture of coordinate and radially covariant constant frames. We have

$$
\begin{aligned}
\bar{\partial} = \bar{\eta}^a \Bigg[&\nabla_{\frac{\partial}{\partial \bar{z}^a}} + \frac{1}{3}R(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial \bar{z}^a}, \frac{\partial}{\partial z^p}, r\frac{\partial}{\partial r}\right) \nabla_{\frac{\partial}{\partial \bar{z}^p}} \\
&+ \frac{1}{3}R(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial \bar{z}^a}, \frac{\partial}{\partial \bar{z}^p}, r\frac{\partial}{\partial r}\right) \nabla_{\frac{\partial}{\partial z^p}} \\
&+ \frac{1}{6}\left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial \bar{z}^a}, \frac{\partial}{\partial z^p}, r\frac{\partial}{\partial r}\right) \nabla_{\frac{\partial}{\partial \bar{z}^p}} \\
&+ \frac{1}{6}\left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial \bar{z}^a}, \frac{\partial}{\partial \bar{z}^p}, r\frac{\partial}{\partial r}\right) \nabla_{\frac{\partial}{\partial z^p}} \Bigg] + O(r^4\nabla).
\end{aligned}
\tag{3.12}
$$

From (3.11) and its conjugate, we can compute the difference between J and J_0. We find

$$
\begin{aligned}
(J - J_0)\frac{\partial}{\partial z^a} = &\frac{2i}{3}R(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^p}, r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial \bar{z}^p} \\
&+ \frac{i}{3}\left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^p}, r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial \bar{z}^p} + O(r^4\nabla).
\end{aligned}
$$

Hence

$$
\begin{aligned}
J = J_0 &+ \frac{2i}{3}R(y)\left(z^b\frac{\partial}{\partial z^b}, \frac{\partial}{\partial \bar{z}^\mu}, z^c\frac{\partial}{\partial z^c}, \cdot\right)\frac{\partial}{\partial z^\mu} \\
&- \frac{2i}{3}R(y)\left(\bar{z}^b\frac{\partial}{\partial \bar{z}^b}, \frac{\partial}{\partial z^\mu}, \bar{z}^c\frac{\partial}{\partial \bar{z}^c}, \cdot\right)\frac{\partial}{\partial \bar{z}^\mu} + \delta J,
\end{aligned}
\tag{3.13}
$$

with

$$\begin{aligned}\delta J &= \frac{i}{3}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(z^b\frac{\partial}{\partial z^b},\frac{\partial}{\partial \bar z^\mu},z^c\frac{\partial}{\partial z^c},\cdot\right)\frac{\partial}{\partial z^\mu}\\ &\quad-\frac{i}{3}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(\bar z^b\frac{\partial}{\partial \bar z^b},\frac{\partial}{\partial z^\mu},\bar z^c\frac{\partial}{\partial \bar z^c},\cdot\right)\frac{\partial}{\partial \bar z^\mu}+O(r^4).\end{aligned}\tag{3.14}$$

As a useful special case, we note that

(3.15)

$$\begin{aligned}(J-J_0)r\frac{\partial}{\partial r} &= \frac{2i}{3}R(y)\left(z^b\frac{\partial}{\partial z^b},\frac{\partial}{\partial \bar z^\mu},z^l\frac{\partial}{\partial z^l},\bar z^c\frac{\partial}{\partial \bar z^c}\right)\frac{\partial}{\partial z^\mu}\\ &\quad-\frac{2i}{3}R(y)\left(\bar z^b\frac{\partial}{\partial \bar z^b},\frac{\partial}{\partial z^\mu},\bar z^l\frac{\partial}{\partial \bar z^l},z^c\frac{\partial}{\partial z^c}\right)\frac{\partial}{\partial \bar z^\mu}\\ &\quad+\frac{i}{3}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(z^b\frac{\partial}{\partial z^b},\frac{\partial}{\partial \bar z^\mu},z^l\frac{\partial}{\partial z^l},\bar z^c\frac{\partial}{\partial \bar z^c}\right)\frac{\partial}{\partial z^\mu}\\ &\quad-\frac{i}{3}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(\bar z^b\frac{\partial}{\partial \bar z^b},\frac{\partial}{\partial z^\mu},\bar z^l\frac{\partial}{\partial \bar z^l},z^c\frac{\partial}{\partial z^c}\right)\frac{\partial}{\partial \bar z^\mu}+O(r^5\nabla).\end{aligned}$$

We summarize these computations with the following lemma relating J and J_0.

Lemma 3.1. *There exists $A_{bl}\in C^\infty(B_y,\mathrm{Hom}(T^{0,1},T^{1,0}))$ and $B_{\bar b\bar l}\in C^\infty(B_y,\mathrm{Hom}(T^{1,0},T^{0,1}))$ for which*

$$J=J_0+z^bz^lA_{bl}+\bar z^b\bar z^lB_{\bar b\bar l}+O(r^3).$$

We will also need the following elementary lemma.

Lemma 3.2. *For all vector fields X,*

$$\begin{aligned}&Ric(X,JX)=0,\ \textit{and}\\ &Ric\left(r\frac{\partial}{\partial r},r\frac{\partial}{\partial r}\right)=2z^l\bar z^jRic\left(\frac{\partial}{\partial z^l},\frac{\partial}{\partial \bar z^j}\right)+O(r^4).\end{aligned}$$

Proof. Write $X=X^{1,0}+X^{0,1}$, with $JX^{1,0}=iX^{1,0}$ and $JX^{0,1}=-iX^{0,1}$. Then since Ric is symmetric,

$$Ric(X,JX)=iRic(X^{1,0}+X^{0,1},X^{1,0}-X^{0,1})=0.$$

For the second claim, we write

$$\begin{aligned} Ric\left(z^a\frac{\partial}{\partial z^a}, z^b\frac{\partial}{\partial z^b}\right) &= -iRic\left(z^a\frac{\partial}{\partial z^a}, J_0 z^b\frac{\partial}{\partial z^b}\right) \\ &= -iRic\left(z^a\frac{\partial}{\partial z^a}, J z^b\frac{\partial}{\partial z^b}\right) + O(r^4) = O(r^4). \end{aligned}$$

Similarly $Ric(\bar{z}^a\frac{\partial}{\partial \bar{z}^a}, \bar{z}^b\frac{\partial}{\partial \bar{z}^b}) = O(r^4)$, and the claim follows upon expanding $r\frac{\partial}{\partial r} = z^a\frac{\partial}{\partial z^a} + \bar{z}^a\frac{\partial}{\partial \bar{z}^a}$. □

3.3. Parallel transport

Before we construct an approximate heat kernel, it is useful to make a few observations about parallel translation. Let $S := \Lambda^{0,\mathrm{even}} \otimes E \otimes L^k$, and $\pi_j\colon M \times M \to M$, $j = 1$ or 2, denote the projection onto the first or second factor. Let the j-lift of X be the unique element of $(d\pi_j)^{-1}(X) \cap \ker(d\pi_{j\pm 1})$. When no confusion will result, we will use the same symbol to denote a vector and its lift. A Schwartz kernel, q_t, for an approximate heat kernel is a section of the bundle

$$V := \mathrm{Hom}(\pi_2^* S, \pi_1^* S).$$

Thus $q_t(x, y) \in \mathrm{Hom}(S_y, S_x)$. In describing such kernels it is useful to identify S_x and S_y via parallel translation along distance minimizing geodesics. The kernel we construct will be supported in a small neighborhood of the diagonal in $M \times M$, making such an identification unique. Let $\gamma_{x,y}\colon [0, d(x, y)] \to M$ be the minimal unit speed geodesic from y to x. Let $\psi(x, y)$ denote parallel translation $\psi\colon S_y \to S_x$ along $\gamma_{x,y}$. Let $\frac{\partial}{\partial r_1}$ and $\frac{\partial}{\partial r_2}$ in $T_{x,y}(M \times M)$ denote the canonical 1- and 2- lifts to $M \times M$ of $\gamma'_{x,y}(d(x, y))$ and $-\gamma'_{x,y}(0)$ respectively. By definition,

- $\psi(x, x) =$ identity, and
- $\nabla_{\frac{\partial}{\partial r_j}} \psi = 0, \;\; j = 1, 2.$

It is convenient to factor ψ as $\psi = \psi_{LC} \otimes \psi_E \otimes \psi_L =: \hat{\psi} \otimes \psi_L$, where ψ_{LC}, ψ_E, and ψ_L denote parallel translation of sections of $\Lambda^{0,\mathrm{even}}$, E, and L respectively. The local geometry is largely encoded in ψ; hence we record some of its properties before constructing approximate kernels.

Let (x^i) be geodesic local coordinates on M with center y. We will not distinguish between $\frac{\partial}{\partial x^i}$ and its canonical 1-lift to $M \times M$. We will also set $\frac{\partial}{\partial r} = \frac{\partial}{\partial r_1}$. For a vector field X on $M \times M$, we let $\psi_{;X}$ denote $\nabla_X \psi$. For coordinate vector fields $\frac{\partial}{\partial x^i}$, we abbreviate this covariant derivative to $\psi_{;i}$ (and similarly interpret $\psi_{L;i}$, etc.).

Lemma 3.3. *For derivatives of ψ_L, we have*

$$\psi_L^{-1}\psi_{L;i}(x,y) = -\frac{ik}{2}g\left(Jr\frac{\partial}{\partial r},\frac{\partial}{\partial x^i}\right) + \psi_L^{-1}\delta\psi_{L;i}, \tag{3.16}$$

with

$$\begin{aligned}\psi_L^{-1}\delta\psi_{L;i} = &-k\frac{z^a\bar{z}^b}{12}R(y)\left(\frac{\partial}{\partial x^i},r\frac{\partial}{\partial r},\frac{\partial}{\partial z^a},\frac{\partial}{\partial \bar{z}^b}\right)\\ &-\frac{kz^a\bar{z}^b}{20}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(\frac{\partial}{\partial x^i},r\frac{\partial}{\partial r},\frac{\partial}{\partial z^a},\frac{\partial}{\partial \bar{z}^b}\right) + O(kr^5).\end{aligned} \tag{3.17}$$

For derivatives of $\hat{\psi}$, we have

$$\hat{\psi}^{-1}\hat{\psi}_{;i} = \frac{1}{2}F^E(y)\left(r\frac{\partial}{\partial r},\frac{\partial}{\partial x^i}\right) + \frac{1}{2}R(y)\left(r\frac{\partial}{\partial r},\frac{\partial}{\partial x^i}\right) + \hat{\psi}^{-1}\delta\hat{\psi}_{;i}, \tag{3.18}$$

with

$$\hat{\psi}^{-1}\delta\hat{\psi}_{;i} \in O(r^2). \tag{3.19}$$

Proof. Let $F^V = \pi_1^*F^S - \pi_2^*F^S$ denote the curvature of V induced by the connection on S. Let $\{e_j\}_{j=1}^{2m}$ be the 1-lift of a local tangent frame covariant constant along radial geodesics centered at y, with $e_1 = \frac{\partial}{\partial r}$. The assumption that $\nabla_{\frac{\partial}{\partial r}}\psi(x,y) = 0$ allows us to write

$$\begin{aligned}F^V\left(r\frac{\partial}{\partial r},e_j\right)(x,y)\psi(x,y) &= \nabla_{r\frac{\partial}{\partial r}}\nabla_{e_j}\psi + \nabla_{[e_j,r\frac{\partial}{\partial r}]}\psi\\ &= \nabla_{\frac{\partial}{\partial r}}r\nabla_{e_j}\psi + \nabla_{\Phi(e_j)}\psi,\end{aligned} \tag{3.20}$$

Note because $r\frac{\partial}{\partial r}$ and e_j are 1-lifts, we have

$$F^V\left(r\frac{\partial}{\partial r},e_j\right)(x,y) = F^S\left(r\frac{\partial}{\partial r},e_j\right)(x)$$

if we use the same symbol to denote a vector and its lift. Write $\nabla_X\psi = \psi\Gamma^V(X)$. With this notation, we rewrite (3.20) as

$$\psi^{-1}(x,y)F^S\left(r\frac{\partial}{\partial r},e_j\right)(x)\psi(x,y) = \frac{\partial}{\partial r}\big(r\Gamma^V(e_j)\big) + \Gamma^V(\Phi(e_j)). \tag{3.21}$$

This translates the equation into an ordinary differential equation on a trivial bundle on $M \times \{y\}$, and we can solve recursively by integrating.

$$\Gamma^V(e_j) = \frac{1}{r}\int_0^r \Big(\psi^{-1}F^S(se_1, e_j)\psi - \Gamma^V(\Phi(e_j))\Big)ds. \tag{3.22}$$

Using the recursion relation (3.22), we expand (3.22) as:

$$\begin{aligned}\Gamma^V(e_j) &= \frac{1}{r}\int_0^r \psi^{-1}F^S(se_1, e_j)\psi ds \\ &\quad - \frac{1}{r}\int_0^r \langle\Phi(e_j), e_i\rangle \frac{1}{s}\int_0^s \Big(\psi^{-1}F^S(s_2e_1, e_i)\psi - \Gamma^V(\Phi(e_i))\Big)ds_2 ds.\end{aligned} \tag{3.23}$$

Note that the integral $\frac{1}{r}\int_0^r \langle\Phi(e_j), e_i\rangle \frac{1}{s}\int_0^s \Gamma^V(\Phi(e_i))ds_2 ds$ is $O(kr^5)$, giving for $X \in \ker(d\pi_2)$,

$$\begin{aligned}\Gamma^V(X) = {}& \frac{1}{2}F^S(y)\left(r\frac{\partial}{\partial r}, X\right) + \frac{r}{3}\left(\nabla_{\frac{\partial}{\partial r}}F^S\right)(y)\left(r\frac{\partial}{\partial r}, X\right) \\ & + \frac{r^2}{4}\left(\nabla^2_{\frac{\partial}{\partial r}}F^S\right)(y)\left(r\frac{\partial}{\partial r}, X\right) \\ & - F^S\left(r\frac{\partial}{\partial r}, \frac{r^2}{24}R(y)(\frac{\partial}{\partial r}, X)\frac{\partial}{\partial r}\right) \\ & + O(kr^4).\end{aligned} \tag{3.24}$$

If we now let $\{e_j\}_{j=1}^{2m}$ be the 2-lift of a local tangent frame covariant constant along radial geodesics centered at x, with $e_1 = \frac{\partial}{\partial r_2}$, we may repeat the preceding analysis exchanging x and y and $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial r_2}$ to get for $Y \in \ker(d\pi_1)$,

$$\begin{aligned}\psi_{;Y}(x,y)\psi^{-1}(x,y) = {}& -\frac{1}{2}F^S(x)\left(r\frac{\partial}{\partial r_2}, Y\right) - \frac{r}{3}\left(\nabla_{\frac{\partial}{\partial r_2}}F^S\right)(x)\left(r\frac{\partial}{\partial r_2}, Y\right) \\ & - \frac{r^2}{4}\left(\nabla^2_{\frac{\partial}{\partial r_2}}F^S\right)(x)\left(r\frac{\partial}{\partial r_2}, Y\right) \\ & + F^S\left(r\frac{\partial}{\partial r_2}, \frac{r^2}{24}R(x)\left(\frac{\partial}{\partial r_2}, Y\right)\frac{\partial}{\partial r_2}\right) \\ & + O(kr^4).\end{aligned} \tag{3.25}$$

The preceding discussion did not employ any properties of S beyond the structure of a bundle with connection. Hence we may replace S with any of its factors in order to compute the derivatives of $\hat{\psi}$ or ψ_L. In the case of $\hat{\psi}$, the error term improves from $O(kr^4)$ to $O(r^4)$. Letting $V(L^k)$ denote $\mathrm{Hom}(\pi_1^*L^k, \pi_2^*L^k)$, we note that $F^{V(L^k)} = -ik\pi_1^*\omega + ik\pi_2^*\omega$ is covariant constant. Hence for $X \in \ker(d\pi_2)$,

$$\begin{aligned}\Gamma^{V(L^k)}(X) = &\frac{-ik}{2} g\left(Jr\frac{\partial}{\partial r}, X\right)\\ &- \frac{ik}{24} R(y)\left(X, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, Jr\frac{\partial}{\partial r}\right)\\ &- \frac{ik}{40}\left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(X, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, Jr\frac{\partial}{\partial r}\right)\\ &+ O(kr^5).\end{aligned} \tag{3.26}$$

□

It is useful to record the following special case of (3.17) and (3.24):

$$\begin{aligned}&\psi_L^{-1}\delta\psi_{L;Jr\frac{\partial}{\partial r}} + \hat{\psi}^{-1}\hat{\psi}_{;Jr\frac{\partial}{\partial r}}\\ &= -iz^a\bar{z}^b\Bigg[F^E(y)\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right) + R(y)\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)\\ &+ \frac{2}{3}\left(\nabla_{r\frac{\partial}{\partial r}} F^E\right)\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right) + \frac{2}{3}\left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)\\ &+ \frac{kz^c\bar{z}^e}{6} R(y)\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar{z}^e}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)\\ &+ \frac{kz^c\bar{z}^e}{10}\left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar{z}^e}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)\Bigg]\\ &+ O(kr^6 + r^4).\end{aligned} \tag{3.27}$$

Proposition 3.4.

$$\begin{aligned}\nabla^*\psi^{-1}\nabla\psi = &\frac{ik}{40}(\nabla_{e_j} R)\left(e_j, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, Jr\frac{\partial}{\partial r}\right)\\ &- \frac{1}{3} d_A^* F^E\left(r\frac{\partial}{\partial r}\right) - \frac{1}{3} d_{\nabla^{LC}}^* R\left(r\frac{\partial}{\partial r}\right)\\ &+ O(r^2 + kr^4).\end{aligned}$$

Proof. Let $\{e_j\}_j$ be an orthonormal $\nabla_{\frac{\partial}{\partial r}}$ constant tangent frame, with $\nabla e_j = 0$ at $r = 0$. In this frame we have

$$
\begin{aligned}
&- \nabla^* \psi^{-1} \nabla \psi \\
&= (\nabla_{e_j}(\psi^{-1}\nabla_{e_j}\psi) - \psi^{-1}\nabla_{\nabla_{e_j}e_j}\psi) \\
&= \frac{-ik}{2} g\left(J\nabla_{e_j} r\frac{\partial}{\partial r}, e_j\right) - \frac{ik}{24}R(y)\left(e_j, r\frac{\partial}{\partial r}, e_j, Jr\frac{\partial}{\partial r}\right) \\
&\quad - \frac{ik}{24}R(y)\left(e_j, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, Je_j\right) - \frac{ik}{40}(\nabla_{e_j}R)\left(e_j, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, Jr\frac{\partial}{\partial r}\right) \\
&\quad - \frac{ikr}{40}\left(\nabla_{\frac{\partial}{\partial r}}R\right)\left(e_j, r\frac{\partial}{\partial r}, e_j, Jr\frac{\partial}{\partial r}\right) - \frac{ikr}{40}\left(\nabla_{\frac{\partial}{\partial r}}R\right)\left(e_j, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, Je_j\right) \\
&\quad + \frac{1}{2}F^E(y)\left(\nabla_{e_j} r\frac{\partial}{\partial r}, e_j\right) + \frac{1}{3}(\nabla_{e_j}F^E)(y)\left(r\frac{\partial}{\partial r}, e_j\right) \\
&\quad + \frac{1}{2}R(y)\left(\nabla_{e_j} r\frac{\partial}{\partial r}, e_j\right) + \frac{1}{3}(\nabla_{e_j}R)(y)\left(r\frac{\partial}{\partial r}, e_j\right) \\
&\quad + O(r^2 + kr^4) \\
&= \frac{-ik}{2} g(J\Phi(e_j), e_j) - \frac{ik}{40}(\nabla_{e_j}R)\left(e_j, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, Jr\frac{\partial}{\partial r}\right) \\
&\quad + \frac{1}{3}(\nabla_{e_j}F^E)(y)\left(r\frac{\partial}{\partial r}, e_j\right) + \frac{1}{3}(\nabla_{e_j}R)(y)\left(r\frac{\partial}{\partial r}, e_j\right) \\
&\quad + O(r^2 + kr^4) \\
&= -\frac{ik}{40}(\nabla_{e_j}R)\left(e_j, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}, Jr\frac{\partial}{\partial r}\right) + \frac{1}{3}d_A^* F^E\left(r\frac{\partial}{\partial r}\right) + \frac{1}{3}d^*_{\nabla^{LC}}R\left(r\frac{\partial}{\partial r}\right) \\
&\quad + O(r^2 + kr^4).
\end{aligned}
$$

□

3.4. The inductive construction

We now construct the Schwartz kernel, q_t of our approximation Q_t to $e^{-tD^2_{A(k)}}$. We construct q_t explicitly only in a small neighborhood of the diagonal in $M \times M$. Because q_t is rapidly decreasing away from the diagonal, we will suppress in our discussion the cutoff functions which are needed to extend q_t to the complement of a neighborhood of the diagonal. Fix geodesic normal coordinates, x, centered at y, and defined in some neighborhood of y, B_y. We construct the approximation inductively. Let r denote the geodesic distance from y to x, and let

$$U := e^{-\frac{kr^2}{4\tanh(tk)}}$$
$$\mathcal{U} := \left(\frac{k}{4\pi\sinh(tk)}\right)^m U \sum_q e^{kt(m-4q)} P_{2q}$$

we write

$$q_t(x,y) = \mathcal{U}\psi(x,y)\sum_{l=0}^{N} u_l(x,y)$$

with the u_l's to be determined and N large.

The $u_l(x,y)$'s are sections of $\mathrm{End}(S_y) \simeq \mathrm{End}(\Lambda^{0,\mathrm{even}}T_y^*M \otimes E_y)$ and are constructed inductively so that $Q_0 = I$ and so that $\epsilon_t := (\frac{\partial}{\partial t} + D_A^2)q_t$ is sufficiently small. Write

$$D_A^2 = \nabla^*\nabla + \mathcal{F} + 2e(F^{0,2}) + 2e^*(F_A^{0,2}), \tag{3.28}$$

where, in a local frame $\{Z_j\}_{j=1}^m$ for $T^{1,0}M$ and coframe $\{w^j\}_{j=1}^m$,

$$\mathcal{F} := -[e(w^{\bar{j}}), e^*(w^{\bar{l}})]F_{\bar{j}l} = kg_{\bar{j}l}[e(w^{\bar{j}}), e^*(w^{\bar{l}})] - [e(w^{\bar{j}}), e^*(w^{\bar{l}})]\hat{F}_{\bar{j}l}.$$

On $(0,q)$ forms this becomes

$$\mathcal{F} = k(2q-m) + \hat{\mathcal{F}},$$

where

$$\hat{\mathcal{F}} := -[e(w^{\bar{j}}), e^*(w^{\bar{l}})]\hat{F}_{\bar{j}l}.$$

We now compute

$$\begin{aligned}\epsilon_t &= \left(\frac{\partial}{\partial t} + D_A^2\right) q_t \\ &= \left(\frac{\partial}{\partial t} + \nabla^*\nabla + \mathcal{F} + 2e(F_A^{0,2}) + 2e^*(F_A^{0,2})\right)\left(\mathcal{U}\psi(x,y)\sum_{l=0}^{N} u_l(x,y)\right).\end{aligned}$$

It is convenient to conjugate by $\mathcal{U}\psi$. Note that because $e(F^{0,2})$ raises degree,

$$e(F^{0,2})\mathcal{U}\psi = \mathcal{U}\psi\big(e^{4kt}\psi^{-1}e(F^{0,2})\psi\big),$$

and

$$e^*(F^{0,2})\mathcal{U}\psi = \mathcal{U}\psi\big(e^{-4kt}\psi^{-1}e^*(F^{0,2})\psi\big).$$

Conjugating and recalling that in local coordinates

$$\nabla^*\nabla = -g^{ij}\nabla_i\nabla_j + g^{ij}\Gamma^l_{ij}\nabla_l,$$

we recast ϵ_t as

$$\begin{aligned}\epsilon_t = \psi(x,y)\mathcal{U}\Bigg(&\frac{\partial}{\partial t} + \nabla^*\nabla + \frac{kr}{\tanh(tk)}\nabla_{\frac{\partial}{\partial r}} - \frac{k^2r^2}{4} - \frac{k(4m+\Delta(r^2))}{4\tanh(tk)}\\ &- g^{ij}\Big(2\psi^{-1}\psi_{;i}\nabla_j + (\psi^{-1}\psi_{;j})_{;i} + \psi^{-1}\psi_{;i}\psi^{-1}\psi_{;j}\Big)\\ &+ g^{ij}\Gamma^l_{ij}\psi^{-1}\psi_{;l} + \psi^{-1}\hat{\mathcal{F}}\psi + 2e^{4kt}\psi^{-1}e(F_A^{0,2})\psi\\ &+ 2e^{-4kt}\psi^{-1}e^*(F_A^{0,2})\psi\Bigg)\sum_{l=0}^{N} u_l(x,y).\end{aligned}$$

We use the triviality of the bundle $M \times \mathrm{End}(S_y)$ to replace covariant derivatives by partial derivatives. Expanding $\psi^{-1}\psi_{;l}$ as $\big(-\frac{ik}{2}g(rJ\frac{\partial}{\partial r}, \frac{\partial}{\partial x^l}) + \psi_L^{-1}\delta\psi_{L;l} + \hat{\psi}^{-1}\hat{\psi}_{;l}\big)$ as per Equation (3.16) then gives

$$\begin{aligned}\epsilon_t = \psi(x,y)\mathcal{U}\Bigg(&\frac{\partial}{\partial t} + \Delta + ikrJ\frac{\partial}{\partial r} + \frac{kr}{\tanh(tk)}\frac{\partial}{\partial r} - \frac{k(4m+\Delta(r^2))}{4\tanh(tk)}\\ &- 2g^{ij}(\psi_L^{-1}\delta\psi_{L;i} + \hat{\psi}^{-1}\hat{\psi}_{;i})\frac{\partial}{\partial x^j} + \nabla^*\psi^{-1}\nabla\psi\\ &- g^{ij}(\psi_L^{-1}\delta\psi_{L;i} + \hat{\psi}^{-1}\hat{\psi}_{;i})(\psi_L^{-1}\delta\psi_{L;j} + \hat{\psi}^{-1}\hat{\psi}_{;j})\\ &+ ik\big(\psi_L^{-1}\delta\psi_{L;rJ\frac{\partial}{\partial r}} + \hat{\psi}^{-1}\hat{\psi}_{;rJ\frac{\partial}{\partial r}}\big) + \psi^{-1}\hat{\mathcal{F}}\psi\\ &+ 2e^{4kt}\psi^{-1}e(F_A^{0,2})\psi + 2e^{-4kt}\psi^{-1}e^*(F_A^{0,2})\psi\Bigg)\sum_{l=0}^{N} u_l(x,y).\end{aligned}$$

We now make several strongly coordinate dependent choices in our analysis of ϵ_t. Set $\Delta_E = -\sum_{j=1}^{2m}\frac{\partial^2}{(\partial x^j)^2}$. With this notation, we define

$$L := \partial_t + \frac{kr}{\tanh(tk)}\frac{\partial}{\partial r} + ikrJ_0\frac{\partial}{\partial r} + \Delta_E.$$

This operator will dominate our analysis of ϵ_t. Define H by

$$\epsilon_t = \psi\mathcal{U}(L+H)\sum_{l=0}^{N} u_l.$$

We begin the inductive construction by setting $u_0 = I$. Observe that L annihilates u_0. In the next section, we define an operator L^{-1} that approximately inverts L. We then define u_l, $l > 0$, inductively by setting

$$u_{l+1} = -L^{-1} H u_l. \tag{3.29}$$

We now introduce a filtration on operators which greatly simplifies our later analysis of the magnitude of the Schwartz kernels produced by this algorithm. Define the filtration $W_y^{\cdot}$ on partial differential operators defined in a neighborhood B_y of y as follows. We say that an operator $Z \in W_y^b$ if in geodesic normal coordinates centered at y, Z can be expressed as a finite sum

$$Z = \sum_{2p+|I|-|J|\leq b} k^p (x-y)^J \sum_{d\geq 0} e^{4dtk} a_{I,J,p,d}(x,y,tk) \frac{\partial}{\partial x^I}, \tag{3.30}$$

with $a_{I,J,p,d} = \sum_{j\geq 0} P_{2j+2d} a_{I,J,p,d}$ when $d > 0$, and $a_{I,J,p,d}(x,y,tk)$ smooth, and bounded for $t < 1$. In particular, differentiation by a coordinate vector field has weight $+1$, multiplication by $(x^j - y^j)$ has weight -1, multiplication by k has weight 2, multiplication by t has weight -2, multiplication by $e^{4kt}\psi^{-1} e(F_A^{0,2})\psi$ has weight 0, and $W^i \circ W^j \subset W^{i+j}$. Set

$$H_h := H - \big(2e^{4kt}\psi^{-1} e(F^{0,2})\psi + 2e^{-4kt}\psi^{-1} e^*(F_A^{0,2})\psi\big).$$

Proposition 3.5.

$$H \in W_y^0.$$

Proof. This claim is an immediate consequence of Lemma 3.1 and Equations (3.6), (3.7), (3.9), (3.17), (3.18), and (3.19). $\square$

3.5. L^{-1}

In this subsection, we construct an approximate inverse to L. First compute, for J, K multi-indices,

$$\begin{aligned} L\big(a_{JK}(tk) z^J \bar{z}^K\big) &= \big(\partial_t + \Delta_E + (|J| + |K|)\frac{k}{\tanh(tk)} \\ &\quad + k(|K| - |J|)\big) a_{JK}(tk) z^J \bar{z}^K \\ &= \mu_{JK}^{-1}(tk) \big(\partial_t + \Delta_E\big)\Big(\mu_{JK}(tk) a_{JK}(tk) z^J \bar{z}^K\Big), \end{aligned} \tag{3.31}$$

where

$$\mu_{JK}(tk) = \sinh(tk)^{(|J|+|K|)} e^{(|K|-|J|)tk}.$$

On sections which are polynomial in z and $\bar{z}$, the inverse operator is

$$\begin{aligned} L^{-1}\Big(z^J \bar{z}^K a_{JK}(tk)\Big) &= \int_0^t \int_{\mathbb{R}^{2m}} \frac{e^{-\frac{|z-y|^2}{4(t-s)}}}{\big(4\pi(t-s)\big)^m} \frac{\mu_{JK}(sk)}{\mu_{JK}(tk)} y^J \bar{y}^K a_{JK}(sk) dyds \\ &= \frac{1}{k} \int_0^{tk} \int_{\mathbb{R}^{2m}} e^{-\pi|y|^2} \frac{\mu_{JK}(s)}{\mu_{JK}(tk)} \left(\sqrt{\frac{4\pi}{k}(tk-s)}\, y + z\right)^J \\ &\qquad \cdot \left(\sqrt{\frac{4\pi}{k}(tk-s)}\, \bar{y} + \bar{z}\right)^K a_{JK}(s) dyds \end{aligned}$$

Observe that L^{-1} lowers weight by 2 and increases by 1 the order of vanishing of $a_{JK}(tk)$ at 0.

Proposition 3.6. $u_l(x, y) \in W_y^{-2l}$.

Proof. This proposition follows immediately from Equation (3.29) and Proposition 3.5. □

We extend L^{-1} from polynomials to smooth functions $A(z, \bar{z}, tk)$ by setting

$$L^{-1}A := L^{-1}p_{2N},$$

where p_{2N} is the degree $2N$ Taylor polynomial for A. Then

$$LL^{-1} - I \in W_y^{-2N}. \tag{3.32}$$

Proposition 3.7. *For some constants C_l, c_l, B_l depending on the geometry of M,*

$$\begin{aligned} &\int_{M\times M} |P_{2q} u_l(x,y) P_{2b}|^2 \frac{k^{2m} e^{-\frac{kr^2}{2\tanh(tk)}} e^{2kt(m-4q)}}{\big(4\pi \sinh(tk)\big)^{2m}} dxdy \\ &\le \begin{cases} C_l k^{m-2l}, & \text{for } l \ge b+q,\ t \ge k^{-1} \\ C_l k^{m-2l} e^{-8kt}, & \text{for } l < b+q,\ t \ge k^{-1} \\ c_l t^{2l-m}, & \text{for } l \ge |q-b|,\ t < k^{-1}, \end{cases} \end{aligned}$$

and

$$P_{2q} u_l(x,y) P_{2b} = 0, \quad \forall l < |q-b|.$$

Moreover

$$\int_M \operatorname{tr} P_{2q} u_l(x,x) \left(\frac{k}{4\pi \sinh(tk)}\right)^m e^{kt(m-4q)} dx$$
$$\leq \begin{cases} B_l k^{m-2q} \left(\frac{e^{kt}}{\sinh(tk)}\right)^m, & \textit{for } tk > 1, \\ B_l t^{l-m}, & \textit{for } tk \leq 1. \end{cases}$$

Proof. Since u_l is a zero order partial differential operator, $u_l(x,y) \in W_y^{-2l}$ implies that

$$P_{2q} u_l P_{2b} = \sum_{2p-|J| \leq -2l} \sum_{d \leq q} k^p (x-y)^J e^{4dtk} a_{l,J,p,d}(x,y,tk),$$

with $a_{l,J,p,d}$ smooth and bounded.

When $tk > 1$, there exist constants $c_{l,J,p,d}, C_{l,d}$ so that we estimate

$$\int_{M\times M} |P_{2q} u_l(x,y) P_{2b}|^2 \left(\frac{k}{4\pi \sinh(tk)}\right)^{2m} e^{-\frac{kr^2}{2\tanh(tk)}} e^{2kt(m-4q)} dx dy$$
$$\leq \int_{M\times M} \sum_{2p-|J| \leq -2l} \sum_{d \leq q} c_{l,J,p,d} k^{2p} r^{2|J|} \left(\frac{k}{4\pi \sinh(tk)}\right)^{2m}$$
$$\cdot e^{-\frac{kr^2}{2\tanh(tk)}} e^{2kt(m-4q+4d)} dx dy$$
$$\leq \sum_{d \leq q} C_{l,d} k^{m-2l} e^{8kt(-q+d)}.$$

The only way for u_l to acquire a factor of e^{4qtk} is for $L^{-1} 2e^{4tk} e(F_A^{0,2})$ to occur at least q times in its construction. (We note that $(L^{-1} 2e^{-4tk} e^*(F_A^{0,2}))^j I$ is *not* exponentially decreasing in general.) This raises degree by $2q$. If $b > 0$, then $L^{-1} 2e^{-4tk} e^*(F_A^{0,2})$ must also occur at least b times. This requires $l \geq b+q$. Hence, when $l < b+q$, $e^{8kt(-q+d)} \leq e^{-8kt}$.

When $tk < 1$, we use the fact that L^{-1} increases the order of vanishing in tk by 1 to write

$$a_{l,J,p,d}(x,y,tk) = (tk)^l \hat{a}_{l,J,p,d}(x,y,tk),$$

with $\hat{a}_{l,J,p,d}$ smooth. Hence, there exist constants $\hat{c}_{l,J,p}, \hat{c}_l, c_l$ so that

$$\begin{aligned}
&\int_{M\times M} |P_{2q}u_l(x,y)P_{2b}|^2 \left(\frac{k}{4\pi\sinh(tk)}\right)^{2m} e^{-\frac{kr^2}{2\tanh(tk)}} e^{2kt(m-4q)} dxdy \\
&\leq \int_{M\times M} \sum_{2p-|J|\leq -2l} (tk)^{2l} c_{l,J,p} k^{2p} r^{2|J|} \left(\frac{k}{4\pi\sinh(tk)}\right)^{2m} e^{-\frac{kr^2}{2\tanh(tk)}} dxdy \\
&\leq \hat{c}_l t^{2l-m} \frac{(tk)^m}{\sinh(tk)^m} \leq c_l t^{2l-m}.
\end{aligned}$$

The vanishing of $P_{2q}u_l(x,y)P_{2b}$ for $l < |q-b|$ follows from the observation that the only terms that raise or lower degree in our construction are $2e^{4kt}\psi^{-1}e(F_A^{0,2})\psi$ and $2e^{-4kt}\psi^{-1}e^*(F_A^{0,2})\psi$. To raise or lower degree by $2q-2b$ requires at least $|q-b|$ applications of $2e^{4kt}\psi^{-1}e(F_A^{0,2})\psi$ or $2e^{-4kt}\psi^{-1}e^*(F_A^{0,2})\psi$ and therefore $|q-b|$ applications of H. That many applications of H do not occur in the construction of u_l until $l \geq |q-b|$.

The trace estimate is similar to the Hilbert–Schmidt estimate, with one added complication — the only terms in $P_{2q}u_lP_{2q}$ large enough to cancel the e^{-4tkq} in the integrand are those with $L^{-1}2e^{4kt}\psi^{-1}e(F_A^{0,2})\psi$ entering q times. These must also then have $L^{-1}2e^{4kt}\psi^{-1}e^*(F_A^{0,2})\psi$ entering q times to map $(0,2q)$ forms back to $(0,2q)$ forms. Hence, when tk is large, the trace is exponentially decreasing in tk unless $l \geq 2q$. □

Specializing to $t = sk^{-1/2}$, for say $s \in [\frac{1}{4}, 1]$, (this range merely needs to be k-independent) gives

Corollary 3.8. *For some $\alpha > 0$, and for all $s \in [\frac{1}{4}, 1]$,*

$$\|Q_{sk^{-1/2}}\|_{HS}^2 \leq \alpha^2 k^m.$$

Proposition 3.9. *For some $c > 0$, depending on the geometry of M, we have*

$$\|\epsilon_t\|_{HS}^2 \leq c(t^{2N-m} + k^{m-2N}). \tag{3.33}$$

Proof. By construction,

$$\epsilon_t = \psi\mathcal{U}\left(Hu_N + \sum_{l=0}^{N-1}(I - LL^{-1})Hu_l\right),$$

and $Hu_N + \sum_{l=0}^{N-1}(I - LL^{-1})Hu_l \in W_y^{-2N}$. Moreover, $(I - LL^{-1})Hu_l$ vanishes to order $2N$. Hence

$$\begin{aligned}
&\Big\|\psi\mathcal{U}\sum_{l=0}^{N-1}(I - LL^{-1})Hu_l\Big\|_{HS}^2 \\
&\leq C\sum_{s\geq 2N}\sum_{2p\leq s-2N}\int_{M\times M}\left(\frac{ke^{kt}}{\sinh(tk)}\right)^{2m} e^{-\frac{k|x-y|^2}{2\tanh(tk)}}k^{2p}|x-y|^{2s}dxdy \\
&\leq C'\sum_{s\geq 2N}k^{m-2N}\left(\frac{e^{kt}}{\sinh(kt)}\right)^m\tanh^s(tk) \\
&\leq \begin{cases} C''k^{m-2N} & \text{if } tk\geq 1 \\ C''t^{2N-m} & \text{if } tk\leq 1. \end{cases}
\end{aligned}$$

Because H has weight 0, the estimate for $\|\psi\mathcal{U}Hu_N\|_{HS}^2$ is the same as the first estimate of Proposition 3.7, giving

$$\|\psi\mathcal{U}Hu_N\|_{HS}^2 \leq \begin{cases} \hat{C}k^{m-2N} & \text{if } tk\geq 1, \\ \hat{C}t^{2N-m} & \text{if } tk\leq 1. \end{cases}$$

Combining these estimates gives the result. □

Proposition 3.10. *For some $a > 0$, independent of k large,*

$$\|\Pi - Q_{k^{-1/2}}\|_{HS} \leq ak^{\frac{m-N-1}{2}}.$$

This estimate is also true pointwise for the kernel. Since this pointwise estimate is not necessary for our analysis, we relegate its proof to the appendix.

Proof. We have

$$\|\Pi - Q_{k^{-1/2}}\|_{HS} \leq \|\Pi - e^{-k^{-1/2}D_{A(k)}^2}\|_{HS} + \|e^{-k^{-1/2}D_{A(k)}^2} - Q_{k^{-1/2}}\|_{HS}.$$

By (2.10), we can estimate this quantity by

$$e^{-\frac{k^{1/2}}{2}}\|e^{-\frac{k^{-1/2}}{2}D_{A(k)}^2}\|_{HS} + \|e^{-k^{-1/2}D_{A(k)}^2} - Q_{k^{-1/2}}\|_{HS}. \tag{3.34}$$

Using (2.12) and (3.9), we have, for $s < 1$, the estimate

$$\|e^{-sk^{-1/2}D_{A(k)}^2} - Q_{sk^{-1/2}}\|_{HS} \leq \int_0^{sk^{-1/2}}\|\epsilon_t\|_{HS}dt \leq Ck^{\frac{m-N-1}{2}}. \tag{3.35}$$

Using Corollary (3.8), we get the estimate

$$\begin{aligned} e^{-\frac{k^{1/2}}{2}}\|e^{-\frac{k^{-1/2}}{2}D^2_{A(k)}}\|_{HS} &\le e^{-\frac{k^{1/2}}{2}}\|Q_{k^{-1/2}}\|_{HS} + e^{-\frac{k^{1/2}}{2}}Ck^{\frac{m-N-1}{2}} \\ &\le 2\alpha e^{-\frac{k^{1/2}}{2}}k^{\frac{m}{2}} \end{aligned}$$

for the first summand in (3.34). The desired estimate follows by adding the estimates for the two summands. □

Proposition 3.11. *(See [16, Theorem 4.1.1].)*

$$\mathrm{Tr}\Pi = \frac{k^m}{2^m\pi^m}\mathrm{Vol}(M)\mathrm{rk}(E) + O(k^{m-1}). \tag{3.36}$$

Proof. Because Π is a projection, $\mathrm{Tr}\Pi = \|\Pi\|^2_{HS}$. By the preceding proposition, it suffices to compute the Hilbert–Schmidt norm of $Q_{k^{-1/2}}$. Because we are computing only the leading term, we may ignore all u_j for $j > 0$. In particular,

$$\begin{aligned} \|Q_{k^{-1/2}}\|^2_{HS} &= \int_{M\times M}\left(\frac{ke^{kt}}{4\pi\sinh(tk)}\right)^{2m} e^{-\frac{k|x-y|^2}{2\tanh(tk)}}\mathrm{rk}(E)dydx + O(k^{m-1}) \\ &= \frac{k^m}{2^m\pi^m}\mathrm{Vol}(M)\mathrm{rk}(E) + O(k^{m-1}). \end{aligned}$$

□

4. The asymptotics

4.1. Projections

Let $E \in S^q_{V,p}$, and let A be an $S^q_{V,p}$-compatible connection. Write

$$\Pi = \sum_{a,b=0}^{\lfloor\frac{m}{2}\rfloor}\Pi^{2b}_{2a},$$

where $\Pi^{2b}_{2a} = P_{2b}\Pi P_{2a}$, and $\lfloor\frac{m}{2}\rfloor$ denotes the integer part of $\frac{m}{2}$. Because Π is self adjoint,

$$\Pi^{2a}_{2b} = (\Pi^{2b}_{2a})^*.$$

Because Π is a projection, we have

$$\Pi_{2a}^{2b} = \sum_{\mu=0}^{\lfloor \frac{m}{2} \rfloor} \Pi_{2\mu}^{2b} \Pi_{2a}^{2\mu}.$$

In particular,

$$\Pi_{2a}^{2a} = \Pi_0^{2a} (\Pi_0^{2a})^* + \sum_{\mu > 0} \Pi_{2\mu}^{2a} (\Pi_{2\mu}^{2a})^*.$$

Proposition 4.1. *For $a > 0$,*

$$\mathrm{Tr} \Pi_{2a}^{2a} = \|\Pi_0^{2a}\|_{HS}^2 + O(k^{m-2a-2}).$$

Proof. We know that

$$\mathrm{Tr} \Pi_{2a}^{2a} = \|\Pi_0^{2a}\|_{HS}^2 + \sum_{\mu > 0} \|\Pi_{2\mu}^{2a}\|_{HS}^2.$$

By Proposition 3.10 and Proposition 3.7,

$$\|\Pi_{2\mu}^{2a}\|_{HS} = \|P_{2a} Q_{k^{-1/2}} P_{2\mu}\|_{HS} + O(k^{\frac{m-N-1}{2}}) = O(k^{\frac{m}{2}-a-\mu}), \forall a > 0. \tag{4.1}$$

$\square$

Corollary 4.2. *$E \in S_{V,p}^q$ with A an $S_{V,p}^q$ compatible connection if and only if $\sum_{\mu \leq \frac{p}{2}} \|\Pi_{2\mu}^{2q+2}\|_{HS}^2 = O(k^{m-2q-2-p})$.*

We henceforth focus our attention on the analysis of Π_0^{2q+2}. Proposition 3.10 allows us to consider instead $P_{2q} Q_{k^{-1/2}} P_0$ at the cost of introducing errors with $O(k^{\frac{m-N-1}{2}})$ Hilbert–Schmidt norm. We therefore assume in the following calculations that $t = k^{-1/2}$.

Proposition 4.3. *For $a > 0$,*

$$\|\Pi_0^{2a}\|_{HS}^2 = k^{m-2a} \frac{2^{-2a}}{(2\pi)^m (a!)^2} \|(F_A^{0,2})^a\|_{L^2}^2 + O(k^{m-2a-1}). \tag{4.2}$$

Proof. Since $P_0 u_b^*(x,y) P_{2a} u_l(x,y) P_0 \in W_y^{-2b-2l}$, we have

$$\left| \int_{M\times M} \left(\frac{k}{4\pi \sinh(tk)}\right)^{2m} U^2 e^{2kt(m-4a)} \mathrm{tr} P_0 u_b^*(x,y) P_{2a} u_l(x,y) P_0 \, dydx \right|$$
$$\leq \int_{M\times M} \left(\frac{k}{4\pi \sinh(tk)}\right)^{2m} e^{-\frac{k}{2}r^2} e^{2mkt} \sum_{2p-|J|\leq -2l-2b} C_{p,J} k^p r^{|J|} \, dydx \leq C k^{m-l-b}.$$

Now we estimate

$$\begin{aligned}
\|\Pi_0^{2a}\|_{HS}^2 &= \|P_{2a} Q_{k^{-1/2}} P_0\|_{HS}^2 + O(k^{\frac{m-N-1}{2}}) \\
&= \int_{M\times M} \left(\frac{k}{4\pi \sinh(tk)}\right)^{2m} U^2 e^{2kt(m-4a)} \\
&\qquad \cdot \mathrm{tr} P_0 \sum_{b=a}^{N} u_b^*(x,y) P_{2a} \sum_{l=a}^{N} u_l(x,y) P_0 \, dydx + O(k^{\frac{m-N-1}{2}}) \\
&= \int_{M\times M} \left(\frac{k}{4\pi \sinh(tk)}\right)^{2m} U^2 e^{2kt(m-4a)} \\
&\qquad \cdot \mathrm{tr} P_0 u_a^*(x,y) P_{2a} u_a(x,y) P_0 \, dydx + O(k^{m-2a-1}),
\end{aligned}$$

as long as $N \geq 2a$. (We always, of course, choose N sufficiently large.) Our computation showing that

$$\left| \int_{M\times M} \left(\frac{k}{4\pi \sinh(tk)}\right)^{2m} U^2 e^{2kt(m-2a)} \mathrm{tr} P_0 u_b^*(x,y) P_{2a} u_l(x,y) P_0 \, dydx \right|$$
$$\leq C k^{m-l-b}$$

extends immediately to show that any term in $P_0 u_a^*(x,y) P_{2a} u_a(x,y) P_0$ of weight less than $-2a$ contributes at most $O(k^{m-2a-1})$ to the integral.

Degree considerations show that

$$P_{2a} u_a P_0 = \left(-L^{-1} 2 e^{4kt} \psi^{-1}(x,y) e(F_A^{0,2}(x)) \psi(x,y)\right)^a I. \tag{4.3}$$

Observe that

$$\psi^{-1}(x,y) e(F_A^{0,2}(x)) \psi(x,y) = e(F_A^{0,2}(y)) + O(x-y),$$

and the $O(x-y)$ term is weight -1. Hence we may replace $P_{2a} u_a P_0$ by $(-L^{-1} 2 e^{4kt} e(F_A^{0,2}(y)))^a I$ in the computation of the Hilbert–Schmidt norm, introducing at most an $O(k^{m-2a-1})$ error. Referring to Lemma 4.4 below for a computation of $(-L^{-1}2e^{4kt}e(F_A^{0,2}(y)))^aI$, we see that

$$\begin{aligned}
\|\Pi_0^{2a}\|_{HS}^2 = \int_{M\times M} &\left(\frac{k}{4\pi\sinh(tk)}\right)^{2m} U^2 e^{2kt(m-4a)} \\
&\operatorname{tr} P_0 \frac{2^{-2a}e^{8atk}}{k^{2a}(a!)^2} e^*(F_A^{0,2}(y))^a e(F_A^{0,2}(y))^a P_0 dydx \\
&+ O(k^{m-2a-1}) \\
= k^{m-2a} \int_M &(\frac{1}{2\pi})^m \operatorname{tr} P_0 \frac{2^{-2a}}{(a!)^2} e^*(F_A^{0,2}(y))^a e(F_A^{0,2}(y))^a P_0 dy \\
&+ O(k^{m-2a-1}),
\end{aligned}$$

proving the asserted equality. $\square$

Recall double-factorial notation:

$$j!! = \begin{cases} j\cdot(j-2)\cdots 1, & \text{if } j \text{ is odd,} \\ j\cdot(j-2)\cdots 2, & \text{if } j \text{ is even.} \end{cases}$$

Lemma 4.4. *Suppose $t<1$, $k \gg 1$ and $tk \gg 1$. For $p \ge 0$, we have*

$$\begin{aligned}
\left(-L^{-1}2e^{4kt}e\big(F_A^{0,2}(y)\big)\right)^a I &= (-1)^a \frac{e^{4atk}}{2^a k^a a!} e\big(F_A^{0,2}(y)\big)^a + O(e^{(4a-1)tk}), \\
L^{-1}e^{4pkt}z^J\bar{z}^K &= \frac{e^{4ptk}}{2k(2p+|K|)} z^J\bar{z}^K + O\Bigg(r^{|J|+|K|}e^{(4p-1)tk} \\
&\quad + r^{|J|+|K|}e^{4ptk}\sum_{j=0}^{\min\{|J|,|K|\}} \frac{1}{r^{2j}k^j}\Bigg), \\
(-L^{-1}2e^{4kt})^a e^{4pkt}z^J\bar{z}^K &= \frac{(-1)^a(2p+|K|)!!}{k^a(2p+|K|+2a)!!} e^{4(a+p)tk}z^J\bar{z}^K \\
&\quad + O\Bigg(r^{|J|+|K|}e^{(4(a+p)-1)tk} \\
&\quad + r^{|J|+|K|}e^{4ptk}\sum_{j=0}^{\min\{|J|,|K|\}} \frac{1}{r^{2j}k^j}\Bigg).
\end{aligned}$$

Proof. We compute

$$L^{-1}2e^{4ftk}a(y) = \frac{e^{4ftk}-1}{2fk}a(y).$$

Hence

$$(-L^{-1}2e^{4kt}e(F_A^{0,2}(y)))^a I = (-1)^a \frac{e^{4atk}}{2^a k^a a!} e(F_A^{0,2}(y))^a + O(e^{(4a-1)tk}).$$

The proof of the second equality is similarly a direct application of the definition of L^{-1}. $\square$

Corollary 4.5. *$E \in S_{V,1}^q$ with A an $S_{V,1}^q$ compatible connection if and only if*

$$(F_A^{0,2})^{q+1} = 0.$$

Corollary 4.6. *If $E \in S_{V,1}^q$, then $ch_p(E) \in (S_H^{p-q} \cap \bar{S}_H^{p-q})H^{2p}(M, \mathbb{Q})$, $\forall p < q+3$.*

Proof. Let A be an $S_{V,1}^q$ compatible connection on A. We will treat the case $p = q+2$. The other cases follow from similar, albeit simpler, considerations. By Hodge theory, it suffices to show that $\operatorname{tr} F_A^p$ is a sum of $(s, p-s)$ forms with $2 \le s \le p-2$. Expanding $\operatorname{tr} F_A^p = \operatorname{tr}(F_A^{2,0} + F_A^{1,1} + F_A^{0,2})^p$ as the sum of the trace of a word in the letters $F_A^{2,0}$, $F_A^{1,1}$, and $F_A^{0,2}$, we see that it suffices to show that the letter $F_A^{0,2}$ occurs at most $p-2$ times in any word with nonzero trace (and symmetrically $F_A^{2,0}$ occurs at most $p-2$ times in any word with nonzero trace). Clearly, $\operatorname{tr}(F_A^{0,2})^p = 0$ by Corollary 4.5. By the cyclic invariance of the trace, we also have $\operatorname{tr}(F_A^{0,2})^a(sF_A^{1,1} + tF_A^{2,0})(F_A^{0,2})^{q+1-a} = \operatorname{tr}(F_A^{0,2})^{q+1}(sF_A^{1,1} + tF_A^{2,0}) = 0$, as desired. $\square$

Corollary 4.7. *Let $E \in S_{V,1}^q$ with $S_{V,1}^q$ compatible connection A. Then*

$$\|\Pi_{2\mu}^{2q+2}\|^2 = O(k^{m-2q-3-2\mu}).$$

Proof. Using Proposition 3.7 and our approximation $Q_{k^{-1/2}}$ for Π, we have

$$\begin{aligned}
\|\Pi_{2\mu}^{2q+2}\| &= \|P_{2q+2}Q_{k^{-1/2}}P_{2\mu}\| + O(k^{\frac{k-N-1}{2}}) \\
&\le \sum_{l \ge 2q+2\mu+2} \|P_{2q+2}\psi \mathcal{U} u_l P_{2\mu}\| \\
&\quad + \underbrace{\sum_{l < 2q+2\mu+2} \|P_{2q+2}\psi \mathcal{U} u_l P_{2\mu}\|}_{\text{Prop 3.7} \implies \text{exp. decay}} + O(k^{\frac{k-N-1}{2}}) \\
&= \|P_{2q+2}\psi \mathcal{U} u_{2q+2\mu+2} P_{2\mu}\| + O(k^{m-2q-2\mu-3}),
\end{aligned}$$

given N sufficiently large. The leading order term of the highest weight term in $u_{2q+2\mu+2}$ arises from

$$\Big(-L^{-1}2e^{4kt}e(F_A^{0,2}(y))\Big)^{q+1}\Big(-L^{-1}2e^{-4kt}e^*(F_A^{0,2}(y))\Big)^{\mu},$$

which vanishes by Corollary 4.5. The remaining terms have weight less than or equal to $-2q-3-2\mu$. □

Proposition 4.8. *For $E \in S^q_{V,1}$ with $S^q_{V,1}$ compatible connection A, we have*

$$\begin{aligned}&\|\Pi_0^{2q+2}\|^2_{HS}\\ &= \frac{k^{m-2q-3}}{2^{2q+2-m}(4\pi)^m}\left\|\sum_{b=0}^{q}\frac{2^{b+1}(2q-2b+1)!!(F_A^{0,2})^b(\nabla^{0,1}F_A^{0,2})(F_A^{0,2})^{q-b}}{(2q+3)!!(q-b)!}\right\|^2_{L^2}\\ &\quad + O(k^{m-2q-4}).\end{aligned}$$

Proof. We have seen in the proof of Proposition 4.3 that for $t = k^{-1/2}$, we have

$$\begin{aligned}\|\Pi_0^{2a}\|^2_{HS} &= \int_{M\times M}\left(\frac{k}{4\pi\sinh(tk)}\right)^{2m}U^2e^{2kt(m-4a)}\\ &\qquad\cdot \operatorname{tr}P_0\sum_{b=a}^{N}u_b^*(x,y)P_{2a}\sum_{l=a}^{N}u_l(x,y)P_0\,dydx + O(k^{\frac{m-N-1}{2}})\\ &= \int_{M\times M}\left(\frac{k}{4\pi\sinh(tk)}\right)^{2m}U^2e^{2kt(m-4a)}\operatorname{tr}P_0\Big(u_a^*(x,y)+u_{a+1}^*(x,y)\Big)\\ &\qquad\cdot P_{2a}\Big(u_a(x,y)+u_{a+1}(x,y)\Big)P_0\,dydx\\ &\quad + O(k^{m-2a-2}).\end{aligned}$$

Setting $a = q+1$, Corollary 4.5 implies the vanishing of $e(F_A^{0,2}(y))^a$. Consequently, from the discussion in the proof of Proposition 4.3, we see that $u_a(x,y) \in W_y^{-2a-1}$. This implies

$$\begin{aligned}\|\Pi_0^{2a}\|^2_{HS} &= \int_{M\times M}\left(\frac{k}{4\pi\sinh(tk)}\right)^{2m}U^2e^{2kt(m-4a)}\\ &\qquad\cdot \operatorname{tr}P_0u_a^*(x,y)P_{2a}u_a(x,y)P_0\,dydx + O(k^{m-2a-2}).\end{aligned}\tag{4.4}$$

We Taylor expand in the radial direction:

$$\psi^{-1}(x,y)e(F_A^{0,2}(x))\psi(x,y) = e(F_A^{0,2}(y)) + z^\mu e\big(\nabla_{\frac{\partial}{\partial z^\mu}} F_A^{0,2}(y)\big) \\ + \bar{z}^\mu e\big(\nabla_{\frac{\partial}{\partial \bar{z}^\mu}} F_A^{0,2}(y)\big) + O(|x-y|^2).$$

With the vanishing of $e(F_A^{0,2}(y))^a$, the leading order term in $P_{2a}u_aP_0$, as per Equation (4.3), becomes

$$\begin{aligned}
&(-1)^a \sum_{b=0}^{a-1} \Big(L^{-1}2e^{4kt}e\big(F_A^{0,2}(y)\big)\Big)^b \\
&\qquad \cdot \Big(L^{-1}2e^{4kt}\Big[z^\mu e\big(\nabla_{\frac{\partial}{\partial z^\mu}} F_A^{0,2}(y)\big) + \bar{z}^\mu e\big(\nabla_{\frac{\partial}{\partial \bar{z}^\mu}} F_A^{0,2}(y)\big)\Big]\Big) \\
&\qquad \cdot \Big(L^{-1}2e^{4kt}e\big(F_A^{0,2}(y)\big)\Big)^{a-b-1} \\
&= (-1)^a \sum_{b=0}^{a-1} \big(L^{-1}2e^{4kt}e\big(F_A^{0,2}(y)\big)\big)^b L^{-1} \\
&\qquad \cdot \frac{2e^{4(a-b)kt}\bar{z}^\mu e\big(\nabla_{\frac{\partial}{\partial \bar{z}^\mu}} F_A^{0,2}(y)\big)e\big(F_A^{0,2}(y)\big)^{a-b-1}}{2^{a-b-1}k^{a-b-1}(a-b-1)!} \\
&\quad + (-1)^a \sum_{b=0}^{a-1} \frac{e^{4akt}e(F_A^{0,2}(y))^b z^\mu e(\nabla_{\frac{\partial}{\partial z^\mu}} F_A^{0,2}(y))e(F_A^{0,2}(y))^{a-b-1}}{k^a 2^a a!} \\
&\qquad \bmod W_y^{-2a-2} \\
&= (-1)^a \sum_{b=0}^{a-1} \frac{2^{b+1}(2a-2b-1)!!}{k^a(2a+1)!!2^a(a-b-1)!} \\
&\qquad \cdot e^{4akt}e\big(F_A^{0,2}(y)\big)^b \bar{z}^\mu e\big(\nabla_{\frac{\partial}{\partial \bar{z}^\mu}} F_A^{0,2}(y)\big)e\big(F_A^{0,2}(y)\big)^{a-b-1} \\
&\quad + (-1)^a \sum_{b=0}^{a-1} \frac{e^{4akt}e\big(F_A^{0,2}(y)\big)^b z^\mu e\big(\nabla_{\frac{\partial}{\partial z^\mu}} F_A^{0,2}(y)\big)e\big(F_A^{0,2}(y)\big)^{a-b-1}}{k^a 2^a a!} \\
&\qquad \bmod W_y^{-2a-2}.
\end{aligned}$$

The coefficient of z^μ is a multiple of the $\frac{\partial}{\partial z^\mu}$ covariant derivative of $0 = e(F_A^{0,2})^a$, and therefore vanishes. (The vanishing of the $\bar z$-linear term in the Taylor expansion of $0 = e(F_A^{0,2}(y))^a$ can also be used to modify the coefficients of $\bar z^\mu$.) Hence

$$\begin{aligned} P_{2a}u_aP_0 = (-1)^a \sum_{b=0}^{a-1} &\frac{2^{b+1}(2a-2b-1)!!}{k^a(2a+1)!!2^a(a-b-1)!} \\ &\cdot e^{4akt} e\big(F_A^{0,2}(y)\big)^b \bar z^\mu e\big(\nabla_{\frac{\partial}{\partial \bar z^\mu}} F_A^{0,2}(y)\big) e\big(F_A^{0,2}(y)\big)^{a-b-1} \\ &\bmod W_y^{-2a-2}. \end{aligned}$$

Inserting this equality into Equation (4.4) gives

$$\begin{aligned} \|\Pi_0^{2a}\|_{HS}^2 &= \int_{M\times M} \left(\frac{k}{4\pi}\right)^{2m} U^2 \Bigg| \sum_{b=0}^{a-1} \frac{2^{b+1}(2a-2b-1)!! e\big(F_A^{0,2}(y)\big)^b \bar z^\mu}{k^a(2a+1)!!2^a(a-b-1)!} \\ &\qquad \cdot e(\nabla_{\frac{\partial}{\partial \bar z^\mu}} F_A^{0,2}(y)) e(F_A^{0,2}(y))^{a-b-1} P_0 \Bigg|_{HS}^2 dydx \\ &\quad + O(k^{m-2a-2}) \\ &= \frac{k^{m-2a-1}}{2^{2a-m}(4\pi)^m} \int_M \sum_\mu \Bigg| \sum_{b=0}^{a-1} \frac{2^{b+1}(2a-2b-1)!! e(F_A^{0,2}(y))^b}{(2a+1)!!(a-b-1)!} \\ &\qquad \cdot e(\nabla_{\frac{\partial}{\partial \bar z^\mu}} F_A^{0,2}(y)) e(F_A^{0,2}(y))^{a-b-1} P_0 \Bigg|_{HS}^2 dy \\ &\quad + O(k^{m-2a-2}) \end{aligned}$$

To complete the proof, we set $a = q+1$. $\square$

Corollary 4.9. *$E \in S^q_{V,2}$ with A an $S^q_{V,2}$ compatible connection if and only if*

$$0 = (F_A^{0,2})^{q+1}, \quad \text{and} \tag{4.5}$$

$$0 = \sum_{b=0}^{q} \frac{2^{b+1}(2q-2b+1)!!(F_A^{0,2})^b(\nabla^{0,1}F_A^{0,2})(F_A^{0,2})^{q-b}}{(2q+3)!!(q-b)!}. \tag{4.6}$$

5. Metric variation

In this section we begin investigating the additional constraints placed on an $S^q_{V,p}$ compatible connection A by $IS^q_{V,p}$ compatibility. These new constraints arise by computing the metric variation of the constraints imposed by $S^q_{V,p}$ compatibility.

Let $g(t)$ be a smooth 1 parameter family of Kahler metrics. Suppose that $\dot{g}(0) = \eta$, for some hermitian 2 tensor η. Then at $t = 0$,

$$\dot{\Gamma}^{\bar{c}}_{\bar{a}\bar{b}} = g^{\bar{c}e}\eta_{\bar{b}e;\bar{a}}, \tag{5.1}$$

Suppose henceforth that

$$\eta_{a\bar{c}} = \frac{\partial^2 H}{\partial z^a \partial \bar{z}^c}.$$

This is the form of the metric variation when we vary the line bundle metric h in the polarizing data (L, h). At the origin of a Kähler normal coordinate system we now have

$$\dot{\Gamma}^{\bar{c}}_{\bar{a}\bar{b}} = H_{,\bar{a}\bar{b}c}. \tag{5.2}$$

Proposition 5.1. *Let A be an $IS^1_{V,2}$ compatible connection and $dim_{\mathbb{C}} M > 3$. Then for every vector field Z,*

$$F^{0,2} \wedge i_Z F^{0,2} = 0.$$

Proof. Corollary 4.5 specialized to $q = 1$ gives

$$F^{0,2} \wedge F^{0,2} = 0.$$

The derivative of this equality combined with the second equation of Corollary 4.9 gives,

$$0 = F^{0,2}_A \wedge \nabla^{0,1} F^{0,2}_A, \tag{5.3}$$

for A an $S^1_{V,2}$ compatible connection. Equation (5.3) couples the connection to the metric via the Levi–Civita action of $\nabla_{\bar{Z}}$. As A is $IS^1_{V,2}$ compatible, the equation holds for all polarizations, and we may differentiate it to obtain

$$- H_{,\bar{a}\bar{b}c} d\bar{z}^b \wedge F^{0,2} \wedge F^{0,2}_{\bar{c}\bar{f}} d\bar{z}^f = 0. \tag{5.4}$$

The 1-form $-H_{,\bar{a}\bar{b}c} d\bar{z}^b$ can take any $(0,1)$ value at a point. Hence, we deduce that for every vector field Z,

$$F^{0,2} \wedge i_Z F^{0,2} = 0. \tag{5.5}$$

□

Proposition 5.2. *If $E \in IS^1_{V,2}$ with compatible connection A and $\dim_{\mathbb{C}} M > 3$, then $F^{0,2}_A$ is a 2-form taking values in a commutative subalgebra of* $\mathrm{ad}(E)$.

Proof. Let A be an $IS^1_{V,2}$ compatible connection. Proposition 5.1 gives

$$d\bar{z}^c \wedge F^{0,2}_{\bar{a}\bar{c}} \wedge F^{0,2}_A = 0, \quad \forall a. \tag{5.6}$$

Expanding this equation in coordinates gives

$$F^{0,2}_{\bar{a}\bar{c}} F^{0,2}_{\bar{b}\bar{f}} + F^{0,2}_{\bar{a}\bar{b}} F^{0,2}_{\bar{f}\bar{c}} + F^{0,2}_{\bar{a}\bar{f}} F^{0,2}_{\bar{c}\bar{b}} = 0, \quad \forall a, b, c, f. \tag{5.7}$$

When $a = f$ this reduces to

$$[F^{0,2}_{\bar{a}\bar{b}}, F^{0,2}_{\bar{a}\bar{c}}] = 0, \quad \forall a, b, c. \tag{5.8}$$

A change of coordinates (replacing a by $a + f$) then implies

$$[F^{0,2}_{\bar{a}\bar{b}}, F^{0,2}_{\bar{c}\bar{f}}] = [F^{0,2}_{\bar{f}\bar{b}}, F^{0,2}_{\bar{a}\bar{c}}], \quad \forall a, b, c, f. \tag{5.9}$$

The left hand side of this equality is invariant under the simultaneous exchanges $a \leftrightarrow b$ and $c \leftrightarrow f$, but the right hand side is multiplied by -1. Hence

$$[F^{0,2}_{\bar{a}\bar{b}}, F^{0,2}_{\bar{c}\bar{f}}] = 0, \quad \forall a, b, c, f. \tag{5.10}$$

□

6. Further asymptotics

Although we will not do so here, we note that probing the restrictions on $S^q_{V,p}$ compatible connections is sometimes simplified by using the identity $\|\Pi^0_{2a}\|^2_{HS} = \|\Pi^{2a}_0\|^2_{HS}$ to shift our computations to Π^0_{2a} where certain simplifications arise. To see these simplifications, first consider the model computation

$$\begin{aligned}
&L^{-1}z^J\bar{z}^K e^{-4ptk}\\
&= \frac{k^{-1}e^{tk(|J|-|K|)}}{\sinh(tk)^{(|J|+|K|)}}\int_0^{tk}\int_{\mathbb{R}^n} e^{-\pi|y|^2}\sinh(s)^{(|J|+|K|)}e^{s(|K|-|J|-4p)}\\
&\qquad\qquad \cdot\left(\sqrt{\frac{4\pi}{k}(tk-s)}y+z\right)^J\\
&\qquad\qquad \cdot\left(\sqrt{\frac{4\pi}{k}(tk-s)}\bar{y}+\bar{z}\right)^K dyds.
\end{aligned}$$

In particular, for $p>0$, $L^{-1}z^J\bar{z}^K e^{-4ptk}$ is exponentially decreasing if $|K|\neq 0$. Hence, in estimating

$$\begin{aligned}
\|\Pi^0_{2a}\|^2_{HS} &= \|P_0 Q_{k^{-1/2}}P_{2a}\|^2_{HS} + O\big(k^{\frac{m-N-1}{2}}\big)\\
&= \int_{M\times M}\left(\frac{k}{4\pi\sinh(tk)}\right)^n U^2 e^{2kt(m-4a)}\\
&\qquad \cdot \operatorname{tr}P_{2a}\sum_{b=a}^N u_b^*(x,y)P_0\sum_{l=a}^N u_l(x,y)P_{2a}dydx + O(k^{\frac{m-N-1}{2}}),
\end{aligned}$$

we may discard terms in $u_f = (-L^{-1}H)^f I$ arising from exponentially decreasing terms in $H(-L^{-1}H)^{f-1}I$ with $\bar{z}^K$ factors, $|K|>0$. Because L^{-1} has terms lowering the degree of a polynomial, we cannot simply remove any term with a $\bar{z}^b$ factor. Nonetheless, this suggests we analyze the polynomials arising in H.

6.1. The fine structure of H

It is convenient to say a monomial differential operator $z^A\bar{z}^B\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha\partial\bar{z}^\beta}$ is of type $(|A|-|\alpha|,|B|-|\beta|)$ and to define the *charge* of a monomial differential operator of type (p,q) to be $p-q$. We identify the terms in H which raise or lower homogeneity in $\bar{z}$. First we determine the $O(r^3)$ terms arising in $\Delta r^2 +$

$4m - \frac{2}{3}Ric(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r})$. Let $\{e_j\}_{j=1}^n$ be an orthonormal tangent frame parallel along radial geodesics emanating from y, with $(\nabla e_j)(y) = 0$. We choose the frame so that $e_j - \frac{\partial}{\partial x^j} = O(r^2\nabla)$, and therefore $e_j r^2 = 2(x^j - y^j) + O(r^3)$.

Then

$$
\begin{aligned}
&r\frac{\partial}{\partial r}\left(4m + \Delta r^2\right)\\
&= 2\Delta r^2 - \left[r\frac{\partial}{\partial r}, e_je_j - \nabla_{e_j}e_j\right] r^2\\
&= 2\Delta r^2 + \Big(\big(e_j + \Phi(e_j)\big)e_j + e_j\big(e_j + \Phi(e_j)\big)\\
&\quad + \nabla_{r\frac{\partial}{\partial r}}\nabla_{e_j}e_j - \nabla_{e_j}e_j - \Phi(\nabla_{e_j}e_j)\Big)r^2\\
&= \left(\Phi(e_j)e_j + e_j\Phi(e_j) + R\left(r\frac{\partial}{\partial r}, e_j\right)e_j - \nabla_{\Phi(e_j)}e_j - \Phi(\nabla_{e_j}e_j)\right) r^2\\
&= \frac{4}{3}Ric(y)\left(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}\right) - \frac{r}{2}\left(\nabla_{\frac{\partial}{\partial r}}Ric\right)\left(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}\right) + O(r^4).
\end{aligned}
$$

Hence

$$
\begin{aligned}
(6.1)\qquad \left(4m + \Delta r^2\right) = {}& \frac{2}{3}Ric(y)\left(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}\right) + \frac{r}{6}\left(\nabla_{\frac{\partial}{\partial r}}Ric\right)\left(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}\right)\\
&+ O(r^4).
\end{aligned}
$$

We now examine H. We have

$$
\begin{aligned}
H &= H_h + 2e^{4kt}\psi^{-1}e(F_A^{0,2})\psi + 2e^{-4kt}\psi^{-1}e^*(F_A^{0,2})\psi, \text{ and}\\
H_h &= \Delta - \Delta_E + ikr(J - J_0)\frac{\partial}{\partial r} - \frac{k(4m + \Delta(r^2))}{4\tanh(tk)}\\
&\quad - 2g^{ij}(\psi_L^{-1}\delta\psi_{L;i} + \hat{\psi}^{-1}\hat{\psi}_{;i})\frac{\partial}{\partial x^j} + \nabla^*\psi^{-1}\nabla\psi\\
&\quad - g^{ij}(\psi_L^{-1}\delta\psi_{L;i} + \hat{\psi}^{-1}\hat{\psi}_{;i})(\psi_L^{-1}\delta\psi_{L;j} + \hat{\psi}^{-1}\hat{\psi}_{;j})\\
&\quad + ik\big(\psi_L^{-1}\delta\psi_{L;rJ\frac{\partial}{\partial r}} + \hat{\psi}^{-1}\hat{\psi}_{;rJ\frac{\partial}{\partial r}}\big) + \psi^{-1}\hat{\mathcal{F}}\psi.
\end{aligned}
$$

A relatively straightforward but lengthy computation involving numerous identities given so far allows one to get

$$
\begin{aligned}
H_h = &-\frac{8}{3}R(y)\left(\frac{\partial}{\partial z^i}, \bar{z}^a\frac{\partial}{\partial \bar{z}^a}, z^b\frac{\partial}{\partial z^b}, \frac{\partial}{\partial \bar{z}^j}\right)\frac{\partial^2}{\partial \bar{z}^i \partial z^j}\\
&-\frac{4}{3}R(y)\left(\frac{\partial}{\partial z^i}, \bar{z}^a\frac{\partial}{\partial \bar{z}^a}, \bar{z}^b\frac{\partial}{\partial \bar{z}^b}, \frac{\partial}{\partial z^j}\right)\frac{\partial^2}{\partial \bar{z}^i \partial \bar{z}^j}\\
&-\frac{4}{3}R(y)\left(\frac{\partial}{\partial \bar{z}^i}, z^a\frac{\partial}{\partial z^a}, z^b\frac{\partial}{\partial z^b}, \frac{\partial}{\partial \bar{z}^j}\right)\frac{\partial^2}{\partial z^i \partial z^j}\\
&+\frac{4}{3}Ric\left(\bar{z}^a\frac{\partial}{\partial \bar{z}^a}, \frac{\partial}{\partial z^j}\right)\frac{\partial}{\partial \bar{z}^j}+\frac{4}{3}Ric\left(z^a\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^j}\right)\frac{\partial}{\partial z^j}\\
&-\frac{2k}{3}R(y)\left(z^b\frac{\partial}{\partial z^b}, \frac{\partial}{\partial \bar{z}^\mu}, z^l\frac{\partial}{\partial z^l}, \bar{z}^c\frac{\partial}{\partial \bar{z}^c}\right)\frac{\partial}{\partial z^\mu}\\
&+\frac{2k}{3}R(y)\left(\bar{z}^b\frac{\partial}{\partial \bar{z}^b}, \frac{\partial}{\partial z^\mu}, \bar{z}^l\frac{\partial}{\partial \bar{z}^l}, z^c\frac{\partial}{\partial z^c}\right)\frac{\partial}{\partial \bar{z}^\mu}\\
&+\frac{kz^a\bar{z}^b}{3}R(y)\left(\frac{\partial}{\partial z^j}, \bar{z}^c\frac{\partial}{\partial \bar{z}^c}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)\frac{\partial}{\partial \bar{z}^j}\\
&+\frac{kz^a\bar{z}^b}{3}R(y)\left(\frac{\partial}{\partial \bar{z}^j}, z^c\frac{\partial}{\partial z^c}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)\frac{\partial}{\partial z^j}\\
&-2\left(F^E(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial \bar{z}^j}\right)+z^cR(y)\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar{z}^j}\right)\right)\frac{\partial}{\partial z^j}\\
&-2\left(F^E(y)\left(r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^j}\right)+\bar{z}^cR(y)\left(\frac{\partial}{\partial \bar{z}^c}, \frac{\partial}{\partial z^j}\right)\right)\frac{\partial}{\partial \bar{z}^j}\\
&+kz^a\bar{z}^b\left(F^E(y)\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)+R(y)\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)\right)\\
&+\frac{k^2z^az^c\bar{z}^b\bar{z}^e}{6}R(y)\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar{z}^e}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)+\psi^{-1}\hat{\mathcal{F}}\psi\\
&-\frac{k\bar{z}^az^bRic(y)\left(\frac{\partial}{\partial \bar{z}^a}, \frac{\partial}{\partial z^b}\right)}{3\tanh(tk)}+\delta H_h, \qquad \text{with}
\end{aligned}
$$

$$
\begin{aligned}
\delta H_h = &-\frac{k}{3}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(z^b\frac{\partial}{\partial z^b}, \frac{\partial}{\partial \bar{z}^\mu}, z^l\frac{\partial}{\partial z^l}, \bar{z}^c\frac{\partial}{\partial \bar{z}^c}\right)\frac{\partial}{\partial z^\mu}\\
&+\frac{k}{3}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(\bar{z}^b\frac{\partial}{\partial \bar{z}^b}, \frac{\partial}{\partial z^\mu}, \bar{z}^l\frac{\partial}{\partial \bar{z}^l}, z^c\frac{\partial}{\partial z^c}\right)\frac{\partial}{\partial \bar{z}^\mu}\\
&+\frac{kz^a\bar{z}^b}{5}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(\frac{\partial}{\partial z^j}, \bar{z}^c\frac{\partial}{\partial \bar{z}^c}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)\frac{\partial}{\partial \bar{z}^j}\\
&+\frac{kz^a\bar{z}^b}{5}\left(\nabla_{r\frac{\partial}{\partial r}}R\right)\left(\frac{\partial}{\partial \bar{z}^j}, z^c\frac{\partial}{\partial z^c}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)\frac{\partial}{\partial z^j}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2kz^a\bar z^b}{40}(\nabla_{e_j} R)\left(e_j, r\frac{\partial}{\partial r}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar z^b}\right) - \frac{1}{3}d_A^* F^E\left(r\frac{\partial}{\partial r}\right)\\
&- \frac{1}{3}d^*_{\nabla^{LC}} R\left(r\frac{\partial}{\partial r}\right) - \frac{k\bar z^a z^b r\left(\nabla_{\frac{\partial}{\partial r}} Ric\right)\left(\frac{\partial}{\partial \bar z^a}, \frac{\partial}{\partial z^b}\right)}{12\tanh(tk)}\\
&+ \frac{2}{3}kz^a\bar z^b\left(\left(\nabla_{r\frac{\partial}{\partial r}} F^E\right)\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar z^b}\right) + \left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar z^b}\right)\right)\\
&+ \frac{k^2 z^a z^c \bar z^b \bar z^e}{10}\left(\nabla_{r\frac{\partial}{\partial r}} R\right)\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar z^e}, \frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar z^b}\right)\\
&+ O(r^2\nabla + r^3\nabla^2) + O\left(r^2 + kr^4 + kr^5\nabla + \frac{kr^4}{\tanh(tk)}\right).
\end{aligned}
$$

This decomposition reflects the fact that $\delta H_h \in W_y^{-1}$. Except for the term $-2F^E(y)(\bar z^a \frac{\partial}{\partial \bar z^a}, \frac{\partial}{\partial \bar z^j})\frac{\partial}{\partial z^j}$ which is of type $(-1,1)$ (and thus of charge -2), and the term $-2F^E(y)(z^a \frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^j})\frac{\partial}{\partial \bar z^j}$ which is of type $(1,-1)$ (and thus of charge 2), the rest of H (modulo terms of weight -1) consists of terms of charge 0 and has no terms of type (p,q) with $p<0$ or $q<0$.

Proposition 6.1. *If $E \in IS^1_{V,3}$ with A an $IS^1_{V,3}$ compatible connection and $dim_{\mathbb{C}} M > 3$, then for all vector fields Z,*

$$0 = F^{0,2}_{A;a} \wedge i_Z F^{0,2}, \tag{6.2}$$

and

$$0 = F^{0,2}_A \wedge F^{1,1}_A \wedge F^{0,2}_A. \tag{6.3}$$

Proof. The assumption that A is $S^1_{V,3}$ compatible implies that $P_4\sum_l u_l P_0$ vanishes modulo weight -7. If r_1 and r_2 are two polynomials of charge q_1 and q_2 respectively, then

$$r_1 \perp r_2 \text{ in } L_2(e^{-|z|^2/2}), \quad \forall q_1 \neq q_2.$$

Hence, the terms of $P_4\sum_l u_l P_0$ and $P_0\sum_l u_l P_4$ of charge q_i each vanish modulo weight -7, for $q_i \in \{-2,-1,0,1,2\}$. The assumption that A is $S^1_{V,2}$ compatible implies that

$$e(F^{0,2}_A)^2 = e(F^{0,2}_A)e(\nabla^{0,1}F^{0,2}_A) = 0. \tag{6.4}$$

The only remaining terms of charge 0 in $P_4 u_2 P_0$, modulo weight -7 are, modulo $O(e^{7kt})$,

$$
\begin{aligned}
& L^{-1} 2e(F_A^{0,2}) e^{4kt} L^{-1} z^a \bar{z}^b e(F^{0,2}_{A;a\bar{b}+\bar{b}a}) e^{4kt} \\
&= \frac{e^{8kt}}{30k^2} e(F_A^{0,2}) z^a \bar{z}^b e(F^{0,2}_{A;a\bar{b}+\bar{b}a}) + \frac{37e^{8kt}}{900k^3} e(F_A^{0,2}) e(F^{0,2}_{A;a\bar{a}+\bar{a}a}) \\
&= \frac{e^{8kt}}{15k^2} e(F_A^{0,2}) z^a \bar{z}^b e(F^{0,2}_{A;\bar{b}a}) + \frac{e^{8kt}}{30k^2} e(F_A^{0,2}) z^a \bar{z}^b (F^E_{\bar{b}a} + R_{\bar{b}a}) e(F_A^{0,2}) \\
&\quad + \frac{37e^{8kt}}{450k^3} e(F_A^{0,2}) e(F^{0,2}_{A;\bar{a}a}) + \frac{37e^{8kt}}{900k^3} e(F_A^{0,2}) (F^E_{\bar{a}a} + R_{\bar{a}a}) e(F_A^{0,2}),
\end{aligned}
$$

$$
\begin{aligned}
& L^{-1} z^a \bar{z}^b e(F^{0,2}_{A;a\bar{b}+\bar{b}a}) e^{4kt} L^{-1} 2e(F_A^{0,2}) e^{4kt} \\
&= \frac{e^{8kt}}{20k^2} z^a \bar{z}^b e(F^{0,2}_{A;a\bar{b}+\bar{b}a}) e(F_A^{0,2}) + \frac{e^{8kt}}{50k^3} e(F^{0,2}_{A;a\bar{a}+\bar{a}a}) e(F_A^{0,2}) \\
&= \frac{e^{8kt}}{10k^2} z^a \bar{z}^b e(F^{0,2}_{A;\bar{b}a}) e(F_A^{0,2}) + \frac{e^{8kt}}{20k^2} z^a \bar{z}^b e(R_{\bar{b}a} F_A^{0,2}) e(F_A^{0,2}) \\
&\quad - \frac{e^{8kt}}{20k^2} z^a \bar{z}^b e(F_A^{0,2}) F^E_{\bar{b}a} e(F_A^{0,2}) + \frac{e^{8kt}}{25k^3} e(F^{0,2}_{A;\bar{a}a}) e(F_A^{0,2}) \\
&\quad + \frac{e^{8kt}}{50k^3} e(R_{\bar{a}a} F_A^{0,2}) e(F_A^{0,2}) - \frac{e^{8kt}}{50k^3} e(F_A^{0,2}) F^E_{\bar{a}a} e(F_A^{0,2}),
\end{aligned}
$$

$$
\begin{aligned}
& L^{-1} 2z^a e(F^{0,2}_{A;a}) e^{4kt} L^{-1} 2\bar{z}^b e(F^{0,2}_{A;\bar{b}}) e^{4kt} \\
&= \frac{e^{8kt}}{15k^2} z^a e(F^{0,2}_{A;a}) \bar{z}^b e(F^{0,2}_{A;\bar{b}}) + \frac{2e^{8kt}}{75k^3} e(F^{0,2}_{A;a}) e(F^{0,2}_{A;\bar{a}}),
\end{aligned}
$$

$$
\begin{aligned}
& L^{-1} 2\bar{z}^b e(F^{0,2}_{A;\bar{b}}) e^{4kt} L^{-1} 2z^a e(F^{0,2}_{A;a}) e^{4kt} \\
&= \frac{e^{8kt}}{10k^2} \bar{z}^b e(F^{0,2}_{A;\bar{b}}) z^a e(F^{0,2}_{A;a}) + \frac{e^{8kt}}{25k^3} e(F^{0,2}_{A;\bar{a}}) e(F^{0,2}_{A;a}).
\end{aligned}
$$

The only remaining term of charge 0 in $P_4 u_3 P_0$, modulo weight -7 is, modulo $O(e^{7kt})$,

$$
\begin{aligned}
& - L^{-1} 2e(F_A^{0,2}) e^{4kt} L^{-1} (\hat{\mathcal{F}}(y) + k z^a \bar{z}^b [F^E_{a\bar{b}} + R_{a\bar{b}}]) L^{-1} 2e(F_A^{0,2}) e^{4kt} \\
&= -\frac{e^{8kt}}{32k^3} e(F_A^{0,2}) \hat{\mathcal{F}}(y) e(F_A^{0,2}) - \frac{e^{8kt}}{60k^3} e(F_A^{0,2}) k z^a \bar{z}^b [F^E_{a\bar{b}} + R_{a\bar{b}}] e(F_A^{0,2}) \\
&\quad - \frac{37e^{8kt}}{1800k^3} e(F_A^{0,2}) [F^E_{a\bar{a}} + R_{a\bar{a}}] e(F_A^{0,2}).
\end{aligned}
$$

Differentiating (6.4), we get that $F^{0,2}_{A;a}F^{0,2}_{A;\bar b} = -F^{0,2}_{A}F^{0,2}_{A;\bar b a}$ and $F^{0,2}_{A;\bar b}F^{0,2}_{A;a} = -F^{0,2}_{A;\bar b a}F^{0,2}_{A}$. These equations allow us to cancel some of the contributions above. The total leading order contribution from charge zero is then

$$\begin{aligned}\nu_0 := {} & \frac{e^{8kt}}{16k^3}e(F^{0,2}_A)e(d\bar z^b)e^*(d\bar z^a)(F^E_{\bar b a} + R_{\bar b a})e(F^{0,2}_A) + \frac{e^{8kt}}{18k^3}e(F^{0,2}_A)e(F^{0,2}_{A;\bar a a}) \\ & + \frac{e^{8kt}}{24k^3}e(F^{0,2}_A)F^E_{\bar a a}e(F^{0,2}_A) + \frac{e^{8kt}}{24k^3}e(F^{0,2}_A)R_{\bar a a}e(F^{0,2}_A).\end{aligned}$$

The assumption that A is $S^1_{V,3}$ compatible implies $\nu_0 = 0$. Note that $[R_{\bar a b}, F^{0,2}_A] = 2g^{s\bar t}R_{\bar a b t \bar l}d\bar z^l \wedge i_{\frac{\partial}{\partial \bar z^s}}F^{0,2}$. When A is $IS^1_{V,2}$-compatible then the identities in Proposition 5.1 force $F^{0,2}_A[R_{\bar a b}, F^{0,2}_A] = 0$.

Therefore, under the assumption that A is $IS^1_{V,2}$-compatible, ν_0 reduces to

$$\begin{aligned}\check\nu_0 := {} & -\frac{e^{8kt}}{18k^3}e(F^{0,2}_{A;a})e(F^{0,2}_{A;\bar a}) - \frac{e^{8kt}}{24k^3}e(F^{0,2}_A)F^E_{\bar a a}e(F^{0,2}_A) \\ & + \frac{e^{8kt}}{16k^3}e(F^{0,2}_A)e(d\bar z^b)e^*(d\bar z^a)F^E_{\bar b a}e(F^{0,2}_A).\end{aligned}$$

Finally, we exploit the full assumption that A is $IS^1_{V,3}$-compatible. Thus we assume $\check\nu_0$ vanishes for all polarizations. First we rewrite $0 = -144k^3e^{-8kt}\check\nu_0$ with metric terms explicit rather than hidden in orthonormal coordinate systems:

$$\begin{aligned}0 = {} & 8g^{a\bar b}e(F^{0,2}_{A;a})e(F^{0,2}_{A;\bar b}) + 6e(F^{0,2}_A)g^{a\bar b}F^E_{\bar b a}e(F^{0,2}_A) \\ & - 9e(F^{0,2}_A)e(d\bar z^b)g^{a\bar\mu}F^E_{\bar b a}i_{\frac{\partial}{\partial\bar z^\mu}}e(F^{0,2}_A).\end{aligned}$$

Writing $F^{0,2}$ in an anti-holomorphic frame, we see that $F^{0,2}_{;a}$ is independent of the metric. Varying the metric gives

$$\begin{aligned}0 = {} & 8\dot g^{a\bar b}F^{0,2}_{A;a}F^{0,2}_{A;\bar b} + 32g^{a\bar b}F^{0,2}_{A;a}H_{,\bar b\bar\mu c}d\bar z^\mu \wedge F_{\bar c\bar f}d\bar z^f + 6F^{0,2}_A\dot g^{a\bar b}F^E_{\bar b a}F^{0,2}_A \\ & - 9F^{0,2}_A \wedge d\bar z^b\dot g^{a\bar\mu}F^E_{\bar b a} \wedge i_{\frac{\partial}{\partial\bar z^\mu}}F^{0,2}_A.\end{aligned}$$

At a fixed point, we may simultaneously choose $\dot g = 0$ and $\bar\partial H_{,\bar b c}$ an arbitrary $(0,1)$ form. For such a choice,

$$0 = \bar\partial H_{,\bar b c} \wedge F^{0,2}_{A;b} \wedge F^{0,2}_{\bar c\bar f}d\bar z^f.$$

Hence when $m > 3$, for all vector fields Z,

$$0 = F^{0,2}_{A;a} \wedge i_Z F^{0,2}_A.$$

Now choose $\dot{g}$ arbitrary at a fixed point, and the remaining terms in the variation of $\check{\nu}_0$ give

$$0 = 8F^{0,2}_{A;a} \wedge F^{0,2}_{A;\bar{b}} + 6F^{0,2}_A \wedge F^E_{\bar{b}a} F^{0,2}_A - 9F^{0,2}_A \wedge F^E_{\bar{\mu}a} d\bar{z}^\mu \wedge F^{0,2}_{\bar{b}\bar{c}} d\bar{z}^c.$$

Wedging with $dz^a \wedge d\bar{z}^b$ and summing over a and b yields Equation (6.3). $\square$

Proposition 6.2. *If $E \in S^1_{V,1}$, then*

$$\mathrm{ch}_p(E) \in (S_H^{p-\lfloor \frac{p}{2} \rfloor} \cap \bar{S}_H^{p-\lfloor \frac{p}{2} \rfloor}) H^{2p}(M, \mathbb{Q}), \text{ for all } p.$$

If $E \in IS^1_{V,3}$, then

$$\mathrm{ch}_p(E) \in (S_H^{p-1} \cap \bar{S}_H^{p-1}) H^{2p}(M, \mathbb{Q}) \text{ for all } p < 7.$$

Proof. Let $S^a_H C^p(M, \mathbb{C})$ denote the $p-$forms which can be written as a sum of $(s, p-s)$ forms, $s \geq a$. Let $\bar{S}^a_H C^p(M, \mathbb{C})$ denote the conjugate filtration. To show that $\mathrm{ch}_p(E) \in (S_H^{p-a} \cap \bar{S}_H^{p-a}) H^{2p}(M, \mathbb{Q})$, it suffices to show that $\mathrm{tr}(F^{2,0}_A + F^{1,1}_A + F^{0,2}_A)^p$ is cohomologous to an element of $(S_H^{p-a} \cap \bar{S}_H^{p-a})$ $C^p(M, \mathbb{C})$. Expand this trace as the sum of traces of words in the letters $F^{2,0}_A$, $F^{1,1}_A$, and $F^{0,2}_A$. Let A be an $S^1_{V,1}$ compatible connection. Then Corollary 4.5 implies that $F^{0,2}_A \wedge F^{0,2}_A = 0$. Hence, after any cyclic rearrangement of a monomial with nonzero trace, there must be an $F^{2,0}_A$ or $F^{1,1}_A$ factor between any two $F^{0,2}_A$ factors. Consequently, at most $\lfloor \frac{p}{2} \rfloor$ $F^{0,2}_A$ factors may appear in any monomial of degree p with nonzero trace. This proves $\mathrm{ch}_p(E) \in S_H^{p-\lfloor \frac{p}{2} \rfloor} H^{2p}(M, \mathbb{Q})$ for all p. The conjugate inclusion follows similarly, proving the first assertion.

Now assume that A is an $IS^1_{V,3}$ compatible connection. To prove the second assertion, we consider the case $p = 6$, as the case $p < 6$ follows from similar but simpler arguments. We have seen that the trace of any nonzero monomial of degree 6 in the curvature components must have at most 3 $F^{0,2}_A$ factors, and (for every cyclic rearrangement) there must be an $F^{2,0}_A$ or $F^{1,1}_A$ factor between any two such factors. If there are at least 2 $F^{2,0}_A$ factors, then the trace of the monomial lies in $S^5_H C^{12}(M, \mathbb{C})$. By Proposition 6.1, $F^{0,2}_A \wedge F^{1,1}_A \wedge F^{0,2}_A = 0$ for $IS^1_{V,3}$ compatible A . Hence between any two $F^{0,2}_A$

factors there must be an $F_A^{2,0}$ factor or an $(F_A^{1,1})^2$ factor. The case of 3 $F_A^{0,2}$ factors, 1 or 2 $F_A^{1,1}$ factors and 2 or 1 (respectively) $F_A^{2,0}$ is therefore excluded. Thus the only monomials with nonzero trace and 3 $F_A^{0,2}$ factors are cyclic rearrangements of $(F_A^{0,2} \wedge F_A^{2,0})^3$. These terms yield $(6,6)$ forms. Terms with at most 1 $F_A^{0,2}$ factor lie in $S_H^5 C^{12}(M,\mathbb{C})$, as do terms with 1 $F_A^{2,0}$ factor and 2 $F_A^{0,2}$ factors. Hence we are left to consider terms with exactly 2 $F_A^{0,2}$ factors and 4 $F_A^{1,1}$ factors. The only monomials of this form with nonzero trace are cyclic rearrangements of $(F_A^{1,1})^2 \wedge F_A^{0,2} \wedge (F_A^{1,1})^2 \wedge F_A^{0,2}$. We need now to show that the trace of this term is cohomologous to an element of $S_H^5 C^{12}(M,\mathbb{C})$. We can use Proposition 6.1 to set $F_A^{0,2} \wedge \partial_A F_A^{0,2} = 0$ and the Bianchi identity to replace $\bar\partial_A F_A^{1,1}$ by $-\partial_A F_A^{0,2}$. We thus have

$$\begin{aligned}
&\operatorname{tr}(F_A^{1,1})^2 \wedge F_A^{0,2} \wedge (F_A^{1,1})^2 \wedge F_A^{0,2}\\
&= \operatorname{tr}(F_A^{1,1})^2 \wedge F_A^{0,2} \wedge F_A^{1,1} \wedge \bar\partial_A \partial_A F_A^{0,2}\\
&= \operatorname{tr}(F_A^{1,1})^2 \wedge \bar\partial_A [F_A^{0,2} \wedge F_A^{1,1} \wedge \partial_A F_A^{0,2}]\\
&\quad - \operatorname{tr}(F_A^{1,1})^2 \wedge F_A^{0,2} \wedge \bar\partial_A F_A^{1,1} \wedge \partial_A F_A^{0,2}\\
&= \operatorname{tr}\bar\partial_A [(F_A^{1,1})^2 \wedge F_A^{0,2} \wedge F_A^{1,1} \wedge \partial_A F_A^{0,2}]\\
&\quad - \operatorname{tr}\bar\partial_A F_A^{1,1} \wedge F_A^{1,1} \wedge F_A^{0,2} \wedge F_A^{1,1} \wedge \partial_A F_A^{0,2}\\
&\quad - \operatorname{tr}F_A^{1,1} \wedge \bar\partial_A F_A^{1,1} \wedge F_A^{0,2} \wedge F_A^{1,1} \wedge \partial_A F_A^{0,2}\\
&\quad + \operatorname{tr}(F_A^{1,1})^2 \wedge F_A^{0,2} \wedge \partial_A F_A^{0,2} \wedge \partial_A F_A^{0,2}\\
&= \bar\partial \operatorname{tr}(F_A^{1,1})^2 \wedge F_A^{0,2} \wedge F_A^{1,1} \wedge \partial_A F_A^{0,2}\\
&= d\operatorname{tr}(F_A^{1,1})^2 \wedge F_A^{0,2} \wedge F_A^{1,1} \wedge \partial_A F_A^{0,2} \quad \bmod S_H^5 C^{12}(M,\mathbb{C}).
\end{aligned}$$

The result follows. □

Having exploited the terms in $P_4 \sum_l u_l P_0$ of charge 0, we now turn to those of charge -2.

Proposition 6.3. *Let $E \in S_{V,3}^1$ with compatible connection A. Then for all a, b,*

$$F_{A;\bar a}^{0,2} \wedge F_{A;\bar b}^{0,2} + F_{A;\bar b}^{0,2} \wedge F_{A;\bar a}^{0,2} = 0. \tag{6.5}$$

For $E \in IS_{V,3}^1$ with compatible connection A, for all a, c,

$$i_{\frac{\partial}{\partial \bar z^c}} F_A^{0,2} \wedge F_{A;\bar a}^{0,2} = 0, \text{ and} \tag{6.6}$$

$$(i_{\frac{\partial}{\partial \bar z^c}} F_A^{0,2}) \wedge (i_{\frac{\partial}{\partial \bar z^e}} F_A^{0,2}) = 0. \tag{6.7}$$

Proof. The terms of charge -2 in $P_4u_2P_0$, modulo weight -7 and $O(e^{7kt})$ are

$$\begin{aligned}
L^{-1}2\bar{z}^a e^{4kt}e(F^{0,2}_{A;\bar{a}})L^{-1}2\bar{z}^b e^{4kt}e(F^{0,2}_{A;\bar{b}}) &= \frac{e^{8kt}}{18k^2}\bar{z}^a e(F^{0,2}_{A;\bar{a}})\bar{z}^b e(F^{0,2}_{A;\bar{b}}),\\
L^{-1}2e^{4kt}e(F^{0,2}_{A})L^{-1}e^{4kt}\bar{z}^a\bar{z}^b e(F^{0,2}_{A;\bar{a}\bar{b}}) &= \frac{e^{8kt}}{48k^2}e(F^{0,2}_{A})\bar{z}^a\bar{z}^b e(F^{0,2}_{A;\bar{a}\bar{b}}), \text{ and}\\
L^{-1}e^{4kt}\bar{z}^a\bar{z}^b e(F^{0,2}_{A;\bar{a}\bar{b}})L^{-1}2e^{4kt}e(F^{0,2}_{A}) &= \frac{e^{8kt}}{24k^2}\bar{z}^a\bar{z}^b e(F^{0,2}_{A;\bar{a}\bar{b}})e(F^{0,2}_{A}).
\end{aligned}$$

The assumption that $E \in S^1_{V,2}$ implies that

$$0 = e(F^{0,2}_{A;\bar{a}})e(F^{0,2}_{A}).$$

Using these equalities, we write the charge -2 contribution (modulo weight -7 and $O(e^{7kt})$) as

$$\begin{aligned}
&\frac{e^{8kt}\bar{z}^a\bar{z}^b}{6k^2}\left(\frac{1}{3}e(F^{0,2}_{A;\bar{a}})e(F^{0,2}_{A;\bar{b}}) + \frac{1}{8}e(F^{0,2}_{A})e(F^{0,2}_{A;\bar{a}\bar{b}}) + \frac{1}{4}e(F^{0,2}_{A;\bar{a}\bar{b}})e(F^{0,2}_{A})\right)\\
&= -\frac{e^{8kt}\bar{z}^a\bar{z}^b}{144k^2}e(F^{0,2}_{A;\bar{a}})e(F^{0,2}_{A;\bar{b}}).
\end{aligned}$$

This term must vanish for $E \in S^1_{V,3}$. Hence

$$F^{0,2}_{A;\bar{a}} \wedge F^{0,2}_{A;\bar{b}} + F^{0,2}_{A;\bar{b}} \wedge F^{0,2}_{A;\bar{a}} = 0. \tag{6.8}$$

Suppose now that A is $IS^1_{V,3}$ compatible. Then we may take the first variation of the preceding equation with $b = a$ to obtain (no a-sum)

$$\sum_c \bar{\partial}H_{,\bar{a}c} \wedge [(i_{\frac{\partial}{\partial\bar{z}^c}}F^{0,2}_{A}) \wedge F^{0,2}_{A;\bar{a}} + F^{0,2}_{A;\bar{a}} \wedge i_{\frac{\partial}{\partial\bar{z}^c}}F^{0,2}_{A}] = 0.$$

Hence for all (a, c),

$$(i_{\frac{\partial}{\partial\bar{z}^c}}F^{0,2}_{A}) \wedge F^{0,2}_{A;\bar{a}} + F^{0,2}_{A;\bar{a}} \wedge i_{\frac{\partial}{\partial\bar{z}^c}}F^{0,2}_{A} = 0. \tag{6.9}$$

Since by Proposition 5.2, $F^{0,2}_{A}$ is a form taking values in a commutative subalgebra of $\mathrm{ad}(E)$, this equation implies Equation (6.6). In turn, taking the first variation of Equation (6.6), we find

$$(i_{\frac{\partial}{\partial\bar{z}^c}}F^{0,2}_{A}) \wedge (i_{\frac{\partial}{\partial\bar{z}^e}}F^{0,2}_{A}) = 0. \tag{6.10}$$

$\square$

Corollary 6.4. *Let $E \in IS^1_{V,3}$ with compatible connection A. Then $F_A^{0,2}$ takes values in a commutative nilpotent subalgebra of* $\mathrm{End}(E)$ *whose elements all square to zero.*

Appendix: From Hilbert–Schmidt to pointwise

In this appendix, we show how to derive pointwise bounds from Hilbert–Schmidt bounds. See [17] for an alternate derivation. For simplicity, the constant c is allowed to change value from line to line.

Let $b(T, x, y)$ denote the Schwartz kernel for the operator $B_T := e^{-TD_k^2} - \Pi$. On page 567, we claim that

$$\sup_{x,y\in M} |b(k^{-1/2}, x, y)| = O(k^m e^{-\frac{k^{1/2}}{2}}). \tag{6.11}$$

We also claim on page 584 that the estimate of Proposition 3.10 is also valid for the kernel of $\Pi - Q_{k^{-1/2}}$. Given (6.11), this claim follows from

$$\sup_{x,y\in M} |q_{k^{-1/2}}(x, y) - k_{k^{-1/2}}(x, y)| = O(k^{\frac{m-N-1}{2}}). \tag{6.12}$$

Note that Equation (6.11) is an estimate about $\|b(k^{-1/2}, \cdot, \cdot)\|_{\sup} = \|b(k^{-1/2}, \cdot, \cdot)\|_{L^\infty}$, not about $\|B_{k^{-1/2}}\|_{\sup} = \|B_{k^{-1/2}}\|_{op}$. Indeed, the last one is easy to estimate: suppose λ_1^2 is the first non-zero eigenvalue of $D^2_{A(k)}$, then $\|B_T\|_{\sup} \leq e^{-T\lambda_1^2}$.

We now prove Equation (6.11). Since $B^2_{\frac{T}{2}} = B_T$, we have

$$b(T, x, y) = \int_M b\left(\frac{T}{2}, x, z\right) b\left(\frac{T}{2}, z, y\right) dz,$$

and therefore

$$\begin{aligned} |b(T, x, y)| &\leq \sup_{p\in M} \int_M \left|b(\frac{T}{2}, p, z)\right|^2 dz = \sup_{p\in M} \operatorname{tr} b(T, p, p) \\ &= \sup_{p\in M} \lim_{\epsilon\to 0} \operatorname{Tr} \frac{\chi_{B_\epsilon(p)}}{Vol(B_\epsilon(p))} B_T, \end{aligned} \tag{6.13}$$

where χ_Y denotes the characteristic function of Y. Now we estimate using (2.7) and (2.8):
(6.14)

$$\begin{aligned}
\mathrm{Tr}\frac{\chi_{B_\epsilon(p)}}{Vol(B_\epsilon(p))}B_T &\le \left\|\frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}}B_{\frac{T}{2}}\right\|_{HS}\left\|e^{-\frac{T}{2}D^2_{A(k)}}\frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}}\right\|_{HS}\\
&\le \left\|\frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}}e^{-\frac{T}{4}D^2_{A(k)}}\right\|_{HS}\left\|B_{\frac{T}{4}}\right\|_{\sup}\\
&\quad\cdot\left\|e^{-\frac{T}{2}D^2_{A(k)}}\frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}}\right\|_{HS}\\
&\le e^{-\frac{Tk}{2}}\left\|\frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}}e^{-\frac{T}{4}D^2_{A(k)}}\right\|_{HS}\\
&\quad\cdot\left\|e^{-\frac{T}{2}D^2_{A(k)}}\frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}}\right\|_{HS}
\end{aligned}$$

For $t=\frac{k^{-1/2}}{2}$ or $t=\frac{k^{-1/2}}{4}$, we are left to estimate
(6.15)

$$\begin{aligned}
\left\|e^{-tD_k^2}\frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}}\right\|_{HS} &\le \left\|Q_t\frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}}\right\|_{HS}\\
&\quad+\left\|\int_0^t e^{-(t-s)D^2_{A(k)}}\epsilon_s ds\frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}}\right\|_{HS}.
\end{aligned}$$

For the first term, we start with a pointwise equivalent to Proposition 3.7. Suppose that $tk>1$. Then

$$\begin{aligned}
|P_{2q}\psi\mathcal{U}u_l| &\le |P_{2q}u_l(x,y)|\frac{k^m e^{-\frac{kr^2}{4\tanh(tk)}}e^{kt(m-4q)}}{\big(4\pi\sinh(tk)\big)^m}\\
&\le \sum_{2p-|J|\le -2l}\sum_{d\le q}k^p|x-y|^{|J|}e^{4dtk}|a_{I,J,p,d}(x,y,tk)|\\
&\qquad\cdot\frac{k^m e^{-\frac{k|x-y|^2}{4\tanh(tk)}}e^{kt(m-4q)}}{\big(4\pi\sinh(tk)\big)^m}\\
&\le c\sideset{}{'}\sum_{2p-|J|\le -2l}k^{m+p}e^{-\frac{k|x-y|^2}{4}}|x-y|^{|J|},
\end{aligned}$$

where the sum is over the finite number of indices for which $|a_{I,J,p,d}(x,y,tk)| \neq 0$.

We then have

$$\begin{aligned}
&\left\| P_{2q}\psi\mathcal{U}u_l \frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}} \right\|_{HS}^2 \\
&\leq \frac{1}{Vol(B_\epsilon(p))} \int_{M\times B_\epsilon(p)} |P_{2q}\mathcal{U}u_l(x,y)|^2 dxdy \\
&\leq \frac{c}{Vol(B_\epsilon(p))} \sum_{2p-|J|\leq -2l}{}' k^{2m+2p} \int_{\mathbb{R}^{2m}\times B_\epsilon(p)} e^{-\frac{k|x-y|^2}{2}} |x-y|^{2|J|} dxdy \\
&\leq c \sum_{2p-|J|\leq -2l}{}' k^{2m+2p} \int_{\mathbb{R}^{2m}} e^{-\frac{k|x|^2}{2}} |x|^{2|J|} dx \\
&\leq ck^{m-2l}.
\end{aligned}$$

Thus for both $t = \frac{k^{-1/2}}{2}$ or $t = \frac{k^{-1/2}}{4}$, we have

$$\left\| Q_t \frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}} \right\|_{HS} \leq ck^{\frac{m}{2}}. \tag{6.16}$$

Similarly modifying the computations of the proof of Prop. 3.9, we find that

$$\left\| \epsilon_s \frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}} \right\|_{HS} \leq \begin{cases} ck^{\frac{m}{2}-2} & \text{if } sk \geq 1 \\ cs^{N-\frac{m}{2}} & \text{if } sk \leq 1. \end{cases}$$

Since $\|e^{-(t-s)D^2_{A(k)}}\|_{\sup} \leq 1$ for all $s \leq t$, we have that

$$\begin{aligned}
&\left\| \left(Q_t - e^{-tD^2_{A(k)}}\right) \frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}} \right\|_{HS} \\
&= \left\| \int_0^t e^{-(t-s)D^2_{A(k)}} \epsilon_s \frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}} ds \right\|_{HS} \\
&\leq \int_0^t \left\| \epsilon_s \frac{\chi_{B_\epsilon(p)}}{\sqrt{Vol(B_\epsilon(p))}} \right\|_{HS} ds \\
&\leq c\left(k^{\frac{m}{2}-N-1} + k^{\frac{m}{2}-N}(t-k^{-1})\right) \\
&\leq ck^{\frac{m}{2}}.
\end{aligned} \tag{6.17}$$

Setting $T = k^{-1/2}$ and combining (6.13), (6.14), (6.15), (6.16), and (6.17), we get (6.11), as desired.

We now proceed to prove Equation (6.12). Observe that the derivation of the pointwise estimate of $|b(T,x,y)|$ can be applied to obtain an estimate for $|k_T(x,y)|$. The only difference is that we do not have an exponentially decaying bound for $\|e^{-\frac{T}{4}D^2}\|_{\text{sup}}$, merely the bound $\|e^{-\frac{T}{4}D^2}\|_{\text{sup}} \leq 1$. Hence we only obtain the bound $|k_T(x,y)| \leq c\frac{k^m e^{Tkm}}{\sinh(Tk)^m}$. (In the $|b(T,x,y)|$ estimate we absorbed the $\frac{e^{Tkm}}{\sinh(Tkm)}$ factor into our constant since we were considering the case Tk large.)

Recall that $q_t(x,y) - k_t(x,y) = \int_0^t f_s(t,x,y)ds$, where

$$f_s(t,x,y) = \int_M k_{t-s}(x,z)\epsilon_s(z,y)dz.$$

By a slight modification of the proof of Prop 3.9, we find

$$|\epsilon_s(x,y)| \leq \begin{cases} cs^{N-m}, & \text{when } s \leq \frac{1}{k} \\ ck^{m-N}, & \text{when } s \geq \frac{1}{k}. \end{cases}$$

Hence

$$\begin{aligned} |q_{k^{-1/2}}(x,y) - k_{k^{-1/2}}(x,y)| &= \left| \int_0^{k^{-1/2}} f_s(k^{-1/2},x,y)ds \right| \\ &\leq \int_0^{k^{-1/2}} c\frac{k^m e^{skm}}{\sinh(sk)^m} \sup_{z\in M} |\epsilon_s(z,y)|ds \\ &\leq \int_0^{k^{-1}} c\frac{k^m}{\sinh(sk)^m} s^{N-m}ds + ck^{2m-N-\frac{1}{2}} \\ &\leq ck^{2m-N-1} + ck^{2m-N-\frac{1}{2}} \\ &\leq ck^{2m-N-\frac{1}{2}} \end{aligned}$$

for N big enough. The proof of estimate (6.12) is now complete.

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