

The moduli space of maps with crosscaps: Fredholm theory and orientability

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Just as a symmetric surface with separating fixed locus halves into two oriented bordered surfaces, an arbitrary symmetric surface halves into two oriented symmetric half-surfaces, i.e. surfaces with crosscaps. Motivated in part by the string theory view of real Gromov-Witten invariants, we introduce moduli spaces of maps from surfaces with crosscaps, develop the relevant Fredholm theory, and resolve the orientability problem in this setting. In particular, we give an explicit formula for the holonomy of the orientation bundle of a family of real Cauchy-Riemann operators over Riemann surfaces with crosscaps. Special cases of our formulas are closely related to the orientability question for the space of real maps from symmetric Riemann surfaces to an almost complex manifold with an anti-complex involution and in fact resolve this question in genus 0. In particular, we show that the moduli space of real J -holomorphic maps from the sphere with a fixed-point free involution to a simply connected almost complex manifold with an even canonical class is orientable. In a sequel, we use the results of this paper to obtain a similar orientability statement for genus 1 real maps.

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1. Introduction

The theory of J -holomorphic maps plays a prominent role in symplectic topology, algebraic geometry, and string theory. The foundational work of [7, 12, 16, 20, 26] has established the theory of (closed) Gromov-Witten invariants, i.e. counts of J -holomorphic maps from closed Riemann surfaces to symplectic manifolds. In contrast, the theory of open and real Gromov-Witten invariants, i.e. counts of J -holomorphic maps from bordered Riemann surfaces with boundary mapping to a Lagrangian submanifold and of J -holomorphic maps from symmetric Riemann surfaces commuting with the involutions on the domain and the target, has been under development over the past 10–15 years and still is today.

The two main obstacles to defining the open invariants are the potential non-orientability of the moduli space and the existence of real codimension-one boundary strata. The orientability problem in open Gromov-Witten theory is addressed by the first author in [9]. Some approaches [5, 17, 25] to dealing with the codimension one boundary have raised the issue of orientability in real Gromov-Witten theory. Symmetric Riemann surfaces, however, have convoluted degenerations, making the orientability of their moduli spaces difficult to study. Physical considerations [1, 27, 30] suggest that oriented surfaces with crosscaps provide a suitable replacement for symmetric Riemann surfaces in real Gromov-Witten theory. In this paper, we introduce moduli spaces of J -holomorphic maps from oriented surfaces with crosscaps, develop the necessary Fredholm theory, and study the orientability of these moduli spaces. In particular, we combine the principles of [9] with

equivariant cohomology and give an explicit criterion specifying whether the determinant line bundle of a loop of real Cauchy-Riemann operators over Riemann surfaces with crosscaps is trivial. As explained after Corollary 1.7 and in [11], this last issue is related to the orientability problem in real Gromov-Witten theory via the doubling constructions of (1.6) and Section 3. In a future paper, we will study compactifications of the moduli spaces of maps with crosscaps and use them to define real Gromov-Witten invariants in the style of [30].

A symmetric surface $(\hat{\Sigma}, \sigma)$ consists of a closed connected oriented smooth surface $\hat{\Sigma}$ (manifold of real dimension 2) and an orientation-reversing involution $\sigma: \hat{\Sigma} \rightarrow \hat{\Sigma}$. Every anti-holomorphic involution σ on $\hat{\Sigma} = \mathbb{P}^1$ such that \mathbb{P}^1/σ is not orientable is conjugate to

$$(1.1) \quad \eta: \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \quad [u, v] \longrightarrow [-\bar{v}, \bar{u}].$$

An approach to orienting indices of real Cauchy-Riemann operators on real bundle pairs over (\mathbb{P}^1, η) is introduced in [5, Section 2.1]. We reinterpret this construction in terms of real Cauchy-Riemann operators on Riemann surfaces with orientation-preserving involutions on the boundary components in a way that reduces to the orienting construction of [8, Proposition 8.1.4] whenever the boundary involutions are trivial. This allows us to extend the principles used in [9], which treats the case with trivial boundary involutions, to the case with any number of crosscaps, corresponding to the nontrivial boundary involutions. Theorem 1.1 below specifies whether the index bundle of a loop of real Cauchy-Riemann operators over Riemann surfaces with orientation-preserving involutions on the boundary components is trivial. As a corollary, we show that the moduli space of real J -holomorphic maps from (\mathbb{P}^1, η) to a simply connected almost complex manifold with an even canonical bundle is orientable; see Corollary 1.8. As another corollary, we conclude that the local system of orientations on the moduli space of J -holomorphic maps from bordered Riemann surfaces that commute with the involutions on the boundary and on the target is isomorphic to the pull-back of a local system defined on a product of the equivariant free loop space of the target, of the fixed point locus of the involution on the target, and of its free loop space; see Corollary 6.2. Along the way, we establish the necessary Fredholm theory for bordered surfaces with crosscaps, discuss topological issues that crosscaps introduce, and include examples illustrating a number of subtle points. In [11], we built on the results of this paper to orient moduli spaces of real genus 1 maps; this is a step in our project to construct real Gromov-Witten invariants in positive genera.

An involution on a topological space (resp. smooth manifold) M is a homeomorphism (resp. diffeomorphism) $\phi: M \rightarrow M$ such that $\phi \circ \phi = \text{id}_M$; in particular, the identity map on M is an involution. Let

$$M^\phi = \{x \in M: \phi(x) = x\}$$

denote the fixed locus. An involution ϕ determines an action of \mathbb{Z}_2 on M ; we denote by $H_\phi^*(M)$, $H_*^\phi(M)$, and $H_*^\phi(M; \mathbb{Z})$ the corresponding \mathbb{Z}_2 -equivariant cohomology and homology of M with \mathbb{Z}_2 -coefficients and the \mathbb{Z}_2 -equivariant homology of M with \mathbb{Z} -coefficients, respectively; see Section 2. If the fixed-point locus of ϕ is empty, there are canonical isomorphisms

$$\begin{aligned} H_\phi^*(M) &\approx H^*(M/\mathbb{Z}_2), & H_*^\phi(M) &\approx H_*(M/\mathbb{Z}_2), \\ H_*^\phi(M; \mathbb{Z}) &\approx H_*(M/\mathbb{Z}_2; \mathbb{Z}). \end{aligned}$$

If in addition M is a compact manifold (and thus so is M/\mathbb{Z}_2), we denote by $[M]^\phi \in H_*^\phi(M)$ the fundamental homology class of M/\mathbb{Z}_2 with \mathbb{Z}_2 -coefficients. A conjugation on a complex vector bundle $V \rightarrow M$ lifting an involution ϕ is a vector bundle homomorphism $\tilde{\phi}: V \rightarrow V$ covering ϕ (or equivalently a vector bundle homomorphism $\tilde{\phi}: V \rightarrow \phi^*V$ covering id_M) such that the restriction of $\tilde{\phi}$ to each fiber is anti-complex linear and $\tilde{\phi} \circ \phi = \text{id}_V$. We will call the conjugation

$$\tilde{\phi}_n: M \times \mathbb{C}^n \rightarrow M \times \mathbb{C}^n, \quad (x, v) \rightarrow (\phi(x), \bar{v}) \quad \forall (x, v) \in M \times \mathbb{C}^n,$$

the trivial lift of ϕ . For any conjugation $\tilde{\phi}$ in $V \rightarrow M$ lifting ϕ , $V^{\tilde{\phi}} \rightarrow M^\phi$ is a maximal totally real subbundle of $V|_{M^\phi}$. We denote by

$$\Lambda_{\mathbb{C}}^{\text{top}}(V, \tilde{\phi}) = (\Lambda_{\mathbb{C}}^{\text{top}}V, \Lambda_{\mathbb{C}}^{\text{top}}\tilde{\phi})$$

the top exterior power of V over \mathbb{C} with the induced conjugation and by

$$w_i^{\tilde{\phi}}(V) \in H_\phi^i(M)$$

the i -th \mathbb{Z}_2 -equivariant Stiefel-Whitney class of V . Moreover, if M is a manifold, possibly with boundary, or a (possibly nodal) surface, and ϕ is an involution on a submanifold $M' \subseteq M$, a real bundle pair $(V, \tilde{\phi}) \rightarrow (M, \phi)$ consists of a complex vector bundle $V \rightarrow M$ and a conjugation $\tilde{\phi}$ on $V|_{M'}$ lifting ϕ .

A boundary involution on a surface Σ with boundary $\partial\Sigma$ is an orientation-preserving involution c preserving each component of $\partial\Sigma$. The restriction of

such an involution to a boundary component is either the identity or given by

$$(1.2) \quad \alpha: S^1 \longrightarrow S^1, \quad z \longrightarrow -z \quad \forall z \in S^1 \subset \mathbb{C},$$

for a suitable identification $(\partial\Sigma)_i \approx S^1$; the latter type of boundary structure is called **crosscap** in the string theory literature. We define

$$c_i = c|_{(\partial\Sigma)_i}, \quad |c_i| = \begin{cases} 0, & \text{if } c_i = \text{id}; \\ 1, & \text{otherwise;} \end{cases}$$

$$|c|_k = |\{(\partial\Sigma)_i \subset \Sigma : |c_i| = k\}| \quad k = 0, 1.$$

Thus, $|c|_0$ is the number of standard boundary components of $(\Sigma, \partial\Sigma)$ and $|c|_1$ is the number of crosscaps.

An **oriented symmetric half-surface** (or simply **oriented sh-surface**) is a pair (Σ, c) consisting of an oriented bordered smooth surface Σ and a boundary involution c on Σ . Such a pair doubles to a symmetric surface $(\hat{\Sigma}, \hat{c})$; see (1.6). We denote by \mathcal{J}_Σ the space of all complex structures on Σ compatible with the orientation and by \mathcal{J}_c the subspace of \mathcal{J}_Σ consisting of the complex structures j so that c is real-analytic with respect to j ; see Section 3. In the standard case of open Gromov-Witten theory, $c = \text{id}_{\partial\Sigma}$ and $\mathcal{J}_c = \mathcal{J}_\Sigma$.

A **real Cauchy-Riemann operator on a real bundle pair** $(V, \tilde{c}) \longrightarrow (\Sigma, c)$, where (Σ, c) is an oriented sh-surface, is a linear map of the form

$$(1.3) \quad D = \bar{\partial} + A: \Gamma(\Sigma; V)^{\tilde{c}} \equiv \{\xi \in \Gamma(\Sigma; V) : \xi \circ c = \tilde{c} \circ \xi|_{\partial\Sigma}\} \\ \longrightarrow \Gamma_j^{0,1}(\Sigma; V) \equiv \Gamma(\Sigma; (T^*\Sigma, j)^{0,1} \otimes_{\mathbb{C}} V),$$

where $\bar{\partial}$ is the holomorphic $\bar{\partial}$ -operator for some $j \in \mathcal{J}_\Sigma$ and a holomorphic structure in V and

$$A \in \Gamma(\Sigma; \text{Hom}_{\mathbb{R}}(V, (T^*\Sigma, j)^{0,1} \otimes_{\mathbb{C}} V))$$

is a zeroth-order deformation term. A real Cauchy-Riemann operator on a real bundle pair need not be Fredholm in the appropriate completions; see Remark 3.7. However, it is Fredholm if the boundary involution c is real-analytic with respect to j ; see Proposition 3.6.

Let $\mathbb{I} = [0, 1]$. For an orientation-preserving diffeomorphism $\psi: \Sigma \longrightarrow \Sigma$, let

$$(M_\psi, \partial M_\psi) = (\mathbb{I} \times (\Sigma, \partial\Sigma_\psi)) / (1, x) \sim (0, \psi(x))$$

be the mapping torus of ψ and $\pi: M_\psi \longrightarrow S^1$ be the projection map. For each $t \in S^1$, let $\Sigma_t = \pi^{-1}(t)$ be the fiber over t . An involution c on $\partial\Sigma$ commuting

with ψ induces a fiber-preserving involution on ∂M_ψ , which we continue to denote by c . In such a case, a continuous family of real Cauchy-Riemann operators on a real bundle pair (V, \tilde{c}) over (M_ψ, c) is a collection of real Cauchy-Riemann operators

$$D_t : \Gamma(\Sigma_t; V|_{\Sigma_t})^{\tilde{c}} \longrightarrow \Gamma_{j_t}^{0,1}(\Sigma_t; V|_{\Sigma_t})$$

which varies continuously with $t \in S^1$. If $j_t \in \mathcal{J}_c$, so that D_t is Fredholm, we denote by $\det D \longrightarrow S^1$ the determinant line bundle corresponding to this family; see [21, Section A.2] and [31] for a construction.

Theorem 1.1. *Let (Σ, c) be an oriented sh-surface, $\psi : \Sigma \longrightarrow \Sigma$ be a diffeomorphism preserving the orientation and each boundary component and commuting with c when restricted to $\partial\Sigma$, and (V, \tilde{c}) be a real bundle pair over (M_ψ, c) . For each boundary component $(\partial\Sigma)_i$ of Σ , choose a section α_i of*

$$(\partial M_\psi)_i \equiv M_\psi|_{(\partial\Sigma)_i} \longrightarrow S^1.$$

If D is a continuous family of real Cauchy-Riemann operators on (V, \tilde{c}) such that each D_t is compatible with some $j_t \in \mathcal{J}_c$, then

$$\begin{aligned} \langle w_1(\det D), S^1 \rangle &= \sum_{|c_i|=0} \left(\langle w_1(V^{\tilde{c}}), (\partial\Sigma)_i \rangle + 1 \right) \langle w_1(V^{\tilde{c}}), [\alpha_i] \rangle \\ (1.4) \qquad &+ \langle w_2(V^{\tilde{c}}), (\partial M_\psi)_i \rangle \\ &+ \sum_{|c_i|=1} \langle w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{c}}(\Lambda_{\mathbb{C}}^{\text{top}} V), [(\partial M_\psi)_i]^c \rangle, \end{aligned}$$

where the sums are taken over the connected components $(\partial M_\psi)_i$ of ∂M_ψ .

This theorem extends [9, Theorem 1.1] to bordered Riemann surfaces with crosscaps and is the key step to the remaining results in this paper, analogously to [9, Theorem 1.1] being the key step to the remaining results in [9]. By Lemma 2.6, $w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{c}}(\Lambda_{\mathbb{C}}^{\text{top}} V)$ in (1.4) can be replaced by the simpler looking $w_2^{\tilde{c}}(V)$. However, as [5, Section 2.1] suggests, $\Lambda_{\mathbb{C}}^{\text{top}}(V, \tilde{c})$ is in fact the simpler object to work with. By Proposition 2.1,

$$(1.5) \quad w_2^{\Lambda_{\mathbb{C}}^{\text{top}}(\tilde{\phi}_1 \oplus \tilde{\phi}_2)}(\Lambda_{\mathbb{C}}^{\text{top}}(V_1 \oplus V_2)) = w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{\phi}_1}(\Lambda_{\mathbb{C}}^{\text{top}} V_1) + w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{\phi}_2}(\Lambda_{\mathbb{C}}^{\text{top}} V_2)$$

for all real bundle pairs $(V_1, \tilde{\phi}_1), (V_2, \tilde{\phi}_2) \rightarrow (M, \phi)$, but in general

$$w_2^{\tilde{\phi}_1 \oplus \tilde{\phi}_2}(V_1 \oplus V_2) \neq w_2^{\tilde{\phi}_1}(V_1) + w_2^{\tilde{\phi}_2}(V_2).$$

The last equality fails even for the trivial rank 1 real bundle pairs $(V_i, \tilde{\phi}_i) \rightarrow (M, \phi)$ over the Klein bottle with a natural fixed-point-free involution. We show in [11] that the equivariant w_2 of $\Lambda_{\mathbb{C}}^{\text{top}}(V, \tilde{\phi})$, and not of $(V, \tilde{\phi})$ itself, enters into orientability considerations in real Gromov-Witten theory (i.e. when interchanges of halves are considered).

We prove Theorem 1.1 in Section 4 by separating off the contributions of the individual crosscaps, following one of the principles in the proof of [9, Theorem 1.1]; the contribution from the remainder of Σ is then given by [9, Theorem 1.1]. We determine the contributions from the crosscaps in Section 4 by combining some of the ideas in the proof of [9, Theorem 1.1] with equivariant cohomology.

Remark 1.2. Families of real Cauchy-Riemann operators often arise by pulling back data from a target manifold by smooth maps as follows. Suppose (X, J) is an almost complex manifold with an anti-complex involution $\phi: X \rightarrow X$ and $(V, \tilde{\phi}) \rightarrow (X, \phi)$ is a real bundle pair. Let ∇ be a connection in V and

$$A \in \Gamma(X; \text{Hom}_{\mathbb{R}}(V, (T^*X, J)^{0,1} \otimes_{\mathbb{C}} V)).$$

For any map $u: \Sigma \rightarrow X$ and $j \in \mathcal{J}_{\Sigma}$, let ∇^u denote the induced connection in u^*V and

$$A_{j;u} = A \circ \partial_j u \in \Gamma(\Sigma; \text{Hom}_{\mathbb{R}}(u^*V, (T^*\Sigma, j)^{0,1} \otimes_{\mathbb{C}} u^*V)).$$

If c is a boundary involution on Σ and $u \circ c = \phi \circ u$ on $\partial\Sigma$, the homomorphisms

$$\bar{\partial}_u^{\nabla} = \frac{1}{2}(\nabla^u + i \circ \nabla^u \circ j), \quad D_u \equiv \bar{\partial}_u^{\nabla} + A_{j;u}: \Gamma(\Sigma; u^*V)^{u^*\tilde{\phi}} \rightarrow \Gamma_j^{0,1}(\Sigma; u^*V)$$

are real Cauchy-Riemann operators on $(u^*V, u^*\tilde{\phi}) \rightarrow (\Sigma, c)$ that form families of real Cauchy-Riemann operators over families of maps.

The double of an oriented sh-surface (Σ, c) is the closed oriented topological surface

$$(1.6) \quad \hat{\Sigma} \equiv (\Sigma^+ \sqcup \Sigma^-) / \sim \equiv \{+, -\} \times \Sigma / \sim, \quad (+, z) \sim (-, c(z)) \quad \forall z \in \partial\Sigma.$$

The involution c on $\partial\Sigma$ naturally extends to the involution

$$\hat{c}: \hat{\Sigma} \longrightarrow \hat{\Sigma}, \quad [\pm, z] \longrightarrow [\mp, z] \quad \forall z \in \Sigma.$$

Similarly, if (X, ϕ) is a manifold with an involution, a map $u: \Sigma \longrightarrow X$ such that $u \circ c = \phi \circ u$ on $\partial\Sigma$ **doubles** to a map $\hat{u}: \hat{\Sigma} \longrightarrow X$ such that $\hat{u} \circ \hat{c} = \phi \circ \hat{u}$. A complex structure j on $\Sigma^+ = \Sigma$ extends to a complex structure \hat{j} on $\hat{\Sigma}$ so that $\hat{c}^* \hat{j} = -\hat{j}$ if and only if $c: \partial\Sigma \longrightarrow \partial\Sigma$ is real-analytic with respect to j ; see Corollary 3.3. If c is real-analytic with respect to j , J is an almost complex structure on X such that $\phi^* J = -J$, and u as above is (J, j) -holomorphic, then \hat{u} is (J, \hat{j}) -holomorphic.

Remark 1.3. We note that (1.6) does not specify a smooth structure on $\hat{\Sigma}$ across the boundary of Σ . Whenever c is real-analytic with respect to j , there is a natural doubled smooth complex structure \hat{j} so that the image of $\partial\Sigma$ in Σ is a real-analytic curve; see (1) in the proof of Corollary 3.3. However, there can be other smooth and complex structures on $\hat{\Sigma}$ that are compatible with \hat{c} and restrict to j on Σ ; see Remark 3.4.

Let (Σ, c) be a genus g oriented sh-surface with orderings

$$(\partial\Sigma)_{1, \dots, (\partial\Sigma)|_{c|_0}} \quad \text{and} \quad (\partial\Sigma)|_{c|_0+1}, \dots, (\partial\Sigma)|_{c|_0+|c|_1}$$

of the boundary components with $|c_i|=0$ and with $|c_i|=1$, respectively. Denote by \mathcal{D}_c the group of diffeomorphisms of Σ preserving the orientation and each boundary component and commuting with the involution c on $\partial\Sigma$. If (X, ϕ) is a smooth manifold with an involution and

(1.7)

$$\mathbf{b} = (B, b_1, \dots, b_{|c|_0+|c|_1}) \in H_2(X; \mathbb{Z}) \oplus H_1(X^\phi; \mathbb{Z})^{\oplus |c|_0} \oplus H_1^\phi(X; \mathbb{Z})^{\oplus |c|_1},$$

let $\mathfrak{B}_g(X, \mathbf{b})^{\phi, c}$ denote the space of maps $u: \Sigma \longrightarrow X$ such that

- $u \circ c = \phi \circ u$ on $\partial\Sigma$,
- $\hat{u}_*[\hat{\Sigma}] = B$, $u_*[(\partial\Sigma)_i] = b_i$ for $i = 1, \dots, |c|_0$, and
- $[u|_{(\partial\Sigma)_i}]^{c_i} = b_i$ for $i = |c|_0+1, \dots, |c|_0+|c|_1$, where $[u|_{(\partial\Sigma)_i}]^{c_i}$ is the equivariant pushforward of $[(\partial\Sigma)_i]^{c_i}$ by $u|_{(\partial\Sigma)_i}$.

We define

$$\mathcal{H}_g(X, \mathbf{b})^{\phi, c} = (\mathfrak{B}_g(X, \mathbf{b})^{\phi, c} \times \mathcal{J}_c) / \mathcal{D}_c.$$

By Lemma 3.1, the action of \mathcal{D}_c on \mathcal{J}_Σ given by $h \cdot j = h^* j$ preserves \mathcal{J}_c ; so, the above quotient is well-defined.

Remark 1.4. As discussed in detail at the beginning of Section 2.3, $H_1^\phi(X; \mathbb{Z})$ provides a finer invariant than $H_1(X; \mathbb{Z})$.

Remark 1.5. For simplicity, we will assume that the action of \mathcal{D}_c has no fixed points on the relevant subspaces of $\mathfrak{B}_g(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c$. This happens for example if sufficiently many marked points are added to Σ . In applications to more general cases, this issue can be avoided by working with Prym structures on Riemann surfaces; see [18].

The determinant line bundle of a family of real Cauchy-Riemann operators $D_{(V, \tilde{\phi})}$ on

$$\mathfrak{B}_g(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c$$

induced by a real bundle pair as in Remark 1.2 descends to a line bundle over $\mathcal{H}_g(X, \mathbf{b})^{\phi,c}$, which we still denote by $\det D_{(V, \tilde{\phi})}$. As a direct corollary of Theorem 1.1, we obtain the following result on its orientability.

Corollary 1.6. *Suppose γ is a loop in $\mathcal{H}_g(X, \mathbf{b})^{\phi,c}$ and $\tilde{\gamma}$ is a path in $\mathfrak{B}_g(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c$ lifting γ such that $\tilde{\gamma}_1 = \psi \cdot \tilde{\gamma}_0$ for some $\psi \in \mathcal{D}_c$ with $\psi|_{\partial\Sigma} = \text{id}$. For each boundary component $(\partial\Sigma)_i$ of Σ , denote by*

$$\alpha_i: S^1 \longrightarrow X \quad \text{and} \quad \beta_i: S^1 \times (\partial\Sigma)_i \longrightarrow X$$

the paths traced by a fixed point on $(\partial\Sigma)_i$ and by the entire boundary component $(\partial\Sigma)_i$ along $\tilde{\gamma}$. Then,

$$\begin{aligned} \langle w_1(\det D_{(V, \tilde{\phi})}), \gamma \rangle &= \sum_{i=1}^{|\mathcal{C}_0|} \left(\langle w_1(V^{\tilde{\phi}}), b_i \rangle + 1 \right) \langle w_1(V^{\tilde{\phi}}), [\alpha_i] \rangle \\ (1.8) \quad &= \sum_{i=1}^{|\mathcal{C}_0|} \left(+ \langle w_2(V^{\tilde{\phi}}), [\beta_i] \rangle \right) + \sum_{i=|\mathcal{C}_0|+1}^{|\mathcal{C}_0|+|\mathcal{C}_1|} \langle w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{\phi}}(\Lambda_{\mathbb{C}}^{\text{top}} V), [\beta_i]^{\text{id} \times c_i} \rangle, \end{aligned}$$

where $[\beta_i]^{\text{id} \times c_i} \in H_2^\phi(X)$ is the equivariant push-forward of $[S^1 \times (\partial\Sigma)_i]^{\text{id} \times c_i}$ by β_i .

The first assumption in this corollary imposes no restriction on γ ; see Lemma 4.5. Lemma 2.7 simplifies the computation of the second line in (1.8) in some cases. In particular, combining Corollary 1.6 and Lemma 2.7, we obtain Corollary 1.7 below, which concerns the orientability problem for families of real Cauchy-Riemann operators over surfaces without standard boundary components (i.e. with crosscaps only).

If $(V, \tilde{\phi}) \rightarrow (X, \phi)$ is a rank 1 real bundle pair, a real square root for $(V, \tilde{\phi})$ is a rank 1 real bundle pair $(L, \tilde{\phi}') \rightarrow (X, \phi)$ and an isomorphism

$$(V, \tilde{\phi}) \approx (L, \tilde{\phi}')^{\otimes 2}$$

of real bundle pairs. As shown in [5, Section 2.1], real square roots canonically induce orientations on the determinant lines of real Cauchy-Riemann operators over disks with crosscaps. Thus, Corollary 1.7 explains and extends this key observation in [5].

Corollary 1.7. *Let (X, ϕ) be a manifold with an involution, $(V, \tilde{\phi}) \rightarrow (X, \phi)$ be a real bundle pair, and (Σ, c) be an oriented sh-surface with $|c|_0 = 0$. If $\pi_1(X) = 0$ and $c_1(V)$ is an even class or $\Lambda_{\mathbb{C}}^{\text{top}}(V, \tilde{\phi})$ admits a real square root, then the determinant line bundle*

$$\det D_{V, \tilde{\phi}} \rightarrow \mathcal{H}_g(X, \mathbf{b})^{\phi, c}$$

is orientable.

By Corollary 2.4, the two sets of completely different conditions in Corollary 1.7 are in fact two different specializations of the natural vanishing condition on the w_2 -terms in (1.4) and (1.8) that first came up in the case $|c|_1 = 0$ in [9]: $w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{\phi}}(\Lambda_{\mathbb{C}}^{\text{top}} V)$ is a square of a class in $H^1_{\phi}(X)$. Since there are no non-trivial square top cohomology classes on a closed orientable surface, each summand on the second lines in (1.4) and (1.8) vanishes if $w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{\phi}}(\Lambda_{\mathbb{C}}^{\text{top}} V)$ is a square class. Thus, the two sets of conditions in Corollaries 1.7 and 1.8 can be replaced by the single requirement that the equivariant w_2 is a square class.

Let σ be an orientation-reversing involution on a compact closed oriented genus g surface $\hat{\Sigma}$. Denote by \mathcal{J}_{σ} and \mathcal{D}_{σ} the space of complex structures \hat{j} on $\hat{\Sigma}$ such that $\sigma^* \hat{j} = -\hat{j}$ and the group of orientation-preserving diffeomorphisms ψ of $\hat{\Sigma}$ such that $\sigma \circ \psi = \psi \circ \sigma$, respectively. Let (X, ϕ) be a smooth manifold with an involution and J be an almost complex structure on X such that $\phi^* J = -J$. Of a particular interest in real Gromov-Witten theory is the problem of the orientability of the moduli spaces

$$\mathfrak{M}_g(X, J, B)^{\phi, \sigma} \\ \equiv \{(\hat{u}, \hat{j}) \in C^{\infty}(\hat{\Sigma}, X) \times \mathcal{J}_{\sigma} : \hat{u}_*[\hat{\Sigma}] = B, \hat{u} \circ \sigma = \phi \circ \hat{u}, \bar{\partial}_{J, \hat{j}} \hat{u} = 0\} / \mathcal{D}_{\sigma},$$

where

$$(1.9) \quad \bar{\partial}_{J,\hat{j}}\hat{u} = \frac{1}{2}(d\hat{u} + J \circ d\hat{u} \circ \hat{j}).$$

This problem is closely related to the problem of orienting the index bundle for a family of real Cauchy-Riemann operators over $\mathfrak{M}_g(X, J, B)^{\phi,\sigma}$ induced by the bundle $(TX, d\phi) \rightarrow (X, \phi)$ as in Remark 1.2. Since $\hat{\Sigma}$ can be decomposed into two conjugate oriented sh-surfaces, (Σ, c) and $(\bar{\Sigma}, c)$, the latter problem is in turn closely related to the orientability of $\det D_{(TX, d\phi)}$ over the space of holomorphic maps from Σ to X that commute with the involutions on the boundary.

Let $\eta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be as in (1.1). As shown in [5, Section 2.3], the orientability problem for $\mathfrak{M}_0(X, J, B)^{\phi,\eta}$ is precisely equivalent to the orientability problem for $D_{(TX, d\phi)}$ over

$$\begin{aligned} \mathcal{P}(X, J, B) &\equiv \{ \hat{u} \in C^\infty(\mathbb{P}^1, X) : \hat{u}_*[\mathbb{P}^1] = B, \hat{u} \circ \eta = \phi \circ \hat{u}, \bar{\partial}_{J,\hat{j}}\hat{u} = 0 \} \\ &\equiv \{ u \in C^\infty(D^2, X) : \hat{u}_*[\mathbb{P}^1] = B, u \circ \mathbf{a}|_{S^1} = \phi \circ u|_{S^1}, \bar{\partial}_{J,j}u = 0 \}, \end{aligned}$$

where \hat{j} and j are the standard complex structures on \mathbb{P}^1 and the unit disk $D^2 \subset \mathbb{P}^1$, respectively. The reason is that

$$\mathfrak{M}_0(X, J, B)^{\phi,\eta} = \mathcal{P}(X, J, B) / \text{Aut}(\mathbb{P}^1, \eta) \quad \text{and} \quad \text{Aut}(\mathbb{P}^1, \eta) \approx \mathbb{R}\mathbb{P}^3.$$

Thus, the orientability of $\mathfrak{M}_0(X, J, B)^{\phi,\eta}$ along a loop γ in this moduli space is described by the last term in (1.8) with $(V, \tilde{\phi}) = (TX, d\phi)$. This allows us to immediately deduce the following conclusion about the orientability of $\mathfrak{M}_0(X, J, B)^{\phi,\eta}$ from Corollary 1.7.

Corollary 1.8. *Let (X, ϕ, J) be a manifold with an involution and an almost complex structure J such that $\phi^*J = -J$. If $\pi_1(X) = 0$ and $c_1(TX, J)$ is an even class or $\Lambda_{\mathbb{C}}^{\text{top}}(TX, \tilde{\phi})$ admits a real square root, the moduli space $\mathfrak{M}_0(X, J, B)^{\phi,\eta}$ is orientable for every $B \in H_2(X; \mathbb{Z})$.*

The orientability of $\mathfrak{M}_0(X, J, B)^{\phi,\eta}$ under the last assumption is shown directly in [5, Section 2.1]. This in particular implies that the moduli spaces $\mathfrak{M}_0(\mathbb{P}^{4m-1}, B)^{\phi,\eta}$ are orientable. All moduli spaces $\mathfrak{M}_0(\mathbb{P}^{2m-1}, B)^{\phi,\eta}$ with the two standard involutions ϕ are shown to be orientable in [5, Section A.1], as implied by the first case of our assumptions. By Example 2.10 below, the

moduli spaces $\mathfrak{M}_0(\mathbb{P}^n, B)^{\tau_n, \eta}$, where

$$(1.10) \quad \tau_n: \mathbb{P}^n \longrightarrow \mathbb{P}^n, \quad [Z_0, Z_1, \dots, Z_n] \longrightarrow [\bar{Z}_0, \bar{Z}_1, \dots, \bar{Z}_n],$$

is not orientable if n and B are even, i.e. the condition $\pi_1(X) = 0$ alone does not suffice for the orientability of $\mathfrak{M}_0(X, J, B)^{\phi, \eta}$. By Example 2.9, which builds on [5, Example 2.5], no divisibility condition on $c_1(TX)$ can suffice by itself either.

This paper is organized as follows. Section 2 reviews \mathbb{Z}_2 -equivariant cohomology and homology and obtains a number of related results that are used in the proof and applications of Theorem 1.1. The examples of Section 2.4 indicate the delicate nature of the equivariant w_2 -terms in (1.8) and show that Corollary 1.8 is sharp in a sense. Section 3 describes doubling constructions for oriented surfaces with boundary involutions, develops the necessary Fredholm theory, and obtains a Riemann-Roch theorem for Cauchy-Riemann operators over such surfaces. The somewhat technical Sections 2 and 3 enable us to extend the principles from [9] to surfaces with crosscaps. The proof of Theorem 1.1 is the subject of Section 4. Section 5 reinterprets Corollary 1.6 in terms of local systems. In Section 6, we define moduli spaces of J -holomorphic maps from oriented surfaces with crosscaps and apply the reinterpretation of Section 5 to describe their local systems of orientations. In Appendix A, we show that the notion of almost complex structure on a bordered Riemann surface used in this paper is equivalent to the notion of analytic structure used in [2, 15, 17].

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2. Equivariant cohomology

We begin this section by recalling basic notions in equivariant cohomology, in the case the group is \mathbb{Z}_2 , and establishing some key properties of the equivariant w_2 of real vector bundle pairs; see Proposition 2.1 and Corollary 2.4. We then make a key observation, Lemma 2.5, concerning the \mathbb{Z}_2 -equivariant second Stiefel-Whitney class of real bundle pairs over the torus $(S^1 \times S^1, \text{id} \times \mathfrak{a})$; it is used in the proof of Theorem 1.1. Lemma 2.7 simplifies the computation of the second line in (1.8) in some cases and immediately

leads to Corollary 1.7 from Corollary 1.6. We conclude with three examples intended to give the flavor of the equivariant w_2 -term which plays a central role in the orientability problems studied in this paper.

2.1. Basic notions

The group \mathbb{Z}_2 acts freely on the contractible space $\mathbb{E}\mathbb{Z}_2 \equiv S^\infty$ with the quotient $\mathbb{B}\mathbb{Z}_2 \equiv \mathbb{R}\mathbb{P}^\infty$. An involution $\phi: M \rightarrow M$ corresponds to a \mathbb{Z}_2 -action on M . We denote by

$$\mathbb{B}_\phi M = \mathbb{E}\mathbb{Z}_2 \times_{\mathbb{Z}_2} M$$

the corresponding Borel construction and by

$$\begin{aligned} H_\phi^*(M) &\equiv H^*(\mathbb{B}_\phi M; \mathbb{Z}_2), & H_*^\phi(M) &\equiv H_*(\mathbb{B}_\phi M; \mathbb{Z}_2), \\ H_\phi^*(M; \mathbb{Z}) &\equiv H^*(\mathbb{B}_\phi M; \mathbb{Z}), & H_*^\phi(M; \mathbb{Z}) &\equiv H_*(\mathbb{B}_\phi M; \mathbb{Z}) \end{aligned}$$

the corresponding \mathbb{Z}_2 -equivariant cohomology and homology of M . The projection map $p_1: \mathbb{E}\mathbb{Z}_2 \times M \rightarrow \mathbb{E}\mathbb{Z}_2$ descends to a fibration

$$(2.1) \quad M \rightarrow \mathbb{B}_\phi M \rightarrow \mathbb{B}\mathbb{Z}_2 = \mathbb{R}\mathbb{P}^\infty.$$

If $(V, \tilde{\phi}) \rightarrow (M, \phi)$ is a real bundle pair,

$$\mathbb{B}_{\tilde{\phi}} V \equiv \mathbb{E}\mathbb{Z}_2 \times_{\mathbb{Z}_2} V \rightarrow \mathbb{B}_\phi M$$

is a real vector bundle; this is the quotient of the vector bundle

$$p_2^* V \rightarrow \mathbb{E}\mathbb{Z}_2 \times M$$

by the natural lift of the free \mathbb{Z}_2 -action on the base. Let

$$w_i^{\tilde{\phi}}(V) \equiv w_i(\mathbb{B}_{\tilde{\phi}} V) \in H_\phi^i(M)$$

be the \mathbb{Z}_2 -equivariant Stiefel-Whitney classes of $V \rightarrow M$. For example, if M is a point and $V = \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$,

$$\mathbb{B}_{\tilde{\phi}} V = \mathbb{R}\mathbb{P}^\infty \times \mathbb{R} \oplus \mathcal{O}_{\mathbb{R}\mathbb{P}^\infty}(-1) \rightarrow \mathbb{R}\mathbb{P}^\infty,$$

where $\mathcal{O}_{\mathbb{R}\mathbb{P}^\infty}(-1)$ is the tautological line bundle; thus, $w_1^{\tilde{\phi}}(V)$ is the generator of $H_\phi^1(M)$ in this case. The non-equivariant Stiefel-Whitney classes of V are recovered from the equivariant Stiefel-Whitney classes of V by restricting to

the fiber of the fibration (2.1). If $f: \Sigma \rightarrow M$ is a continuous map commuting with involutions c on Σ and ϕ on M , the involution $\tilde{\phi}$ on V induces an involution $f^*\tilde{\phi}$ on f^*V lifting c and

$$(2.2) \quad w_i^{f^*\tilde{\phi}}(f^*V) = \{\mathbb{B}_{\phi,c}f\}^*w_i^{\tilde{\phi}}(V) \in H_c^i(\Sigma),$$

where

$$\mathbb{B}_{\phi,c}f: \mathbb{B}_c\Sigma \rightarrow \mathbb{B}_\phi M, \quad \{\mathbb{B}_{\phi,c}f\}([e, z]) = [e, f(z)],$$

is the map induced by f .

If an involution $c: \Sigma \rightarrow \Sigma$ has no fixed points, the projection

$$p_2: \mathbb{E}\mathbb{Z}_2 \times \Sigma \rightarrow \Sigma$$

descends to a fibration

$$(2.3) \quad \mathbb{E}\mathbb{Z}_2 \rightarrow \mathbb{B}_c\Sigma \xrightarrow{q} \Sigma/\mathbb{Z}_2.$$

Since $\mathbb{E}\mathbb{Z}_2$ is contractible, this fibration is a homotopy equivalence, with a homotopy inverse provided by any section of q . In particular, q induces isomorphisms

$$(2.4) \quad q^*: H^*(\Sigma/\mathbb{Z}_2) \rightarrow H_c^*(\Sigma), \quad q_*: H_*^c(\Sigma; \mathbb{Z}) \rightarrow H_*(\Sigma/\mathbb{Z}_2; \mathbb{Z}).$$

Any section of q embeds Σ/\mathbb{Z}_2 as a homotopy retract, and every two such sections are homotopic. Thus, if $f: \Sigma \rightarrow M$ is a continuous map commuting with the involutions c on Σ and ϕ on M , we also denote by

$$\mathbb{B}_{\phi,c}f: \Sigma/\mathbb{Z}_2 \rightarrow \mathbb{B}_\phi M$$

the composition of $\mathbb{B}_{\phi,c}f: \mathbb{B}_c\Sigma \rightarrow \mathbb{B}_\phi M$ with any section of q ; this is well-defined and unambiguous up to homotopy. If $(V, \tilde{c}) \rightarrow (\Sigma, c)$ is a real bundle pair, $V/\mathbb{Z}_2 \rightarrow \Sigma/\mathbb{Z}_2$ is a real vector bundle and

$$\begin{aligned} \mathbb{B}_{\tilde{c}}V \rightarrow q^*(V/\mathbb{Z}_2) &\equiv \{([e, x], [v]) \in \mathbb{B}_c\Sigma \times (V/\mathbb{Z}_2) : [x] = [p(v)]\}, \\ [e, v] &\rightarrow ([e, p(v)], [v]), \end{aligned}$$

is a vector bundle isomorphism covering the identity on $\mathbb{B}_c\Sigma$. Thus,

$$(2.5) \quad w_i^{\tilde{c}}(V) = w_i(q^*(V/\mathbb{Z}_2)) = q^*w_i(V/\mathbb{Z}_2) \in H_c^i(\Sigma; \mathbb{Z}_2).$$

2.2. Tensor products of real line bundle pairs

We now establish some important properties of the equivariant w_2 of real vector bundle pairs.

Proposition 2.1. *Let (X, ϕ) be a paracompact topological space with an involution.*

(1) *If $(L_1, \tilde{\phi}_1), (L_2, \tilde{\phi}_2) \rightarrow (X, \phi)$ are rank 1 real bundle pairs, then*

$$(2.6) \quad w_2^{\tilde{\phi}_1 \otimes_{\mathbb{C}} \tilde{\phi}_2}(L_1 \otimes_{\mathbb{C}} L_2) = w_2^{\tilde{\phi}_1}(L_1) + w_2^{\tilde{\phi}_2}(L_2).$$

(2) *If $(V_1, \tilde{\phi}_1), (V_2, \tilde{\phi}_2) \rightarrow (X, \phi)$ are any real bundle pairs, then*

$$w_2^{\Lambda_{\mathbb{C}}^{\text{top}}(\tilde{\phi}_1 \oplus \tilde{\phi}_2)}(\Lambda_{\mathbb{C}}^{\text{top}}(V_1 \oplus V_2)) = w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{\phi}_1}(\Lambda_{\mathbb{C}}^{\text{top}} V_1) + w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{\phi}_2}(\Lambda_{\mathbb{C}}^{\text{top}} V_2).$$

Proof. The first statement of this proposition implies the second, since

$$\Lambda_{\mathbb{C}}^{\text{top}}((V_1, \tilde{\phi}_1) \oplus (V_2, \tilde{\phi}_2)) = (\Lambda_{\mathbb{C}}^{\text{top}}(V_1, \tilde{\phi}_1)) \otimes (\Lambda_{\mathbb{C}}^{\text{top}}(V_2, \tilde{\phi}_2)).$$

Below we establish (2.6).

(1) Let \mathbb{P}^{∞} denote the infinite-dimensional complex projective space with the standard involution,

$$\tau_{\infty}: \mathbb{P}^{\infty} \rightarrow \mathbb{P}^{\infty}, \quad [z_1, z_2, \dots] \rightarrow [\bar{z}_1, \bar{z}_2, \dots],$$

$p_1, p_2: \mathbb{P}^{\infty} \times \mathbb{P}^{\infty} \rightarrow \mathbb{P}^{\infty}$ be the projection maps, and

$$(\mathbb{X}, \Phi) = (\mathbb{P}^{\infty} \times \mathbb{P}^{\infty}, \tau_{\infty} \times \tau_{\infty}).$$

The homotopy exact sequence for the fibration (2.1) with M replaced by \mathbb{X} gives an exact sequence

$$\pi_2(\mathbb{X}) \rightarrow \pi_2(\mathbb{B}_{\Phi} \mathbb{X}) \rightarrow 0 \rightarrow 0 \rightarrow \pi_1(\mathbb{B}_{\Phi} \mathbb{X}) \rightarrow \pi_1(\mathbb{R}\mathbb{P}^{\infty}) \rightarrow 0.$$

In particular, $H_1(\mathbb{B}_{\Phi} \mathbb{X}; \mathbb{Z}) = \mathbb{Z}_2$. By [13, Satz II], $\pi_2(\mathbb{B}_{\Phi} \mathbb{X})$ surjects onto $H_2^{\Phi}(\mathbb{X}; \mathbb{Z})$, since

$$H_2(\pi_1(\mathbb{B}_{\Phi} \mathbb{X}); \mathbb{Z}) = H_2(\mathbb{Z}_2; \mathbb{Z}) \cong H_2(\mathbb{B}\mathbb{Z}_2; \mathbb{Z}) = H_2(\mathbb{R}\mathbb{P}^{\infty}; \mathbb{Z}) = 0.$$

Thus, by the Universal Coefficient Theorem for Homology [23, Theorem 55.1], $H_2^\Phi(\mathbb{X})$ is generated by

$$\begin{aligned} \mathbb{P}_1^1 &\equiv \{[1, z, [1]] : z \in \mathbb{P}^1\}, & \mathbb{P}_2^1 &\equiv \{[1, [1], z] : z \in \mathbb{P}^1\}, \\ \mathbb{R}\mathbb{P}^2 &\equiv \{[z, [1], [1]] : z \in S^2\}, \end{aligned}$$

where $1 \equiv [1, 0, \dots] \in S^\infty$, $[1] \in \mathbb{P}^\infty$ denotes the S^1 -equivalence class of 1 in S^∞ , and $[z_1, z_2, z_3]$ is the \mathbb{Z}_2 -equivalence class of (z_1, z_2, z_3) in $S^\infty \times \mathbb{P}^\infty \times \mathbb{P}^\infty$; the spheres \mathbb{P}_1^1 and \mathbb{P}_2^1 are generators of $H_2^\Phi(\mathbb{X}; \mathbb{Z})$ coming from $\pi_2(\mathbb{X})$ via $\pi_2(\mathbb{B}_\Phi \mathbb{X})$, while $\mathbb{R}\mathbb{P}^2$ generates

$$\text{Tor}(H_1(\mathbb{B}_\Phi \mathbb{X}; \mathbb{Z}), \mathbb{Z}_2) \approx \mathbb{Z}_2$$

in the split short exact sequence for $H_2^\Phi(\mathbb{X})$ provided by [23, Theorem 55.1].

(2) The involution τ_∞ naturally lifts to a conjugation $\tilde{\tau}_\infty$ on the tautological line bundle

$$\mathcal{O}_{\mathbb{P}^\infty}(-1) \longrightarrow \mathbb{P}^\infty.$$

We first verify (2.6) for the rank 1 real bundle pairs

$$(\mathbb{L}_1, \tilde{\Phi}_1) \equiv p_1^*(\mathcal{O}_{\mathbb{P}^\infty}(-1), \tilde{\tau}_\infty), \quad (\mathbb{L}_2, \tilde{\Phi}_2) \equiv p_2^*(\mathcal{O}_{\mathbb{P}^\infty}(-1), \tilde{\tau}_\infty) \longrightarrow (\mathbb{X}, \Phi).$$

Since the homology classes of \mathbb{P}_1^1 and \mathbb{P}_2^1 are the images of classes in a fiber of (2.1) with M replaced by \mathbb{X} ,

$$\begin{aligned} \langle w_2^{\tilde{\Phi}_1 \otimes \tilde{\Phi}_2}(\mathbb{L}_1 \otimes \mathbb{L}_2), [\mathbb{P}_i^1]_{\mathbb{Z}_2} \rangle &= \langle w_2(\mathbb{L}_1 \otimes \mathbb{L}_2), [\mathbb{P}_i^1]_{\mathbb{Z}_2} \rangle \\ &= \langle c_1(\mathbb{L}_1 \otimes \mathbb{L}_2), [\mathbb{P}_i^1]_{\mathbb{Z}} \rangle + 2\mathbb{Z} = \langle c_1(\mathbb{L}_1), [\mathbb{P}_i^1]_{\mathbb{Z}} \rangle + \langle c_1(\mathbb{L}_2), [\mathbb{P}_i^1]_{\mathbb{Z}} \rangle + 2\mathbb{Z} \\ &= \langle w_2(\mathbb{L}_1), [\mathbb{P}_i^1]_{\mathbb{Z}_2} \rangle + \langle w_2(\mathbb{L}_2), [\mathbb{P}_i^1]_{\mathbb{Z}_2} \rangle \\ &= \langle w_2^{\tilde{\Phi}_1}(\mathbb{L}_1), [\mathbb{P}_i^1]_{\mathbb{Z}_2} \rangle + \langle w_2^{\tilde{\Phi}_2}(\mathbb{L}_2), [\mathbb{P}_i^1]_{\mathbb{Z}_2} \rangle. \end{aligned}$$

The restrictions of $\mathbb{B}_{\tilde{\Phi}_1} \mathbb{L}_1$, $\mathbb{B}_{\tilde{\Phi}_2} \mathbb{L}_2$, and $\mathbb{B}_{\tilde{\Phi}_1 \otimes_{\mathbb{C}} \tilde{\Phi}_2}(\mathbb{L}_1 \otimes_{\mathbb{C}} \mathbb{L}_2)$ to

$$\mathbb{R}\mathbb{P}^\infty \equiv \mathbb{R}\mathbb{P}^\infty \times [1] \times [1] \subset \mathbb{B}_\Phi \mathbb{X}$$

are

$$S^\infty \times_{\mathbb{Z}_2} \mathbb{C} \approx \mathbb{R}\mathbb{P}^\infty \times \mathbb{R} \oplus \mathcal{O}_{\mathbb{R}\mathbb{P}^\infty}(-1) \longrightarrow \mathbb{R}\mathbb{P}^\infty.$$

Thus,

$$\begin{aligned} \langle w_2^{\tilde{\Phi}_1 \otimes_{\mathbb{C}} \tilde{\Phi}_2}(\mathbb{L}_1 \otimes_{\mathbb{C}} \mathbb{L}_2), [\mathbb{R}\mathbb{P}^2]_{\mathbb{Z}_2} \rangle &= 0 = 0 + 0 \\ &= \langle w_2^{\tilde{\Phi}_1}(\mathbb{L}_1), [\mathbb{R}\mathbb{P}^2]_{\mathbb{Z}_2} \rangle + \langle w_2^{\tilde{\Phi}_2}(\mathbb{L}_2), [\mathbb{R}\mathbb{P}^2]_{\mathbb{Z}_2} \rangle. \end{aligned}$$

Since \mathbb{P}_1^1 , \mathbb{P}_2^1 , and $\mathbb{R}\mathbb{P}^2$ generate $H_2^\Phi(\mathbb{X})$, this establishes (2.6) for $(L_i, \tilde{\phi}_i) = (\mathbb{L}_i, \tilde{\Phi}_i)$.

(3) Let $(L_i, \tilde{\phi}_i) \rightarrow (X, \phi)$ be as in the statement of the proposition. By the proof of [22, Lemma 5.6], there exist continuous maps

$$f_1, f_2 : (X, \phi) \rightarrow (\mathbb{P}^\infty, \tau_\infty) \quad \text{s.t.} \quad (L_i, \tilde{\phi}_i) = f_i^*(\mathcal{O}_{\mathbb{P}^\infty}(-1), \tilde{\tau}_\infty).$$

Thus,

$$\begin{aligned} w_2^{\tilde{\phi}_1 \otimes_{\mathbb{C}} \tilde{\phi}_2}(L_1 \otimes_{\mathbb{C}} L_2) &= \{f_1 \times f_2\}^* w_2^{\tilde{\Phi}_1 \otimes_{\mathbb{C}} \tilde{\Phi}_2}(\mathbb{L}_1 \otimes_{\mathbb{C}} \mathbb{L}_2) \\ &= \{f_1 \times f_2\}^* (w_2^{\tilde{\Phi}_1}(\mathbb{L}_1) + w_2^{\tilde{\Phi}_2}(\mathbb{L}_2)) \\ &= \{f_1 \times f_2\}^* (p_1^* w_2^{\tilde{\tau}_\infty}(\mathcal{O}_{\mathbb{P}^\infty}(-1)) + p_2^* w_2^{\tilde{\tau}_\infty}(\mathcal{O}_{\mathbb{P}^\infty}(-1))) \\ &= w_2^{\tilde{\phi}_1}(L_1) + w_2^{\tilde{\phi}_2}(L_2); \end{aligned}$$

the second equality above follows from (2). □

Lemma 2.2. *Let Σ be a compact connected unorientable surface and $b_\Sigma \in H_1(\Sigma; \mathbb{Z})$ be the nontrivial torsion class. If $\kappa \in H^1(\Sigma; \mathbb{Z}_2)$,*

$$(2.7) \quad \langle \kappa^2, [\Sigma]_{\mathbb{Z}_2} \rangle = \langle \kappa, b_\Sigma \rangle,$$

where $[\Sigma]_{\mathbb{Z}_2} \in H_2(\Sigma; \mathbb{Z}_2)$ is the fundamental class with \mathbb{Z}_2 -coefficients.

Proof. By [24, Theorem 77.5], Σ is the connected sum of m copies of $\mathbb{R}\mathbb{P}^2$ and

$$H_2(\Sigma; \mathbb{Z}) \approx \mathbb{Z}^{m-1} \oplus \mathbb{Z}_2$$

for some $m \in \mathbb{Z}$. By [24, Theorem 77.5], Σ can be represented by the labeling scheme $a_1 a_1 a_2 a_2 \cdots a_m a_m$; see Figure 1. From the labeling scheme, it is immediate that the torsion element is given by

$$b_\Sigma = a_1 + a_2 + \cdots + a_m.$$

From the diagram, we see that the \mathbb{Z}_2 -homology intersection product is given by $a_i \cdot a_i = 1$ and $a_i \cdot a_j = 0$ if $i \neq j$. By the unoriented Poincaré duality [23, Theorem 67.1], κ is thus Poincaré dual to the sum of some number of a_1, a_2, \dots, a_m and κ^2 is Poincaré dual to the sum of the same number of $a_1^2, a_2^2, \dots, a_m^2$. Thus, each side of (2.7) vanishes if and only if κ is Poincaré dual to an even number of a_1, a_2, \dots, a_m . □

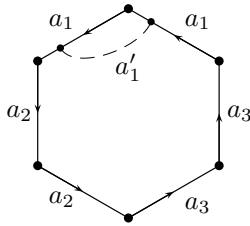


Figure 1: Labeling scheme for $\Sigma = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ and a deformation of the loop a_1 used to compute the intersection product on $H_1(\Sigma; \mathbb{Z})$

Corollary 2.3. *For any topological space M ,*

$$(2.8) \quad \begin{aligned} \{w \in H^2(M; \mathbb{Z}_2) : w(B) = 0 \ \forall B \in H_2(M; \mathbb{Z})\} \\ \supset \{\kappa^2 : \kappa \in H^1(M; \mathbb{Z}_2)\}. \end{aligned}$$

If $H_1(M; \mathbb{Z})$ is finitely generated, the reverse inclusion holds if and only if $H_1(M; \mathbb{Z})$ has no 4-torsion.

Proof. (1) If $B \in H_2(M; \mathbb{Z})$, there exists a continuous map $f : \Sigma \rightarrow M$ from a closed oriented surface Σ such that

$$f_*[\Sigma]_{\mathbb{Z}} = B \in H_2(M; \mathbb{Z}).$$

Since every square class in $H^2(\Sigma; \mathbb{Z}_2)$ is trivial,

$$\langle \kappa^2, B \rangle = \langle \kappa^2, f_*[\Sigma]_{\mathbb{Z}_2} \rangle = \langle (f^* \kappa)^2, [\Sigma]_{\mathbb{Z}_2} \rangle = 0 \quad \forall \kappa \in H^1(M; \mathbb{Z}_2).$$

This establishes (2.8).

(2) If $H_1(M; \mathbb{Z})$ is finitely generated,

$$(2.9) \quad H_1(M; \mathbb{Z}) \approx \mathbb{Z}^{r_0} \oplus \mathbb{Z}_{m_1}^{r_1} \oplus \dots \oplus \mathbb{Z}_{m_k}^{r_k}$$

for some $m_1, \dots, m_k \geq 2$ and $r_0, r_1, \dots, r_m \geq 0$. In this case,

$$\text{Ext}(H_1(M; \mathbb{Z}), \mathbb{Z}_2) \approx \bigoplus_{2|m_i} \mathbb{Z}_2^{r_i};$$

see [23, p331]. We denote by $b_{i,j}$, with $i = 1, \dots, k$ and $j = 1, \dots, r_i$, loops representing the generators of the torsion part in (2.9). For each i with $2|m_i$ and $j = 1, \dots, r_i$, there exist a compact oriented surface $\Sigma_{i,j}$ with two boundary

components $(\partial\Sigma_{i,j})_1$ and $(\partial\Sigma_{i,j})_2$, a continuous map $F_{i,j}: \Sigma_{i,j} \rightarrow M$, and an orientation-preserving diffeomorphism $\varphi_{i,j}: (\partial\Sigma_{i,j})_2 \rightarrow (\partial\Sigma_{i,j})_1$ such that

$$[F|_{(\partial\Sigma_{i,j})_1}] = (m_i/2)[b_{i,j}] \quad \text{and} \quad F_{i,j}|_{(\partial\Sigma_{i,j})_2} = F_{i,j}|_{(\partial\Sigma_{i,j})_1} \circ \varphi_{i,j}.$$

The map $F_{i,j}$ descends to a continuous map $\hat{F}_{i,j}$ from the unorientable surface

$$\hat{\Sigma}_{i,j} \equiv \Sigma_{i,j}/\sim, \quad z \sim \varphi_{i,j}(z) \quad \forall z \in (\partial\Sigma_{i,j})_2.$$

By Lemma 2.2,

$$(2.10) \quad \begin{aligned} \langle \kappa^2, \{\hat{F}_{i,j}\}_*[\hat{\Sigma}_{i,j}]_{\mathbb{Z}_2} \rangle &= \langle F_{i,j}^* \kappa, [(\partial\Sigma_{i,j})_1]_{\mathbb{Z}_2} \rangle \\ &= \frac{m_i}{2} \langle \kappa, b_{i,j} \rangle \quad \forall \kappa \in H^1(M; \mathbb{Z}_2). \end{aligned}$$

By the Universal Coefficient Theorem for Cohomology [23, Theorem 53.1], the natural homomorphism

$$H^1(M; \mathbb{Z}_2) \rightarrow \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}_2) \approx \mathbb{Z}_2^{r_0} \oplus \bigoplus_{2|m_i} \mathbb{Z}_2^{r_i}$$

is an isomorphism. By (2.10), the homomorphism

$$(2.11) \quad \bigoplus_{2|m_i, 4 \nmid m_i} \mathbb{Z}_2^{r_i} \rightarrow \text{Ext}(H_1(M; \mathbb{Z}), \mathbb{Z}_2) \subset H^2(M; \mathbb{Z}_2), \quad \kappa \rightarrow \kappa^2,$$

is injective. On the other hand, this homomorphism vanishes on the factors corresponding to \mathbb{Z}_{m_i} with $4|m_i$, since their image is contained in the image of the homomorphism

$$H^1(M; \mathbb{Z}_{m_i}) \rightarrow H^2(M; \mathbb{Z}_{m_i}) \rightarrow H^2(M; \mathbb{Z}_2)$$

and $2\kappa^2 = 0$ in $H^2(M; \mathbb{Z}_{m_i})$ for all $\kappa \in H^1(M; \mathbb{Z}_{m_i})$. Thus, the cokernel of the homomorphism (2.11) is isomorphic to $\bigoplus_{4|m_i} \mathbb{Z}_2^{r_i}$. In particular, every element of $H^2(M; \mathbb{Z}_2)$ vanishing on the image of $H_2(M; \mathbb{Z})$ in $H_2(M; \mathbb{Z}_2)$ is a square class if and only if $H_1(M; \mathbb{Z}_2)$ has no 4-torsion. \square

Corollary 2.4. *Let (X, ϕ) be a topological space with an involution and $(L, \tilde{\phi}) \rightarrow (X, \phi)$ be a rank 1 real bundle pair.*

(1) *If X is simply connected and $w_2(L) = 0$, $w_2^{\tilde{\phi}}(L)$ is a square class.*

(2) *If X is paracompact and $(L, \tilde{\phi})$ admits a real square root, $w_2^{\tilde{\phi}}(L) = 0$.*

Proof. (1) The homotopy exact sequence for the fibration (2.1) with M replaced by X gives an exact sequence

$$\pi_2(X) \longrightarrow \pi_2(\mathbb{B}_\phi X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \pi_1(\mathbb{B}_\phi X) \longrightarrow \pi_1(\mathbb{R}\mathbb{P}^\infty) \longrightarrow 0.$$

In particular, $H_1(\mathbb{B}_\phi X; \mathbb{Z}) = \mathbb{Z}_2$. Thus, by Corollary 2.3,

$$(2.12) \quad \{w \in H_\phi^2(X) : w(b) = 0 \ \forall b \in H_2^\phi(X; \mathbb{Z})\} = \{\kappa^2 : \kappa \in H_\phi^1(X)\}.$$

By [13, Satz II], $\pi_2(\mathbb{B}_\phi X)$ surjects onto $H_2^\phi(X; \mathbb{Z})$, since

$$H_2(\pi_1(\mathbb{B}_\phi X); \mathbb{Z}) = H_2(\mathbb{Z}_2; \mathbb{Z}) \cong H_2(\mathbb{B}\mathbb{Z}_2; \mathbb{Z}) = H_2(\mathbb{R}\mathbb{P}^\infty; \mathbb{Z}) = 0.$$

Since the diagram

$$\begin{array}{ccc} \pi_2(X) & \longrightarrow & \pi_2(\mathbb{B}_\phi X) \\ \downarrow \approx & & \downarrow \\ H_2(X; \mathbb{Z}) & \longrightarrow & H_2(\mathbb{B}_\phi X; \mathbb{Z}) \end{array}$$

commutes, $H_2(X; \mathbb{Z})$ surjects onto $H_2^\phi(X; \mathbb{Z})$. Thus, we can replace $H_2^\phi(X; \mathbb{Z})$ in (2.12) by $H_2(X; \mathbb{Z})$, i.e.

$$\{w \in H_\phi^2(X) : w(b) = 0 \ \forall b \in H_2(X; \mathbb{Z})\} = \{\kappa^2 : \kappa \in H_\phi^1(X)\}.$$

Since the restriction of $w_2^{\tilde{\phi}}(L)$ to the fiber $X \subset \mathbb{B}_\phi X$ is $w_2(L) = 0$, it follows that $w_2^{\tilde{\phi}}(L)$ is a square class.

(2) This follows immediately from the first statement of Proposition 2.1. □

2.3. Applications to real bundle pairs

The antipodal involution $\mathbf{a} : S^1 \rightarrow S^1$ given by (1.2) has no fixed points. A section of the projection q in (2.3) in this case is given by

$$S^1 \longrightarrow S^\infty \times_{\mathbb{Z}_2} S^1 \subset (\mathbb{C}^\infty - 0) \times_{\mathbb{Z}_2} S^1, \quad e^{i\theta} \longrightarrow [(e^{i\theta/2}, 0, 0, \dots), e^{i\theta/2}].$$

Thus, if (X, ϕ) is a topological space with an involution and $\alpha : S^1 \rightarrow X$ is any map such that $\alpha \circ \mathbf{a} = \phi \circ \alpha$, then

$$\mathbb{B}_{\phi, \mathbf{a}} \alpha : S^1 \longrightarrow \mathbb{B}_\phi X, \quad e^{i\theta} \longrightarrow [(e^{i\theta/2}, 0, 0, \dots), \alpha(e^{i\theta/2})].$$

The composition of $\mathbb{B}_{\phi, \mathbf{a}}\alpha$ with the projection in (2.1) is a generator of $\pi_1(\mathbb{B}\mathbb{Z}_2) \approx \mathbb{Z}_2$, and so $[\alpha]^\mathbf{a} \neq 0 \in H_1^\phi(X)$. Furthermore, every loop in $\mathbb{B}_\phi X$ which projects to a generator of $\pi_1(\mathbb{B}\mathbb{Z}_2)$ is homotopic to $\mathbb{B}_{\phi, \mathbf{a}}\alpha$ for some \mathbb{Z}_2 -equivariant map $\alpha: S^1 \rightarrow X$. If $\alpha, \beta: S^1 \rightarrow X$ are two maps commuting with the involutions \mathbf{a} on S^1 and ϕ on X , a homotopy $h: \mathbb{I} \times S^1 \rightarrow \mathbb{B}_\phi X$ between $\mathbb{B}_{\phi, \mathbf{a}}\alpha$ and $\mathbb{B}_{\phi, \mathbf{a}}\beta$ lifts to a homotopy

$$\tilde{h}: \mathbb{I} \times S^1 \rightarrow \mathbb{E}\mathbb{Z}_2 \times X$$

commuting with the \mathbb{Z}_2 -actions. Thus, $\mathbb{B}_{\phi, \mathbf{a}}\alpha$ and $\mathbb{B}_{\phi, \mathbf{a}}\beta$ are homotopic if and only if α and β are homotopic through maps $S^1 \rightarrow X$ intertwining \mathbf{a} and ϕ .

The composition of $\mathbb{B}_{\phi, \mathbf{a}}\alpha$ with the projection $\mathbb{B}_\phi X \rightarrow X/\mathbb{Z}_2$ is the loop $e^{i\theta} \rightarrow [\alpha(e^{i\theta/2})]$, i.e. the composition of the projection $q_X: X \rightarrow X/\mathbb{Z}_2$ with the restriction of α to the upper half S^1_+ of S^1 ; since α is \mathbb{Z}_2 -invariant, $\alpha(1) = \phi(\alpha(-1))$ and so the endpoints of this semi-circle map to the same point in X/\mathbb{Z}_2 . Thus, if ϕ acts on X without fixed point, the loop $\mathbb{B}_{\phi, \mathbf{a}}\alpha$ in $\mathbb{B}_\phi X$ corresponds to the loop $q_X \circ \alpha|_{S^1_+}$ in X/\mathbb{Z}_2 under the isomorphism q_* in (2.4), with Σ replaced by X . For example, let $n \geq 2$,

$$X = (S^n \times S^n) / \sim, \quad (x_1, x_2) \sim (-x_1, -x_2), \\ \phi([x_1, x_2]) = [-x_1, x_2] = [x_1, -x_2].$$

If $\alpha: S^1 \rightarrow S^n$ is any map intertwining the antipodal involutions, the homotopically trivial loops

$$\alpha_1, \alpha_2: S^1 \rightarrow X, \quad \alpha_1 = [\alpha, x^*], \quad \alpha_2 = [x^*, \alpha],$$

where $x^* \in S^n$ is any point, are \mathbb{Z}_2 -equivariant. Since $\mathbb{B}_{\phi, \mathbf{a}}\alpha_1$ and $\mathbb{B}_{\phi, \mathbf{a}}\alpha_2$ correspond to the two standard generators of $\pi_1(\mathbb{R}\mathbb{P}^n \times \mathbb{R}\mathbb{P}^n)$ by the above,

$$[\alpha_1]^\mathbf{a} \neq [\alpha_2]^\mathbf{a} \in H_1^\phi(X; \mathbb{Z}),$$

illustrating the statement made in Remark 1.4.

If $c: S^1 \rightarrow S^1$ is an orientation-preserving involution, denote by

$$(V_\pm, \tilde{c}_\pm) \rightarrow (S^1 \times S^1, \text{id} \times c)$$

the real bundle pairs with

$$V_\pm = (\mathbb{I} \times S^1 \times \mathbb{C}) / \sim, \quad (0, z, v) \sim (1, z, \pm v) \quad \forall z \in S^1, v \in \mathbb{C},$$

and with the involutions induced by the standard conjugation on \mathbb{C} .

Lemma 2.5. *Let $c: S^1 \rightarrow S^1$ be an orientation-preserving involution different from the identity. For every $n \geq 1$,*

$$\begin{aligned} \langle w_2^{n\tilde{c}_+}(nV_+), [S^1 \times S^1]^{\text{id} \times c} \rangle &= 0, \\ \langle w_2^{\tilde{c}_- \oplus (n-1)\tilde{c}_+}(V_- \oplus (n-1)V_+), [S^1 \times S^1]^{\text{id} \times c} \rangle &\neq 0 \in \mathbb{Z}_2. \end{aligned}$$

Proof. We can assume that $c(z) = -z$ on $S^1 \subset \mathbb{C}$. The real bundle pairs

$$(V, \tilde{c}) = (nV_+, n\tilde{c}_+), (V_- \oplus (n-1)V_+, \tilde{c}_- \oplus (n-1)\tilde{c}_+)$$

canonically decompose into two \mathbb{Z}_2 -equivariant real vector bundles, induced by the real and imaginary axes in \mathbb{C} . By (2.5),

$$\begin{aligned} \langle w_2^{\tilde{c}}(V), [S^1 \times S^1]^{\text{id} \times c} \rangle &= \langle w_2(V/\mathbb{Z}_2), [S^1 \times \mathbb{R}\mathbb{P}^1] \rangle \\ &= \langle w_2(V_{\mathbb{R}} \oplus V_{i\mathbb{R}}), [S^1 \times \mathbb{R}\mathbb{P}^1] \rangle, \end{aligned}$$

where $\mathbb{R}\mathbb{P}^1 = S^1/\mathbb{Z}_2 \cong \mathbb{I}/0 \sim 1$ and

$$V_{\mathbb{R}} = n(V_+)_{\mathbb{R}}, (V_-)_{\mathbb{R}} \oplus (n-1)(V_+)_{\mathbb{R}}, \quad V_{i\mathbb{R}} = n(V_+)_{i\mathbb{R}}, (V_-)_{i\mathbb{R}} \oplus (n-1)(V_+)_{i\mathbb{R}}$$

are the \mathbb{Z}_2 -quotients of the real and imaginary parts of V . Since

$$\begin{aligned} (V_{\pm})_{\mathbb{R}} &= (\mathbb{I} \times \mathbb{I} \times \mathbb{R}) / \sim, \quad (0, t, v) \sim (1, t, \pm v), \quad (s, 0, v) \sim (s, 1, v) \\ &\quad \forall s, t \in \mathbb{I}, v \in \mathbb{R} \\ (V_{\pm})_{i\mathbb{R}} &= (\mathbb{I} \times \mathbb{I} \times \mathbb{R}) / \sim, \quad (0, t, v) \sim (1, t, \pm v), \quad (s, 0, v) \sim (s, 1, -v) \\ &\quad \forall s, t \in \mathbb{I}, v \in \mathbb{R} \end{aligned}$$

we find that

$$(V_+)_{\mathbb{R}} = \tau, \quad (V_-)_{\mathbb{R}} = \gamma_1, \quad (V_+)_{i\mathbb{R}} = \gamma_2, \quad (V_-)_{i\mathbb{R}} = \gamma_1 \otimes \gamma_2,$$

where $\tau \rightarrow S^1 \times \mathbb{R}\mathbb{P}^1$ is the trivial real line bundle and $\gamma_1, \gamma_2 \rightarrow S^1 \times \mathbb{R}\mathbb{P}^1$ are the pull-backs of the Mobius/tautological line bundle by the projection maps. Thus,

$$w_2(n(V_+/\mathbb{Z}_2)) = w_2(n(\tau \oplus \gamma_2)) = \binom{n}{2} w_1(\gamma_2)^2 = 0 \in H^2(S^1 \times \mathbb{R}\mathbb{P}^1, \mathbb{Z}_2),$$

$$\begin{aligned}
 w_2((V_- \oplus (n-1)V_+)/\mathbb{Z}_2) &= w_1(\gamma_1 \oplus \gamma_1 \otimes \gamma_2)w_1((n-1)\gamma_2) \\
 &\quad + w_2(\gamma_1 \oplus \gamma_1 \otimes \gamma_2) + w_2((n-1)\gamma_2) \\
 &= w_1(\gamma_1)(w_1(\gamma_1) + w_1(\gamma_2)) \\
 &\quad + w_1(\gamma_2) \cdot (n-1)w_1(\gamma_2) + 0 \\
 &= w_1(\gamma_1)w_1(\gamma_2) \neq 0 \in H^2(S^1 \times \mathbb{RP}^1, \mathbb{Z}_2).
 \end{aligned}$$

This implies the claim. □

Lemma 2.6. *Let $c: S^1 \rightarrow S^1$ be an orientation-preserving involution different from the identity. If $(V, \tilde{c}) \rightarrow (S^1 \times S^1, \text{id} \times c)$ is a real bundle pair,*

$$(2.13) \quad \langle w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{c}}(\Lambda_{\mathbb{C}}^{\text{top}} V), [S^1 \times S^1]^{\text{id} \times c} \rangle = \langle w_2^{\tilde{c}}(V), [S^1 \times S^1]^{\text{id} \times c} \rangle.$$

Proof. We continue with the notation of Lemma 2.5 and its proof. By the proof of [5, Lemma 2.2], every real bundle pair (\tilde{V}, \tilde{c}) over $\mathbb{I} \times S^1$ admits a trivialization, i.e. a fiber-preserving bundle isomorphism

$$\begin{aligned}
 &\Psi: \tilde{V} \rightarrow \mathbb{I} \times S^1 \times \mathbb{C}^n \\
 \text{s.t. } &\Psi(\tilde{c}(\Psi^{-1}(t, z, w))) = (t, c(z), \bar{w}) \quad \forall (t, z, w) \in \mathbb{I} \times S^1 \times \mathbb{C}^n,
 \end{aligned}$$

where \bar{w} denotes the standard complex conjugate of w . Thus, the real bundle pairs (V, \tilde{c}) over

$$(S^1 \times S^1, \text{id} \times c) = ((\mathbb{I} \times S^1)/(0, z) \sim (1, z), \text{id} \times c)$$

are classified by the homotopy classes of the clutching maps $S^1 \rightarrow \text{GL}_n \mathbb{C}$ satisfying the condition $A(c(z)) = \overline{A(z)}$ for all $z \in S^1$. By [5, Lemma 2.2], there are two homotopy classes of such maps; they are represented by the constant maps with values in the diagonal matrices, with at most one diagonal entry -1 and the remaining diagonal entries 1. Thus,

$$(V, \tilde{c}) \approx n(V_+, \tilde{c}_+) \quad \text{or} \quad (V, \tilde{c}) \approx (V_1, \tilde{c}_-) \oplus (n-1)(V_+, \tilde{c}_+).$$

Since

$$\begin{aligned}
 (V, \tilde{c}) = n(V_+, \tilde{c}_+) &\implies \Lambda_{\mathbb{C}}^{\text{top}}(V, \tilde{c}) = (V_+, \tilde{c}_+), \\
 (V, \tilde{c}) = (V_-, \tilde{c}_-) \oplus (n-1)(V_+, \tilde{c}_+) &\implies \Lambda_{\mathbb{C}}^{\text{top}}(V, \tilde{c}) = (V_-, \tilde{c}_-),
 \end{aligned}$$

the claim thus follows from Lemma 2.5. □

Lemma 2.7. *Let (X, ϕ) be a manifold with an involution, $(V, \tilde{\phi}) \rightarrow (X, \phi)$ be a real bundle pair, and $\beta: S^1 \times S^1 \rightarrow X$ be a continuous map commuting with $\text{id} \times c$ and ϕ , where $c: S^1 \rightarrow S^1$ is an orientation-preserving involution different from the identity. The real bundle pair*

$$\beta^* \Lambda_{\mathbb{C}}^{\text{top}}(V, \tilde{\phi}) \rightarrow (S^1 \times S^1, \text{id} \times c)$$

admits a real square root if and only if

$$(2.14) \quad \langle w_2^{\tilde{\phi}}(V), [\beta]^{\text{id} \times c} \rangle = 0.$$

If $\pi_1(X) = 0$, then $[\beta]^{\text{id} \times c}$ is the image of the homology class of a map $\beta': S^2 \rightarrow X$ under the inclusion of $X \rightarrow \mathbb{B}_{\phi} X$ and

$$\langle w_2^{\tilde{\phi}}(V), [\beta]^{\text{id} \times c} \rangle = \langle w_2(V), [\beta'] \rangle.$$

Proof. By Lemma 2.6, we can assume that $\text{rk}_{\mathbb{C}} V = 1$. Suppose

$$(L, \tilde{c}) \rightarrow (S^1 \times S^1, \text{id} \times c)$$

is a real bundle pair such that

$$\beta^*(V, \tilde{\phi}) \approx (L, \tilde{c})^{\otimes 2}.$$

By the proof of Lemma 2.6,

$$L = (\mathbb{I} \times S^1 \times \mathbb{C}) / \sim, \quad (0, z, v) \sim (1, z, \pm v) \quad \forall z \in S^1, v \in \mathbb{C},$$

with the sign \pm fixed and the conjugation \tilde{c} induced by the standard conjugation in \mathbb{C} . Thus,

$$\beta^*(V, \tilde{\phi}) \approx (\mathbb{I} \times S^1 \times \mathbb{C}) / \sim, \quad (0, z, v) \sim (1, z, v) \quad \forall z \in S^1, v \in \mathbb{C},$$

i.e. $\beta^*(V, \tilde{\phi}) \approx (V_+, \tilde{c}_+)$. Along with Lemma 2.5, this implies (2.14). On the other hand, by the proof of Lemma 2.6 and (2.2),

$$\langle w_2^{\tilde{\phi}}(V), [\beta]^{\text{id} \times c} \rangle = 0 \quad \implies \quad \beta^*(V, \tilde{\phi}) \approx (V_+, \tilde{c}_+).$$

Thus, $\beta^*(V, \tilde{\phi})$ is isomorphic to the square of (V_+, \tilde{c}_+) if $\langle w_2^{\tilde{\phi}}(V), [\beta]^{\text{id} \times c} \rangle = 0$.

If $\pi_1(X)=0$, the fibration (2.1) gives rise to an exact sequence

$$\cdots \longrightarrow \pi_2(X) \longrightarrow \pi_2(\mathbb{B}_\phi X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \pi_1(\mathbb{B}_\phi X) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Thus, the restriction of $\mathbb{B}_{\phi, \text{id} \times c} \beta: S^1 \times \mathbb{R}P^1 \longrightarrow \mathbb{B}_\phi X$ to at least one simple circle is homotopically trivial (specifically, the circle $S^1 \times x$). Therefore, $[\beta]^{\text{id} \times c}$ is a spherical class and thus equals $\iota_*[\beta']$ for some $\beta' \in H_2(X)$, with \mathbb{Z}_2 or \mathbb{Z} coefficients, where $\iota: X \longrightarrow \mathbb{B}_\phi X$ is the inclusion of a fiber in (2.1). Thus,

$$\langle w_2^{\tilde{c}}(V), [\beta]^{\text{id} \times c} \rangle = \langle \iota^* w_2^{\tilde{c}}(V), [\beta'] \rangle = \langle w_2(V), [\beta'] \rangle.$$

This establishes the last claim. □

2.4. Some examples

We now give three concrete examples. By Example 2.8, real bundle pairs $(V, \tilde{\phi}) \longrightarrow (X, \phi)$ which induce non-orientable determinant bundle are quite common over non-simply connected spaces. Examples 2.9 and 2.10 illustrate the significance of the vanishing requirements on $\pi_1(X)$ and $w_2(X)$ in Corollary 1.8, showing that neither requirement by itself suffices for the orientability of the moduli space $\mathfrak{M}_0(X, J, b)^{\phi, \eta}$. The orientability in the former example in fact fails due to the twisting phenomenon of Example 2.8. The $m, n = 1$ case of Example 2.9 is [5, Example 2.5].

Example 2.8. Let $(V, \tilde{\phi}) \longrightarrow (X, \phi)$ be the trivial bundle pair of rank 1, i.e.

$$V = X \times \mathbb{C}, \quad \tilde{\phi}: V \longrightarrow V, \quad \tilde{\phi}(x, v) = (\phi(x), \bar{v}) \quad \forall (x, v) \in X \times \mathbb{C},$$

and $L \longrightarrow Y$ be any real line bundle. If $\pi_X, \pi_Y: X \times Y \longrightarrow X, Y$ are the projection maps,

$$(V_L, \tilde{\phi}_L) \equiv (\pi_X^* V \otimes_{\mathbb{R}} \pi_Y^* L, \pi_X^* \tilde{\phi} \otimes_{\mathbb{R}} \pi_Y^* \text{id}_L) \longrightarrow (X \times Y, \phi \times \text{id}_Y)$$

is a real bundle pair. If

$$\pi_X^\phi, \pi_Y: \mathbb{B}_{\phi \times \text{id}_Y}(X \times Y) = (\mathbb{B}_\phi X) \times Y \longrightarrow \mathbb{B}_\phi X, Y$$

are the projection maps,

$$\begin{aligned} \mathbb{B}_{\tilde{\phi}_L}(V_L) &= \pi_X^{\phi} \mathbb{B}_{\tilde{\phi}} V \otimes_{\mathbb{R}} \pi_Y^* L \quad \implies \\ w_2^{\tilde{\phi}_L}(V_L) &= \pi_X^{\phi} w_2^{\tilde{\phi}}(V) + \pi_X^{\phi} w_1^{\tilde{\phi}}(V) \cdot \pi_Y^* w_1(L) + \pi_Y^* w_1(L)^2. \end{aligned}$$

In particular, if $X = \{\text{pt}\}$ and $L \rightarrow Y = S^1$ is the Mobius band line bundle,

$$u: S^1 \times S^1 \rightarrow X \times Y, \quad (s, t) \rightarrow (\text{pt}, s),$$

is a continuous map intertwining the involutions $\phi \times \text{id}_Y$ and $\text{id} \times \mathbf{a}$ on $S^1 \times S^1$, where \mathbf{a} is the antipodal map, such that

$$\langle w_2^{\tilde{\phi}_L}(V_L), [u]^{\text{id} \times \mathbf{a}} \rangle \neq 0.$$

Example 2.9. Let $m, n \in \mathbb{Z}^+$ and τ_n be as in (1.10). We define

$$\begin{aligned} \eta_{2m-1}: \mathbb{P}^{2m-1} &\rightarrow \mathbb{P}^{2m-1} && \text{by} \\ [W_1, W_2, \dots, W_{2m-1}, W_{2m}] &\rightarrow [-\overline{W}_2, \overline{W}_1, \dots, -\overline{W}_{2m}, \overline{W}_{2m-1}]. \end{aligned}$$

With $S^1 \subset \mathbb{C}$ denoting the unit circle as before, we define

$$\begin{aligned} \cdot: S^1 \times \mathbb{P}^n &\rightarrow \mathbb{P}^n, & v \cdot [Z_0, Z_1, \dots, Z_{n-1}, Z_n] &= [Z_0, Z_1, \dots, Z_{n-1}, vZ_n], \\ X &= S^1 \times S^1 \times \mathbb{P}^n \times \mathbb{P}^{2m-1}, \\ \phi: X &\rightarrow X, & \phi(u, v, z, w) &= (\bar{u}, v, \tau_n(v \cdot z), \eta_{2m-1}(w)), \\ Y &= \{(v, z) \in S^1 \times \mathbb{P}^n : \tau_n(v \cdot z) = z\}. \end{aligned}$$

Since the non-trivial deck transformation of the double cover

$$S^1 \times \mathbb{R}\mathbb{P}^n \rightarrow Y, \quad (v, z) \rightarrow (v^2, v^{-1} \cdot z),$$

is orientation-reversing if n is odd, Y is not orientable for every n (if n is even, the covering space is not orientable). Let $B \in H_2(X; \mathbb{Z})$ denote the homology class of a line in the last factor. Since the projections

$$\pi_1 \times \pi_2 \times \pi_3, \pi_4: X \rightarrow S^1 \times S^1 \times \mathbb{P}^n, \mathbb{P}^{2m-1}$$

induce an isomorphism

$$\mathfrak{M}_0(X, B)^{\phi, \eta} \approx \{\pm 1\} \times Y \times \mathfrak{M}_0(\mathbb{P}^{2m-1}, B)^{\eta_{2m-1}, \eta},$$

the space $\mathfrak{M}_0(X, B)^{\phi, \eta}$ is not orientable. Thus, the condition $\pi_1(X) = 0$ in Corollary 1.8 cannot be dropped, even at the cost of requiring $c_1(TX)$ to be divisible by an arbitrarily high integer.

Example 2.10. If $n, d \in \mathbb{Z}^+$ and τ_n is as in (1.10), $\mathcal{P}(\mathbb{P}^n, d)^{\tau_n, \eta}$ consists of maps of the form

$$u: \mathbb{P}^1 \longrightarrow \mathbb{P}^n, \quad [x, y] \longrightarrow [p_0(x, y), p_1(x, y), \dots, p_n(x, y)],$$

where p_0, p_1, \dots, p_n are degree d homogeneous polynomials in two variables without a common factor. The commutativity condition on u implies that this space is empty if d is odd. For d even, this commutativity condition is equivalent to

$$p_i(x, y) = A_i \prod_{r=1}^{d/2} ((a_{i;r}x - b_{i;r}y)(\bar{b}_{i;r}x + \bar{a}_{i;r}y)),$$

for some $A_i \in \mathbb{C}$ and $[a_{i;r}, b_{i;r}] \in \mathbb{P}^1$ such that

$$[A_0, A_1, \dots, A_n] = [\bar{A}_0, \bar{A}_1, \dots, \bar{A}_n] \in \mathbb{P}^n.$$

Let $\Delta_{n+1;d}^\eta \subset (\text{Sym}^{d/2}\mathbb{C})^{n+1}$ denote the image of the set

$$\left\{ ((b_{0;1}, \dots, b_{0;\frac{d}{2}}), \dots, (b_{n;1}, \dots, b_{n;\frac{d}{2}})) \in (\mathbb{C}^{d/2})^{n+1} : \bigcap_{i=0}^n \{b_{i;1}, -\bar{b}_{i;1}^{-1}, \dots, b_{i;\frac{d}{2}}, -\bar{b}_{i;\frac{d}{2}}^{-1}\} \neq \emptyset \right\}$$

under the quotient map $(\mathbb{C}^{d/2})^{n+1} \longrightarrow (\text{Sym}^{d/2}\mathbb{C})^{n+1}$. The map

$$\begin{aligned} & \mathbb{R}\mathbb{P}^n \times ((\text{Sym}^{d/2}\mathbb{C})^{n+1} - \Delta_{n+1;d}^\eta) \longrightarrow \mathcal{P}(\mathbb{P}^n, d)^{\tau_n, \eta}, \\ & ([A_0, \dots, A_n], [b_{0;1}, \dots, b_{0;d/2}], \dots, [b_{n;1}, \dots, b_{n;d/2}]) \\ & \longrightarrow \left[A_0 \prod_{r=1}^{d/2} ((x - b_{0;r}y)(\bar{b}_{0;r}x + y)), \dots, A_n \prod_{r=1}^{d/2} ((x - b_{n;r}y)(\bar{b}_{n;r}x + y)) \right], \end{aligned}$$

is an isomorphism over the open subset of $\mathcal{P}(\mathbb{P}^n, d)^{\tau_n, \eta}$ consisting of maps u such that $u([1, 0])$ does not lie in any of the coordinate subspaces of \mathbb{P}^n ; the subset $\Delta_{n+1;d}^\eta$ corresponds to polynomials with common factors and thus does not correspond to maps to \mathbb{P}^n . Since $\mathbb{R}\mathbb{P}^n$ is not orientable if n is even, it follows that $\mathcal{P}(\mathbb{P}^n, d)^{\tau_n, \eta}$ is not orientable and neither is $\mathfrak{M}_0(\mathbb{P}^n, d)^{\tau_n, \eta}$. Thus, the condition $w_2(TX) = 0$ in Corollary 1.8 cannot simply be dropped.

3. Fredholm theory

This section describes doubling constructions for oriented sh-surfaces and shows that real Cauchy-Riemann operators over such surfaces are Fredholm if the complex structure on the domain is compatible with the involution. In the case the boundary involution is trivial, the results in this section specialize to results in [15, Section 3]. However, in contrast to the situation in [15, Section 3], not every complex structure on an oriented sh-surface can be doubled and not every real Cauchy-Riemann operator is Fredholm. Corollary 3.3 below describes a necessary and sufficient condition for doubling a complex structure; it can be viewed as directly capturing the bianalytic nature of the doubling construction for Klein surfaces in [2, Section 1.6]. Remark 3.7 provides examples of real Cauchy-Riemann operators that are not Fredholm.

Let j be an almost complex structure on a bordered Riemann surface Σ . We call a smooth chart

$$\psi: (U, U \cap \partial\Sigma) \longrightarrow (\mathbb{H}, \mathbb{R}), \quad \text{where } \mathbb{H} = \{z \in \mathbb{C} : \text{Im } z \geq 0\},$$

j -holomorphic if $j = \psi^* j_0$, where j_0 is the standard complex structure on \mathbb{C} . By Corollary A.2, Σ can be covered by such charts, and so (Σ, j) is a Riemann surface in the sense of [15, Definition 3.1.4] and [17, Definition 2.5]. We thus call j simply a complex structure on Σ .

Let (Σ, c) be an oriented sh-surface. If j is a complex structure on Σ , we call c real-analytic with respect to j if for every $z \in \partial\Sigma$ there exist j -holomorphic charts

$$(3.1) \quad \psi_z: U_z \longrightarrow U'_z \quad \text{and} \quad \psi_{c(z)}: U_{c(z)} \longrightarrow U'_{c(z)},$$

where U_z and $U_{c(z)}$ are open subsets of Σ containing z and $c(z)$, respectively, and U'_z and $U'_{c(z)}$ are open subsets of \mathbb{H} , such that

$$(3.2) \quad \psi_{c(z)} \circ c \circ \psi_z^{-1}: \psi_z(U_z \cap c(U_{c(z)} \cap \partial\Sigma)) \longrightarrow \mathbb{R}$$

is a real-analytic function on an open subset of $\mathbb{R} \subset \mathbb{C}$. In particular, $\text{id}_{\partial\Sigma}$ is a real-analytic involution with respect to any complex structure j on Σ and so $\mathcal{J}_{\text{id}_\Sigma} = \mathcal{J}_\Sigma$. If c is real-analytic with respect to j , then (3.2) is real-analytic for any choice of j -holomorphic charts as in (3.1). The following lemma describes an important property of the collection \mathcal{J}_c of complex structures on Σ for which c is real-analytic with respect to j .

Lemma 3.1. *Let (Σ, c) be an oriented sh-surface. If $\mathfrak{j} \in \mathcal{J}_c$ and $h \in \mathcal{D}_c$, then $h^*\mathfrak{j} \in \mathcal{J}_c$.*

Proof. If $\{\psi_z : U_z \rightarrow \mathbb{H}\}$ are the analytic charts for (Σ, \mathfrak{j}) , then

$$h^*\psi_z \equiv \psi_z \circ h : h^{-1}(U_z) \rightarrow \mathbb{H}$$

are the analytic charts for $(\Sigma, h^*\mathfrak{j})$. Since

$$h^*\psi_{c(z)} \circ c \circ \{h^*\psi_z\}^{-1} = \psi_{c(z)} \circ c \circ \psi_z^{-1} : \psi_z(U_z \cap c(U_{c(z)} \cap \partial\Sigma)) \rightarrow \mathbb{R}$$

is real-analytic (because c is real-analytic with respect to \mathfrak{j}), it follows that c is real-analytic with respect to $h^*\mathfrak{j}$. \square

Lemma 3.2. *Let (Σ, c) be an oriented sh-surface. If \mathfrak{j} is a complex structure on Σ such that c is real-analytic with respect to \mathfrak{j} , then for every $z \in \partial\Sigma$ there exist \mathfrak{j} -holomorphic charts as in (3.1) such that*

$$(3.3) \quad c(U_z \cap \partial\Sigma) = U_{c(z)} \cap \partial\Sigma \quad \text{and} \quad \psi_z|_{U_z \cap \partial\Sigma} = \psi_{c(z)} \circ c.$$

Proof. The first condition in (3.3) can be achieved by shrinking the charts. If ψ_c and $\psi_{c(z)}$ are \mathfrak{j} -holomorphic charts as in (3.1) which satisfy the first equation in (3.3),

$$g \equiv \psi_{c(z)} \circ c \circ \psi_z^{-1} : \psi_z(U_z \cap \partial\Sigma) \rightarrow \psi_{c(z)}(U_{c(z)} \cap \partial\Sigma)$$

is a real-analytic orientation-preserving diffeomorphism between open subsets of \mathbb{R} . Let $\tilde{g} : W' \rightarrow \mathbb{C}$ be an extension of g to a holomorphic map on a neighborhood W' of $\psi_z(U_z \cap \partial\Sigma) \times 0$ in $\psi_z(U_z \cap \partial\Sigma) \times \mathbb{R}$. Since g is an orientation-preserving diffeomorphism between open subsets of \mathbb{R} (because the involution c is orientation-preserving), we can assume that \tilde{g} is a biholomorphic map taking W' onto a neighborhood W'' of $\psi_{c(z)}(U_{c(z)} \cap \partial\Sigma) \times 0$ in $\psi_{c(z)}(U_{c(z)} \cap \partial\Sigma) \times \mathbb{R}$ and $W' \cap \mathbb{H}$ onto $W'' \cap \mathbb{H}$. Replacing the chart ψ_z with $\tilde{g} \circ \psi_z|_{\psi_z^{-1}(W')}$, we obtain a \mathfrak{j} -holomorphic chart around z on Σ satisfying (3.3). \square

Let $(\hat{\Sigma}, \hat{c})$ be the double of (Σ, c) as in (1.6).

Corollary 3.3. *Let (Σ, c) be an oriented sh-surface and $\mathfrak{j} \in \mathcal{J}_\Sigma$. There exists a complex structure $\hat{\mathfrak{j}}$ on $\hat{\Sigma}$ so that $\hat{\mathfrak{j}}|_\Sigma = \mathfrak{j}$ and $\hat{c}^*\hat{\mathfrak{j}} = -\hat{\mathfrak{j}}$ if and only if c is real-analytic with respect to \mathfrak{j} .*

Proof. (1) Suppose c is real-analytic with respect to j . Given $z \in \partial\Sigma$, choose j -holomorphic charts on Σ as in Lemma 3.2. The first condition in (3.3) implies that

$$(3.4) \quad W_{[z]} \equiv (\{+\} \times U_z \sqcup \{-\} \times U_{c(z)}) / \sim \subset \hat{\Sigma}$$

is an open subset. The second condition in (3.3) implies that the map

$$\Psi_{[z]}: W_{[z]} \longrightarrow \mathbb{C}, \quad x \longrightarrow \begin{cases} \psi_z(z'), & \text{if } x = [+ , z'], \ z' \in U_z; \\ \overline{\psi_{c(z)}(z')}, & \text{if } x = [- , z'], \ z' \in U_{c(z)}; \end{cases}$$

is well-defined (agrees on the overlap of the two cases, which is when $z' \in U_z \cap \partial\Sigma$). This map is a homeomorphism onto the open subset $\psi_z(U_z) \cup \overline{\psi_{c(z)}(U_{c(z)})}$ of \mathbb{C} . If $(\psi'_z, \psi'_{c(z)})$ is another pair of charts as above with the same domains, the overlap map is given by

$$\begin{aligned} \Psi'_{[z]} \circ \Psi_{[z]}^{-1}: \Psi_{[z]}(W_{[z]}) &\longrightarrow \Psi'_{[z]}(W'_{[z]}), \\ x &\longrightarrow \begin{cases} \psi'_z(\psi_z^{-1}(x)), & \text{if } x \in \Psi_{[z]}(W_{[z]}) \cap \mathbb{H}; \\ \overline{\psi'_{c(z)}(\psi_{c(z)}^{-1}(\bar{x}))}, & \text{if } x \in \Psi_{[z]}(W_{[z]}) \cap \overline{\mathbb{H}}. \end{cases} \end{aligned}$$

Thus, the collection of our charts induces a complex structure on $\hat{\Sigma}$ that agrees with j on Σ^+ and $-j$ on Σ^- , as required.

(2) Suppose there exists a complex structure \hat{j} on $\hat{\Sigma}$ so that $\hat{j}|_{\Sigma} = j$ and $\hat{c}^* \hat{j} = -\hat{j}$. By deforming j away from $\partial\Sigma$ and collapsing circles close to $\partial\Sigma$, we can assume that $\Sigma = D^2$ and so $\hat{\Sigma} = \mathbb{P}^1$. Since $j = \hat{j}|_{\Sigma}$, the standard $\bar{\partial}$ -operator $\bar{\partial}_0$ on the trivial real bundle pair $(D^2 \times \mathbb{C}, \tilde{c}_1) \longrightarrow (D^2, c)$ is surjective and Fredholm, by the commutativity property used in the proof of Proposition 3.6. Remark 3.7 then implies that c is real-analytic with respect to j . □

Remark 3.4. The image of $\partial\Sigma$ in $\hat{\Sigma}$ is an analytic curve with respect to the doubled complex structure \hat{j} constructed in (1) of the proof of Corollary 3.3: there are charts on $\hat{\Sigma}$ taking this curve to $\mathbb{R} \subset \mathbb{C}$. There can be other complex structures \hat{j} satisfying the requirements of Corollary 3.3 for which the image of $\partial\Sigma$ is not analytic; they induce different smooth structures on $\hat{\Sigma}$ across $\partial\Sigma$. For example, let $\eta: \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be as in (1.1). Choose any simple curve in the upper-hemisphere of \mathbb{P}^1 with ends at a pair of antipodal points on the equator. Using η to double the curve, we obtain a simple closed curve which splits \mathbb{P}^1 into two halves interchanged by η ; each half is a disk with boundary

involution induced by η . For a generic choice of the arc, the closed curve is not real-analytic with respect to the standard complex structure on \mathbb{P}^1 . One can ensure that this curve is smooth at the junction by requiring it to run along the equator near its ends.

A real bundle pair $(V, \tilde{c}) \longrightarrow (\Sigma, c)$ doubles to a complex bundle

$$\hat{V} \equiv (\{+\} \times V \sqcup \{-\} \times \bar{V}) / \sim, \quad (+, v) \sim (-, \tilde{c}(v)) \quad \forall v \in V|_{\partial\Sigma},$$

over $\hat{\Sigma}$ with conjugation $\check{c}: \hat{V} \longrightarrow \hat{V}$ lifting $\hat{c}: \hat{\Sigma} \longrightarrow \hat{\Sigma}$, where \bar{V} denotes the same real vector bundle over Σ as V , but with the opposite complex structure on the fibers. We define the Maslov index of (V, \tilde{c}) by

$$\mu(V, \tilde{c}) = \langle c_1(\hat{V}), [\hat{\Sigma}] \rangle.$$

By [21, Theorem C.3.5 and (C.3.4)], this agrees with the usual definition of the Maslov index of $(V, V^{\tilde{c}})$ if $c = \text{id}_{\partial\Sigma}$. By [3, Propositions 4.1, 4.2], real bundle pairs $(V, \tilde{c}) \longrightarrow (\Sigma, c)$ are in fact classified by their rank, the Maslov index, and the orientability of $V^{\tilde{c}}$ over each boundary component $(\partial\Sigma)_i$ with $|c_i|=0$. For the sake of completeness, we confirm this for $\Sigma = D^2$, which is the only case needed for the purposes of this paper.

Lemma 3.5. *Let c be an orientation-preserving involution on $\partial D^2 = S^1$. The Maslov index classifies the real bundle pairs $(V, \tilde{c}) \longrightarrow (D^2, c)$. In particular, a rank n real bundle pair $(V, \tilde{c}) \longrightarrow (D^2, c)$ is isomorphic to the trivial one, i.e.*

$$(D^2 \times \mathbb{C}^n, \tilde{c}_n) \longrightarrow (D^2, c), \quad \tilde{c}_n(z, v) = (c(z), \bar{v}) \quad \forall (z, v) \in S^1 \times \mathbb{C}^n,$$

if and only if $\mu(V, \tilde{c}) = 0$.

Proof. If $c = \text{id}_{S^1}$, this statement follows immediately from [21, Lemma C.3.8, Corollary C.3.9] and the Normalization Property of the Maslov index in [21, Theorem C.3.5]. Thus, we can assume that c is the antipodal map on $S^1 \subset \mathbb{C}$, $V = D^2 \times \mathbb{C}^n$, and

$$\tilde{c}: S^1 \times \mathbb{C}^n \longrightarrow S^1 \times \mathbb{C}^n$$

is a conjugation covering c .

By [5, Lemma 2.2], there exists $A: S^1 \rightarrow \mathrm{GL}_n \mathbb{C}$ such that

$$\tilde{c}(z, v) = (-z, A(-z)\overline{A(z)^{-1}v}) \quad \forall (z, v) \in S^1 \times \mathbb{C}^n.$$

The loops $A_d: S^1 \rightarrow \mathrm{GL}_n \mathbb{C}$ sending $z \in S^1$ to the diagonal matrix with the first entry z^d and the remaining entries 1 represent the elements of $\pi_1(\mathrm{GL}_n \mathbb{C}) \approx \mathbb{Z}$. Thus, there exists $d \in \mathbb{Z}$ so that the map

$$(S^1, 1) \rightarrow (\mathrm{GL}_n(\mathbb{C}), \mathbb{I}_n), \quad z \rightarrow A_d(z)A(z)^{-1},$$

is homotopically trivial (with basepoints fixed) and therefore extends to a smooth map $\Psi: D^2 \rightarrow \mathrm{GL}_n \mathbb{C}$. The bundle isomorphism

$$D^2 \times \mathbb{C}^n \rightarrow D^2 \times \mathbb{C}^n, \quad (z, v) \rightarrow (z, \Psi(z)v),$$

identifies the real bundle pair $(D^2 \times \mathbb{C}^n, \tilde{c})$ with $(D^2 \times \mathbb{C}^n, \tilde{c}_{n,d})$, where

$$\tilde{c}_{n,d}(z, v) = (-z, A_d(-z)\overline{A_d(z)^{-1}v}) \quad \forall (z, v) \in S^1 \times \mathbb{C}^n.$$

The double of $(D^2 \times \mathbb{C}^n, \tilde{c}_{n,d})$,

$$\hat{V} \equiv (D_+^2 \times \mathbb{C}^n \sqcup D_-^2 \times \overline{\mathbb{C}^n}) / \sim, \quad (+, z, v) \sim (-, -z, A_d(-z)\overline{A_d(z)^{-1}v}) \\ \forall (z, v) \in S^1 \times \mathbb{C}^n,$$

has trivializations

$$\hat{V}|_{D_+^2} \rightarrow D_+^2 \times \mathbb{C}^n, \quad [+ , z, v] \rightarrow ([+ , z], v), \\ \hat{V}|_{D_-^2} \rightarrow D_-^2 \times \mathbb{C}^n, \quad [- , z, v] \rightarrow ([- , z], \bar{v}).$$

The overlap between these trivializations is given by

$$D_+^2 \cap D_-^2 \times \mathbb{C}^n \rightarrow D_+^2 \cap D_-^2 \times \mathbb{C}^n, \\ ([+ , z], v) \rightarrow ([+ , z], \overline{A_d(-z)A_d(z)^{-1}v}) = ([+ , z], (-1)^d A_{-2d}(z)v).$$

Thus, $\hat{V} \approx \mathcal{O}(2d) \oplus (n-1)\mathcal{O}$, where $\mathcal{O}, \mathcal{O}(2d) \rightarrow \mathbb{P}^1$ are the trivial complex line bundle and the $2d$ -th power of the hyperplane line bundle, respectively. It follows that

$$\mu(D^2 \times \mathbb{C}^n, \tilde{c}_{n,d}) \equiv \langle c_1(\hat{V}), [\mathbb{P}^1] \rangle = 2d.$$

This establishes both claims of the proposition. □

Let j be a complex structure on Σ so that c is real-analytic with respect to j . Every real Cauchy-Riemann operator D on $(V, \tilde{c}) \rightarrow (\Sigma, c)$ as in (1.3) compatible with j doubles to a real Cauchy-Riemann operator

$$\hat{D}: W^{1,p}(\hat{\Sigma}; \hat{V}) \rightarrow W^{0,p}(\hat{\Sigma}; (T^*\hat{\Sigma}, \hat{j}) \otimes_{\mathbb{C}} \hat{V}),$$

where $p > 2$ and $W^{1,p}$ and $W^{0,p}$ denote Sobolev completions with respect to some metrics on Σ and V (doubled to $\hat{\Sigma}$ and \hat{V}) on the appropriate spaces of bundle sections, by

$$\hat{D}\xi|_{\Sigma^+} = D\xi, \quad \hat{D}\xi|_{\Sigma^-} = \check{c} \circ D(\check{c} \circ \xi \circ \hat{c}) \circ d\hat{c} \quad \forall \xi \in W^{1,p}(\hat{\Sigma}; \hat{V}).$$

Since the image of \hat{D} lies in $W^{0,p}$, there is no overlap condition for \hat{D} along $\partial\Sigma$ to be checked. This operator satisfies

$$\hat{D}(\check{c} \circ \xi \circ \hat{c}) = \check{c} \circ \{\hat{D}\xi\} \circ d\hat{c}.$$

In particular, \hat{D} takes the complementary subspaces

$$\begin{aligned} W^{1,p}(\hat{\Sigma}; \hat{V})^{\check{c}} &\equiv \{\xi \in W^{1,p}(\hat{\Sigma}; \hat{V}) : \check{c} \circ \xi \circ \hat{c} = \xi\}, \\ i W^{1,p}(\hat{\Sigma}; \hat{V})^{\check{c}} &= \{\xi \in W^{1,p}(\hat{\Sigma}; \hat{V}) : \check{c} \circ \xi \circ \hat{c} = -\xi\} \end{aligned}$$

of $W^{1,p}(\hat{\Sigma}; \hat{V})$ to the complementary subspaces

$$\begin{aligned} W^{0,p}(\hat{\Sigma}; (T^*\hat{\Sigma}, \hat{j}) \otimes_{\mathbb{C}} \hat{V})^{\check{c}} &\equiv \{\eta \in W^{0,p}(\hat{\Sigma}; (T^*\hat{\Sigma}, \hat{j}) \otimes_{\mathbb{C}} \hat{V}) : \check{c} \circ \eta \circ d\hat{c} = \eta\}, \\ i W^{0,p}(\hat{\Sigma}; (T^*\hat{\Sigma}, \hat{j}) \otimes_{\mathbb{C}} \hat{V})^{\check{c}} &= \{\eta \in W^{0,p}(\hat{\Sigma}; (T^*\hat{\Sigma}, \hat{j}) \otimes_{\mathbb{C}} \hat{V}) : \check{c} \circ \eta \circ d\hat{c} = -\eta\}, \end{aligned}$$

respectively, of $W^{0,p}(\hat{\Sigma}; (T^*\hat{\Sigma}, \hat{j}) \otimes_{\mathbb{C}} \hat{V})$.

Proposition 3.6. *Let (Σ, c) be an oriented sh-surface, $(V, \tilde{c}) \rightarrow (\Sigma, c)$ be a real bundle pair, $j \in \mathcal{J}_c$, and $p > 2$. A real Cauchy-Riemann operator D on (V, \tilde{c}) compatible with j induces a Fredholm operator between $W^{1,p}$ and W^p -completions of its domain and target, respectively, with*

$$\text{ind}_{\mathbb{R}} D = \mu(V, \tilde{c}) + (1 - g(\hat{\Sigma}))(\text{rk}_{\mathbb{C}} V),$$

where $g(\hat{\Sigma})$ is the genus of $\hat{\Sigma}$. Furthermore, the kernel of the standard $\bar{\partial}$ -operator on the real bundle pair $(\Sigma \times \mathbb{C}^n, \tilde{c}) \rightarrow (\Sigma, c)$ with \tilde{c} induced by the standard conjugation on \mathbb{C}^n consists of constant \mathbb{R}^n -valued functions on Σ ; this operator is surjective if $\Sigma = D^2$.

Proof. Since c is real-analytic with respect to the complex structure j corresponding to D , we have a commutative diagram

$$\begin{CD} W^{1,p}(\hat{\Sigma}; \hat{V})^{\tilde{c}} @>\hat{D}>> W^{0,p}(\hat{\Sigma}; (T^*\hat{\Sigma}, \hat{j}) \otimes_{\mathbb{C}} \hat{V})^{\tilde{c}} \\ @VVV @VVV \\ W^{1,p}(\Sigma; V)^c @>D>> W^{0,p}(\Sigma; (T^*\Sigma, j) \otimes_{\mathbb{C}} V) \end{CD}$$

where the vertical arrows are the restriction isomorphisms $\xi \rightarrow \xi|_{\Sigma^+}$. Since \hat{D} preserves the ± 1 -eigenspaces of \tilde{c} , $W^{k,p}$ and $iW^{k,p}$ above, this diagram induces isomorphisms

$$\ker \hat{D}^+ \approx \ker D, \quad \text{Im } \hat{D}^+ \approx \text{Im } D, \quad \text{cok } \hat{D}^+ \approx \text{cok } D,$$

where \hat{D}^{\pm} is the restriction of \hat{D} to the ± 1 -eigenspace of \tilde{c} . Since \hat{D} is Fredholm, it follows that so is D . The index of D is the same as the index of its \mathbb{C} -linear part $D^{1,0}$. Since multiplication by i commutes with $\hat{D}^{1,0}$, it induces isomorphisms

$$\ker(\hat{D}^{1,0})^+ \rightarrow \ker(\hat{D}^{1,0})^-, \quad \text{cok}(\hat{D}^{1,0})^+ \rightarrow \text{cok}(\hat{D}^{1,0})^-.$$

Thus,

$$\begin{aligned} \text{ind}_{\mathbb{R}} D &= \text{ind}_{\mathbb{R}} D^{1,0} = \text{ind}_{\mathbb{R}}(\hat{D}^{1,0})^+ = \frac{1}{2} \text{ind}_{\mathbb{R}} \hat{D}^{1,0} \\ &= \langle c_1(\hat{V}), [\hat{\Sigma}] \rangle + (1 - g(\hat{\Sigma}))(\text{rk}_{\mathbb{C}} V); \end{aligned}$$

the last equality follows from Riemann-Roch for a closed complex curve; see [21, Theorem C.1.10(ii)], for example.

If D is the standard $\bar{\partial}$ -operator on the trivial real bundle pair $(\Sigma \times \mathbb{C}^n, \tilde{c}_n) \rightarrow (\Sigma, c)$, \hat{D} is the standard $\bar{\partial}$ -operator in a trivial vector bundle over $\hat{\Sigma}$ with the standard conjugation. This implies the last claim. \square

Remark 3.7. We now show that the standard $\bar{\partial}$ -operator $\bar{\partial}_0$ on $(D^2 \times \mathbb{C}, \tilde{c}) \rightarrow (D^2, c)$, where \tilde{c} is the lift of the involution c on $S^1 = \partial \Sigma$ induced by the standard conjugation in \mathbb{C} , has infinite-dimensional cokernel if c is not real-analytic with respect to the standard complex structure j_0

on D^2 . Specifically, we show that

$$\{\bar{z}^{2k-1}d\bar{z} : k \in \mathbb{Z}^+\} \cap \bar{\partial}_0(W^{1,p}(D^2)^{\tilde{c}}) = \{0\} \subset W^{0,p}(D^2; (T^*D^2, j_0))$$

if c is not real-analytic. If $k \in \mathbb{Z}^+$,

$$\{f \in W^{1,p}(D^2) : \bar{\partial}f = k\bar{z}^{2k-1}d\bar{z}\} = \{\operatorname{Re} z^{2k} + h : h \in \operatorname{Hol}(D^2)\},$$

where $\operatorname{Hol}(D^2)$ is the space of continuous maps on the closed disk D^2 that are holomorphic in the interior. The condition that $\operatorname{Re} z^{2k} + h$ lies in $W^{1,p}(D^2)^{\tilde{c}}$ is equivalent to

$$\operatorname{Re}(z^{2k} + h(z)) = \operatorname{Re}(c(z)^{2k} + h(c(z))), \quad \operatorname{Im} h(z) = -\operatorname{Im} h(c(z)).$$

The functions $\operatorname{Re}(z^{2k} + h(z))$ and $\operatorname{Im} h(z)$ are real-analytic on S^1 . If c is not real-analytic, the above conditions imply that

$$(3.5) \quad \operatorname{Re}(z^{2k} + h(z)) = C, \quad \operatorname{Im} h(z) = 0 \quad \forall z \in S^1,$$

for some $C \in \mathbb{R}$, which we can take to be 0. Indeed, if $f(z) = \operatorname{Re}(z^{2k} + h(z))$, $\operatorname{Im} h(z)$ were not constant on S^1 , we could choose an analytic coordinate θ near any point θ_0 on S^1 and an analytic coordinate ϑ near the point $\vartheta_0 = c(\theta_0)$ on S^1 so that

$$f(\vartheta) - f(\vartheta_0) = \pm \vartheta^m, \quad f(c(\theta)) - f(\vartheta_0) = \pm \theta^n$$

for some $m, n \in \mathbb{Z}^+$. Since c is smooth, $n|m$ and so $\vartheta(c(\theta)) = \pm \theta^{n/m}$ is real-analytic at θ_0 . This confirms (3.5). Finally, (3.5) with $C = 0$ implies that

$$h^{(m)}(0) = \frac{m!}{2\pi i} \oint_{|z|=1} \frac{-\operatorname{Re} z^{2k} dz}{z^{m+1}} = \begin{cases} -\frac{(2k)!}{2} & \text{if } m = 2k; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $h(z) = -\frac{1}{2}z^{2k}$, which contradicts (3.5).

4. Proofs of main statements

We begin this section by recalling some standard facts concerning determinant lines of real Cauchy-Riemann operators, rephrasing the first half of [9, Section 2] in terms of real bundle pairs $(V, \tilde{c}) \rightarrow (\Sigma, c)$, instead of bundles $(V, V^{\tilde{c}}) \rightarrow (\Sigma, \partial\Sigma)$ of the $|c|_1 = 0$ case. We then deduce Theorem 1.1 from [9,

Theorem 1.1] and Lemma 4.2. The latter treats a very special case of Theorem 1.1 and is the analogue of [9, Lemmas 3.4, 3.6] for the non-trivial involutions c_i on $(\partial\Sigma)_i$. We conclude this section with a set of lemmas extending [9, Lemmas 2.2-2.4] to arbitrary boundary involutions c .

A short exact sequence of Fredholm operators

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & & & \downarrow D' & & \downarrow D & & \downarrow D'' \\ 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0 \end{array}$$

determines a canonical isomorphism

$$(4.1) \quad \det D \approx (\det D') \otimes (\det D'').$$

For a continuous family of Fredholm operators $D_t : X_t \rightarrow Y_t$ parametrized by a topological space B , the determinant lines $\det D_t$ form a line bundle over B ; see [21, Section A.2] and [31]. For a short exact sequence of such families, the isomorphisms (4.1) give rise to a canonical isomorphism between determinant line bundles.

Let (Σ, c) be an oriented, possibly nodal, sh-surface, with nodes away from the boundary and $j \in \mathcal{J}_c$. Let $\pi : \tilde{\Sigma} \rightarrow \Sigma$ be the normalization of Σ . Fix an ordering of the boundary components of Σ (and thus of $\tilde{\Sigma}$) and of the nodes of Σ . A real Cauchy-Riemann operator D on a real bundle pair $(V, \tilde{c}) \rightarrow (\Sigma, c)$ corresponds to a real Cauchy-Riemann operator $\tilde{D} = \bigoplus_i D^i$ on $\pi^*(V, c) \rightarrow (\tilde{\Sigma}, c)$, where the sum is taken over the components of $\tilde{\Sigma}$. Thus, by (4.1), there is a canonical isomorphism

$$\det \tilde{D} \approx \bigotimes_i (\det D^i).$$

On the other hand, gluing together punctured disks around the nodes z_j of Σ , we obtain a smooth surface Σ^ε and a real Cauchy-Riemann operator D^ε over (Σ^ε, c) for a gluing parameter ε . Similarly to [14, Appendix D.4] and [4, Section 3.2], for every sufficiently small ε there is a canonical (up to homotopy) isomorphism

$$(4.2) \quad \det D^\varepsilon \approx (\det \tilde{D}) \otimes \Lambda_{\mathbb{R}}^{\text{top}} \left(\bigoplus_j V_{z_j} \right)^*.$$

Moreover, the gluing maps satisfy an associativity property: the isomorphism (4.2) is independent of the order in which we smooth the nodes.

Remark 4.1. The space of real Cauchy-Riemann operators on $(V, \tilde{c}) \rightarrow (\Sigma, c)$ is contractible; thus, a choice of orientation on one determinant line canonically induces orientations on the rest. Any two families of real Cauchy-Riemann operators on a family $(V_t, \tilde{c}_t) \rightarrow (\Sigma_t, c_t)$ are fiberwise homotopic. This implies that their determinant bundles have the same Stiefel-Whitney classes.

Proof of Theorem 1.1. By Lemma 4.4, we can assume that ψ restricts to the identity in a neighborhood of the boundary. For each boundary component $(\partial\Sigma)_i$ of Σ with $|c_i|=1$, let

$$(4.3) \quad U_i = S^1 \times (\partial\Sigma)_i \times [0, 2\epsilon] \approx S^1 \times \text{Cyl}$$

be a neighborhood of $S^1 \times (\partial\Sigma)_i$ in M_ψ and

$$\tilde{U}_i = \mathbb{I} \times (\partial\Sigma)_i \times [0, 2\epsilon]$$

be the corresponding neighborhood of $\mathbb{I} \times (\partial\Sigma)_i \subset \mathbb{I} \times \Sigma$. By the proof of [9, Proposition 3.1], we can assume that the identification (4.3) commutes with the complex structures on the fibers over S^1 .

By [5, Lemma 2.2],

$$(V, \tilde{c})|_{S^1 \times (\partial\Sigma)_i} \approx (\mathbb{I} \times (S^1 \times \mathbb{C}^n, \tilde{c}_i)) / \sim, \quad (0, z, v) \sim (1, z, g_i(z)v) \\ \forall (z, v) \in (\partial\Sigma)_i \times \mathbb{C}^n,$$

where \tilde{c}_i denotes the lift of c_i induced by the standard conjugation on \mathbb{C}^n and

$$g_i : (\partial\Sigma)_i \rightarrow \text{GL}_n\mathbb{C} \quad \text{s.t.} \quad g_i(c(z)) = \overline{g_i(z)} \quad \forall z \in (\partial\Sigma)_i;$$

in fact, g_i can be taken to be a constant function with values in the diagonal matrices, with at most one diagonal entry -1 and the remaining diagonal entries 1 . Since such g_i is homotopically trivial, it can be extended to a map

$$g_i : (\partial\Sigma)_i \times [0, 2\epsilon] \rightarrow \text{GL}_n\mathbb{C} \quad \text{s.t.} \quad g_i|_{(\partial\Sigma)_i \times [\epsilon/2, 2\epsilon]} = \text{Id.}$$

For all i with $|c_i|=1$ and $t \in S^1$, pinch $t \times \Sigma$ along the curve $t \times (\partial\Sigma)_i \times \epsilon$ to obtain a nodal curve Σ^s with normalization consisting of a disjoint union of disks D_i^2 with $|c_i|=1$ and a Riemann surface Σ' , whose boundary components are the boundary components $(\partial\Sigma)_i$ of Σ with $|c_i|=0$, with special points $0 \in D_i^2$ and $p_i \in \Sigma'$ with $|c_i|=1$. The real bundle pair (V, \tilde{c}) descends

to a real bundle pair over the family of nodal curves as in [9, Remark 2.1], inducing bundle pairs

$$(V', \tilde{c}') \longrightarrow S^1 \times (\Sigma', \partial\Sigma'),$$

$$(V_i, \tilde{c}_i) \equiv \mathbb{I} \times_{g_i} (D^2 \times \mathbb{C}^n, \tilde{c}_i) \longrightarrow S^1 \times (D_i^2, c_i),$$

with $\mu(V_i, \tilde{c}_i) = 0$ and with isomorphisms $V'|_{t \times p_i} \approx \mathbb{C}^n \approx V_i|_{t \times 0}$ for every $t \in S^1$.

Taking a family of real Cauchy-Riemann operators D' on (V', \tilde{c}') and gluing it to a family of real Cauchy-Riemann operators D_i on (V_i, \tilde{c}_i) , we obtain a family of real Cauchy-Riemann operators D^ε on (V, \tilde{c}) . By Remark 4.1 and (4.2),

$$(4.4) \quad \det D_{(V, \tilde{c})} \approx \det D^\varepsilon \approx (\det D') \otimes \bigotimes_{|c_i|=1} ((\det D_i) \otimes \Lambda_{\mathbb{R}}^{\text{top}}(V'|_{(t, p_i)})).$$

Thus,

$$w_1(\det D) = w_1(\det D') + \sum_{|c_i|=1} (w_1(\det D_i) + w_1(V'|_{S^1 \times p_i})).$$

The complex structure on $V'|_{S^1 \times p_i}$ induces a canonical orientation on this space; in particular, $w_1(V'|_{S^1 \times p_i}) = 0$. The term $w_1(\det D')$ is given by [9, Theorem 1.1]. Therefore, the problem reduces to the families of operators D_i on (V_i, \tilde{c}_i) over $S^1 \times (D_i^2, c_i)$. Theorem 1.1 now follows from Lemma 4.2. \square

Lemma 4.2. *Let $c: S^1 \times \partial D^2 \longrightarrow S^1 \times \partial D^2$ be a fiberwise orientation-preserving involution different from the identity and $(V, \tilde{c}) \longrightarrow (S^1 \times D^2, c)$ be a real bundle pair with $\mu(V, \tilde{c}) = 0$ on each fiber. If D is any family of real Cauchy-Riemann operators on (V, \tilde{c}) over S^1 , then*

$$\langle w_1(\det D), S^1 \rangle = \langle w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{c}}(\Lambda_{\mathbb{C}}^{\text{top}} V), [S^1 \times \partial D^2]^c \rangle.$$

Proof. Let $n = \text{rk}_{\mathbb{C}} V$. Denote by

$$(V_{\pm}, \tilde{c}_{\pm}) \longrightarrow (S^1 \times D^2, c)$$

the real bundle pairs with

$$V_{\pm} = (\mathbb{I} \times D^2 \times \mathbb{C}) / \sim, \quad (0, z, v) \sim (1, z, \pm v) \quad \forall z \in D^2, v \in \mathbb{C},$$

with the involutions induced by the standard conjugation on \mathbb{C} .

By Proposition 3.6, the standard $\bar{\partial}_0$ -operator on the trivial bundle pair

$$(t \times D^2 \times \mathbb{C}^n, \tilde{c}_n) \longrightarrow (t \times D^2, c|_{t \times S^1}), \quad t \in S^1,$$

is surjective and its kernel consists of constant real-valued functions. Thus, the index bundle of the family of the standard $\bar{\partial}_0$ -operators on the trivial rank n real bundle pair nV_+ (direct sum of n copies of V_+) is isomorphic to $S^1 \times \mathbb{R}^n$ by evaluation at a boundary point and in particular is orientable. On the other hand, the index bundle of the family of the standard $\bar{\partial}_0$ -operators on

$$V_- \oplus (n-1)V_+ \longrightarrow S^1 \times D^2$$

is the direct sum of the Mobius line bundle over S^1 and $n-1$ copies of the trivial real line bundle; in particular, it is non-orientable. By Remark 4.1, the determinant bundle of any family of real Cauchy-Riemann operators on a trivializable real bundle pair as in the statement of the lemma is thus orientable and on a real bundle isomorphic to $V_- \oplus (n-1)V_+$ is not.

By Lemma 3.5, every bundle pair (V, \tilde{c}) as in the statement of the lemma is isomorphic the bundle pair

$$(\mathbb{I} \times D^2 \times \mathbb{C}) / \sim, \quad (0, z, v) \sim (1, z, A(z)v) \quad \forall z \in D^2, v \in \mathbb{C},$$

for some smooth map $A: D^2 \longrightarrow \text{GL}_n \mathbb{C}$ such that $A(c(z)) = \overline{A(z)}$ for all $z \in S^1$. By [5, Lemma 2.2], there are two homotopy classes of such maps; they are represented by the constant maps with values in the diagonal matrices, with at most one diagonal entry -1 and the remaining diagonal entries 1. Lemmas 2.5 and 2.6 then imply that $w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{c}}(\Lambda_{\mathbb{C}}^{\text{top}} V)$ classifies the rank n real bundle pairs (V, \tilde{c}) as in the statement of Lemma 4.2. Thus, if $w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{c}}(\Lambda_{\mathbb{C}}^{\text{top}} V) = 0$, (V, \tilde{c}) is trivializable; by the previous paragraph, $\det D$ is orientable in this case. On the other hand, if $w_2^{\Lambda_{\mathbb{C}}^{\text{top}} \tilde{c}}(\Lambda_{\mathbb{C}}^{\text{top}} V) \neq 0$, (V, \tilde{c}) is isomorphic to the twisted pair of the previous paragraph and thus $\det D$ is not orientable. Combining the two cases, we obtain the claim. \square

The next three lemmas are used in the proof of Theorem 1.1 and in some of its applications. In particular, in some situations they allow us to replace arbitrary diffeomorphisms of (Σ, c) by those that restrict to the identity near $\partial \Sigma$.

Lemma 4.3. *Let (Σ, c) be an oriented sh-surface. For every $h_0 \in \mathcal{D}_c$, there exists a path h_t in \mathcal{D}_c starting at h_0 such that h_1 restricts to the identity on a neighborhood of $\partial \Sigma$ in Σ .*

Proof. Fix a component $(\partial\Sigma)_i \simeq S^1$ of $\partial\Sigma$, an identification of a neighborhood of $(\partial\Sigma)_i$ in Σ with $S^1 \times [0, 2\delta]$, and $\epsilon \in (0, \delta/2)$ such that

$$h_0(S^1 \times [0, 2\epsilon]) \subset S^1 \times [0, \delta].$$

After composing h_0 with a path of diffeomorphisms on Σ which restrict to the identity outside $S^1 \times (0, 2\delta)$, we can assume that $h_0(S^1 \times [0, 2\epsilon]) = S^1 \times [0, 2\epsilon]$.

By [6, Proposition 2.4] and [19, (1.1)], the group of diffeomorphisms of the cylinder preserving the orientation and each boundary component is path-connected. In particular, there exists a path of diffeomorphisms

$$(4.5) \quad f_t : S^1 \times [0, 2\epsilon] \longrightarrow S^1 \times [0, 2\epsilon] \quad \text{s.t.} \quad f_0 = h_0, \quad f_1 = \tilde{h}_0,$$

where $\tilde{h}_0(z, s) = (\pi_1(h_0(z, 0)), s) \quad \forall (z, s) \in S^1 \times [0, 2\epsilon].$

Replacing f_t with $\tilde{h}_0 \circ \tilde{f}_t^{-1} \circ f_t$, with \tilde{f}_t defined analogously to \tilde{h}_0 , we obtain a path of diffeomorphisms f_t as in (4.5) that restrict to h_0 on $S^1 \times 0$. Thus, after composing h_0 with a path of diffeomorphisms on Σ that restrict to the identity outside $S^1 \times (0, 2\epsilon)$, we can assume that $h_0 = \tilde{h}_0$ on $S^1 \times [0, \epsilon]$; such a path is constructed from a path of diffeomorphisms on $S^1 \times [0, 2\epsilon]$ using vector fields as below.

Since $z \rightarrow h_0(z, 0)$ commutes with the involution c_i , h_0 descends to a diffeomorphism h'_0 on the quotient $(S^1/c_i) \times [0, \epsilon]$. By [6, Proposition 2.4] and [19, (1.1)], there is a path of diffeomorphisms

$$f'_t : (S^1/c_i) \times [0, \epsilon] \longrightarrow (S^1/c_i) \times [0, \epsilon] \quad \text{s.t.} \quad f'_0 = \text{id}, \quad f'_1 = h'^{-1}_0.$$

This path lifts to a path of diffeomorphisms

$$f_t : S^1 \times [0, \epsilon] \longrightarrow S^1 \times [0, \epsilon]$$

s.t. $f_0 = \text{id}, \quad f_1 = h_0^{-1}|_{S^1 \times [0, \epsilon]}, \quad c_i \circ f_t|_{S^1 \times 0} = f_t \circ c_i.$

The path f_t generates a time-dependent vector field X_t . Multiplying X_t by a bump function on Σ vanishing outside $[0, \epsilon]$ and restricting to 1 on $S^1 \times [0, \epsilon/2]$, we obtain a time-dependent vector field \tilde{X}_t on Σ . This new vector field gives rise to diffeomorphisms \tilde{f}_t of Σ which restrict to the identity outside $S^1 \times [0, \epsilon]$, while \tilde{f}_1 restricts to h_0^{-1} on $S^1 \times [0, \epsilon/2]$. Then $h_0 \circ \tilde{f}_t$ is a path in \mathcal{D}_c connecting h_0 with a diffeomorphism which restricts to the identity in a neighborhood of $(\partial\Sigma)_i$. □

Lemma 4.4. *Let (Σ, c) be an oriented sh-surface and $\psi \in \mathcal{D}_c$. Every family of real Cauchy-Riemann operators on a real bundle pair (V, \tilde{c}) over M_ψ*

with D_t compatible with some $j_t \in \mathcal{J}_c$ for each $t \in \mathbb{I}$ can be smoothly deformed through such families to a family of real Cauchy-Riemann operators on a bundle pair (V', \tilde{c}') over $M_{\psi'}$ for some $\psi' \in \mathcal{D}_c$ such that ψ' restricts to the identity on a neighborhood of $\partial\Sigma$.

Proof. By Lemma 4.3, there exists a path h_s in \mathcal{D}_c such that $h_0 = \psi$ and h_1 restricts to the identity on a neighborhood of $\partial\Sigma$ in Σ . Set $f_s = \psi^{-1} \circ h_s$. Let $(j_t, V_t, \tilde{c}_t, D_t)$, with $t \in \mathbb{I}$, be any family of tuples such that $j_t \in \mathcal{J}_c$, D_t is a real Cauchy-Riemann operator on (V_t, \tilde{c}_t) over (Σ, c) compatible with j_t , and

$$(j_1, V_1, \tilde{c}_1, D_1) = \psi^*(j_0, V_0, \tilde{c}_0, D_0).$$

For each $s \in \mathbb{I}$, let

$$(j_{s;t}, V_{s;t}, \tilde{c}_{s;t}, D_{s;t}) = f_{st}^*(j_t, V_t, \tilde{c}_t, D_t).$$

Since $(j_{s;1}, V_{s;1}, \tilde{c}_{s;1}, D_{s;1}) = h_s^*(j_{s;0}, V_{s;0}, \tilde{c}_{s;0}, D_{s;0})$, this defines families of real Cauchy-Riemann operators on the real bundle pairs (V_s, \tilde{c}_s) over M_{h_s} . Since $h_0 = \psi$, we have thus constructed the desired deformation of the original family. \square

Lemma 4.5. *Let (X, ϕ) be a smooth manifold with an involution, (Σ, c) be an oriented sh-surface, and \mathbf{b} be a tuple as in (1.7). Every loop γ in $\mathcal{H}_g(X, \mathbf{b})^{\phi, c}$ lifts to a path $\tilde{\gamma}$ in $\mathfrak{B}_g(X, \mathbf{b})^{\phi, c} \times \mathcal{J}_c$ such that $\tilde{\gamma}_1 = \psi \cdot \tilde{\gamma}_0$ for some $\psi \in \mathcal{D}_c$ with $\psi|_{\partial\Sigma} = \text{id}$.*

Proof. Under the assumption of Remark 1.5, the projection

$$\mathfrak{B}_g(X, \mathbf{b})^{\phi, c} \times \mathcal{J}_c \longrightarrow \mathcal{H}_g(X, \mathbf{b})^{\phi, c}$$

admits local slices. Thus, there exists a path $\tilde{\gamma}_t = (u_t, j_t)$ in $\mathfrak{B}_g(X, \mathbf{b})^{\phi, c} \times \mathcal{J}_c$ lifting γ . Let $\psi \in \mathcal{D}_c$ be such that $\tilde{\gamma}_1 = \psi \cdot \tilde{\gamma}_0$. By Lemma 4.3, there exists a path h_t in \mathcal{D}_c such that $h_0 = \psi$ and h_1 restricts to the identity on the boundary. The lift $\tilde{\gamma}'_t = h_t \cdot \psi^{-1} \cdot \tilde{\gamma}_t$ of γ then satisfies $\tilde{\gamma}'_1 = h_1 \cdot \tilde{\gamma}'_0$. \square

5. Local systems of orientations

This section extends [9, Section 4] to arbitrary boundary involutions and reformulates Corollary 1.6 in terms of local systems of orientations, making it easier to compare systems of orientations induced from different bundles as in Remark 1.2. For the sake of completeness, we begin by recalling the

basics of local systems following [29]. We continue by constructing a local system $\mathcal{Z}_{(w_1, w_2)}^{\phi, c}$ on the product of $|c|_1$ copies of the equivariant free loop space of X and $|c|_0$ copies of the ϕ -fixed locus X^ϕ and its free loop space $\mathcal{L}(X^\phi)$. We then show that its pull-back is isomorphic to the local system twisted by the first Stiefel-Whitney class of $\det D_{(V, \tilde{\phi})}$.

Definition 5.1. A system of local groups \mathcal{G} on a path-connected topological space L consists of

- a group G_x for every $x \in L$ and
- a group isomorphism $\alpha_{xy} : G_x \rightarrow G_y$ for every homotopy class α_{xy} of paths from x to y

such that the composition $\beta_{yz} \circ \alpha_{xy}$ is the isomorphism corresponding to the path $\alpha_{xy}\beta_{yz}$.

Lemma 5.2 ([29, Theorem 1]). *Let L be a path-connected topological space, $p_0 \in L$, and G be a group. For every group homomorphism $\psi : \pi_1(L, p_0) \rightarrow \text{Aut}(G)$, there is a unique system $\mathcal{G}_\psi = \{G_x\}$ of local groups on L such that $G_{p_0} = G$ and the operations of $\pi_1(L, p_0)$ on G_{p_0} are those determined by ψ .*

Two local systems \mathcal{G} and \mathcal{G}' on L are isomorphic if for every point $x \in L$ there is an isomorphism $h_x : G_x \approx G'_x$ such that $\alpha_{xy}h_x = h_y\alpha_{xy}$ for every path α_{xy} between x and y . Equivalently, two local systems are isomorphic if the groups G and G' are isomorphic and the induced actions of $\pi_1(L, x_0)$ are the same. There are $\text{Aut}(G)$ of such isomorphisms, and one is fixed by a choice of an isomorphism $G_{x_0} \approx G'_{x_0}$.

A continuous map $f : (L_1, p_1) \rightarrow (L_2, p_2)$ naturally pulls back a local system \mathcal{G} on L_2 to a local system $f^*\mathcal{G}$ on L_1 . If L_1 and L_2 are path-connected and \mathcal{G} is induced by a group homomorphism $\psi : \pi_1(L_2, p_2) \rightarrow \text{Aut}(G)$, then $f^*\mathcal{G}$ is induced by the group homomorphism

$$\psi \circ f_\# : \pi_1(L_1, p_1) \rightarrow \text{Aut}(G),$$

where $f_\# : \pi_1(L_1, p_1) \rightarrow \pi_1(L_2, p_2)$. The local system of orientations for a vector bundle $V \rightarrow L$, denoted by $\mathcal{Z}_{w_1(V)}$, is the system induced by the homomorphism

$$\psi : \pi_1(L, p_0) \rightarrow \text{Aut}(\mathbb{Z}) = \mathbb{Z}_2, \quad \alpha \rightarrow \langle w_1(V), \alpha \rangle.$$

Let (X, ϕ) be a topological space with an involution and $(V, \tilde{\phi}) \longrightarrow (X, \phi)$ be a real bundle pair. Fix base points p_i, γ_j , and Γ_k for the connected components $X_i^\phi, \mathcal{L}(X^\phi)_j$, and $\mathcal{L}(\mathbb{B}_\phi X)_k$ of $X^\phi, \mathcal{L}(X^\phi)$, and $\mathcal{L}(\mathbb{B}_\phi X)$, respectively. Let $\mathcal{Z}_{w_1, w_2}^{\tilde{\phi}}$ be the local system on $X^\phi \times \mathcal{L}(X^\phi)$ corresponding to the homomorphism

$$\psi : \pi_1(X_i^\phi \times \mathcal{L}(X^\phi)_j, p_i \times \gamma_j) = \pi_1(X_i^\phi, p_i) \times \pi_1(\mathcal{L}(X^\phi)_j, \gamma_j) \longrightarrow \text{Aut}(\mathbb{Z}) = \mathbb{Z}_2,$$

$$(\alpha, \beta) \longrightarrow (\langle w_1(V^{\tilde{\phi}}), \gamma_j \rangle + 1) \langle w_1(V^{\tilde{\phi}}), [\alpha] \rangle + \langle w_2(V^{\tilde{\phi}}), [\beta] \rangle,$$

and $\mathcal{Z}_{w_2}^{\tilde{\phi}}$ be the local system on $\mathcal{L}(\mathbb{B}_\phi X)$ corresponding to the homomorphism

$$\psi : \pi_1(\mathcal{L}(\mathbb{B}_\phi X)_k, \Gamma_k) \longrightarrow \text{Aut}(\mathbb{Z}), \quad \beta \longrightarrow \langle w_2^{\tilde{\phi}}(V), [\beta] \rangle.$$

If c is a boundary involution on an oriented surface Σ , we define $\mathcal{Z}_{(w_1, w_2)}^{\tilde{\phi}, c}$ on

$$X_{\phi, c} \equiv (X^\phi \times \mathcal{L}(X^\phi))^{|c|_0} \times \mathcal{L}(\mathbb{B}_\phi X)^{|c|_1}$$

to be the pull-back of the local systems $\mathcal{Z}_{w_1, w_2}^{\tilde{\phi}}$ and $\mathcal{Z}_{w_2}^{\tilde{\phi}}$ by the projection maps. Thus, the restriction of this system to a component of $X_{\phi, c}$ with a basepoint

$$(5.1) \quad (\vec{p}, \vec{\gamma}, \vec{\Gamma}) \equiv (p_1, \gamma_1, \dots, p_{|c|_0}, \gamma_{|c|_0}, \Gamma_{|c|_0+1}, \dots, \Gamma_{|c|_0+|c|_1})$$

is given by the homomorphism

$$\psi : \pi_1(X_{\phi, c}, (\vec{p}, \vec{\gamma}, \vec{\Gamma})) \longrightarrow \text{Aut}(\mathbb{Z}) = \mathbb{Z}_2,$$

$$(5.2) \quad (\alpha_1, \beta_1, \dots, \alpha_{|c|_0}, \beta_{|c|_0}, \beta_{|c|_0+1}, \dots, \beta_{|c|_0+|c|_1})$$

$$\longrightarrow \sum_{i=1}^{|c|_0} \left((\langle w_1(V^{\tilde{\phi}}), \gamma_i \rangle + 1) \langle w_1(V^{\tilde{\phi}}), [\alpha_i] \rangle + \langle w_2(V^{\tilde{\phi}}), [\beta_i] \rangle \right)$$

$$+ \sum_{i=|c|_0+1}^{|c|_0+|c|_1} \langle w_2^{\tilde{\phi}}(V), [\beta_i] \rangle.$$

If (X, ϕ) and (Σ, c) are as above, g is the genus of Σ , \mathbf{b} is a tuple of homology classes as in (1.7), and $\mathbf{k} = (k_1, \dots, k_{|c|_0+|c|_1})$ is a tuple of nonnegative integers, let

$$\mathfrak{B}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} = \mathfrak{B}_g(X, \mathbf{b})^{\phi,c} \times \prod_{i=1}^{|c|_0+|c|_1} ((\partial\Sigma)_i^{k_i} - \Delta_{i,k_i}),$$

$$\mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} = (\mathfrak{B}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c) / \mathcal{D}_c,$$

where

$$\Delta_{i,k_i} = \{ (x_{i,1}, \dots, x_{i,k_i}) \in (\partial\Sigma)_i^{k_i} : x_{i,j'} \in \{x_{i,j}, c(x_{i,j})\} \text{ for some } j, j' = 1 \dots, k_i, j \neq j' \}$$

is the big c -symmetrized diagonal. In the case X is a point and \mathbf{b} is the zero tuple, we denote $\mathcal{H}_{g,0}(X, \mathbf{b})^{\phi, \text{id}_{\partial\Sigma}}$ by \mathcal{M}_Σ ; this is the usual Deligne-Mumford moduli space of stable bordered Riemann surfaces with ordered boundary components if Σ is not a disk or a cylinder (for stability reasons). In all cases, let

$$\mathfrak{f} : \mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} \longrightarrow \mathcal{H}_g(X, \mathbf{b})^{\phi,c}$$

be the map forgetting the marked points.

Proposition 5.3. *Let (X, ϕ) , (Σ, c) , g , \mathbf{b} , \mathbf{k} , and $X_{\phi,c}$ be as above. If $\mathbf{k} \in (\mathbb{Z}^+)^{|c|_0+|c|_1}$, there is a map*

$$\text{ev} : \mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} \longrightarrow X_{\phi,c}$$

such that for every real bundle pair $(V, \tilde{\phi}) \longrightarrow (X, \phi)$ the local system $\mathcal{Z}_{w_1(\det D_{V, \tilde{\phi}})}$ over $\mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c}$ is isomorphic to $\text{ev}^* \mathcal{Z}_{(w_1, w_2)}^{\tilde{\phi}, c}$. If $\mathbf{k} = \mathbf{1}$, this local system pushes down to a local system

$$\tilde{\text{ev}}^* \mathcal{Z}_{(w_1, w_2)}^{\tilde{\phi}, c} \equiv \mathfrak{f}_* \circ \text{ev}^* \mathcal{Z}_{(w_1, w_2)}^{\tilde{\phi}, c}$$

over $\mathcal{H}_g(X, \mathbf{b})^{\phi,c}$ isomorphic to $\mathcal{Z}_{w_1(\det D_{V, \tilde{\phi}})}$.

Proof. The proof is similar to the proofs of [9, Proposition 4.6] and [9, Lemma 4.7]; so we omit some of the details and refer the reader to [9].

The component $\text{ev}_i^{X^\phi}$ of ev to the i -th X^ϕ factor is given by the evaluation at the first marked point $x_{i,1}$ on the i -th boundary component. We now describe the remaining components of ev . Let \mathcal{D}_0 and \mathcal{D}_1 denote the subgroup of \mathcal{D}_c restricting to the identity on $\partial\Sigma$ and the subgroup of \mathcal{D}_c fixing

the first marked point $x_{i,1}$ on each boundary component $(\partial\Sigma)_i$, respectively. The fibration

$$(\mathfrak{B}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c) / \mathcal{D}_0 \longrightarrow (\mathfrak{B}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c) / \mathcal{D}_1$$

has contractible fibers and thus admits a section s . Since the elements of \mathcal{D}_0 fix $\partial\Sigma$ pointwise, there are well-defined maps

$$e_i: (\mathfrak{B}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c) / \mathcal{D}_0 \longrightarrow \mathcal{L}(X^\phi), \quad [u, \mathbf{x}_1, \dots, \mathbf{x}_{|c|_0+|c|_1}] \longrightarrow u|_{(\partial\Sigma)_i},$$

for $i = 1, \dots, |c|_0$. If $i = |c|_0 + 1, \dots, |c|_0 + |c|_1$, $\phi \circ u|_{(\partial\Sigma)_i} = u \circ c|_{(\partial\Sigma)_i}$. Thus, $u|_{(\partial\Sigma)_i}$ determines an element of $\mathcal{L}(\mathbb{B}_\phi X)$,

$$\mathbb{B}_{\phi,c_i}(u|_{(\partial\Sigma)_i}): S^1 \approx (\partial\Sigma)_i / \mathbb{Z}_2 \longrightarrow \mathbb{B}_{c_i}(\partial\Sigma)_i \xrightarrow{\text{id} \times_{\mathbb{Z}_2} u} \mathbb{B}_\phi X,$$

where the middle map is a section of the fiber bundle (2.3); see also the beginning of Section 2.3. So, there are well-defined maps

$$e_i: (\mathfrak{B}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c) / \mathcal{D}_0 \longrightarrow \mathcal{L}(\mathbb{B}_\phi X)$$

for $i = |c|_0 + 1, \dots, |c|_0 + |c|_1$. Let $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^{|c|_0+|c|_1}$. The component ev_i^c of ev to the i -th factor $\mathcal{L}(X^\phi)$ or $\mathcal{L}(\mathbb{B}_\phi X)$ is the composition

$$\begin{aligned} \mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} &\hookrightarrow (\mathfrak{B}_{g,\mathbf{k}-\mathbf{1}}(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c) / \mathcal{D}_1 \\ &\xrightarrow{e_i \circ \mathfrak{s}} \begin{cases} \mathcal{L}(X^\phi), & \text{if } i = 1, \dots, |c|_0; \\ \mathcal{L}(\mathbb{B}_\phi X), & \text{if } i = |c|_0 + 1, \dots, |c|_0 + |c|_1. \end{cases} \end{aligned}$$

It is well-defined up to homotopy.

We now show that the local systems $\mathcal{Z}_{w_1(\det D_{V,\hat{\phi}})}$ and $\text{ev}^* \mathcal{Z}_{(w_1, w_2)}^{\tilde{\phi},c}$ over $\mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c}$ are isomorphic. Let $u_0 \in \mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c}$ be a preimage under ev of a basepoint $(\vec{p}, \vec{\gamma}, \vec{\Gamma})$ in $X_{\phi,c}$ as in (5.1). In particular,

$$\begin{aligned} b_i &= [\gamma_i] \in H_1(X^\phi; \mathbb{Z}) \quad \forall i = 1, \dots, |c|_0, \\ b_i &= [\Gamma_i] \in H_1^\phi(X; \mathbb{Z}) \quad \forall i = |c|_0 + 1, \dots, |c|_0 + |c|_1. \end{aligned}$$

It is enough to show that the action of every element γ of $\pi_1(\mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c}, u_0)$ on the two local systems is the same. The action on $\mathcal{Z}_{w_1(\det D_{V,\hat{\phi}})}$ is given by

$\langle w_1(\det D_{(V,\tilde{\phi})}), \gamma \rangle$. By Corollary 1.6 and Lemma 2.6,

(5.3)

$$\begin{aligned} \langle w_1(\det D_{(V,\tilde{\phi})}), \gamma \rangle &= \sum_{i=1}^{|c|_0} \left((\langle w_1(V^{\tilde{\phi}}), b_i \rangle + 1) \langle w_1(V^{\tilde{\phi}}), [\alpha_i] \rangle + \langle w_2(V^{\tilde{\phi}}), [\beta_i] \rangle \right) \\ &\quad + \sum_{i=|c|_0+1}^{|c|_0+|c|_1} \langle w_2^{\tilde{\phi}}(V), [\beta_i]^{\text{id} \times c_i} \rangle, \end{aligned}$$

where $\alpha_i: S^1 \rightarrow X$ and $\beta_i: S^1 \times (\partial\Sigma)_i \rightarrow X$ are the paths traced by $x_{i,1}$ and by the entire boundary component $(\partial\Sigma)_i$ along $\tilde{\gamma} = s \circ \gamma$. In particular,

$$\begin{aligned} \alpha_i &= \text{ev}_i^{X^\phi} \circ \gamma, & \beta_i &= \text{ev}_i^{\mathcal{L}} \circ \gamma & \forall i &= 1, \dots, |c|_0, \\ \mathbb{B}_{\phi, \text{id}_{S^1} \times c_i} \beta_i &= \text{ev}_i^{\mathcal{L}} \circ \gamma & \forall i &= |c|_0+1, \dots, |c|_0+|c|_1. \end{aligned}$$

The action of γ on $\text{ev}^* \mathcal{Z}_{(w_1, w_2)}^{\tilde{\phi}, c}$ is given by the action of

$$\text{ev} \circ \gamma = (\alpha_1, \beta_1, \dots, \alpha_{|c|_0}, \beta_{|c|_0}, \mathbb{B}_{\phi, \text{id}_{S^1} \times c_i} \beta_{|c|_0+1}, \dots, \mathbb{B}_{\phi, \text{id}_{S^1} \times c_i} \beta_{|c|_0+|c|_1})$$

on $\mathcal{Z}_{(w_1, w_2)}^{\tilde{\phi}, c}$. By (5.2), this action is also given by the right-hand side of (5.3).

For the last claim of the proposition, it is sufficient to show that the system $\text{ev}^* \mathcal{Z}_{(w_1, w_2)}^{\tilde{\phi}, c}$ over $\mathcal{H}_{g,1}(X, \mathbf{b})^{\phi, c}$ pushes down under the map forgetting the boundary points on a particular component $(\partial\Sigma)_i$ of $\partial\Sigma$, i.e. that the system is trivial along each fiber of this map. For the components with $|c_i|=0$, this is done in the proof of [9, Lemma 4.7]; so, we assume that $|c_i|=1$. Let γ be a loop in the fiber (which is homotopic to S^1). The map

$$\mathbb{B}_{\phi, \text{id}_{S^1} \times c_i} (\text{ev}_i^{\mathcal{L}} \circ \gamma): S^1 \times ((\partial\Sigma)_i / \mathbb{Z}_2) \rightarrow \mathbb{B}_\phi X$$

factors through $(\partial\Sigma)_i / \mathbb{Z}_2 \approx S^1$. Thus, the map $\text{ev}_i^{\mathcal{L}} \circ \gamma$ represents the zero class in $H_2^\phi(X)$, and so the reasoning in the proof of [9, Lemma 4.7] still applies. \square

Proposition 5.4. *Let (X, ϕ) , (Σ, c) , g , \mathbf{b} , \mathbf{k} , and $X_{\phi, c}$ be as before and $(V, \tilde{\phi}) \rightarrow (X, \phi)$ be a real bundle pair. An isomorphism between the local systems of Proposition 5.3 is determined by trivializations of*

- (1) $\Lambda_{\mathbb{R}}^{\text{top}}(V^{\tilde{\phi}})$ over a basepoint of each component of X^ϕ ,
- (2) $V^{\tilde{\phi}} \oplus 3\Lambda_{\mathbb{R}}^{\text{top}} V^{\tilde{\phi}}$ over representatives for free homotopy classes of loops in X^ϕ (one representative for each homotopy class), and

(3) $(V, \tilde{\phi})$ over representatives for free homotopy classes of maps $S^1 \rightarrow X$ intertwining ϕ and the antipodal involution \mathbf{a} on S^1 .

The effect on this isomorphism over $\mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c}$ under the changes

$$\begin{aligned} o_{\mathbb{R}}^{\tilde{\phi}} : \pi_0(X^\phi) &\longrightarrow \{0, 1\}, & s_{\mathbb{R}}^{\tilde{\phi}} : \pi_1(X) &\longrightarrow \pi_1(\mathrm{SO}(\mathrm{rk}_{\mathbb{C}}V)), \\ o_{\mathbb{C}}^{\tilde{\phi}} : \pi_1^{\phi}(X) &\longrightarrow \{0, 1\} \end{aligned}$$

in these trivializations is the multiplication by $(-1)^\epsilon$, where

$$(5.4) \quad \epsilon = \sum_{|c_i|=0} ((\langle w_1(V^{\tilde{\phi}}), b_i \rangle + 1) o_{\mathbb{R}}^{\tilde{\phi}}(\langle b_i \rangle_0) + s_{\mathbb{R}}^{\tilde{\phi}}(b_i)) + \sum_{|c_i|=1} o_{\mathbb{C}}^{\tilde{\phi}}(b_i)$$

and $\langle b_i \rangle_0 \in \pi_0(X^\phi)$ is the component determined by $b_i \in H_1(X^\phi; \mathbb{Z})$.¹

Proof. An isomorphism between the two local systems is determined by a trivialization of $\det D_{(V, \tilde{\phi})}$ over a basepoint u_0 for each component of $\mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c}$ that lies in the preimage of a basepoint $(\vec{p}, \vec{\gamma}, \vec{\Gamma})$ of $X_{\phi,c}$. This fixes the group \mathbb{Z} at u_0 and thus an isomorphism between the two systems. By the proof of [9, Theorem 1.7], this isomorphism is independent of the choice of u_0 .

Fix $u_0 \in \mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c}$ as above. By (4.4),

$$\det D_{(V, \tilde{\phi})}|_{u_0} \approx (\det D'_{u_0}) \otimes \bigotimes_{|c_i|=1} ((\det D_{u_0;i}) \otimes \Lambda_{\mathbb{R}}^{\mathrm{top}}(V'_{p_i})),$$

where D'_{u_0} is a real Cauchy-Riemann operator on a bundle pair $(V', \tilde{c}') \rightarrow (\Sigma', c)$ with

$$(\partial\Sigma', c) = \bigsqcup_{|c_i|=0} ((\partial\Sigma)_i, c_i), \quad (V', \tilde{c}')|_{(\partial\Sigma)_i} = u_0^*(V, \tilde{\phi})|_{(\partial\Sigma)_i},$$

$D_{u_0;i}$ is a real Cauchy-Riemann operator on a bundle pair $(V_i, \tilde{c}_i) \rightarrow (D^2, c_i)$ with

$$(V_i|_{S^1}, \tilde{c}_i) = u_0^*(V, \tilde{\phi})|_{(\partial\Sigma)_i},$$

and $p_i \in \Sigma'$. The vector spaces V'_{p_i} are canonically oriented by their complex structures. By [9, Proposition 4.8], an orientation on D'_{u_0} is determined by

¹ $H_1(X^\phi; \mathbb{Z})$ is the direct sum over $\pi_0(X^\phi)$; $\mathcal{H}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} = \emptyset$ unless each b_i lies in a summand of this decomposition.

the trivializations (1) and (2) in the statement of the proposition and the effect of the changes in these choices on the orientation of D'_{u_0} is described by the first sum in (5.4). By the proof of Lemma 4.2, an orientation of $D_{u_0;i}$ is determined by the homotopy class of a trivialization of $u_0^*(V, \tilde{\phi})$ over $(\partial\Sigma)_i$ and changing the homotopy class changes the induced orientation. This implies the claim. \square

6. Moduli spaces of maps with crosscaps

We begin this section by constructing moduli spaces of J -holomorphic maps from oriented sh-surfaces. We then discuss implications of Proposition 5.4 for the local systems of orientations on these spaces, giving an explicit formula for their first Stiefel-Whitney classes; see Corollary 6.2.

As in [9], let \mathcal{J}_Σ and \mathcal{D}_Σ denote the space of all almost complex structures on Σ and the group of diffeomorphisms of Σ preserving the orientation and the boundary components, respectively. The map

$$\mathcal{J}_c/\mathcal{D}_c \longrightarrow \mathcal{J}_\Sigma/\mathcal{D}_\Sigma$$

is surjective, but has infinite-dimensional fibers unless $c = \text{id}_{\partial\Sigma}$ (in which case $\mathcal{J}_c = \mathcal{J}_\Sigma$ and $\mathcal{D}_c = \mathcal{D}_\Sigma$). We define subspaces $\mathcal{J}_c^* \subset \mathcal{J}_c$ and $\mathcal{D}_c^* \subset \mathcal{D}_c$ so that the map

$$(6.1) \quad \mathcal{J}_c^*/\mathcal{D}_c^* \longrightarrow \mathcal{J}_\Sigma/\mathcal{D}_\Sigma$$

induced by the inclusions $\mathcal{J}_c^* \longrightarrow \mathcal{J}_\Sigma$ and $\mathcal{D}_c^* \longrightarrow \mathcal{D}_\Sigma$ is an isomorphism, whenever (Σ, c) is not a disk with an involution different from the identity.

If Σ is a disk, we identify Σ with the unit disk in \mathbb{C} so that c corresponds to either the identity map or the antipodal involution \mathfrak{a} on $S^1 \subset \mathbb{C}$. Let $\mathcal{J}_c^* = \{j_0\}$, where j_0 is the standard complex structure on the disk. If $c = \text{id}_{S^1}$, we take

$$\mathcal{D}_c^* = \text{PGL}_2^0\mathbb{R} \equiv \left\{ z \longrightarrow v \frac{z + \bar{a}}{1 + az} : v \in S^1, a \in \mathbb{C}, |a| < 1 \right\};$$

this is the group of holomorphic automorphisms of the disk. If $c = \mathfrak{a}$, we take \mathcal{D}_c^* to be the subgroup of $\text{PGL}_2^0\mathbb{R}$ consisting of the standard rotations of S^1 . The map (6.1) is then surjective, since for any other complex structure j on the disk compatible with the orientation, there exists an orientation-preserving diffeomorphism h of the disk such that $j = h^*j_0$; see [2, Corollary 1.9.5]. In particular, the map (6.1) is an isomorphism if $c = \text{id}_{S^1}$; if $c = \mathfrak{a}$,

this map takes a point with the trivial S^1 -action to a point with the trivial $\text{PGL}_2^0\mathbb{R}$ -action.

Suppose next that Σ is a cylinder with ordered boundary components $(\partial\Sigma)_1$ and $(\partial\Sigma)_2$. Let $\mathring{\mathbb{I}}=(0,1)$. For each $r \in \mathring{\mathbb{I}}$, we define

$$A_r = \{z \in \mathbb{C} : (|z|-r)(r|z|-1) \leq 0\},$$

$$(\partial A_r)_1 = \{z \in \mathbb{C} : |z|=r\}, \quad (\partial A_r)_2 = \{z \in \mathbb{C} : r|z|=1\}.$$

Choose a smooth map

$$\Psi : \mathring{\mathbb{I}} \times \Sigma \longrightarrow \mathbb{C}^*, \quad \Psi(r, z) \longrightarrow \Psi_r(z),$$

such that each map

$$\Psi_r : (\Sigma, (\partial\Sigma)_1, (\partial\Sigma)_2) \longrightarrow (A_r, (\partial A_r)_1, (\partial A_r)_2), \quad r \in \mathring{\mathbb{I}},$$

is a diffeomorphism so that $\alpha \circ \Psi_r = \Psi_r \circ c_i$ on $(\partial\Sigma)_i$ if $|c_i|=1$ and $i=1,2$ and the diffeomorphisms

$$\Psi_r \circ \Psi_{r'}^{-1} : A_{r'} \longrightarrow A_r, \quad r, r' \in \mathring{\mathbb{I}},$$

commute with the standard action of $S^1 \subset \mathbb{C}^*$ on \mathbb{C} . The last condition implies that the S^1 -action on Σ given by

$$(6.2) \quad S^1 \times \Sigma \longrightarrow \Sigma, \quad u \cdot z = \Psi_r^{-1}(u\Psi_r(z)) \quad \forall z \in \Sigma, u \in S^1 \subset \mathbb{C},$$

is independent of $r \in \mathring{\mathbb{I}}$. In this case, we take

$$\mathcal{J}_c^* = \{\Psi_r^* j_0 : r \in \mathring{\mathbb{I}}\}$$

and $\mathcal{D}_c^* \subset \mathcal{D}_c$ to be the subgroup corresponding to the action (6.2). The latter is the group of automorphisms of each complex structure in \mathcal{J}_c^* that preserve each boundary component of Σ . By the classification of complex structures on the cylinder [2, Section 9], for every $j \in \mathcal{J}_\Sigma$, there exist a unique $r \in \mathring{\mathbb{I}}$ and a diffeomorphism h of Σ preserving the orientation and the boundary components such that $j = h^* \Psi_r^* j_0$. It follows that the map (6.1) is an isomorphism.

If Σ is not a disk or a cylinder, i.e. the genus of its double is at least 2, we identify each boundary component $(\partial\Sigma)_i$ of $\partial\Sigma$ with S^1 in such a way that $c_i \equiv c|_{(\partial\Sigma)_i}$ corresponds to either the identity or the antipodal map on S^1 and denote by \mathcal{D}_i the subgroup of diffeomorphisms of $(\partial\Sigma)_i$ corresponding

to the rotations of S^1 under this identification. For each $j \in \mathcal{J}_\Sigma$, there exists a unique metric \hat{g}_j on the double $(\hat{\Sigma}', \hat{j}')$ of (Σ, j) with respect to the involution $\text{id}_{\partial\Sigma}$ so that \hat{g}_j has constant scalar curvature -1 and is compatible with \hat{j}' . Each boundary component $(\partial\Sigma)_i$ is a geodesic with respect to \hat{g}_j , and each isometry of $(\partial\Sigma)_i$ with respect to \hat{g}_j is real-analytic with respect to j . We denote by $\mathcal{J}_c^* \subset \mathcal{J}_\Sigma$ the subspace of complex structures j so that each \mathcal{D}_i is the group of isometries of $(\partial\Sigma)_i$ with respect to \hat{g}_j and by \mathcal{D}_c^* the subgroup of diffeomorphisms of Σ that preserve the orientation and the boundary components and restrict to elements of \mathcal{D}_i on each boundary component $(\partial\Sigma)_i$ of Σ . Since $c_i \in \mathcal{D}_i$ for each i , $\mathcal{J}_c^* \subset \mathcal{J}_c$ and $\mathcal{D}_c^* \subset \mathcal{D}_c$. We verify in the proof of Lemma 6.1 that the map (6.1) is an isomorphism in this case as well.

If (X, ϕ) and (Σ, c) are as above, g is the genus of Σ , and \mathbf{b} is a tuple of homology classes as in (1.7), and $\mathbf{k} = (k_1, \dots, k_{|c|_0+|c|_1})$ is a tuple of nonnegative integers, we define

$$\mathcal{H}_{g,\mathbf{k}}^*(X, \mathbf{b})^{\phi,c} = (\mathfrak{B}_{g,\mathbf{k}}(X, \mathbf{b})^{\phi,c} \times \mathcal{J}_c^*) / \mathcal{D}_c^* .$$

If in addition J is an almost complex structure on X such that $\phi^* J = -J$, let

$$\mathfrak{M}_{g,\mathbf{k}}(X, J, \mathbf{b})^{\phi,c} = \{ [u, \mathbf{x}_1, \dots, \mathbf{x}_{|c|_0+|c|_1}, j] \in \mathcal{H}_{g,\mathbf{k}}^*(X, \mathbf{b})^{\phi,c} : \bar{\partial}_{J,j} u = 0 \},$$

where $\bar{\partial}_{J,j}$ is as in (1.9), be the moduli space of marked J -holomorphic maps from Riemann sh-surfaces that intertwine the involutions on the boundary. In the case X is a point and \mathbf{b} is the zero tuple, we denote $\mathcal{H}_{g,\mathbf{0}}^*(X, \mathbf{b})^{\phi,c}$ by \mathcal{M}_Σ^c .

Lemma 6.1. *If (Σ, c) is an sh-surface and Σ is not a disk or a cylinder, the map*

$$(6.3) \quad \mathcal{M}_\Sigma^c \longrightarrow \mathcal{M}_\Sigma,$$

induced by the inclusions $\mathcal{J}_c^ \longrightarrow \mathcal{J}_\Sigma$ and $\mathcal{D}_c^* \longrightarrow \mathcal{D}_\Sigma$, is an isomorphism.*

Proof. If $j \in \mathcal{J}_c^*$ and $h \in \mathcal{D}_\Sigma$ are such that $h^* j \in \mathcal{J}_c^*$, then each parametrized boundary component of Σ is a geodesic with respect to the metrics \hat{g}_j and $h^* \hat{g}_j$ on the doubles $(\hat{\Sigma}', \hat{j}')$ and $(\hat{\Sigma}', \widehat{h^* j}')$. Thus, the restriction of h to each boundary component of Σ is an isometry with respect to the metric \hat{g}_j , and so $h|_{(\partial\Sigma)_i} \in \mathcal{D}_i$ and $h \in \mathcal{D}_c^*$. Thus, the map (6.1) is injective and continuous (since it is induced by inclusions before taking the quotients).

Suppose $j \in \mathcal{J}_\Sigma$. For each boundary component $(\partial\Sigma)_i$ of Σ , let $f : (\partial\Sigma)_i \rightarrow (\partial\Sigma)_i$ be an orientation-preserving geodesic parametrization of the target with respect to the metric \hat{g}_j and the chosen parametrization of the domain. Similarly to the proof of Lemma 4.3, f extends to a diffeomorphism h of Σ that preserves the orientation and the boundary components. By the assumption on f , each parametrized boundary component $(\partial\Sigma)_i$ is a geodesic with respect to $h^*\hat{g}_j = \hat{g}_{h^*j}$ and so $h^*j \in \mathcal{J}_c^*$. Thus, the map (6.1) is surjective and open (since h can be chosen to depend continuously on j). \square

Corollary 6.2 (of Proposition 5.4). *Let (X, ϕ) , (Σ, c) , g , \mathbf{b} , \mathbf{k} , and $X_{\phi,c}$ be as before. There is a local system $\mathcal{Z}_{(w_1, w_2)}^{d\phi, c}$ on $X_{\phi,c}$ such that the local system of orientations on the moduli space $\mathfrak{M}_{g, \mathbf{k}}(X, J, \mathbf{b})^{\phi, c}$ is isomorphic to $\tilde{e}\mathbf{v}^* \mathcal{Z}_{(w_1, w_2)}^{d\phi, c}$. An isomorphism between the two systems is determined by trivializations of*

- (1) $\Lambda_{\mathbb{R}}^{\text{top}}(TX^\phi)$ over a basepoint of each component of X^ϕ ,
- (2) $TX^\phi \oplus 3\Lambda_{\mathbb{R}}^{\text{top}}TX^\phi$ over representatives for free homotopy classes of loops in X^ϕ (one representative for each homotopy class), and
- (3) $(TX, d\phi)$ over representatives for free homotopy classes of maps $S^1 \rightarrow X$ intertwining ϕ and the antipodal involution \mathbf{a} on S^1 .

The effect on this isomorphism of the changes in the above trivializations is described as in (5.4).

Proof. Suppose first that Σ is not a disk or a cylinder. The proof of Lemma 4.3 then applies with \mathcal{D}_c replaced by \mathcal{D}_c^* , once f_t in the first part of the third paragraph is chosen to be a path in \mathcal{D}_i . It follows that the statements and proofs of Lemmas 4.4 and 4.5 and Theorem 1.1 with $(\mathcal{J}_c, \mathcal{D}_c)$ replaced by $(\mathcal{J}_c^*, \mathcal{D}_c^*)$ hold as well. Therefore, Corollary 1.6 and Propositions 5.3 and 5.4 apply with $\mathcal{H}_{g, \mathbf{k}}(X, \mathbf{b})^{\phi, c}$ replaced by $\mathcal{H}_{g, \mathbf{k}}^*(X, \mathbf{b})^{\phi, c}$. In light of Lemma 6.1, Corollary 6.2 follows from Proposition 5.4 with $\mathcal{H}_{g, \mathbf{k}}(X, \mathbf{b})^{\phi, c}$ replaced by $\mathcal{H}_{g, \mathbf{k}}^*(X, \mathbf{b})^{\phi, c}$ by the same argument as [9, Corollary 1.8] follows from [9, Theorem 1.7].

If Σ is a disk or a cylinder, the general principles of the proof of [9, Corollary 1.8] still apply. The orientation of $T\mathfrak{M}_{g, \mathbf{k}}(X, J, \mathbf{b})^{\phi, c}$ at each point of $\mathfrak{M}_{g, \mathbf{k}}(X, J, \mathbf{b})^{\phi, c}$ is determined by orientations for the index of a Cauchy-Riemann operator on a real bundle pair and either the appropriate Deligne-Mumford space or the automorphism group of the complex structures. With our choices of \mathcal{J}_c^* and \mathcal{D}_c^* , the last two objects are canonically oriented. \square

By Corollary 6.2, [9, Corollary 1.6], and Lemma 2.7, $\mathfrak{M}_{g,\mathbf{k}}(X, J, \mathbf{b})^{\phi,c}$ is orientable if

- (1) $X^\phi \subset X$ is orientable and $w_2(X^\phi) = \kappa^2 + \varpi|_{X^\phi}$ for some $\kappa \in H^1(X^\phi; \mathbb{Z}_2)$ and $\varpi \in H^2(X; \mathbb{Z}_2)$,
- (2) and $w_2^{\Lambda_{\mathbb{C}}^{\text{top}} d\phi}(\Lambda_{\mathbb{C}}^{\text{top}} TX)$ is a square class, e.g. if either $\pi_1(X) = 0$ and $w_2(TX) = 0$ or $\Lambda_{\mathbb{C}}^{\text{top}}(TX, d\phi)$ admits a real square root;

the first condition is not needed if $|c|_0 = 0$, while the second condition is not needed if $|c|_1 = 0$. For example, these moduli spaces are orientable for a smooth quintic hypersurface X in \mathbb{P}^4 cut out by an equation with real coefficients and with the involution ϕ being the restriction of the standard involution τ_4 on \mathbb{P}^4 . If

$$w_2(X^\phi) = \kappa^2 + \varpi|_{X^\phi} \quad \text{for some } \kappa \in H^1(X^\phi; \mathbb{Z}_2), \varpi \in H^2(X; \mathbb{Z}_2),$$

but X^ϕ is not orientable, and (2) above still holds, the orientation system of $\mathfrak{M}_{g,\mathbf{k}}(X, J, \mathbf{b})^{\phi,c}$ is a pull-back/push-down of several copies of the orientation system of the Lagrangian X^ϕ . The $|c|_1 = 0$ case of these results contains [8, Theorem 8.1.1] and [28, Theorem 1.1]. However, the presence of w_2 in (5.2) means that in general the local system of orientations on $\mathfrak{M}_{g,\mathbf{k}}(X, J, \mathbf{b})^{\phi,c}$ is not the pull-back of a system on X or X^ϕ .

As described in [9, Section 1], the index bundles $D_{V, \tilde{\phi}}$ over the space $\mathfrak{M}_{g,\mathbf{k}}(X, J, \mathbf{b})^{\phi,c}$ constructed as in Remark 1.2 are expected to play a prominent role in the computation of real Gromov-Witten invariants of submanifolds, such as complete intersections in projective spaces. Specifically, given $n \in \mathbb{Z}^+$ and an m -tuple $\mathbf{a} = (a_1, \dots, a_m)$ of positive integers, let

$$V_{n;\mathbf{a}} = \mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(a_m) \longrightarrow \mathbb{P}^n.$$

This bundle admits a natural lift $\tilde{\tau}_{n;\mathbf{a}}$ of the standard involution τ_n on \mathbb{P}^n given by the conjugation of each homogeneous component. If $|\mathbf{b}|$ is sufficiently large, the operators $D_{V_{n;\mathbf{a}}, \tilde{\tau}_{n;\mathbf{a}}}$ over $\mathfrak{M}_{g,\mathbf{k}}(\mathbb{P}^n, \mathbf{b})^{\phi,c}$ are surjective and their kernels form a vector bundle

$$\mathcal{V}_{n;\mathbf{a}} \longrightarrow \mathfrak{M}_{g,\mathbf{k}}(\mathbb{P}^n, \mathbf{b})^{\tau_n,c}$$

whose orientation bundle is $\det D_{V_{n;\mathbf{a}}, \tilde{\tau}_{n;\mathbf{a}}}$. The euler class of this bundle, with suitable boundary conditions, is expected to relate real Gromov-Witten invariants of a complete intersection $X_{n;\mathbf{a}}$ to real Gromov-Witten invariants of \mathbb{P}^n . The following corollary, which extends [9, Corollary 1.10], suggests

that it may indeed be possible to integrate $e(\mathcal{V}_{n;\mathbf{a}})$ over $\mathfrak{M}_{g,\mathbf{k}}(\mathbb{P}^n, \mathbf{b})^{\tau_n,c}$ when $n - |\mathbf{a}|$ is odd. In these cases, the moduli space $\mathfrak{M}_{g,\mathbf{k}}(X_{n;\mathbf{a}}, \mathbf{b})^{\tau_n,c}$ is oriented and in fact has a canonical orientation, constructed using the Euler sequence for \mathbb{P}^n and the normal bundle sequence for $X_{n;\mathbf{a}}$, similarly to the proof of [9, Corollary 1.10]; see also the proof of [10, Proposition 7.5] for similar results for compactified moduli spaces with $\Sigma = D^2$.

Corollary 6.3. *Let $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}^{\geq 0}$, $\mathbf{a} \in (\mathbb{Z}^+)^m$ be such that $n - |\mathbf{a}|$ is odd and (Σ, c) , \mathbf{k} , and \mathbf{b} for $(X, \phi) = (\mathbb{P}^n, \tau_n)$ be as before. If $|\mathbf{b}|$ is sufficiently large, the line bundles*

$$\Lambda_{\mathbb{R}}^{\text{top}} \mathcal{V}_{n;\mathbf{a}}, \Lambda_{\mathbb{R}}^{\text{top}} T\mathfrak{M}_{g,\mathbf{k}}(\mathbb{P}^n, \mathbf{b})^{\tau_n,c} \longrightarrow \mathfrak{M}_{g,\mathbf{k}}(\mathbb{P}^n, \mathbf{b})^{\tau_n,c}$$

are canonically isomorphic up to multiplication by \mathbb{R}^+ in each fiber.

Proof. By Proposition 5.4 and Corollary 6.2, the local systems for the two line bundles are isomorphic to the push-down/pull-back of the local systems corresponding to $T\mathbb{P}^n$ and $V_{n;\mathbf{a}}$. The action of a loop γ in $\mathfrak{M}_{g,\mathbf{k}}(\mathbb{P}^n, \mathbf{b})^{\tau_n,c}$ on the last two local systems is described by (5.2) with $V = T\mathbb{P}^n, V_{n;\mathbf{a}}$. As shown in the proof of [9, Corollary 1.10], the first sum in (5.2) is the same for $V = T\mathbb{P}^n, V_{n;\mathbf{a}}$. Since $\pi_1(\mathbb{P}^n) = 0$, by the last statement of (2.7) this is also the case for the second sum if the usual second Stiefel-Whitney classes of $T\mathbb{P}^n$ and $V_{n;\mathbf{a}}$ are the same; this is indeed so under our assumptions. Thus, the push-down/pull-back of the local systems are the same, and so the two line bundles are isomorphic.

An isomorphism between the local systems for the two line bundles is induced by identifications of choices (1)-(3) in Proposition 5.4 for the two bundles. The proof of [9, Corollary 1.10] describes such identifications for (1) and (2). Identifications for (3) are described similarly. They are specified by the canonical, \mathbb{Z}_2 -equivariant, trivialization over representatives for homotopy classes of loops $(S^1, c) \rightarrow (X, \phi)$, where $c: S^1 \rightarrow S^1$ is the antipodal map, of the vector bundle

$$(n+1)\mathcal{O}_{\mathbb{P}^n}(1) \oplus V_{n;\mathbf{a}} = (n+1)\mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(a_m)$$

with the complex conjugation induced from the natural complex conjugation in $\mathcal{O}_{\mathbb{P}^n}(1)$. This canonical trivialization is obtained by taking either of the \mathbb{Z}_2 -equivariant trivializations of $\mathcal{O}_{\mathbb{P}^n}(1)$ and using it to trivialize all of the bundle components. The effect of changing the trivialization of $\mathcal{O}_{\mathbb{P}^n}(1)$ on

the trivialization of the entire bundle is

$$(-1)^{n+1} \cdot (-1)^{a_1} \cdot \dots \cdot (-1)^{a_m} = 1,$$

by our assumption on \mathbf{a} . □

Appendix A. Almost complex structures on bordered surfaces

In this appendix we show that every bordered Riemann surface (Σ, \mathfrak{j}) can be covered by $(\mathfrak{j}, \mathfrak{j}_0)$ -holomorphic charts

$$\psi: (U, U \cap \partial\Sigma) \longrightarrow (W, W \cap \mathbb{R}),$$

where U is an open subset of Σ , W is an open subset of \mathbb{H} , and \mathfrak{j}_0 is the standard complex structure on \mathbb{C} ; see Corollary A.2. We also show that every symmetric Riemann surface $(\hat{\Sigma}, \mathfrak{j}, \sigma)$ can be covered by holomorphic charts that intertwine σ with the standard conjugation σ_0 on \mathbb{C} ; see Corollary A.3. These statements are likely known, but we could not find them in the literature and thus include them with proofs for the sake of completeness.

Lemma A.1. *If U is an open neighborhood of the origin in \mathbb{H} and \mathfrak{j} is an almost complex structure on U , there exists a diffeomorphism*

$$h: (U', U' \cap \mathbb{R}) \longrightarrow (W, W \cap \mathbb{R})$$

between an open neighborhood of 0 in U and an open subset W of \mathbb{H} such that $\mathfrak{j} = h^ \mathfrak{j}_0$.*

Proof. There exist $a, b \in C^\infty(U, \mathbb{R})$ such that

$$\mathfrak{j}(x, y) \frac{\partial}{\partial x} = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}.$$

By shrinking U if necessary, it can be assumed that $b(x, y) \neq 0$. With

$$(s, t) = (b(0, 0)x - a(0, 0)y, y),$$

we find that

$$\mathfrak{j}(s, t) \frac{\partial}{\partial s} = \frac{b(0, 0)a(x, y) - a(0, 0)b(x, y)}{b(0, 0)} \frac{\partial}{\partial s} + \frac{b(x, y)}{b(0, 0)} \frac{\partial}{\partial t}.$$

Thus, we can assume that

$$(A.1) \quad j(s, t) \frac{\partial}{\partial s} = \alpha(s, t) \frac{\partial}{\partial s} + (1 + \beta(s, t)) \frac{\partial}{\partial t}$$

for some $\alpha, \beta \in C^\infty(U, \mathbb{R})$ with $\alpha(0, 0), \beta(0, 0) = 0$.

The condition $j = h^*j_0$ is equivalent to

$$dh\left(j \frac{\partial}{\partial s}\right) = j_0 dh\left(\frac{\partial}{\partial s}\right),$$

if h is a diffeomorphism. With

$$h(s, t) = (s + x(s, t), t + y(s, t))$$

and j as in (A.1), the latter condition is equivalent to

$$(A.2) \quad P \begin{pmatrix} x \\ y \end{pmatrix} + Q \begin{pmatrix} x \\ y \end{pmatrix} = \zeta,$$

where

$$P \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_t + y_s \\ y_t - x_s \end{pmatrix}, \quad Q \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x_s + \beta x_t \\ \alpha y_s + \beta y_t \end{pmatrix}, \quad \zeta = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Let $\eta: \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that

$$\eta(r) = \begin{cases} 1, & \text{if } r \leq 1; \\ 0, & \text{if } r \geq 2. \end{cases}$$

We will view D^2 as the closure of \mathbb{H} in S^2 , i.e. as the upper hemisphere. For each $\delta \geq 0$, define

$$\eta_\delta: D^2 \rightarrow [0, 1] \quad \text{by} \quad \eta_\delta(z) = \eta(|z|/\delta).$$

In particular, $\|\eta_\delta\|_{C^1(D^2)} \leq C/\delta$ and

$$(A.3) \quad \begin{aligned} \|\eta_\delta \gamma f\|_{L_1^p(D^2)} &\leq \|\eta_\delta \gamma\|_{C^0(D^2)} \|f\|_{L_1^p(D^2)} \\ &\quad + \|\eta_\delta \gamma\|_{L_1^1(D^2)} \|f\|_{C^0(D^2)} \\ &\leq C_\gamma (\delta + \delta^{2/p}) \|f\|_{L_1^p(D^2)}; \end{aligned} \quad \begin{aligned} &\forall p > 2, \quad \gamma = \alpha, \beta, \\ &f \in L_1^p(D^2); \end{aligned}$$

this estimate uses the vanishing of γ at the origin. Since the standard operator

$$\bar{\partial}: \{\xi \in L_2^p(D^2; \mathbb{C}) : \xi|_{\mathbb{R}} \in C^0(\mathbb{R}; \mathbb{R}), \xi(0) = 0\} \rightarrow L_1^p(D^2; (T^*D^2)^{0,1})$$

is an isomorphism, for all $\delta > 0$ sufficiently small there exists a unique element $\xi_\delta \in L^p_2(D^2; \mathbb{C})$ such that

$$(\operatorname{Im} \xi_\delta)|_{\mathbb{R}} = 0, \quad \xi_\delta(0) = 0, \quad -j_0 \bar{\partial} \xi_\delta + \eta_\delta(Q\xi_\delta) d\bar{z} = \eta_\delta \zeta d\bar{z}.$$

Furthermore,

$$(A.4) \quad \|\xi_\delta\|_{L^p_2(D^2)} \leq C \|\eta_\delta \zeta d\bar{z}\|_{L^p_2(D^2)} \leq C' \delta^{2/p}.$$

On the disk of radius δ around the origin, ξ_δ restricts to a solution of (A.2). If $p > 2$, (A.4) implies that ξ'_δ is a continuous function and $|\xi'_\delta(0)| \leq C'' \delta^{2/p}$.

Thus, if δ is sufficiently small, the restriction (x, y) of ξ_δ to a neighborhood U' of the origin induces a diffeomorphism h satisfying $j = h^* j_0$. The condition $(\operatorname{Im} \xi_\delta)|_{\mathbb{R}} = 0$ corresponds to $y|_{t=0} = 0$ and $h(U' \cap \mathbb{R}) \subset \mathbb{R}$. \square

Corollary A.2. *Let Σ be a bordered surface with an almost complex structure j . For every $z \in \Sigma$, there exists a coordinate chart*

$$\psi: (U, U \cap \partial\Sigma) \longrightarrow (\mathbb{H}, \mathbb{R})$$

around z so that $\psi^* j_0 = j$. The overlap map between any two such charts is a restriction of a bi-holomorphic map between open subsets of \mathbb{C} .

Proof. If $z \notin \partial\Sigma$, such a chart exists because j is integrable on $\Sigma - \partial\Sigma$ by Newlander-Nirenberg Theorem; in fact, the proof in the $z \in \partial\Sigma$ case can be easily adapted to this case. If $z \in \partial\Sigma$, a smooth chart

$$\psi: (U, U \cap \partial\Sigma, z) \longrightarrow (\mathbb{H}, \mathbb{R}, 0)$$

induces an almost complex structure j' on a neighborhood of the origin in \mathbb{H} . Composing ψ with a diffeomorphism h provided by Lemma A.1, we obtain a desired chart around z .

If $\psi: U \longrightarrow \mathbb{H}$ and $\psi': U' \longrightarrow \mathbb{H}$ are two charts as in the statement of the corollary,

$$\psi' \circ \psi^{-1}: (\psi(U \cap U'), \psi(U \cap U') \cap \mathbb{R}) \longrightarrow (\psi'(U \cap U'), \psi'(U \cap U') \cap \mathbb{R})$$

is a bi-holomorphic map. By the Schwartz Reflection Principle,

$$\begin{aligned} \{z \in \mathbb{C} : \{z, \bar{z}\} \cap \psi(U \cap U') \neq \emptyset\} &\longrightarrow \{z \in \mathbb{C} : \{z, \bar{z}\} \cap \psi'(U \cap U') \neq \emptyset\}, \\ z &\longrightarrow \begin{cases} \psi'(\psi^{-1}(z)), & \text{if } z \in \psi(U \cap U'); \\ \overline{\psi'(\psi^{-1}(\bar{z}))}, & \text{if } \bar{z} \in \psi(U \cap U'); \end{cases} \end{aligned}$$

is a bi-holomorphic map between open subsets of \mathbb{C} . \square

Corollary A.3. *Let $(\hat{\Sigma}, j, \sigma)$ be a Riemann surface with an involution (and without boundary). For every $z \in \hat{\Sigma}$, there exists a holomorphic coordinate chart $\psi: U \rightarrow W \subset \mathbb{C}$ (W not necessarily connected) such that $\psi \circ \sigma = \sigma_0 \circ \psi$.*

Proof. If $z \notin \hat{\Sigma}^\sigma$ and $\psi: U \rightarrow \mathbb{C}$ is any holomorphic chart around z such that $U \cap \sigma(U) = \emptyset$, the holomorphic chart

$$U \cup \sigma(U) \rightarrow \mathbb{C}, \quad z \rightarrow \begin{cases} \psi(z), & \text{if } z \in U; \\ \overline{\psi(\sigma(z))}, & \text{if } \sigma(z) \in U; \end{cases}$$

has the desired property.

Suppose $z \in \hat{\Sigma}^\sigma$. Since $\hat{\Sigma}^\sigma \subset \hat{\Sigma}$ is a smooth one-dimensional submanifold, there exists a smooth chart

$$\psi: (U, U \cap \hat{\Sigma}^\sigma, z) \rightarrow (\mathbb{C}, \mathbb{R}, 0);$$

we can assume that $\sigma(U) = U$. This chart induces an almost complex structure j' on a neighborhood of the origin in \mathbb{H} . Composing ψ with a diffeomorphism provided by Lemma A.1, we obtain a diffeomorphism

$$\psi': (\psi^{-1}(\mathbb{H}), \psi^{-1}(\mathbb{H}) \cap \sigma(\psi^{-1}(\mathbb{H}))) \rightarrow (\mathbb{H}, \mathbb{R})$$

such that $\psi'^*j_0 = j$. The holomorphic chart

$$U \rightarrow \mathbb{C}, \quad z \rightarrow \begin{cases} \psi'(z), & \text{if } z \in \psi^{-1}(\mathbb{H}); \\ \overline{\psi'(\sigma(z))}, & \text{if } z \in \sigma(\psi^{-1}(\mathbb{H})); \end{cases}$$

intertwines σ and σ_0 . □

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