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Simple Hamiltonian manifolds

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A **simple Hamiltonian manifold** is a compact connected symplectic manifold equipped with a Hamiltonian action of a torus T with moment map $\Phi : M \to \mathfrak{t}^*$, such that M^T has exactly two connected components, denoted M_0 and M_1. We study the differential and symplectic geometry of simple Hamiltonian manifolds, including a large number of examples.

1. Introduction

Let M be a compact connected symplectic manifold equipped with a Hamiltonian action of a torus $T = (S^1)^n$, and let $\Phi : M \to \mathfrak{t}^*$ denote the moment map. The celebrated Atiyah Guillemin–Sternberg convexity theorem states that the image of the moment map Φ is the convex hull of the image of the fixed points, $\Phi(M^T)$. This polytope is a single point, that is, the moment map is constant, if and only if the action is trivial. So long as the action is non-trivial, this polytope $\Phi(M)$ must have at least two extreme points. In this paper, we consider the simplest non-trivial case, when M^T has exactly two components, and so $\Phi(M)$ is a 1-dimensional polytope.

Definition 1.1. A **simple Hamiltonian manifold** is a compact connected symplectic manifold equipped with a Hamiltonian action of a torus T with moment map $\Phi : M \to \mathfrak{t}^*$, such that M^T has exactly two connected components, denoted M_0 and M_1.

As noted above, a simple manifold has the minimum possible number of fixed components. We describe a simple Hamiltonian manifold by the triple (M, M_0, M_1), and let $2m_i$ and $2m$ be the dimensions of M_i and M, respectively, and set $2r_i = \operatorname{codim} M_i = 2m - 2m_i$. As a consequence of some basic results in equivariant symplectic geometry, the torus action on a simple manifold necessarily factors into a trivial action and a **residual** effective circle action (Lemma 2.2). Thus, our results hold for torus actions, but generally require verification only for the residual circle action.

In what follows, we explore the geometry associated to simple Hamiltonian manifolds. We establish the basic topology of a simple Hamiltonian manifold, using the moment map as the key tool, in Section 2. This is where we discuss the residual circle action (Lemma 2.2). Then we turn to cohomology constraints on simple Hamiltonian manifolds in Section 3. The residual moment map is a Morse–Bott function on M, and so the cohomology of M is determined from M_0 and M_1 (Proposition 3.1 and its Corollaries). This allows us to deduce relations among m, m_0, m_1, r_0 and r_1. This section also includes comments about how our work relates to several recent papers on this topic.

In Section 4, we study bundles over the M_i and the gauge groups of these bundles, and prove our first main theorem giving necessary conditions for two simple Hamiltonian manifolds to be T-equivariantly diffeomorphic, Theorem 4.4. Next, in Section 5, we turn to the special case when M_1 has codimension 2 in M, and characterize M in terms of M_0 (Theorem 5.4). In this special case, we must have that M_1 is diffeomorphic to M_1 (Corollary 5.5). In Section 6, we turn to the classification M up to T-equivariant symplectomorphism, with a complete answer in the same special case $r_1 = 1$ (Theorems 6.2 and 6.3). In particular, when $r_0 = r_1 = 1$, then M_0

and M_1 must be T-equivariantly symplectomorphic (Corollary 6.4). Finally, the last section of the paper is devoted to examples of polygon spaces.

There has been a flurry of recent work on Hamiltonian S^1-manifolds that are in some sense minimal. Tolman introduces Betti number constraints in [14], and shows that only a finite number of cohomology rings can occur. These constraints are explored further in [11] when the fixed set has exactly two components, that is the manifold is a simple Hamiltonian manifold. The differential geometry of simple Hamiltonian manifolds with minimal Betti numbers is discussed in [12]; this work may be related to our results in Section 4. Another natural hypothesis is that the circle action be semi-free, as is the case for weight simple Hamiltonian manifolds discussed below in Section 2. The implications of this hypothesis are developed further in [5, 15].

We now conclude this Introduction with a handful of examples of simple Hamiltonian manifolds.

Example 1.2. Let $M = \mathbb{C}P^n$ with a circle action given by

$$g \cdot [z_0 : \cdots : z_n] = [gz_0 : \cdots : gz_k : z_{k+1} : \cdots : z_n],$$

for $g \in S^1$. This is a simple Hamiltonian manifold $(\mathbb{C}P^n, \mathbb{C}P^k, \mathbb{C}P^{n-k-1})$.

Example 1.3. A simple Hamiltonian manifold M with M^T discrete is diffeomorphic to S^2. In this case, the moment map is a Morse function with exactly two critical points, which implies that M is homeomorphic to a sphere S^n. As M is symplectic, it must be diffeomorphic to S^2.

Example 1.4. *The symplectic cut of a weight bundle.* We may use Lerman's symplectic cuts [10] to produce a simple Hamiltonian manifold from a symplectic manifold equipped with a complex vector bundle. Let M_0 be a compact symplectic manifold and let $\nu_0 \colon V \to M_0$ be a complex vector bundle of rank k. Viewing $S^1 \subset \mathbb{C}$ as the unit complex numbers, there is a natural S^1-action on this bundle, namely fiberwise complex multiplication. We assume that the total space V is equipped with a symplectic form so that this S^1-action is Hamiltonian. The moment map $\phi \colon V \to \mathbb{R}$ has only 0 as a critical value. Let M be the symplectic cut of V at a regular value $\ell > 0$ of ϕ. This gives a simple Hamiltonian manifold (M, M_0, M_1) with M_1 the symplectic reduction of V at ℓ. The bundle projection descends to a map $M \to M_0$ with fiber $\mathbb{C}P^k$. Thus, $M = \hat{\mathbb{P}}(\nu_0)$, the total space of the $\mathbb{C}P^k$-bundle associated to ν_0 and $M_1 = \mathbb{P}(\nu_0)$, the total space of the $\mathbb{C}P^{k-1}$-bundle associated to ν_0. The case $k = 1$ is described in [13, Example 5.10].

Example 1.5. Let $M = G_k(\mathbb{C}^r)$ be the Grassmannian manifold of complex k-planes in $\mathbb{C}^r$, endowed with a $U(r)$-invariant symplectic form. As a homogeneous space, $M \cong U(r)/(U(k) \times U(r-k))$. We may endow M with a symplectic form by identifying it with the $U(r)$ coadjoint orbit of Hermitian $r \times r$ matrices with eigenvalues consisting of k ones and $(r-k)$ zeros. The maximal torus T of diagonal matrices in $U(r)$ acts in a Hamiltonian fashion on M, and we consider the last coordinate circle of this torus. Under the identifications we have made, this action has moment map

$$\Phi(A) = a_{r,r},$$

where A is a symmetric matrix and $a_{r,r}$ its bottom right entry. Then M is a simple Hamiltonian manifold with moment map image the interval $[0,1]$. We identify

$$M_0 = \left\{ \begin{pmatrix} & & & 0 \\ & B & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \right\},$$

where B is a symmetric $(r-1) \times (r-1)$ matrix with eigenvalues consisting of k ones and $(r-k-1)$ zeros. Thus, $M_0 \cong G_{k-1}(\mathbb{C}^{r-1})$. The second fixed component is

$$M_1 = \left\{ \begin{pmatrix} & & & 0 \\ & B & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\},$$

where B is a symmetric $(r-1) \times (r-1)$ matrix with eigenvalues consisting of $(k-1)$ ones and $(r-k)$ zeros; so $M_1 \cong G_k(\mathbb{C}^{r-1})$. The real locus (for complex conjugation) of this simple Hamiltonian manifold is discussed in [6, Example 5].

Example 1.6. If M is a simple Hamiltonian manifold and N is a connected compact symplectic manifold, then $M \times N$ is a simple Hamiltonian manifold, where $g \cdot (x,y) = (gx, y)$ for $g \in T$ and $(x,y) \in M \times N$.

Example 1.7. Grassmannian manifold $\tilde{G}_2(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. See figure 1 and its legend, describing moment polytopes for $\tilde{G}_2(\mathbb{R}^5)$ and $\tilde{G}_2(\mathbb{R}^7)$. These simple manifolds play an important role in [11, 12, 14].

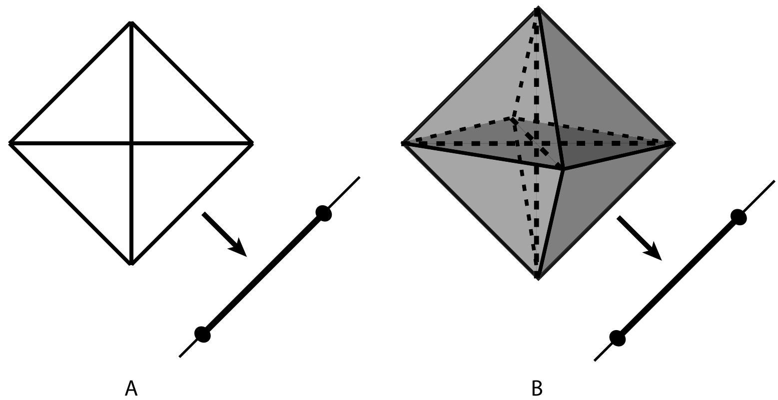

Figure 1: In subfigure (a) there is the T^2-moment polytope for $\tilde{G}_2(\mathbb{R}^5)$, together with a projection for an S^1-moment map for which this manifold is a simple Hamiltonian manifold. Both M_0 and M_1 in this case are diffeomorphic to $\mathbb{P}^1$. Subfigure (b) shows the T^3-moment polytope for $\tilde{G}Sub_2(\mathbb{R}^7)$ and a projection for an S^1-moment map for which this manifold is a simple Hamiltonian manifold. Both M_0 and M_1 in this case are diffeomorphic to $\mathbb{P}^2$.

2. Preliminaries

Standard properties of moment maps, which may be found in [1], immediately imply the following:

Lemma 2.1. *Let* (M, M_0, M_1) *be a simple Hamiltonian manifold with moment map* $\Phi : M \to \mathfrak{t}^*$*. Then*

(i) *the moment polytope* $\Delta = \Phi(M)$ *is a line segment.*

(ii) Φ *is a Morse–Bott function onto* Δ *with exactly two critical values, namely the endpoints of* Δ*.*

We think of the circle S^1 as the complex numbers of norm 1. The Lie algebra $\mathrm{Lie}(S^1)$ may then be identified as $i\,\mathbb{R}$, with basis vector $2\pi i$. We may use the dual basis to identify $\mathrm{Lie}(S^1)^*$ with $\mathbb{R}$. The group of characters of T is $\hat{T} = \mathrm{Hom}(T, S^1)$, the set of smooth homomorphisms. This is isomorphic to the linear maps from $\mathbb{R} = \mathrm{Lie}(S^1)^*$ to $\mathfrak{t}^*$ that send $\mathbb{Z}$ to the weight lattice. Taking the image of 1 identifies $\hat{T}$ with the weight lattice inside $\mathfrak{t}^*$.

Lemma 2.2. *Let* (M, M_0, M_1) *be a simple Hamiltonian* T*-manifold with moment map* $\Phi : M \to \mathfrak{t}^*$*. Then there is a unique character* $\chi \in \hat{T}$ *such that*

(i) *the* T*-action* $\alpha\colon T \to \mathrm{Diff}\,(M)$ *is of the form* $\alpha = \bar{\alpha}\circ\chi$*, where* $\bar{\alpha}\colon S^1 \to \mathrm{Diff}\,(M)$ *is an effective action making* (M, M_0, M_1) *a simple Hamiltonian* S^1*-manifold. We call* $\bar{\alpha}$ *the* **residual action***.*

(ii) *The residual action* $\bar{\alpha}$ *admits a moment map* $\bar{\Phi}\colon M \to \mathbb{R}$ *such that* $\bar{\Phi}(M_0) = 0$ *and* $\bar{\Phi}(M_1) > 0$*.*

Moreover, the above character χ*, seen as an element of the weight lattice, is a positive multiple of* $\Phi(M_1) - \Phi(M_0)$*.*

Remark 2.3. The character χ of Part (ii) of Lemma 2.2 is the **associated character** to the simple Hamiltonian manifold (M, M_0, M_1). The moment map $\bar{\Phi}\colon M \to \mathbb{R}$ is called the **residual moment map**. This lemma reduces the classification of simple Hamiltonian T-manifolds to the case of simple Hamiltonian S^1-manifolds, for effective circle actions.

Proof. As the moment polytope is 1-dimensional, $\bar{T} = T/\ker\alpha$ is a 1-dimensional torus (see e.g. [1, Section III.2.b]). Choosing an identification of $\bar{T}$ with S^1 gives a character χ and a residual action $\bar{\alpha}$ with moment map $\bar{\Phi}\colon M \to \mathbb{R}$ (with $\bar{\Phi}(M_0) = 0$). We denote by $2m_i$ and $2m$ the dimensions of M_i and M and we set $2r_i = \mathrm{codim}\, M_i$. As $\alpha = \bar{\alpha}\circ\chi$, this implies that $\chi\circ\bar{\Phi}$ is a moment map for α, proving the last statement. The uniqueness statement (ii) follows from the fact that the two identifications of $\bar{T}$ with S^1 differ by the sign of $\bar{\Phi}$. □

Let (M, M_0, M_1) be a simple Hamiltonian T-manifold. Recall that M always admits a T-invariant almost complex structure J that is ω-**compatible**: J is an isometry for ω and $\omega(v, Jv) > 0$ for all non-zero tangent vectors v to M (see, for example, [13, Section 2.5] or [2, Part V]). Then $\langle v, w\rangle = \omega(v, Jw)$ defines a Riemannian metric on M and $\langle\,,\rangle + i\omega(\,,)$ is a T-invariant Hermitian metric. The space of T-invariant ω-compatible almost complex structures on M is denoted by $\mathcal{J}(M, \omega)$ and is contractible (see [13, Proposition 4.1 and 2.49] or [2, Proposition 13.1]). Therefore, choosing $J \in \mathcal{J}(M, \omega)$ endows the tangent bundle TM with a $U(r)$-structure whose isomorphism class is well-defined. As the M_i are symplectic submanifolds, the normal bundles $\nu_i = TM|_{M_i}/TM_i$ are also Hermitian bundles, with structure group $U(r_i)$, and these structures are well-defined up to isomorphism. Observe that ν_i is isomorphic to the orthogonal complement to TM_i in $TM|_{M_i}$ with respect to the Riemannian metric associated to J.

The $U(r_i)$ structure on ν_i is T-invariant, so the bundle ν_i decomposes into a Whitney sum of T-weight bundles. It follows from Lemmas 2.1 and 2.2 that the weights which occur are multiples of χ.

Definition 2.4. If ν_0 (or, equivalently, ν_1) is itself a weight bundle, we call M a **weight simple Hamiltonian manifold.**

For instance, M is a weight simple Hamiltonian manifold when codim $M_0 = 2$ or $\operatorname{codim} M_1 = 2$. The Grassmannian manifold $\tilde{G}_2(\mathbb{R}^{m+2})$ of Example 1.1.6 is not a weight simple manifold. Observe that M is a weight simple Hamiltonian manifold if and only if the residual action is semi-free. By [11, Proposition 8.1], a simple Hamiltonian manifold (M, M_0, M_1) with $m = m_0 + m_1 + 1$ is a weight simple Hamiltonian manifold unless $\dim M_0 = \dim M_1$.

Remark 2.5. In the above discussion, the Hermitian bundle ν_i is the underlying bundle of a Hermitian bundle $\hat{\nu}_i$ endowed with a T-action. We do not distinguish these two notions because in the case of interest for us, where (M, M_0, M_1) is a weight simple manifold, $\hat{\nu}_i$ is determined by ν_i. Indeed, T acts on ν_0 via the character $\chi\colon T \to S^1$ composed with complex multiplication on the fibers. The same holds for ν_1, replacing χ by χ^{-1}.

Let (M, M_0, M_1) be a simple Hamiltonian T-manifold with residual moment map $\bar{\Phi}\colon M \to \mathbb{R}$. Let $\ell > 0$ defined by $\{\ell\} = \bar{\Phi}(M_1)$. Define

$$V_0 = \bar{\Phi}^{-1}([0, \ell/2]) \quad\text{and}\quad V_1 = \bar{\Phi}^{-1}([\ell/2, \ell])\,. \tag{2.1}$$

Lemma 2.6. *For $i = 0$ and 1, the subspace V_i of (2.1) is a T-invariant (closed) tubular neighborhood of M_i in M.*

Proof. We prove this for the case $i = 0$, and mention the necessary adaptations to complete the case $i = 1$. The proof introduces techniques which are useful in subsequent sections (see Remark 2.7 for the idea of a more direct argument). Passing to the residual action, we suppose that $T = S^1$.

Choose an S^1-invariant almost complex structure J on M. This makes ν_0 an S^1-equivariant Hermitian bundle with structure group $U(r_0)$. We denote by $E(\nu_0)$ its total space and by $p\colon E(\nu_0) \to M_0$ the bundle projection. Denote by $S(\nu_0) \subset E(\nu_0)$ the associated unit sphere bundle. For $\varepsilon > 0$, let $D_\varepsilon(\nu_0) \subset E(\nu_0)$ the disk bundle formed by the elements of $E(\nu_0)$ of norm $\leq \varepsilon$. An element of $D_\varepsilon(\nu_0)$ may be written under the form rz, with $z \in S(\nu_0)$ and $r \in [0, \varepsilon]$, with the identification $0z = p(z)$.

As ν_0 is a Hermitian bundle, each fiber of $E(\nu_0)$ carries a symplectic form, isomorphic to the standard form on $\mathbb{C}^{r_0}$ via a trivialization. The orthogonal sum with the symplectic form on M_0 provides a symplectic form ω^0 on $E(\nu_0)$. The same construction works for the almost complex structure and the Riemannian metric, so there is a compatible triple $(\omega^0, J^0, \langle\,,\rangle^0)$ over $E(\nu_0)$, extending the given one over M_0.

Let $b\colon D_\varepsilon(\nu_0) \to M$ be the S^1-equivariant tubular neighborhood embedding given by the exponential with respect to the Riemannian metric $\langle\,,\rangle$, for $\varepsilon > 0$ small enough. The two symplectic forms ω^0 and $b^*\omega$ coincide on M_0. By [13, Lemma 3.14], there is a tubular neighborhood embedding $h\colon D_{\varepsilon'}(\nu_0) \to D_\varepsilon(\nu_0)$ such that $h^*b^*\omega = \omega^0$. Based on Moser's argument, the construction of h can be made S^1-invariant (see, e.g. [1, Remark II.1.13]). Thus, replacing b with $b\circ h$ and ε with ε' if necessary, we may assume that $b^*\omega = \omega^0$. Pushing the triple $(\omega^0, J^0, \langle\,,\rangle^0)$ down to M via b, we get a compatible triple $(\omega, J^0, \langle\,,\rangle^0)$ near M_0.

Choose a smooth function $\delta_0\colon [0,\ell] \to [0,1]$ which is equal to 0 near 0 and so that the support of $(1-\delta_0)\circ\bar\Phi$ is contained in the interior of $b(D_\varepsilon(\nu_0))$. Recall that the space $\mathcal{J}(b(D_\varepsilon(\nu_0)),\omega)$ of S^1-invariant ω-compatible almost complex structures on $b(D_\varepsilon(\nu_0))$ is contractible. The standard proof of this, for example in [13, Propositions 4.1 and 2.49], actually provides a path J^s ($s \in [0,1]$) from J^0 to $J^1 = J$. The formula

$$J'_x = J_x^{\delta_0\circ\bar\Phi(x)} \in \mathrm{Aut}_{\mathbb{R}}\, T_xM$$

makes sense for all $x \in M$ and provides a ω-compatible almost complex structure on M. We say that J' is obtained by *straightening* J *around* M_0, using the *straightening function* δ_0. The almost complex structure J' determines a Riemannian metric $\langle\,,\rangle'$ on M, and hence we have an S^1-invariant compatible triple $(\omega, J', \langle\,,\rangle')$ on M.

Let us consider the gradient vector field $\mathrm{Grad}\,\bar\Phi$ for the metric $\langle\,,\rangle'$. This vector field depends only on J', since $\mathrm{grad}\,\bar\Phi = J'X$, where X is the fundamental vector field of the Hamiltonian residual circle action. A J'-gradient line is the closure of a trajectory of $\mathrm{Grad}\,\bar\Phi$.

Suppose that M is a weight simple manifold. We claim that for each vector $z \in S(\nu_0)$, there is a unique J'-gradient line Γ_z that is tangent to z and that hits M_0 at a point $p(z)$. This process parameterizes the gradient lines by $S(\nu_0)$. To see this, we transport ourselves into $D_\varepsilon(\nu_0)$ via b. If M is a weight manifold, the restriction of the moment map $\bar\Phi\circ b$ on each fiber is just the norm square, whose level surfaces of $\bar\Phi\circ b$ are round spheres and the J'-gradient lines are the radial lines to the zero sections. Checking this also makes it clear that the equation

$$\beta_0(rz) = \Gamma_z \cap \bar{\Phi}^{-1}\left(\frac{\ell\, r^2}{2}\right) \tag{2.2}$$

defines a map $\beta_0\colon D_1(\nu_0) \to M$ which is an S^1-equivariant smooth embedding with image V_0. This completes the proof of Lemma 2.6 for $i = 0$ when M is a weight simple manifold. The case $i = 1$ is analogous. We reverse the orientation of the gradient lines, and for $rz \in [0, \sqrt{\ell}] \times \mathbb{D}_1$, we define $\beta_1(rz)$ to be the point $y \in \Gamma_z$ such that $\bar{\Phi}(y) = \frac{\ell - r^2}{2}$.

Finally, when M is not a weight manifold, the level surfaces of $\bar{\Phi}\circ b$ are ellipsoids and the above process does not work: it requires that the Hessian of $\bar{\Phi}\circ b$ be proportional to the metric $\langle\,,\rangle^0$. To get around this difficulty, we precompose b with an automorphism of ν_0 which transforms the ellipsoids into round spheres. We use this new tubular neighborhood $b''\colon D_{\varepsilon''} \to M$ to transport the metric $\langle\,,\rangle^0$ on a neighborhood of M_0 in M, providing a Riemannian metric $\langle\,,\rangle''$ on this neighborhood. This metric may be mixed with $\langle\,,\rangle$ using a function like δ to obtain an S^1-invariant Riemannian metric $\langle\,,\rangle^-$ on M. Then Equation (2.2) together with the metric $\langle\,,\rangle^-$ provides an S^1-invariant smooth tubular neighborhood embedding with image V_0. Note that the metric $\langle\,,\rangle^-$ is no longer compatible with the symplectic form, but this is not necessary for the proof of Lemma 2.6. □

Remark 2.7. The above proof of Lemma 2.6 was designated to introduce techniques useful in subsequent sections. For a more direct proof, recall that the Morse Lemma provides an embedding $\psi\colon D_1(\nu_0) \to M$ with image a tubular neighborhood $\mathcal{D}$ of M_0, such that each gradient line of $\bar{\Phi}$ intersects the boundary of $\mathcal{D}$ transversally in one point. This enables us to construct a diffeomorphism $\beta_0\colon D_1(\nu_0) \to V_0$ as in (2.2). Thus, V_0 is a tubular neighborhood of M_0 (note that V_0 is T^1-invariant by definition).

3. Cohomology constraints

In this paper, $H^*(\cdot)$ denotes the cohomology ring of a space with rational coefficients. Recall that, for (M, M_0, M_1) a simple Hamiltonian manifold, $2m_i$ and $2m$ are the dimensions of M_i and M, respectively, and that $2r_i = \operatorname{codim} M_i$.

Proposition 3.1. *Let (M, M_0, M_1) be a simple Hamiltonian manifold. Then for $i, j \in \{0, 1\}$ and $i \neq j$, there are short exact sequences*

$$0 \to H^{*-2r_j}(M_j) \to H^*(M) \to H^*(M_i) \to 0 \tag{3.1}$$

and

$$0 \to H_{2m-*}(M_j) \to H^*(M) \to H^*(M_i) \to 0, \tag{3.2}$$

where the right hand homomorphisms are induced by inclusion.

Remark 3.2. This is related to the results in [8, Section 3]. Here we do not need to assume that the cohomology of M_0 and M_1 is concentrated in even degrees because the moment map provides a perfect Morse–Bott function that allows us to deduce the result.

Proof. Let V_j be the tubular neighborhood near M_j given by Lemma 2.6 that satisfies $V_i = M - \operatorname{int} V_j$. We first note that the cohomology exact sequence of the pair (M, V_i) splits into short exact sequences

$$0 \to H^*(M, V_i) \to H^*(M) \to H^*(V_i) \to 0\,. \tag{3.3}$$

This is related to the fact that the residual moment map is a perfect Morse–Bott function. A proof of (3.3) for the T-equivariant cohomology is given in [15, Proposition 2.1]. Exactness of (3.3) then follows because M_i is T-fixed, so the map $H^*_T(M_i) \to H^*(M_i)$ is onto. By excision of $\operatorname{int} V_i$ and the Thom isomorphism,

$$H^*(M, V_i) \approx H^*(V_j, \partial V_j) \approx H^{*-2r_j}(M_j)\,. \tag{3.4}$$

Then (3.3) and (3.4) give exactness of Sequence (3.1).

Next, Poincaré duality for V_j implies that

$$H^*(M, V_i) \approx H^*(V_j, \partial V_j) \approx H_{2m-*}(V_j) \approx H_{2m-*}(M_j)\,. \tag{3.5}$$

Thus (3.3) and (3.5) imply exactness of Sequence (3.2). □

Let $P, P_i \in \mathbb{Z}[t]$ be the Poincaré polynomials of M and M_i.

Corollary 3.3. *The Poincaré polynomial P_0 together with r_0 and r_1 determine both P_1 and P by the following equations:*

$$\begin{cases}(1-t^{2r_1})P_1 = (1-t^{2r_0})P_0 \\ (1-t^{2r_1})P = (1-t^{2(r_0+r_1)})P_0\end{cases}. \tag{3.6}$$

Proof. Sequences (3.1) for $i = 0$ and $i = 1$ immediately give the following equations:

$$\begin{cases}P = P_0 + t^{2r_1}P_1 \\ P = t^{2r_0}P_0 + P_1\end{cases}, \tag{3.7}$$

from which we may deduce the equations of Corollary 3.3. Note that Equations (3.7) are just the Morse–Bott equalities for the residual moment map and its opposite. □

Corollary 3.4. *Let (M, M_0, M_1) be a simple Hamiltonian manifold with $r_1 = 2$. Then there are additive isomorphisms*

$$H^*(M_1) \approx_{\text{add}} H^*(M_0) \otimes H^*(\mathbb{C}P^{r_0-1}) \quad \textit{and}$$
$$H^*(M) \approx_{\text{add}} H^*(M_0) \otimes H^*(\mathbb{C}P^{r_0}).$$

Proof. Suppose that M is obtained by a symplectic cut of the trivial bundle $M_0 \times \mathbb{C}^{r_0}$. Then $M_1 = M_0 \times \mathbb{C}P^{r_0-1}$ and $M = M_0 \times \mathbb{C}P^{r_0}$, which proves the lemma in this case. The general case follows from Corollary 3.3. □

Remark 3.5. It is not true that P_0 together with r_1 determines the cohomology ring $H^*(M)$. For instance, for the symplectic cut of a weight bundle ν_0 over M_0 given in Example 1.4, the ring structure on $H^*(M)$ depends on the bundle ν_0. For $M_0 = S^2$ and $r_0 = 1$, M is diffeomorphic to $S^2 \times S^2$ if $c_1(\nu_0)$ is even and to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P}^2$ if $c_1(\nu_0)$ is odd.

The first equation in (3.6) immediately implies the following corollary.

Corollary 3.6. *If $r_0 = r_1$, the Poincaré polynomials of M_0 and M_1 are identical: $P_0 = P_1$.*

The following proposition appears as a special case of the first centered equation in [11]. In the case of a simple Hamiltonian manifold, their inequality is precisely this one.

Proposition 3.7. *Let* (M, M_0, M_1) *be a simple Hamiltonian manifold. Then*

$$m \leq m_0 + m_1 + 1.$$

Proof. Suppose that $2r_1 > 2m_0 + 2$. The first equation of (3.7) then implies that $H^{2m_0+2}(M) = 0$, which is impossible as M is a compact symplectic manifold of dimension $\geq 2m_0 + 2$. Hence, $2r_1 \leq 2m_0 + 2$, which implies that $2m \leq 2m_0 + 2m_1 + 2$. □

Lemma 3.8. *Let* (M, M_0, M_1) *be a simple Hamiltonian manifold. Then*

$$H^1(M_0) \approx H^1(M) \approx H^1(M_1),$$

these isomorphisms being induced by the inclusions $M_i \subset M$*.*

Proof. As $m_i \geq 1$, the abstract isomorphisms come from Equations (3.7). By Proposition 3.1, inclusions $M_i \subset M$ induce surjective homomorphisms, which are then isomorphisms. □

Proposition 3.9. *For a simple Hamiltonian manifold* (M, M_0, M_1)*, the following conditions are equivalent.*

(a) $H^{\text{odd}}(M_0) = 0$.

(b) $H^{\text{odd}}(M_1) = 0$.

(c) $H^{\text{odd}}(M) = 0$.

Proof. By the first equation of (3.6), Conditions (a) and (b) are equivalent. By Equation (3.7), (c) is equivalent to (a) and (b) together. □

Example 3.10. Suppose that M has the cohomology of $\mathbb{C}P^n$. Then M_0 and M_1 have the cohomology ring of a complex projective space. Indeed, their cohomology groups vanish in odd degree by Proposition 3.9. Also, their Betti numbers are ≤ 1 by Proposition 3.1 and they are symplectic manifolds. The first equation of (3.7) implies that $m_0 + m_1 + 1 = m$, as in Example 1.2 (For $M_1 = pt$, this is a result of [6, Theorem 1]).

Remark 3.11. The extreme case in Proposition 3.7, i.e., $m = m_0 + m_1 + 1$, is studied in [11, 12, 14]. Much stronger restrictions than what we prove in this section hold in that special case. In that context, the ring $H^*(M; \mathbb{Z})$ must be isomorphic either to $H^*(\mathbb{C}P^m)$ or to $H^*(\tilde{G}_2(\mathbb{R}^{m+2}))$, and M is necessarily simply connected. Moreover, M_1 and M_2 each have the homotopy type of a complex projective space.

4. Diffeomorphism invariants

Let M_0^a and M_1^a be fixed compact smooth manifolds (the exponent a stands for **abstract**). We also fix two Hermitian vector bundles $\nu_i^a \colon \mathbb{E}_i \to M_i^a$ of complex rank r_i. The isomorphism class $[\nu_i^a]$ of the **abstract normal bundle** may be considered as an element of $[M_i^a, BU(r_i)]$; we write

$$[\nu^a] = ([\nu_0^a], [\nu_1^a]) \in [M_0^a, BU(r_0)] \times [M_1^a, BU(r_1)].$$

Definition 4.1. A $[\nu^a]$**-simple Hamiltonian** T-manifold consists of a weight simple Hamiltonian T-manifold (M, M_0, M_1) together with diffeomorphisms $\alpha_i \colon M_i^a \xrightarrow{\approx} M_i$ for $i = 0, 1$, such that $\alpha_i^*[\nu_i] = [\nu_i^a]$. Here, $\nu_i = TM|_{M_i}/TM_i$ is called the **concrete** normal bundle to M_i in M. It can be endowed with a $U(r_i)$-structure group via the choice of an almost complex structure $J \in \mathcal{J}(M, \omega)$.

The isomorphism class $[\nu_i]$ is well-defined (see the Discussion before Remark 2.5). Two such objects $((M, M_0, M_1), \alpha_i)$ and $((M', M_0', M_1'), \alpha_i')$ are considered equivalent if there is a T-equivariant symplectomorphism $h \colon M \to M'$ such that $h \circ \alpha_i = \alpha_i'$. The set of equivalence classes of $[\nu^a]$-simple Hamiltonian T-manifolds is denoted $\mathcal{H}([\nu^a])$.

The first invariant associated to a class $\mathcal{M} \in \mathcal{H}([\nu^a])$ is the character $\chi(\mathcal{M}) \in \hat{T}$ defined in Lemma 2.2. Note that, since $M \neq M^T$, the map $\chi : T \to S^1$ is surjective. As we are dealing with weight manifolds, the residual action is semi-free, with residual moment map: $\bar{\Phi} \colon M \to [0, \ell]$, that sends M_0 to 0. The number $\ell = \ell(\mathcal{M}) > 0$ is another invariant of the class $\mathcal{M} \in \mathcal{H}([\nu^a])$, called the T**-size** of $\mathcal{M}$.

Note that ν_i^a and the character χ determine unique T-equivariant weight bundles, as discussed in Remark 2.5. Thus, ν_i^a is T-equivariantly isomorphic to the concrete normal bundle ν_i of a representative of $\mathcal{H}([\nu^a])$. Associated to the abstract normal Hermitian bundle ν_i^a, we have the following.

Definition 4.2. For the bundle ν_i^a, denote the total space $\mathbb{E}_i$ with its bundle projection $p_i \colon \mathbb{E}_i \to M_i^a$. This has associated bundles and structure groups:

4.2.1 the **abstract sphere bundle** $\mathbb{S}_i \to M_i^a$ (fiber S^{2r_i-1}), where

$$\mathbb{S}_i = \{z \in \mathbb{E}_i \mid |z| = 1\}.$$

4.2.2 The **abstract disk bundle** $\mathbb{D}_i \to M_i^a$ (fiber the unit disk in $\mathbb{C}^{r_i}$), where $\mathbb{D}_i = \{z \in \mathbb{E}_i \mid |z| \leq 1\}$. We also consider the disk bundle $\mathbb{D}_{i,\varepsilon} = \{z \in \mathbb{E}_i \mid |z| \leq \varepsilon\}$.

4.2.3 The **abstract projective bundle** $\mathbb{P}_i \to M_i^a$ (fiber $\mathbb{C}P^{2r_i-1}$), where $\mathbb{P}_i = \mathbb{S}_i/S^1$. The projection $\eta_i \colon \mathbb{S}_i \to \mathbb{P}_i$ is a principal S^1-bundle with Euler class $e(\eta_i) \in H^2(\mathbb{P}_i; \mathbb{Z})$.

4.2.4 The **extended gauge group** $\hat{\mathcal{G}}(\nu_i^a)$, defined by pairs of isomorphisms that fit into commutative diagrams

$$\begin{array}{ccc} \mathbb{E}_i & \xrightarrow{g} & \mathbb{E}_i \\ \downarrow & & \downarrow \\ M_i^a & \xrightarrow{\bar{g}} & M_i^a \end{array} ,$$

where g is smooth and its restriction to each fiber is an isometry. Those isomorphisms with $\bar{g} = \mathrm{id}$ form the usual **gauge group** $\mathcal{G}(\nu_i^a)$. There is thus an exact sequence

$$1 \to \mathcal{G}(\nu_i^a) \to \hat{\mathcal{G}}(\nu_i^a) \to \mathrm{Diff}\,(M_i^a, [\nu_i^a]) \to 1, \tag{4.1}$$

where $\mathrm{Diff}\,(M_i^a, [\nu_i]^a)$ denotes the group of diffeomorphisms $h \colon M_i^a \to M_i^a$ that satisfy $h^*[\nu_i^a] = [\nu_i^a]$. The group $\hat{\mathcal{G}}(\nu_i^a)$ acts naturally on each of the above-associated bundles.

4.2.5 The **extended gauge group** $\hat{\mathcal{G}}(\eta_i)$, defined by pairs of isomorphisms that fit into commutative diagrams

$$\begin{array}{ccc} \mathbb{S}_i & \xrightarrow{g} & \mathbb{S}_i \\ \downarrow & & \downarrow \\ \mathbb{P}_i & \xrightarrow{\bar{g}} & \mathbb{P}_i \end{array}$$

such that g is smooth and S^1-equivariant. Those isomorphisms with $\bar{g} = \mathrm{id}$ form the usual **gauge group** $\mathcal{G}(\eta_i)$.

The T-action on ν_i^a induces a T-action on all the abstract sphere and disk bundles which commutes with the actions of the extended gauge groups.

Let $((M, M_0, M_1), \alpha_i)$ represent an element of $\mathcal{H}([\nu^a])$. Choose a compatible almost complex structure J on M, and consider its associated Riemannian metric. As discussed above, this endows the concrete normal bundle $\nu_i = TM|_{M_i}/TM_i$ with a T-invariant Hermitian structure, making it isometric to the orthogonal complement of TM_i in TM. Choose Hermitian vector bundle isomorphisms $\gamma_i\colon \mathbb{E}_i \to E(\nu_i)$ covering α_i. These induce isomorphisms on the associated bundles: $\gamma_i\colon \mathbb{S}_1 \to S(\nu_i)$ and so forth. We also get a tubular neighborhood embedding $b\colon \mathbb{D}_{i,\varepsilon} \to M$ of M_i in M. We now proceed as in the proof of Lemma 2.6. We may use the embedding b to straighten the Riemannian metric around M_i, using straightening functions $\delta_i\colon [0,\ell] \to [0,1]$. The gradient lines for the moment map Φ and the straightened metric provide a T-equivariant smooth embedding $\beta_0\colon \mathbb{D}_0 \to M$ by

$$\beta_0(rz) = \Gamma_{\gamma_0(z)} \cap \bar{\Phi}^{-1}\left(\frac{\ell\, r^2}{2}\right),$$

where $\Gamma_{\gamma_0(z)}$ is the unique gradient line starting from $p_0(z)$ in the direction of $\gamma_0(z)$. The T-equivariant embedding $\beta_1\colon \mathbb{D}_1 \to M$ is defined symmetrically. The image of β_0 and β_1 are the T-invariant tubular neighborhoods $V_0 = \phi^{-1}([0,\ell/2])$ and $V_1 = \phi^{-1}([\ell/2,\ell])$.

The map

$$\psi = \beta_0^{-1} \circ \beta_1\colon \mathbb{S}_1 \to \mathbb{S}_0 \tag{4.2}$$

is a diffeomorphism which anti-commutes with the S^1-action. Let $\mathcal{E}(\nu^a)$ be the space of such diffeomorphisms $\psi\colon \mathbb{S}_1 \to \mathbb{S}_0$. Observe that ψ descends to a diffeomorphism $\bar{\psi}\colon \mathbb{P}_1 \to \mathbb{P}_0$. By pre-composition, the extended gauge group $\hat{\mathcal{G}}(\nu_1^a)$ acts on the right on $\mathcal{E}(\nu^a)$ and, by post-composition, $\hat{\mathcal{G}}(\nu_0^a)$ acts on the left on $\mathcal{E}(\nu^a)$. These two actions commute and descend to the isotopy classes, giving actions of $\pi_0(\hat{\mathcal{G}}(\nu_1^a))$ and $\pi_0(\hat{\mathcal{G}}(\nu_0^a))$ on $\pi_0(\mathcal{E}(\nu^a))$. We can restrict these actions to the usual gauge groups. Define the set $\mathcal{E}([\nu^a])$ by

$$\mathcal{E}([\nu^a]) = \pi_0(\mathcal{G}(\nu_0^a))\backslash \pi_0(\mathcal{E}(\nu^a))/\pi_0(\mathcal{G}(\nu_1^a)). \tag{4.3}$$

The notation $\mathcal{E}([\nu^a])$ makes sense because the above double coset *depends only on* $[\nu^a]$. More precisely, let $\nu' = (\nu_0', \nu_1')$ and $\nu'' = (\nu_0'', \nu_1'')$ be two representatives of $[\nu^a]$. Choosing principal bundle isomorphisms $\kappa_i\colon E(\nu_i') \to E(\nu_i'')$ produces a bijection κ between the double quotient (4.3) for ν' and ν''. Since we have divided out by the action of the gauge groups, the bijection κ does not depend on the choice of the κ_i's.

Lemma 4.3. *The above construction provides a well-defined map*

$$\Psi\colon \mathcal{H}([\nu^a]) \to \mathcal{E}([\nu^a])\,.$$

Proof. Let $((M, M_0, M_1), \alpha_i)$ represent a class $\mathcal{M} \in \mathcal{H}([\nu^a])$. The definition of the diffeomorphism ψ of (4.2) involves three choices:

(a) The compatible almost complex structure J on M;

(b) The $U(r_i)$-isomorphism $\gamma_i : \mathbb{E}_i \to E(\nu_i)$; and

(c) The straightening functions $\delta_i\colon [0, \ell] \to [0, 1]$.

Once the choices (a) and (b) have been made, the straightening functions δ_0 and δ_1 belong to convex spaces, so their choice does not change ψ in $\pi_0(\mathcal{E}(\nu))$. If we choose instead $\tilde{\gamma}_i\colon \mathbb{E}_i \to E(\nu_i)$ for (b), then $\tilde{\gamma}_i = g_i \circ \gamma_i$ with $g_i \in \mathcal{G}(\nu_i)$. Hence, $\tilde{\psi} = g_0 \circ \psi \circ g_1$, proving that $\tilde{\psi}$ and ψ represent the same class in $\mathcal{E}([\nu])$. Finally, the choice of (a) does not change the class since compatible almost complex structures on M form a contractible space.

Now, let $((\bar{M}, \bar{M}_0, \bar{M}_1), \bar{\alpha}_i)$ be another representative of $\mathcal{M}$. Let $h\colon M \to \bar{M}$ be a T-equivariant symplectomorphism realizing the equivalence. Choose a compatible almost complex structure $\bar{J}$ on $\bar{M}$. Then $J = Th^{-1} \circ \bar{J} \circ Th$ is a compatible almost complex structure on M, which may be used, together with the above Hermitian bundle isomorphisms γ_i to get a representative ψ of $\Psi(\mathcal{M})$. The construction is transported via h to $\bar{M}$, using $\bar{J}$, setting $\bar{\gamma}_i = Th \circ \gamma_i$, and using the same straightening function. We thus get embeddings $\bar{\beta}_i = h \circ \beta_i \colon \mathbb{D}_i \to M'$ which can be used to define $\bar{\psi}\colon \mathbb{S}_1 \to \mathbb{S}_0$, which then satisfies

$$\bar{\psi} = \bar{\beta}_0^{-1} \circ \bar{\beta}_1 = \beta_0^{-1} \circ h^{-1} \circ h \circ \beta_1 = \psi\,.$$

□

Let us consider the following quotients of the set $\mathcal{E}([\nu^a])$:

$$\mathcal{E}^1([\nu^a]) = \pi_0(\mathcal{G}(\nu_0^a)) \backslash \pi_0(\mathcal{E}(\nu^a)) / \pi_0(\hat{\mathcal{G}}(\nu_1^a)) \tag{4.4}$$

and

$$\mathcal{E}^{01}([\nu^a]) = \pi_0(\hat{\mathcal{G}}(\nu_0^a)) \backslash \pi_0(\mathcal{E}(\nu^a)) / \pi_0(\hat{\mathcal{G}}(\nu_1^a)) \tag{4.5}$$

The compositions of the map $\Psi\colon \mathcal{H}(\nu^a]) \to \mathcal{E}([\nu^a])$ with the projections onto $\mathcal{E}^1([\nu^a])$ and $\mathcal{E}^{01}([\nu^a])$ are denoted by Ψ^1 and Ψ^{01}.

Theorem 4.4. *Let* $((M, M_0, M_1), \alpha_i)$ *and* $((M', M'_0, M'_1), \alpha'_i)$ *represent classes* $\mathcal{M}$ *and* $\mathcal{M}'$ *in* $\mathcal{H}([\nu^a])$*, with* $\chi(\mathcal{M}) = \chi(\mathcal{M}')$*. Denote the reduced moment maps by* $\bar{\Phi}\colon M \to [0, \ell]$ *and* $\bar{\Phi}'\colon M \to [0, \ell']$*, where* ℓ *and* ℓ' *are the* T*-sizes of* $\mathcal{M}$ *and* $\mathcal{M}'$*.*

(a) *If we have* $\Psi(\mathcal{M}) = \Psi(\mathcal{M}')$*, then there is a* T*-equivariant diffeomorphism* $h\colon M \to M'$ *satisfying*

$$\bar{\Phi}' \circ h = \frac{\ell'}{\ell}\,\bar{\Phi} \tag{4.6}$$

and such that $h \circ \alpha_i = \alpha'_i$ *for* $i = 0, 1$*.*

(b) *If we have* $\Psi^1(\mathcal{M}) = \Psi^1(\mathcal{M}')$*, then there is a* T*-equivariant diffeomorphism* $h\colon M \to M'$ *satisfying* (4.6) *and such that* $h \circ \alpha_0 = \alpha'_0$*.*

(c) *If* $\Psi^{01}(\mathcal{M}) = \Psi^{01}(\mathcal{M}')$*, then there is a* T*-equivariant diffeomorphism* $h\colon M \to M'$ *satisfying* (4.6)*.*

Equation (4.6) means that $\Phi' \circ h = \sigma \circ \Phi$, where σ is an affine isomorphism of $\mathfrak{t}^*$ of ratio ℓ'/ℓ.

Proof. For Part (a), choose (J, γ_i) and (J', γ'_i) as above, getting T-equivariant embeddings β_i and β'_i and $\psi, \psi' \in \mathcal{E}(\nu)$. The condition $\Psi(\mathcal{M}) = \Psi(\mathcal{M}')$ implies an equation in $\pi_0(\mathcal{E}(\nu))$ of the form $[\psi'] = g_1[\psi]g_0$ with $g_i \in \mathcal{G}_i$. Changing γ_i into $\gamma_i \circ g_i^{-1}$, we get that $[\psi] = [\psi']$ in $\pi_0(\mathcal{E}(\nu))$. Now, the embeddings β_i produce a T-equivariant diffeomorphism $N_\psi = \mathbb{D}_0 \cup_\psi \mathbb{D}_1 \xrightarrow{q} M$ extending α_0 and α_1. In the same way, the embeddings β'_i produce a T-equivariant diffeomorphism

$$N_{\psi'} = \mathbb{D}_0 \cup_{\psi'} \mathbb{D}_1 \xrightarrow{q'} M'$$

extending α'_0 and α'_1. As $[\psi] = [\psi']$, there is a smooth T-equivariant isotopy

$$b\colon \mathbb{S}_0 \times [1/2, 1] \to \mathbb{S}_0 \times [1/2, 1],$$

preserving the projection onto $[1/2, 1]$, such that $b(z, t) = (z, t)$ for t near $1/2$, and $b(z, t) = (\psi' \circ \psi^{-1}(z), t)$ for t near 1. This isotopy extends, by the identity near the null-section, to a T-equivariant diffeomorphism $b\colon \mathbb{D}_0 \to \mathbb{D}_0$. Now, b together with the identity on $\mathbb{D}_1$ gives a T-equivariant diffeomorphism $B\colon N_\phi \xrightarrow{\approx} N_{\phi'}$. Finally, observe that the level sets of the maps $\bar{\Psi} \circ q$ and $\bar{\Psi} \circ q'$ are the manifolds $|z| = \mathit{constant}$ in $\mathbb{D}_i$. These level sets are preserved by the diffeomorphism B. By the definition of the embeddings β_i and β'_i, this proves Equation (4.6) and completes the proof of (a). Parts (b) and (c) are proven in the same way, but the elements g_i that occur in the above argument are now in $\hat{\mathcal{G}}_i$ instead of $\mathcal{G}_i$. □

In order to get applications of Theorem 4.4, we now provide a different description of $\mathcal{E}([\nu])$ and its quotients. Choose an element $h \in \mathcal{E}([\nu^a])$, if $\mathcal{E}([\nu^a])$ is non-empty. Then any $\tilde{h} \in \mathcal{E}([\nu^a])$ is of the form $\tilde{h} = h \circ (h^{-1} \circ \tilde{h})$ and $h^{-1} \circ \tilde{h} \in \hat{\mathcal{G}}(\eta_1)$. Hence, the map $g \mapsto h \circ g$ provides a bijection from $\hat{\mathcal{G}}(\eta_1)$ onto $\mathcal{E}([\nu])$. Now, there is an injection $\hat{\mathcal{G}}(\nu_0) \hookrightarrow \mathcal{E}([\nu])$ given by $\gamma \mapsto \gamma \circ h$. Composed with the above bijection $\hat{\mathcal{G}}(\eta_1) \xrightarrow{\approx} \mathcal{E}([\nu^a])$ gives an injective homomorphism

$$\hat{\mathcal{G}}(\nu_0^a) \to \hat{\mathcal{G}}(\eta_1)$$

defined by $\gamma \mapsto h \circ \gamma \circ h^{-1}$. We have proven the following proposition.

Proposition 4.5. *If $\mathcal{E}([\nu]^a)$ is not empty, the choice of $h \in \mathcal{E}(\nu^a)$ provides bijections*

$$\mathcal{E}([\nu^a]) \xrightarrow{\approx} \pi_0(\mathcal{G}(\nu_0^a)) \backslash \pi_0(\hat{\mathcal{G}}(\eta_1)) / \pi_0(\mathcal{G}(\nu_1^a))\,,$$
$$\mathcal{E}^1([\nu]^a) \xrightarrow{\approx} \pi_0(\mathcal{G}(\nu_0^a)) \backslash \pi_0(\hat{\mathcal{G}}(\eta_1)) / \pi_0(\hat{\mathcal{G}}(\nu_1^a))$$

and

$$\mathcal{E}^{01}([\nu]^a) \xrightarrow{\approx} \pi_0(\hat{\mathcal{G}}(\nu_0^a)) \backslash \pi_0(\hat{\mathcal{G}}(\eta_1)) / \pi_0(\hat{\mathcal{G}}(\nu_1^a))\,,$$

where the inclusion $\hat{\mathcal{G}}(\nu_0^a) \hookrightarrow \hat{\mathcal{G}}(\eta_1^a)$ is given by $\gamma \mapsto h^{-1} \circ \gamma \circ h$.

5. The case $r_1 = 1$

The results of this section follow from the following proposition.

Proposition 5.1. *Let M_i^a be compact smooth manifolds for $i = 0, 1$. Let*

$$[\nu^a] = ([\nu_0^a], [\nu_1^a]) \in [M_0^a, BU(r_0)] \times [M_1^a, BU(r_1)].$$

Suppose that $r_1 = 1$. Then $\mathcal{E}^1([\nu^a])$ is either empty or contains a single element.

Proof. If $\mathcal{E}^1([\nu^a])$ is not empty, then it is, by Proposition 4.5, in bijection with

$$\pi_0(\mathcal{G}(\nu_0^a))\backslash\pi_0(\hat{\mathcal{G}}(\eta_1))/\pi_0(\hat{\mathcal{G}}(\nu_1^a)).$$

As $r_1 = 1$, ν_1^a is isomorphic to the complex line bundle associated to η_1. Hence, $\hat{\mathcal{G}}(\nu_1^a) = \hat{\mathcal{G}}(\eta_1)$ which implies that $\mathcal{E}^1([\nu])$ consists of a single element. □

We now provide a criterion to determine, in Proposition 5.1, whether $\mathcal{E}^1([\nu^a])$ is non-empty. Let $\eta_i : \mathbb{S}_i \to \mathbb{P}_i$ be the S^1-principal bundle associated to ν_i^a. Let $\mathbb{L}_i \to \mathbb{P}_i$ be the Hermitian line bundle associated to η_i. Let $\mathbb{L}_i^- \to \mathbb{P}_i$ be the conjugate line bundle, and denote its isomorphism class by $[\eta_i^-]$.

Proposition 5.2. *Let M_i^a and $[\nu^a]$ as in Proposition 5.1. The set $\mathcal{E}^1([\nu^a])$ is non-empty if and only if there exists a diffeomorphism $\kappa\colon M_1^a \to \mathbb{P}_0$ such that $\kappa^*[\eta_0^-] = [\nu_1^a]$.*

Proof. The diffeomorphism κ would be covered by a diffeomorphism $\tilde{\kappa}\colon \mathbb{S}_1 \to \mathbb{S}_0$ which anti-commutes with the S^1-action. Such a $\tilde{\kappa}$ defines a class in $\mathcal{E}^1([\nu^a])$.

Conversely, a class in $\mathcal{E}^1([\nu^a])$ is represented by a diffeomorphism $h : \mathbb{S}_1 \to \mathbb{S}_0$ which anti-commutes with the S^1-action. This descends to $\bar{h}\colon \mathbb{P}_1 \to \mathbb{P}_0$ satisfying $\bar{h}^*[\eta_0^-] = [\eta_1]$. As $r_1 = 1$, there is a bundle isomorphism between $\mathbb{E}_1$ and $\mathbb{L}(\eta_1)$ (over the identity of M_1^a). Hence, $\kappa = \bar{h}$ is the desired diffeomorphism. □

We now describe in details a basic example.

Example 5.3. Let N be a compact symplectic manifold. Let $\xi\colon E \to N$ be a Hermitian vector bundle of complex rank r. Each fiber of ξ is equipped with a symplectic form coming from the standard symplectic form on $\mathbb{C}^r$ via a trivialization. Then the symplectic form on N as well as those on the fibers of ξ are the restriction of a unique symplectic form ω on E. The action of S^1 by complex multiplication is Hamiltonian, with moment map $\bar{\Phi}(z) = \frac{1}{2}||z||^2$. Any $\ell > 0$ is a regular value, so we may take the symplectic cut $\hat{P}_\ell(\xi)$ of E at ℓ. We thus get a simple S^1-Hamiltonian manifold $(\hat{P}_\ell(\xi), N, P_\ell(\xi))$, where $P_\ell(\xi)$ is the symplectic reduction of E at ℓ. Using a non-trivial character $\chi\colon T \to S^1$, we thus get a weight simple T-Hamiltonian manifold with residual moment map $\bar{\phi}$. We denote this simple Hamiltonian manifold by $\mathcal{C}_\chi(N, \xi, \ell)$.

Let us define abstract manifolds and normal bundles for $\mathcal{C}_\chi(N,\xi,\ell)$. We can take $M_0^a = N$, $\alpha_0 = \mathrm{id}$ and $\nu_0^a = \xi$. Then there is a canonical diffeomorphism $\alpha_1 \colon \mathbb{P}_0 \xrightarrow{\approx} P_\ell(\xi)$ obtained by following the real vector lines in $\mathbb{E}_0 = E$. Hence, together with α_0 and α_1, $\mathcal{C}_\chi(N,\xi,\ell)$ is a $(N,\mathbb{P}_0)$-simple Hamiltonian T-manifold. As seen in Proposition 5.2, $[\nu_1] = [\eta_0^-]$.

The T-embedding β_0 is induced by the embedding $\tilde\beta_0 \colon \mathbb{D}_0 \to \mathbb{E}_0$ defined by $\tilde\beta_0(rz) = r\sqrt{\ell}\, z$. Using the identification $\mathbb{S}_1 = \mathbb{S}_0^-$, the elements of $\mathbb{D}_1$ may be written under the form $r\, z$ with $r \in [0,1]$ and $z \in \mathbb{S}_0$, with the identification $0\, z = 0\, z' = p(z) = p(z')$ when the projection of z and z' onto $\mathbb{P}_0$ coincide. The T-embedding β_1 is then induced by the T-map $\tilde\beta_1 \colon \mathbb{D}_0^- \to \mathbb{E}_0$ defined by $\tilde\beta_1(rz) = r(\sqrt{\ell} - \sqrt{2\ell})\, \bar z$. Hence,

$$\Psi(\mathcal{P}) = [\mathrm{id}]$$

(the identity from $\mathbb{S}_0$ to $\mathbb{S}_0^- = \mathbb{S}_1$ anti-commuting with the S^1-multiplication, as expected).

Theorem 5.4. *Let (M, M_0, M_1) be a simple Hamiltonian T-manifold with T-size ℓ, and associated character χ. Suppose that $r_1 = 1$. Then there exits a T-equivariant diffeomorphism*

$$F \colon \mathcal{C}_\chi(M_0,\nu_0,\ell) \xrightarrow{\approx} M$$

commuting with the residual moment maps and such that $F|_{M_0} = \mathrm{id}$.

Proof. As $r_1 = 1$, we know M is a weight simple Hamiltonian manifold. Define $M_0^a = M_0$ and set $\alpha_0 = \mathrm{id}$. Fix an almost complex structure on M compatible with the symplectic form and let ν_0^a be the orthogonal complement of TM_0 in TM for the associated metric. By Proposition 5.2, there exists a diffeomorphism $\alpha_1 \colon \mathbb{P}_0 \to M_1$ such that $\alpha_1^*[\nu_1^a] = [\eta_0^-]$. Hence, $((M, M_0, M_1), \alpha_i)$ represents a class in $\mathcal{H}([\nu])$ for $[\nu] = ([\nu_0^a], [\eta_0^-])$. So does the simple Hamiltonian manifold $\mathcal{C}_\chi(M_0,\nu_0,\ell)$ of Example 5.3, with its own α_i's. By Theorem 4.4 and Proposition 5.1, this completes the proof of Theorem 5.4. □

Theorem 5.4 implies that M_1 is diffeomorphic to $\mathbb{P}_0$. If, in addition $r_0 = 1$, then $\mathbb{P}_0$ is diffeomorphic to M_0 and we have the following corollary, also found in [4, Lemma 3.2].

Corollary 5.5. *Let (M, M_0, M_1) be a simple Hamiltonian manifold with $r_0 = r_1 = 1$. Then M_1 is diffeomorphic to M_0.*

6. Classification up to T-equivariant symplectomorphism

The philosophy of this section is slightly different from that in Section 4. We fix a single compact smooth manifold M_0^a and a Hermitian vector bundle $\nu_0^a\colon \mathbb{E}_0 \to M_0^a$ of complex rank r_0, whose isomorphism class is denoted by $[\nu_0^a] \in [M_0^a, BU(r_0)]$. The associated bundles $\mathbb{S}_0 \to M_0$ and so forth, as well as η_0, are defined as in Section 4.

Definition 6.1. A $[\nu_0^a]$**-simple Hamiltonian** T-manifold consists of a weight simple Hamiltonian T-manifold (M, M_0, M_1) together with a diffeomorphism $\alpha_0\colon M_0^a \xrightarrow{\approx} M_0$ such that $\alpha_0^*[\nu_0] = [\nu_0^a]$.

Here, ν_0 is the **concrete** normal bundle to M_0 in M, represented by the orthogonal complement of TM_0 in TM for the Riemannian metric associated to a T-invariant almost complex structure on M compatible with the symplectic form. In particular, $\omega^a = \alpha_0^*\omega_0$ is a symplectic form on M_0^a. Two such objects $((M, M_0, M_1), \alpha_0)$ and $((M', M_0', M_1'), \alpha_0')$ are considered as equivalent if there is a T-equivariant symplectomorphism $h\colon M \to M'$ such that $h \circ \alpha_0 = \alpha_0'$. The following are invariants of an equivalence class:

- The associated character χ and the residual action, which is semi-free, since we are in the case of weight simple Hamiltonian manifolds;
- The T-size $\ell > 0$;
- The symplectic form ω_0^a on M_0^a; and
- The codimensions r_0 and r_1.

Fixing $[\nu_0^a]$, ω_0^a, ℓ and r_1, we get a set of equivalence classes denoted by

$$\mathcal{S}^0([\nu_0^a], \omega_0^a, r_1, \ell).$$

We are especially interested in the case $r_1 = 1$. By Theorem 5.4, elements of $\mathcal{S}^0([\nu_0], \omega_0^a, 1, \ell)$ are in bijection with classes of symplectic forms on $\mathcal{C}_\chi(M_0^a, \nu_0^a, \ell)$ coinciding with ω_0^a on M_0^a and for which the T-action is Hamiltonian. Two such forms ω and ω' are equivalent if there is a self-diffeomorphism F of $\mathcal{C}_\chi(M_0^a, \nu_0, \ell)$, commuting with the reduced moment maps, such that $F^*\omega = \omega'$ and $F|_{M_0^a} = \mathrm{id}$.

Let $\Omega^{\rm sym}(M_0^a)$ be the space of symplectic forms on M_0^a, with the topology induced by the C^∞-topology in $\Omega^2(M_0^a)$. Define

$$\mathcal{D}((M_0^a,\omega_0^a),[\nu_0^a],\ell) = \Big\{\omega\colon [0,\ell]\to\Omega^{\rm sym}(M_0^a)\ \Big|\ \omega(0)=\omega_0^a \text{ and } [\omega(\lambda)]=[\omega_0^a]+\lambda e(\eta_0)\Big\},$$

where the last equation holds in de Rham cohomology $H^2_{dr}(M_0^a)$.

Theorem 6.2. *Suppose that $r_1 = 1$. Then there exists a bijection*

$$\Theta : \mathcal{S}^0([\nu_0^a],\omega_0^a,1,\ell) \xrightarrow{\approx} \pi_0\big(\mathcal{D}((M_0^a,\omega_0^a),[\nu_0^a],\ell)\big)\,.$$

Proof. Let $M=\mathcal{C}_\chi(M_0^a,\nu_0,\ell)$. As noted above, a class of $a\in\mathcal{S}^0([\nu_0^a],\omega_0^a,1,\ell)$ is represented by a symplectic form ω on M. Observe that there is a diffeomorphism from M/S^1 to $[0,\ell]\times M_0^a$. The first component is given by the residual moment map and the second one is induced by the projection $\mathbb{E}_0\to M_0^a$. Each slice $\{\lambda\}\times M_0$ is then endowed with a symplectic form $\omega(\lambda)$ given by the symplectic reduction of $\mathbb{E}_0$ at λ. This provides a map $\omega\colon[0,\ell]\to\Omega^{\rm sym}(M_0^a)$ with $\omega(0)=\omega_0^a$. The equation $[\omega(\lambda)]=[\omega_0^a]+\lambda e(\eta_0)$ holds in $H^2(M_0^a)$ by the Duistermaat–Heckman theorem. Hence, $\omega()$ defines a class in $\mathcal{D}((M_0^a,\omega_0^a),[\nu_0^a],\ell)$ which we define to be $\Theta(a)$.

To see that Θ is well-defined, suppose that ω' is a symplectic form on M equivalent to ω. Let F be a self-diffeomorphism of M realizing the equivalence, so $F^*\omega()=\omega'()$. The map F descends to a self-diffeomorphism $\bar F$ of $[0,\ell]\times M_0^a$ commuting with the projection onto $[0,\ell]$. Hence, $\bar F$ is of the form $\bar F(\lambda,x)=(\lambda,\bar F_\lambda(x))$ where $\bar F_\lambda$ is a self-diffeomorphism of M_0^a such that $\bar F_\lambda^*\omega(\lambda)=\omega'(\lambda)$ and $\bar F_0=\mathrm{id}$. For $t\in[0,1]$, let $\omega_t\colon[0,\ell]\to\Omega^{\rm sym}(M_0^a)$ be defined by $\omega_t(\lambda)=\bar F_{t\lambda}^*\omega$. The map $t\mapsto\omega_t()$ is a path in $\Omega^{\rm sym}(M_0^a)$ from $\omega()$ to $\omega'()$. This shows that the two forms are cohomologuous and so Θ is well-defined.

Let us now prove that Θ is surjective. Let $\omega()$ represent a class in $\mathcal{D}\big((M_0^a,\omega_0^a),[\nu_0^a],\ell)\big)$. Let $\mathbb{S}_0^a\to M_0^a$ be the S^1-bundle associated to ν_0^a. Using the normal form for reduced spaces [2, Section 30.3], we can extend the map $\omega()$ to a smooth map $\omega\colon[-\varepsilon,1+\varepsilon]\to\Omega^{\rm sym}(M_0^a)$. For such map there is a symplectic form $\tilde\omega$ on $\mathbb{S}_0^a\times[-\varepsilon,1+\varepsilon]$ such that the S^1-action is Hamiltonian with moment map the projection onto $[-\varepsilon,1+\varepsilon]$, as shown in [13, Proposition 5.8]. Performing symplectic cuts at 0 and 1 provides a simple Hamiltonian manifolds $(N;M_0^a,N_1)$ defining a class $a\in\mathcal{S}^0([\nu_0^a],\omega_0^a,1,\ell)$ and using $\alpha=\mathrm{id}$ so that $\Theta(a)=[\omega()]$.

To prove the injectivity of Θ, suppose $a, a' \in \mathcal{S}^0([\nu_0^a], \omega_0^a, 1, \ell)$ are represented by symplectic forms $\tilde{\omega}$ and $\tilde{\omega}'$ on $M = \mathcal{C}_\chi(M_0, \nu_0, \ell)$. These give rise to $\omega()$ and $\omega'()$ in $\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$ representing $\Theta(a)$ and $\Theta(a')$. If $\Theta(a) = \Theta(a')$, there exists a path $\omega_t() \in \mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$ joining $\omega()$ to $\omega'()$. Because of the cohomology constraint in the definition of $\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$, the cohomology class of $\omega_t(\lambda)$ is independent of t. By Moser's theorem [2, Theorem 7.3], there exists an isotopy $\rho_t \colon M_0^a \times [0, \ell] \to M_0^a \times [0, \ell]$, with $\rho_0 = \mathrm{id}$, such that $\omega_t() = \rho_t^* \omega()$. This isotopy may be covered by an isotopy $\tilde{\rho}_t \colon M \to M$ with $\tilde{\rho}_0 = \mathrm{id}$. Let $\tilde{\omega}_t = \tilde{\rho}_t^* \tilde{\omega}$. By [13, Proposition 5.8], we may deduce that $\tilde{\omega}_1 = \tilde{\omega}'$. This proves that $a = a'$, completing the proof. □

Theorem 6.2 reduces the identification of $\mathcal{S}^0([\nu_0^a], \omega_0^a, 1, \ell)$ to computing $\pi_0\big(\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)\big)$. We only have results when the latter is reduced to one element.

Theorem 6.3. *Suppose that $r_1 = 1$. Then $\pi_0\big(\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)\big) = *$ if $[\omega_0^a]$ and $e(\nu_0^a)$ are linearly dependent in the vector space $H^2_{dr}(M_0^a)$ of de Rham cohomology.*

The linear dependence condition is automatically fulfilled when $H^2_{dr}(M_0^a) \approx \mathbb{R}$, as when M_0^a is a complex Grassmannian or $\tilde{G}_2(\mathbb{R}^{m+2})$ of Example 1.1.6.

Proof. Let $\omega \colon [0, \ell] \to \Omega^{\mathrm{sym}}(M_0^a)$ represent an element of $\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$. As $[\omega_0^a] \neq 0$, our hypothesis of linear dependence implies that there is a unique $s \in \mathbb{R}$ such that $e(\nu_0^a) = s\,[\omega_0^a]$. Hence,

$$[\omega(\lambda)] = [\omega_0^a] + \lambda\, e(\nu_0^a) = (1 + \lambda s)[\omega_0^a]\,.$$

As $[\omega(\lambda)] \neq 0$, we know that $(1 + \lambda s) > 0$. The symplectic form $(1 + \lambda s)^{-1}$ $\omega(\lambda)$ thus satisfies $[(1 + \lambda s)^{-1} \omega(\lambda)] = [\omega_0^a]$. By Moser's theorem [2, Theorem 7.3], there exists an isotopy

$$\rho_\lambda \colon M_0^a \to M_0^a$$

with $\rho_0 = \mathrm{id}$, such that $\omega(\lambda) = \rho_\lambda^* \omega_0^a$. Hence, the formula

$$\omega_t(\lambda) = (1 + \lambda s) \rho_{t\lambda}^* \omega_0^a \quad (t \in [0, 1])$$

defines a path in $\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$ joining ω to $(1 + \lambda s)\omega_0^a$. This shows that $\pi_0\big(\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)\big)$ has only one element. □

Using Theorem 6.2, Theorem 6.3 and its proof have the following corollary.

Corollary 6.4. *Let (M, M_0, M_1) be a $[\nu_0^a]$-simple Hamiltonian T-manifold with T-size ℓ and associated character χ. Suppose that $r_0 = r_1 = 1$ and that $e(\nu_0^a) = s\,[\omega_0^a]$ for some $s \in \mathbb{R}$. Then there exits a T-equivariant symplectomorphism $\alpha\colon \mathcal{C}_\chi(M_0^a, \nu_0^a, \ell) \xrightarrow{\approx} M$ such that $\alpha|_{M_0} = \alpha_0$. Moreover (M_1, ω_1) is symplectomorphic to $(M_0^a, (1 + s\ell)\,\omega_0^a)$.*

As a corollary below, we may reproduce Delzant's result [3, Theorem 1.2] in a slightly more precise way, with essentially the same proof rephrased in our framework. For the diagonal action of S^1 on $\mathbb{C}^{m+1}$, with moment map $\tilde{\Phi}(z) = \frac{1}{2}|z|^2$, denote by $(\mathbb{C}P^m)_\ell$ the symplectic reduction at ℓ:

$$(\mathbb{C}P^m)_\ell = \mathbb{C}^{m+1} \underset{\ell}{/\!/} S^1 .$$

We also consider the symplectic cut $(\widehat{\mathbb{C}P^m})_\ell$ of $\mathbb{C}^{m+1}$ at ℓ, equipped with the induced S^1-action and induced moment map $\hat{\phi}\colon (\widehat{\mathbb{C}P^m})_\ell \to [0, \ell]$. Observe that $(\widehat{\mathbb{C}P^m})_\ell$ is symplectomorphic to $(\mathbb{C}P^{m+1})_\ell$. Indeed, as the symplectic forms vary linearly in ℓ, it is enough to prove this for $\ell = 1$. But $(\widehat{\mathbb{C}P^m})_1$ and $(\mathbb{C}P^{m+1})_1$ are both toric manifolds admitting as moment polytope an $(m+1)$-simplex intersecting the weight lattice at its vertices.

Corollary 6.5. *Let (M^{2m}, M_0, M_1) be a simple Hamiltonian S^1-manifold of S^1-size ℓ, with M_0 a single point. Then*

1. *M is S^1-equivariantly symplectomorphic to $(\mathbb{C}P^m)_\ell$, endowed with a standard S^1-action (multiplication on a single coordinate).*
2. *M_1 is symplectomorphic to $(\mathbb{C}P^{m-1})_\ell$.*

Proof. Let $(W, W_0, W_1) = ((\widehat{\mathbb{C}P^m})_\ell, pt, (\mathbb{C}P^m)_\ell)$ and, for $A \subset \mathbb{R}$, let $X_A = \{z \in \mathbb{C} \mid |z| \in A\}$. Let $0 < \varepsilon < \varepsilon' < \ell$. Performing a symplectic cut to W at ε gives rise to two simple manifolds, the "lower" one (W_-, pt, V_ϵ) and the "upper" one (W_+, V_ϵ, W_1), together with symplectic S^1- equivariant embeddings $h_-\colon X_{(0,\varepsilon)} \to W_-$ and $h_+\colon X_{\varepsilon,\varepsilon'} \to W_+$. From these, one can recover W. The quotient map $p\colon X_{[0,\ell]}$ induces to an S^1- equivariant symplectomorphism

$$W \approx \big(W_- - V_\varepsilon\big) \cup_{h_-} X_{(0,\varepsilon')} \cup_{h_+} \big(W_+ - V_\varepsilon\big) . \tag{6.1}$$

In the same way, performing a symplectic cut of M at ε gives rise to two simple manifolds (M_-, pt, N_ϵ) and (M_+, N_ϵ, M_1). If $\Phi\colon M \to [0, \ell]$ denotes the moment map, we get S^1-symplectomorphisms $\alpha_-\colon \Phi^{-1}([0,\varepsilon)) \to M_- - N_\varepsilon$ and $\alpha_+\colon \Phi^{-1}((\varepsilon,\ell]) \to M + -N_\varepsilon$. By the local forms around a fixed point, there is an S^1- equivariant symplectomorphism $q\colon U \to U'$ between neighborhoods U and U' of W_0 and M_0, respectively. Choose ε' small enough so that $p(X_{(0,\varepsilon')}) \subset U$. We thus get two symplectic S^1- equivariant embeddings

$$g_- = \alpha_-\circ q\circ p\colon X_{(0,\varepsilon)} \to W_- \quad \text{and} \quad g_+ = \alpha_+\circ q\circ p\colon X_{\varepsilon,\varepsilon'} \to W_+,$$

and an S^1- equivariant symplectomorphism

$$M \approx \big(M_- - N_\varepsilon\big) \cup_{g_-} X_{(0,\varepsilon')} \cup_{g_+} \big(M_+ - N_\varepsilon\big)\,. \tag{6.2}$$

The symplectomorphism q induces an S^1- equivariant symplectomorphism $q_-\colon W_- \to M_-$ such that $q_-\circ h_- = g_-$, and also a symplectomorphism $q_\varepsilon\colon V_\varepsilon \to N_\varepsilon$.

In order to get an S^1- equivariant symplectomorphism from W to M it is then enough, given (6.1) and (6.2), to construct an S^1- equivariant symplectomorphism $q_+\colon W_+ \to M_+$ such that $q_+\circ h_+ = g_+$.

The problem may be reformulated as follows. Let Y be the upper manifold of the symplectic cut of $X_{(0,\varepsilon')}$ at ε. The embedding h_+ extends to a symplectic S^1- equivariant embedding $\hat{h}_+\colon Y \to W_+$ onto a tubular neighborhood of V_ε in W_+; h_+ and $\bar{h}_+$ determine each other. In the same way, g_+ extends to a symplectic S^1- equivariant embedding $\hat{g}_+\colon Y \to M_+$ onto a tubular neighborhood of N_ε in M_+; g_+ and $\bar{g}_+$ determine each other. We are then looking for an S^1- equivariant symplectomorphism $q_+\colon W_+ \to M_+$ such that $q_+\circ\hat{h}_+ = \hat{g}_+$. As $\hat{g}_+$ coincides with q_ε on V_ε, it actually suffices to construct an S^1- equivariant symplectomorphism $q_+\colon W_+ \to M_+$ extending q_ε. Indeed, by the uniqueness of S^1-invariant tubular neighborhood of N_ϵ up to symplectomorphism, it will be possible, taking ε' smaller if necessary, to modify q_+ by an isotopy so that the condition $q_+\circ\hat{h}_+ = \hat{g}_+$ remains true.

By construction, W_+ is identified with $(\mathcal{C}_{\mathrm{id}}((\mathbb{C}P^{m-1})_\varepsilon, \eta, \ell-\varepsilon))$, where η is the Hopf bundle. The simple manifold (M_+, N_ϵ, M_1) has S^1-size $\ell - \varepsilon$ and the existence of the diffeomorphism $\hat{g}_+$ implies that $g_\varepsilon^*(\nu(N_\varepsilon)) = \eta$. By Corollary 6.4, g_ε extends to an S^1-equivariant symplectomorphism $q_+\colon W_+ \to M_+$ as required. □

7. Examples of polygon spaces

This section provides examples using polygon spaces. We recall below some minimal theory to state the results. For more developments, classification and references; see e.g. [7, 8].

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{\nearrow}$, where $\mathbb{R}^n_{\nearrow} := \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \mid 0 < \alpha_1 \leq \cdots \leq \alpha_n\}$. Let $S^2_{\alpha_i}$ denote the sphere in $\mathbb{R}^3$ with radius α_i. We identify $\mathbb{R}^3$ with $so(3)^*$ so that the Lie–Kirillov–Kostant–Souriau symplectic structure gives $S^2_{\alpha_i}$ the symplectic volume $2\alpha_i$.

Definition 7.1. The **polygon space** $\mathcal{N}_\alpha$ is the symplectic reduction at 0

$$\mathcal{N}_\alpha = \left(\prod_{i=1}^m S^2_{\alpha_i}\right) /\!\!/_0 \; SO_3$$

for the the diagonal co-adjoint action of $SO(3)$.

The moment map for the co-adjoint action on the product of spheres maps $\rho \mapsto \sum \rho_i$, so we get
(7.1)
$$\mathcal{N}_\alpha = \left\{\rho = (\rho_1, \dots, \rho_m) \in (\mathbb{R}^3)^m \;\middle|\; \forall i, |\rho_i| = \alpha_i \text{ and } \sum_{i=1}^m \rho_i = 0\right\} \Big/ SO_3$$

as the moduli space of spatial configurations of a polygon with length-side vector α. Note that $\mathcal{N}_\alpha$ is denoted by $\mathrm{Pol}\,(\alpha)$ in [8] and by $\mathcal{N}_3^n(\alpha)$ in [7]). The origin is a regular value for the moment map if and only if there is no aligned configuration, that is the equation

$$\sum_{i=1}^n \epsilon_i \alpha_i = 0$$

has no solution with $\epsilon_i = \pm 1$. Such length vectors α are called **generic**.

When $\alpha_i \neq \alpha_j$ for some i, j, then $\Phi_{i,j}(\rho) = |\rho_i + \rho_j|$ defines a smooth function $\Phi_{i,j} \colon \mathcal{N}_\alpha \to \mathbb{R}$. This is the moment map of a Hamiltonian S^1-action on $\mathcal{N}_\alpha$, a particular case of a **bending flow** [9]. It acts on ρ by rotating ρ_i and ρ_j at constant speed around the axis $\rho_i + \rho_j$. The critical points for $\Phi_{i,j}$ are those configurations ρ for which $\{\rho_k \mid k \neq i, j\}$ generate a one-dimensional space.

If $\alpha \in \mathbb{R}^n_{\nearrow}$ satisfies the inequalities

$$\alpha_n < \sum_{i<n} \alpha_i \;\text{ and }\; \alpha_n + \alpha_1 > \sum_{i=2}^{n-1} \alpha_i\,, \tag{7.2}$$

then $\mathcal{N}_\alpha$ is known to be diffeomorphic to $\mathbb{C}P^{n-3}$, as shown in [7, Example 2.6]. Using Corollary 6.5, we get a precise symplectic description.

Proposition 7.2. *Let* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{\nearrow}$ *($n \geq 4$) satisfying* (7.2). *Then* $\mathcal{N}_\alpha$ *is symplectomorphic to* $(\mathbb{C}P^{n-3})_\ell$ *for* $\ell = \alpha_1 + \cdots + \alpha_{n-1} - \alpha_n$.

Proof. Since $n \geq 4$, the second equation in (7.2) implies that

$$\alpha_n - \alpha_{n-1} > \alpha_2 + \cdots + \alpha_{n-2} - \alpha_1 > 0\,. \tag{7.3}$$

Hence, the bending flow $\Phi = \Phi_{n,n-1}$ is defined, with image

$$I = [\alpha_n - \alpha_{n-1}, \alpha_1 + \cdots + \alpha_{n-2}]\,,$$

an interval of length $\ell = \alpha_1 + \cdots + \alpha_{n-1} - \alpha_n$. The fact that $\alpha \in \mathbb{R}^n_{\nearrow}$ together with the second inequality of (7.3) imply that there are no critical points for Φ in the interior of I. Hence, Φ makes $\mathcal{N}_\alpha$ a simple Hamiltonian manifold with S^1-size equal to ℓ. The manifold $\Phi^{-1}(\alpha_1 + \cdots + \alpha_{n-2})$ is equal to a point. Proposition 7.2 then follows from Corollary 6.5 (exchanging the role of M_0 and M_1). □

We now study the operation of adding a tiny edge to a polygon. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{\nearrow}$ be generic. If $\varepsilon > 0$ is small enough, then, for all integer $j \in \{1, \dots, n\}$, the n-tuple

$$\alpha(j, \delta) = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j + \delta, \alpha_{j+1}, \dots, \alpha_n)$$

belongs to $\mathbb{R}^n_{\nearrow}$ and is generic when $|\delta| \leq \varepsilon$. We say that ε is α**-tiny**. The manifolds $\mathcal{N}_{\alpha(j,\delta)}$ are then canonically diffeomorphic to $\mathcal{N}_\alpha$, see [7, Lemma 1.2 and its proof].

We shall now describe the symplectic manifold $\mathcal{N}_{\alpha^\varepsilon}$ where

$$\alpha^\varepsilon = (\varepsilon, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1}_{\nearrow} \tag{7.4}$$

and ε is α-tiny. For convenience, we will now index the coordinates by 0 to n. We check that the bending flow

$$\Phi_{j,0} \colon \mathcal{N}_{\alpha^\varepsilon} \to I_j = [\alpha_j - \varepsilon, \alpha_j + \varepsilon]$$

is well-defined and makes $\mathcal{N}_{\alpha^\varepsilon}$ a simple Hamiltonian S^1-manifold of S^1-size equal to 2ε, with $M_0 = \mathcal{N}_{\alpha(j,-\varepsilon)}$ and $M_1 = \mathcal{N}_{\alpha(j,\varepsilon)}$.

For $i = 0, \dots, n$, consider the space E_i of configurations ρ as in (7.1) such that $\rho_i = (0, 0, \alpha_i)$. This is the total space of a principal S^1-bundle ξ over $\mathcal{N}_{\alpha^\varepsilon}$, or over $\mathcal{N}_{\alpha(i,-\varepsilon)}$ if $1 \leq i \leq n$. We also denote by ξ_i its associated complex line bundle.

Proposition 7.3. *Let $\alpha \in \mathbb{R}^n_{\nearrow}$ and let $\varepsilon > 0$ be tiny for α. Then for each $1 \leq j \leq n$, the bending flow $\Phi_{j,0}$ makes the manifold $\mathcal{N}_{\alpha^\varepsilon}$ S^1-equivariantly symplectomorphic to $\mathcal{C}_{\mathrm{id}}(\mathcal{N}_{\alpha(j,-\varepsilon)}, \xi_j, 2\varepsilon)$.*

For two descriptions of $\mathcal{N}_{\alpha^\varepsilon}$ as a smooth manifold, see [7, Proposition 2.2].

Proof. Choose an S^1-invariant almost complex structure on $\mathcal{N}_{\alpha^\varepsilon}$ compatible with the symplectic form and let ν be the normal bundle to $\mathcal{N}_{\alpha(j,-\varepsilon)}$ in $\mathcal{N}_{\alpha^\varepsilon}$.As $r_0 = r_1 = 1$, Corollary 6.4 implies that there is an S^1-equivariant symplectomorphism from $\mathcal{C}_{\mathrm{id}}(\mathcal{N}_{\alpha(j,-\varepsilon)}, \nu, 2\varepsilon)$ to $\mathcal{N}_{\alpha^\varepsilon}$. We have to identify ν with ξ_j.

The symplectic reduction of $\mathcal{N}_{\alpha^\varepsilon}$ at $\lambda \in I_j$ is $\mathcal{N}_{\alpha(j,\lambda)}$. Identifying the latter with $\mathcal{N}_\alpha$ gives a symplectic form $\omega_\lambda \in \Omega^2(\mathcal{N}_\alpha)$ which, by the Duistermaat–Heckman theorem satisfies the equation

$$[\omega(\lambda)] = [\omega_0^a] + \lambda e(\nu)$$

in $H^2_{dr}(\mathcal{N}_\alpha)$. Hence,

$$e(\nu) = \frac{d}{\lambda}[\omega(\lambda)]\,. \tag{7.5}$$

But, by [8, Remark 7.5.d],

$$[\omega(0)] = \sum_{i=1}^n \alpha_i e(\xi_i)\,. \tag{7.6}$$

By (7.5) and (7.5), we deduce that ν is isomorphic to ξ_j. □

Proposition 7.4. *Let $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}_{\nearrow}$ satisfying*

$$\alpha_n + \alpha_0 < \sum_{i<n} \alpha_i \quad \text{and } \alpha_n + \alpha_1 > \sum_{i=2}^{n-1} \alpha_i\,, \tag{7.7}$$

let $\ell = \alpha_0 + \cdots + \alpha_{n-1} - \alpha_n$. Then $\mathcal{N}_\alpha$ is symplectomorphic to a symplectic cut of $(\mathbb{C}P^{m-2})_\ell$ so that the symplectic slice has size $\ell - 2\alpha_0$.

In particular, $\mathcal{N}_\alpha$ is diffeomorphic to $\mathbb{C}P^{m-2} \sharp \overline{\mathbb{C}P}^{m-2}$. For a generalization of this fact, see [7, Example 2.12].

Proof. We note that $\alpha = \beta^{\alpha_0}$ in the sense of (7.4), where $\beta = (\alpha_1, \ldots, \alpha_n)$ satisfies (7.2). We use Proposition 7.3 and its notations, with the bending flow $\Phi_{n,0}$. Hence, $\mathcal{N}_\alpha$ is symplectomorphic to $\mathcal{C}_{\mathrm{id}}(\mathcal{N}_{\alpha(n,-\alpha_0)}, \xi_n, 2\alpha_0)$. Using (7.5) and [8, Proposition 7.3], we deduce that $e(\xi_n) = -1$.

Thus, $\Phi_{n,0}$ makes $\mathcal{N}_\alpha$ a simple Hamiltonian manifold $(\mathcal{N}_\alpha, M_0, M_1)$ with $M_0 = (\mathbb{C}P^{n-3})_\ell$ and $M_1 = (\overline{\mathbb{C}P}^{n-3})_{\ell-2\alpha_0}$, using Proposition 7.2 to identify M_0. □

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