COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 23, Number 2, 377–387, 2015

Uniqueness of de Sitter and Schwarzschild–de Sitter spacetimes

A.K.M. Masood-ul-Alam and Wenhua Yu

We give a simple proof of the uniqueness of de Sitter and Schwarzschild–de Sitter spacetime without assuming extra conditions on the conformal boundary at infinity. Such spacetimes are the only solutions in the static class satisfying Einstein equations $\overset{4}{R}_{\alpha\beta} = \Lambda \overset{4}{g}_{\alpha\beta}$, where the cosmological constant Λ is positive, under appropriate boundary conditions. In the absence of black holes, that is, when the event horizon has only one component the unique solution is de Sitter solution. In the presence of a black hole, we get Schwarzschild–de Sitter spacetime. The problem has important relevance in differential geometry.

1. Introduction

Boucher and Gibbons [1, 2] gave an elliptic formulation for the uniqueness problem of de Sitter spacetime. Boucher, Gibbons and Horowitz [3] proved the uniqueness of the anti-de Sitter spacetime ($\Lambda < 0$). In the same paper, an inequality was proved for the case $\Lambda > 0$, where equality implies the uniqueness. Since then the uniqueness of de Sitter spacetime has been proved by many authors under various extra conditions and in different dimensions (for a somewhat detailed discussion see [4]). However, the elliptic problem originally elaborated by Boucher and Gibbons [1] was unsolved. The elliptic problem has important relevance in the study of constant scalar curvature Riemannian metrics and the critical points of the scalar curvature map (see Kobayashi [5], Shen [6, 7], Moncrief [8], Lafontaine [9] and Hwang [10]). Our present method uses the positive mass theorem of Schoen and Yau [11] to show that the spatial 3-metric in the usual decomposition is conformally flat.

Representing the static metric $\overset{4}{g}$ by $-V^2dt^2 + g$, where g is the induced Riemannian 3-metric on an open orientable spacelike hypersurface Σ^+, and

V and g are independent of time t, the field equations become equivalent to

$$R_{ij} = V^{-1}V_{;ij} + \Lambda g_{ij}, \quad \Delta V = -\Lambda V. \tag{1}$$

Here ";" denotes covariant derivative and Δ denotes the Laplacian relative to g. V and g are assumed to be regular on the compact manifold with boundary $\Sigma^+ \cup \partial\Sigma^+$. $V > 0$ in Σ^+ and $V = 0$ on the boundary $\partial\Sigma^+$. It is known that $\partial\Sigma^+$ is then totally geodesic. We shall attach another copy of (Σ^+, g) along the totally geodesic boundary $\partial\Sigma^+$. Under the assumption that the 4-geometry is regular g extends at least in $C^{1,1}$ fashion across $\partial\Sigma^+$. Taking $V < 0$ in the attached Σ^-, we assume that V and g are C^3 in $\Sigma^\pm$ and they are globally $C^{1,1}$ in the compact manifold $\Sigma \overset{def}{=} \Sigma^+ \cup \partial\Sigma^+ \cup \Sigma^-$. Then elliptic regularity theory applied to Equations (1) in harmonic coordinates in the metric g makes V and g, C^∞. Although these harmonic coordinate functions may be only $C^{2,\alpha}$ relative to the original coordinates from now on we can select only C^∞ compatible charts so that the double Σ is a smooth manifold and V and g are smooth functions on it. However, apart from the fact that the existence theorems we have referred for a Green function and conformal normal coordinates use a smooth 3-metric, all our calculations are done assuming V and g only C^4. As in the case of black hole uniqueness theorems we are not worried about the required minimum regularity. For the uniqueness of de Sitter spacetime we also assume that the maximum value of V is 1. We shall first solve the following uniqueness problem.

Theorem 1.1. *Suppose $\partial\Sigma^+$ is diffeomorphic to a 2-sphere. Then the only solution (V, g) of the field equations Equations (1) on $\Sigma^+ \cup \partial\Sigma^+$ is de Sitter solution. That is, de Sitter spacetime metric*

$$ds^2 = -(1 - \Lambda r^2/3)dt^2 + (1 - \Lambda r^2/3)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

is the only static solution of $\overset{4}{R}_{\alpha\beta} = \Lambda \overset{4}{g}_{\alpha\beta}$, $\Lambda > 0$ such that the interior of the event horizon is diffeomorphic to the product of an open ball in $\mathbb{R}^3$ with the time line.

Kobayashi [5] and Shen [7] considered a general situation when the set $V = 0$ has more than one component. If g is conformally flat all complete solutions have been found by Kobayashi [5] and Lafontaine [12]. After proving the uniqueness of de Sitter spacetime we shall consider the situation when the set $V = 0$ has more than one component. Again it follows that g is conformally flat so that the results of [5] and [12] apply.

From the field equations Equations (1), we find that the scalar curvature of g is

$$R_g = 2\Lambda. \tag{3}$$

Conformal flatness gives spherical symmetry via the following identity (see Lindblom [13]):

$$V^4 R_{ijk} R^{ijk} = 8|\nabla V|^4 ||\Psi||^2 + |\nabla_{\mathrm{T}} |\nabla V|^2|^2,$$

where $R_{ijk} = R_{ij;k} - R_{ik;j} + (1/4)(g_{ik} R_{;j} - g_{ij} R_{;k})$ and Ψ is the trace-free part of the extrinsic curvature of the $V =$ constant two-surfaces, and ∇_{T} $|\nabla V|^2$ is the gradient of $|\nabla V|^2$ on these two-surfaces. The tensor R_{ijk} vanishes iff the 3-manifold is locally conformally flat. $R_{ijk} = 0$ implies that $|\nabla V|$ is a function of V only and the $V =$ constant surfaces are umbilic. One can then show that g and V are spherically symmetric (see Künzle [14], Avez [15]).

2. Proof of the new results

Since $\Lambda > 0$, (Σ, g) is a compact Riemannian manifold with positive scalar curvature. Hence we have a unique Green's function by a theorem due to Lee and Parker (Theorem 2.8 [16]).

Existence of the Green's Function (Theorem 3.5 on page 213 in [17] for dimension 3): For each $P \in \Sigma$ there exists a unique smooth function G on $\Sigma \backslash \{P\}$ such that $(R_g - 8\Delta_g)G = \delta_P$ in the distribution sense where δ_P is the Dirac delta function at P. The metric $\Pi = G^4 g$ is scalar flat and asymptotically flat. With respect to the geodesic normal coordinates $\{x^i\}$ of some conformal metric $h = \Omega^2 g$, G is of the form

$$G = \frac{1}{r} + \frac{m}{2} + v, \tag{4}$$

where v is $O(r)$, $v \in C^{2,\mu}$ and $r = \sqrt{\sum (x^i)^2}$.

Relative to coordinates $\left\{X^i = x^i r^{-2}\right\}$, $\Pi = G^4 g$ has expansion

$$\widetilde{\Pi}_{ij} = \left(1 + 2m\mathfrak{R}^{-1}\right)\delta_{ij} + O\left(\mathfrak{R}^{-2}\right), \tag{5}$$

where $\mathfrak{R} = r^{-1}$. The constant m is the mass of the metric Π. $m \geq 0$ by the positive mass theorem of Schoen and Yau. An easy computation shows that relative to the $\{x^i\}$ system components of g have expansions $g_{lk} = \delta_{lk} + O(r^2)$. We have on $\Sigma \backslash \{P\}$

$$2\Lambda G - 8\Delta_g G = 0. \tag{6}$$

Following lemma is proved rigorously by Yu in his undergraduate thesis [18]. This lemma is not necessary for the proof of the main result although it may provide alternate justification of some claims used in the proof by giving the equation of the surface $\partial\Sigma^+$ locally as a graph over the $z = 0$ coordinate plane of the chosen conformal normal coordinate system (see also Remark 2.1 below).

Lemma 2.1. *Let $q \in \partial\Sigma^+$. There exists a Green's function G satisfying Equation (6) and having its singularity at q such that on $\partial\Sigma^+ \backslash \{q\}$ the normal derivative of G vanishes.*

Proof. Here we omit the details. The idea is to show that if $\mathcal{G}$ is the Green's function of the Yamabe operator with singularity at q that exists by the existence theorem, then its reflection under the doubling $\widetilde{\mathcal{G}}$ and $(\mathcal{G} + \widetilde{\mathcal{G}})/2$ are also Green's functions having their singularities at q. By definition $\widetilde{\mathcal{G}}(\widetilde{p}) = \mathcal{G}(p)$ where $p, \widetilde{p} \in \Sigma$ are any pair of points related by the doubling. Then by the uniqueness all three Green's functions are the same. Now $\langle \nabla\mathcal{G}, n\rangle = -\left\langle \nabla\widetilde{\mathcal{G}}, n\right\rangle$. So the lemma follows. □

We shall need the following expansion of V in a coordinate system having origin at a point $q \in \partial\Sigma^+$ such that $g_{lk} = \delta_{lk} + O(r^2)$. This will be the same conformal normal coordinate system in which the Green's function having singularity at q is expanded in Equation (4). We suppose that the $z = x^3$-axis is perpendicular to $\partial\Sigma^+$ at q.

$$V = cz + b_{jl}x^l x^j + O(r^3),$$

where $c \neq 0$, and b_{jl} are constant. $c \neq 0$ because it is well-known that $\nabla V \neq 0$ on the $V = 0$ surface. The first equation of Equations (1) gives $b_{ik} = 0$ since R_{ij} is defined at q. We now write

$$V = cz + C_{ijk}x^i x^j x^k + \omega, \qquad \text{where } \omega = O(r^4). \tag{7}$$

The second equation and Equation (6) give $\Delta_{\Pi} V = -G^{-4}\Lambda V + 2G^{-5} \langle \nabla G, \nabla V\rangle_g$. Referring to Ω introduced before Equation (4) we write $\Omega = 1 + (1/2)D_{ij}x^i x^j + O(r^3)$ where D_{ij} are constant depending possibly on q. We shall use Ω given in the following lemma proved in the appendix.

Lemma 2.2. *In 3-dimension and for our field equations Equations (1) it is possible to choose* Ω *such that* D_{ij} *is diagonal. Furthermore. we have (summation implied)*

$$6C_{izi} = -\Lambda c + cD_{zz}, \quad C_{iAi} = 0 \quad \textit{for } x^A = x \textit{ or } y.$$

In the $\{X^i\}$-coordinates about q,

$$\begin{aligned}\Delta_\Pi V = &-2cZ\mathfrak{R}^{-4} + 5mc\mathfrak{R}^{-5}Z - \Lambda cZ\mathfrak{R}^{-6} - (15/2)m^2 Z\mathfrak{R}^{-6}c \\ &+ 10cvZ\mathfrak{R}^{-5} + 2c(\partial v/\partial z)\mathfrak{R}^{-5} - 6C_{jml}X^jX^lX^m\mathfrak{R}^{-8} \\ &- 2cZD_{ij}X^iX^j\mathfrak{R}^{-8} + O\left(\mathfrak{R}^{-6}\right).\end{aligned} \tag{8}$$

Here Z is the coordinate X^3. Since $c \neq 0$, the surface $\partial\Sigma^+$ is locally a graph over the $z = 0$ coordinate plane of a function $z = z(x, y)$.

Remark 2.1. $(\partial v/\partial z)(q)$ can be evaluated using Lemma 2.1. However, we do not need it. Terms involving v disappear from our final equations namely Equations (9) and (10) below.

Let $\widetilde{\Gamma}$ denote the Christoffel symbol of Π in the $\{X^i\}$ system. Then using Equation (8) with $\delta^{ik} - \widetilde{\Pi}^{ik} = (1 - \Omega^2(1 - 2mr - 4vr + (5/2)m^2r^2))\delta_{ij} + O(\mathfrak{R}^{-3})$, $(\partial\Omega/\partial z) = D_{zj}x^j = D_{zz}Z\mathfrak{R}^{-2}$ and $\Delta_E V \equiv \delta^{ik}\frac{\partial^2 V}{\partial X^k \partial X^i} = (\delta^{ik} - \widetilde{\Pi}^{ik})\frac{\partial^2 V}{\partial X^k \partial X^i} + \widetilde{\Pi}^{ik}\widetilde{\Gamma}^m_{ik}\frac{\partial V}{\partial X^m} + \Delta_\Pi V$ we get about q,

$$\begin{aligned}\Delta_E V = &-2cm^2 Z\mathfrak{R}^{-6} + cD_{zz}Z\mathfrak{R}^{-6} - 2cZ\mathfrak{R}^{-4} - \Lambda cZ\mathfrak{R}^{-6} \\ &- 6C_{ijk}X^iX^jX^k\mathfrak{R}^{-8} + O\left(\mathfrak{R}^{-6}\right).\end{aligned} \tag{9}$$

Since the computation of the above equation is tedious we include the expression for $\widetilde{\Pi}^{ik}\widetilde{\Gamma}^m_{ik}$ below. We recall $\widetilde{\Pi}_{kl} = \dfrac{\partial x^i}{\partial X^k}\dfrac{\partial x^j}{\partial X^l}\Pi_{ij}$, where $\Pi_{ij} = G^4 g_{ij}$ in xyz system.

$$\begin{aligned}\widetilde{\Pi}^{ik}\widetilde{\Gamma}^m_{ik} = &\, m\mathfrak{R}^{-3}X^m - (1/2)\, m^2\mathfrak{R}^{-4}X^m + (\partial\Omega/\partial x^m)\mathfrak{R}^{-2} \\ &- 2X^mX^a(\partial\Omega/\partial x^a)\mathfrak{R}^{-4} + 2v\mathfrak{R}^{-3}X^m - 2(\partial v/\partial x^m)\mathfrak{R}^{-3} \\ &+ 4X^mX^a(\partial v/\partial x^a)\mathfrak{R}^{-5} + O\left(\mathfrak{R}^{-4}\right).\end{aligned}$$

But we can also obtain $\Delta_E V$ in the straightforward way from Equation (7) by computing $(\partial^2 V/\partial X^k \partial X^i)$ using the coordinate transformation formula $x^i = x^i(X^j)$. This way we first get

$$\Delta_E V = 6C_{iki}X^k\mathfrak{R}^{-6} - 2cZ\mathfrak{R}^{-4} - 6C_{ikl}X^lX^kX^i\mathfrak{R}^{-8} + O(\mathfrak{R}^{-6}).$$

Putting $6C_{izi} = -\Lambda c + cD_{zz}, C_{iAi} = 0$ from Lemma 2.2 we get

$$\begin{aligned} \Delta_E V = &-\Lambda cZ\mathfrak{R}^{-6} + cD_{zz}Z\mathfrak{R}^{-6} - 2cZR^{-4} \\ &- 6C_{ijk}X^iX^jX^k\mathfrak{R}^{-8} + O(\mathfrak{R}^{-6}) \end{aligned} \tag{10}$$

Equation (10) contradicts Equation (9) unless $m = 0$, because both the equations hold in an open 3-ball about q. Hence the mass m of Π is 0 and by the positive mass theorem of Schoen and Yau, Π is Euclidean. Thus g is conformally flat. This proves Theorem 1.1.

For the uniqueness of Schwarzschild–de Sitter solution we simply need to paste the two copies across the identical components of the totally geodesic boundary $\partial\Sigma^+$ so as to form a manifold which is, in the simplest case, diffeomorphic to $S^2 \times S^1$. Since we are assuming $V > 0$ in Σ^+ the proof actually shows nonexistence of complicated topology inside a compact set away from the boundary and also nonexsistence of more than two components of the boundary. As before we show that g is conformally flat. Then we invoke the works of Kobayashi [5] and Lafontaine [12] as explained before. $V > 0$ in Σ^+ then implies that the 3-metric is that of Schwarzschild–de Sitter spacetime. Thus, assuming $V > 0$ in Σ^+, we have the following theorem.

Theorem 2.3. *Suppose the totally geodesic boundary $\partial\Sigma^+$, which is also the set $V = 0$, has more than one component each diffeomorphic to a 2-sphere. Then the only static solution (V, g) of the field equations $\overset{4}{R}_{\alpha\beta} = \Lambda\overset{4}{g}_{\alpha\beta}$, $\Lambda > 0$ on $\Sigma^+ \cup \partial\Sigma^+$ is Schwarzschild–de Sitter solution*

$$\begin{aligned} ds^2 = &-\left(1 - \frac{\Lambda r^2}{3} - \frac{2m}{r}\right)dt^2 \\ &+ \left(1 - \frac{\Lambda r^2}{3} - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned}$$

In particular, the $V = 0$ set has only two connected components.

3. Conclusion

Bunting and Masood-ul-Alam [20] used the positive mass theorem technique to prove the uniqueness of Schwarzschild spacetime. That proof also used suitable coordinates relative to a conformal metric for better control. The method of using a conformal metric for better regularity has been used earlier by Künzle [14]. Our present proof is essentially 3-dimensional. Gibbons, Hartnoll and Pope [19] provided counterexamples showing non-uniqueness of de Sitter solution in some higher dimensions. In the appendix we used special conformal coordinates valid for 3-dimension. Possibly a more serious issue is that even our starting point of the problem is 3-dimensional in the following sense. In general dimensions the component of the boundary set $V = 0$ representing the horizon may not have the topology of higher dimensional spheres. In future we hope to undertake the study of these issues and non-uniqueness in higher dimensions in view of the present technique.

Acknowledgments

We thank an anonymous referee for suggesting the possibility of an error in the earlier version of Equation (10). The detection and correction of the error forced the next order calculation of the present paper. We also thank the referees for their suggestions and comments for improving the presentation.

Appendix A

In this appendix, we prove Lemma 2.2 used in the proof of the main theorems. Thus we show that in 3-dimension and for our field equations it is possible to choose Ω such that in the conformal normal coordinates about $q \in \partial\Sigma^+$, Equation (7) holds and D_{ij} is diagonal. We also show that $6C_{izi} = -\Lambda c + cD_{zz}, C_{iAi} = 0$ for $x^A = x$ or y. Finally as a by-product we prove an extra result.

First we note that it is possible to choose the conformal function Ω such that the metric $h = \Omega^2 g$ has Ricci curvature 0 at the origin of the conformal coordinate system (see Equation (3.6), page 212 in Schoen and Yau [17]). In 3-dimension this makes $\Omega^2 g_{ij}$ to be $\delta_{ij} + O\left(r^3\right)$. Conformal transformation formula and $\mathrm{Ric}(h) = 0$ gives at the origin $R_{ij} = D_{ij} + D_{kk}\delta_{ij}$. Contracting we get $D_{kk} = (1/2)\Lambda$. So that

$$R_{ij} = D_{ij} + (1/2)\Lambda\delta_{ij} \text{ at } q. \tag{A.1}$$

$\partial\Sigma^+$ and $z = 0$ coordinate plane share the same unit normal vector n^i at q relative to g. Let x^A denotes x or y. Since $\partial\Sigma^+$ is totally geodesic and the normal n^i is parallel to $(\partial/\partial z)$ we can apply Codacci's equation for g to get $R_{zA} = D_{zA} = 0$ at q. Similarly using Gauss' equation and the fact that the

Riemann curvature of $h = \Omega^2 g$ vanishes at q we get D_{AB} is diagonal. Thus D_{ij} is diagonal as claimed.

Now we prove the second claim. Let $\overline{R}$ denote the scalar curvature of $\partial\Sigma^+$ in the metric induced from g. Doubly contracted Gauss–Codacci equation gives at q

$$\overline{R} = \Lambda - 2D_{zz}. \tag{A.2}$$

Let Δ_e denote the Euclidean Laplacian for xyz. Since

$$(\Gamma_g)^k_{ij} = -\delta^k_i D_{jm}x^m - \delta^k_j D_{im}x^m + \delta_{ij}D_{km}x^m + O\left(r^2\right),$$

$\Delta V = -\Lambda V$ and $(\partial^2 V/\partial x^i \partial x^j) = 6C_{ijk}x^k + O(r^2)$ give

$$\begin{aligned} &6C_{kxk}x + 6C_{kyk}y + 6C_{kzk}z - cD_{zz}z - 6D_{CE}x^E C_{CAB}x^B x^A \\ &\quad - 6D_{zz}C_{zzz}z^3 - 6D_{AB}C_{Azz}x^B z^2 - 12D_{zz}C_{zzA}x^A z^2 \\ &\quad - 12D_{EB}x^B C_{EzA}x^A z - 6D_{zz}C_{zAB}x^B x^A z + O(r^2) = -\Lambda V. \end{aligned} \tag{A.3}$$

The above equation is satisfied in a 3-ball about q including a neighborhood of q on $\partial\Sigma^+$. For $V \neq 0$ on the z-axis (with $z \neq 0$), Equation (A.3) gives

$$6C_{kzk}z - cD_{zz}z - 6D_{zz}C_{zzz}z^3 = -\Lambda cz - \Lambda C_{zzz}z^3 - \Lambda\omega(0,0,z) \tag{A.4}$$

Dividing by z and taking limit $z \to 0$, we get

$$6C_{kzk} - cD_{zz} = -\Lambda c. \tag{A.5}$$

Generally (with $z \neq 0$) Equation (A.3) gives

$$\begin{aligned} &6C_{kxk}x/z + 6C_{kyk}y/z + 6C_{kzk} - cD_{zz} - 6D_{CE}x^E C_{CAB}x^B x^A/z \\ &\quad - 6D_{zz}C_{zzz}z^2 - 6D_{AB}C_{Azz}x^B z - 12D_{zz}C_{zzA}x^A z \\ &\quad - 12D_{EB}x^B C_{EzA}x^A - 6D_{zz}C_{zAB}x^B x^A + O(r^2)/z = -\Lambda(V/z), \end{aligned} \tag{A.6}$$

$\lim(V/z)$ exists. In case the $z = 0$ set does not coincide with the $V = 0$ set in a neighborhood of q we can put $V = 0$ and then taking limit as $(x, y, z) \to q$ on $\partial\Sigma^+$ we get

$$6C_{kzk} - cD_{zz} - 6\lim D_{CE}x^E C_{CAB}x^B x^A/z = 0, \qquad C_{kAk} = 0. \tag{A.7}$$

If the $z = 0$ set coincides with the $V = 0$ set in a neighborhood of q then Equation (A.3) gives $D_{AB}C_{ACE} = 0 = C_{ACE}$. In this case, we do not put

$V = 0$ in Equation (A.3) to derive Equation (A.6). In stead in the RHS of Equation (A.6) we use $\lim(V/z)$. Since this limit exists in the LHS we must have $C_{kAk} = 0$. This completes the proof of the second claim and Lemma 2.2.

As a by-product of the argument above we shall now show that if in these coordinates $\partial\Sigma^+$ is locally a graph over the $z = 0$ coordinate plane of a function $z = z(x, y)$ about the origin $q \in \partial\Sigma^+$ with $z = O(r^3)$ and the $z = 0$ set does not coincide with $\partial\Sigma^+$ on an open subset of $\partial\Sigma^+$ about q for any q then g saturates the Boucher–Gibbons–Horowitz [3] inequality namely $\Lambda(\text{Area } \partial\Sigma^+) \leq 12\pi$. This inequality has been established assuming the boundary $\partial\Sigma^+$ to be a single component and in this case equality gives the uniqueness of the de Sitter spacetime.

Using $V = cz + C_{zzz}z^3 + 3C_{zzA}x^Az^2 + 3C_{zxx}x^2z + 3C_{zyy}y^2z + 6C_{zxy} xyz + C_{ABC}x^Cx^Bx^A + \omega$ in the right-hand side of Equation (A.3) we have

(A.8)
$$6C_{kzk} - cD_{zz} - 6\lim D_{CE}x^EC_{CAB}x^Bx^A/z = -\Lambda(c + \lim C_{ABC}x^Cx^Bx^A/z).$$

Thus

$$\lim C_{ABC}x^Cx^Bx^A/z = -c. \tag{A.9}$$

In particular on $\partial\Sigma^+$, $\lim \omega/z = 0$ and all C_{ABC} cannot vanish unless $z = 0$ on the $V = 0$ set in a neighborhood of q. Equations (A.7) and (A.5) give

$$6\lim D_{CE}x^EC_{CAB}x^Bx^A/z = -\Lambda c. \tag{A.10}$$

Since by Equation (A.9) $\lim_{\text{on } y=0,\partial\Sigma^+} C_{xxx}x^3/z = -c$, we have $\lim D_{AB} C_{ACE}x^Cx^Bx^E/z = \lim_{\text{on } y=0,\partial\Sigma^+} D_{xx}C_{xxx}x^3/z = -cD_{xx} = -cD_{yy}$. Thus, $D_{xx} = D_{yy} = D_{zz} = (1/6)\Lambda$. Equation (A.2) now gives $\overline{R} = (2/3)\Lambda$ at q. Thus if for no $q \in \partial\Sigma^+$, the $z = 0$ set coincides with an open subset of $\partial\Sigma^+$ about q, we have $\overline{R} = (2/3)\Lambda$ everywhere on $\partial\Sigma^+$. Then Gauss–Bonnet gives $\Lambda(\text{Area}\partial\Sigma^+) = 12\pi$ which is the equality part of Boucher–Gibbons–Horowitz inequality mentioned above. Thus when $\partial\Sigma^+$ has a single component g is that of de Sitter metric by a result in [3]. Assuming, if possible, that the $z = 0$ set coincides with an open subset of $\partial\Sigma^+$ about $q \in \partial\Sigma^+$ for some q one still possibly can show that $\overline{R} = (2/3)\Lambda$ everywhere on the sphere $\partial\Sigma^+$ is the only possibility. Since this method is becoming more complicated than the method based on the positive mass theorem and it will not work for the case with multiple boundary components we shall not pursue it further.

References

[1] W. Boucher and G.W. Gibbons, *Cosmic baldness* in ‘The Very Early Universe,’ eds. G.W. Gibbons, S.W. Hawking and S.T.C. Siklos, CUP, Cambridge, 1983.

[2] W. Boucher, *Cosmic no-hair theorems* in ‘Classical General Relativity’, eds. W.B. Bonnor, J.N. Islam and M.A.H. MacCallum, CUP, Cambridge, 1984.

[3] W. Boucher, G.W. Gibbons and G. Horowitz, *Uniqueness theorem for anti-de Sitter spacetime*, Phys. Rev. D **30**(12) (1984), 2447–2451.

[4] A.K.M. Masood-ul-Alam, *Static equations with positive cosmological constant*, `http://msc.tsinghua.edu.cn/upload/news_201241144839.pdf` (1211).

[5] O. Kobayashi, *A differential equation arising from scalar curvature function*, J. Math. Soc. Japan **34** (1982), 665–675.

[6] Y. Shen, *PhD Thesis*, Chapter 5, Standford University, 1992.

[7] Y. Shen, *A Note on Fischer–Marsden’s Conjecture*, Proc. Amer. Math. Soc. **125** (1997), 901–905.

[8] V. Moncrief, *Boost symmetries in spatially compact spacetimes with a cosmological constant*, Class. Quantum Gravit. **9** (1992), 2515–2520.

[9] J. Lafontaine, *A remark about static space times*, J. Geom. Phys. **59** (2009), 50–53.

[10] S. Hwang, *The critical point equation on a three-dimensional compact manifold*, Proc. Amer. Math. Soc. **131** (2003), 3221–3230.

[11] R. Schoen, S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Commun. Math. Phys. **65** (1979), 45–76.

[12] J. Lafontaine, *Sur la géométrie d’une géné ralisation de l’équation différentielle d’Obata*, J. Math. Pures Appl. **62**(9) (1983), 63–72.

[13] L. Lindblom, *Some properties of static general relativistic stellar models*, J. Math. Phys. **21**(6) (1980), 1455–1459.

[14] H.P. Künzle, *On the spherical symmetry of a static perfect fluid*, Commun. Math. Phys. **20** (1971), 85–100.

[15] A. Avez, *Le* ds^2 *de Schwarzschild parmi les* ds^2 *stationnaires*, Ann. Inst. Henri Poincarè A **1** (1964), 291–300.

[16] J. Lee and T. Parker, *The Yamabe problem*, Bull. Am. Math. Soc. **17**(1) (1987), 37–91.

[17] R. Schoen and S.-T. Yau, Lectures on Differential Geometry, Vol. 1, International Press (1994).

[18] W.-H. Yu, *A note on the normal derivative of the Green's function of the Yamabe operator*, Undergraduate Thesis in Mathematics, Tsinghua University (2013).

[19] G.W. Gibbons, S.A. Hartnoll and C.N. Pope, *Bohm and Einstein–Sasaki metrics, black holes and cosmological event horizons*, *Phys. Rev. D* **67** (2003), 084024 (24 pages).

[20] G.L. Bunting and A.K.M. Masood-ul-Alam, *Nonexistence of multiple black holes in asymptotically euclidean static vacuum spacetimes*, Gen. Relativ. Gravit. **19** (1987), 147–154.

Mathematical Sciences Center and
Department of Mathematical Sciences
Tsinghua University
Haidian District
Beijing 100084
People's Republic of China
E-mail address: abulm@math.tsinghua.edu.cn
E-mail address: wenhuayu19920402@gmail.com

Received December 5, 2013