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Surfaces that become isotopic after Dehn filling

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We show that after generic filling along a torus boundary component of a 3-manifold, no two closed, 2-sided, essential surfaces become isotopic, and no closed, 2-sided, essential surface becomes inessential. That is, the set of essential surfaces (considered up to isotopy) survives unchanged in all suitably generic Dehn fillings. Furthermore, for all but finitely many non-generic fillings, we show that two essential surfaces can only become isotopic in a constrained way.

1. Introduction

Let M be a compact, orientable, irreducible 3-manifold, and T a torus component of ∂M. Let $M(\alpha)$ be the result of Dehn filling M along a slope α on T. How does the set of 2-sided, closed, essential surfaces[1] in M relate to the set of 2-sided, closed, essential surfaces in $M(\alpha)$? For "most" slopes α, we expect these sets to be the same.

While it is possible that $M(\alpha)$ contains an essential surface that is not (isotopic to) an essential surface in M, this yields a bounded essential surface in M with boundary slope α on T. Hatcher [4, 5] demonstrated that the set of such slopes is finite, so this phenomenon is suitably restricted.

It may also be that a closed essential surface $F \subset M$ compresses in the filled manifold $M(\alpha)$. This is constrained even more precisely by [3, 7]: if there is an incompressible annulus running between F and T, then F clearly compresses when filling along the slope of the annulus, call it β. Moreover, if F compresses when filling along any other slope α, then α and β intersect once and F compresses in $M(\alpha')$ for precisely those α' for which α' and β intersect once. If there is no such annulus, and F compresses in both $M(\alpha)$ and $M(\alpha')$, then α and α' intersect once. Thus, in the the non-annular case there are at most three compressing slopes in total.

[1] All surfaces in this paper are embedded.

Here we answer two remaining questions. While we can avoid slopes for which any particular surface compresses, M may contain infinitely many essential surfaces. Can we determine slopes α where *every* essential surface in M remains incompressible in $M(\alpha)$? We answer this question via the following theorem:

Theorem 1.1. *Let M be a compact, orientable, irreducible 3-manifold with a torus boundary component, T. Then there is a finite set of slopes Ω on T such that for any slope α on T and any closed, connected, 2-sided, essential surface F in M, at least one of the following holds:*

(1) *F is incompressible in $M(\alpha)$.*

(2) *α intersects some slope $\omega \in \Omega$ once.*

(3) *$\alpha \in \Omega$.*

The second conclusion is governed by the annular case of [3] cited above, and the surface in question will compress when filling along any slope that meets ω once.

It is also possible that two non-isotopic closed essential surfaces become isotopic after Dehn filling. Can the set of slopes for which this occurs be restricted? This second question is answered affirmatively by the following:

Theorem 1.2. *Let M be a compact, orientable, irreducible 3-manifold with a torus boundary component, T. Then there is a finite set of slopes Ω on T such that for any slope α on T and any pair of non-isotopic, closed, connected, 2-sided surfaces F and G in M where F is essential in M, at least one of the following holds:*

(1) *F and G are not isotopic in $M(\alpha)$.*

(2) *α intersects some slope $\omega \in \Omega$ once and there is a level isotopy in $M(\alpha)$ between F and G.*

(3) *$\alpha \in \Omega$.*

The *level isotopy* of the second conclusion refers to an isotopy $\gamma \colon F \times I \to M(\alpha)$ such that each component of $\gamma^{-1}(K)$ is contained in a level $F \times \{t\}$, where K is the core of the solid torus attached to M to form $M(\alpha)$. In particular, in parallel with [3], this implies that there is an annulus running between an essential surface F and the boundary component T. Moreover, if such a level isotopy exists, it will exist for any slope that meets the slope of the annulus once.

Together these theorems answer the main questions raised above. If F is any essential surface in M, then for any slope α meeting every slope in Ω at least twice, F cannot become compressible in $M(\alpha)$ by Theorem 1.1 and cannot become isotopic to any other (essential or peripheral) surface in $M(\alpha)$ by Theorem 1.2. We summarize this in the following corollary:

Corollary 1.3. *Let M be a compact, orientable, irreducible 3-manifold with a torus boundary component, T. Then there is a finite set of slopes Ω on T such that for any slope α on T that meets every slope $\omega \in \Omega$ at least twice, there is a bijection between the set of closed, connected, 2-sided, essential surfaces of M and the set of of closed, connected, 2-sided, essential surfaces of $M(\alpha)$, where the elements of each set are considered unique up to isotopy.*

Our proofs are conducted in the filled manifold $M(\alpha)$, where we consider separately the cases that an essential surface F compresses to a surface G, in the case of Theorem 1.1, or becomes isotopic to a different surface G, in the case of Theorem 1.2. In Section 2, we construct a *compressing sequence*, a sequence of surfaces that starts with F, ends with G, and both encodes compressions and discretizes isotopies that pass through the core of the attached solid torus. Certain elements of a minimal compressing sequence behave like the thick levels for a knot in thin position. In particular, there are compressing disks on both sides of such a surface, and every compressing disk on one side meets every compressing disk on the other (Corollary 2.14). By [2], such a thick level must therefore be strongly irreducible and ∂-strongly irreducible (and hence its boundary slope is in a finite set by [1] and [6]), or its boundary slope meets the boundary slope of an incompressible, ∂-incompressible surface (a finite set by [5]) at most once. It follows that α is therefore in a predetermined finite set of slopes Ω, or meets one of the slopes in Ω once. In Section 3, we show that in the latter case the core of the attached solid torus can be isotoped into every thick level, giving a level isotopy from F to G.

2. Compressing sequences

For the remainder of this paper, M will denote a compact, orientable, irreducible 3-manifold, T a component of ∂M which is homeomorphic to a torus, and $M(\alpha)$ the result of Dehn filling along α, i.e., attaching a solid torus to M along T so that a loop representing α bounds a disk. The results of this section will be used in the case that $M(\alpha)$ is irreducible, and hence we

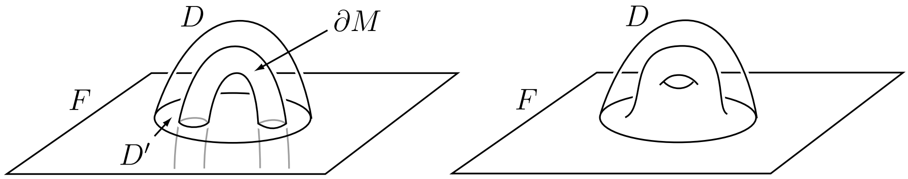

Figure 1: A dishonest compressing disk D and its witness D' (left) and an honest compressing disk D (right) for $F \cap M$.

assume that here. We will always denote the core of the attached solid torus as K.

Definition 2.1. Let F be a (possibly empty) embedded surface in $M(\alpha)$. If F is empty, then we define the *width* $w(F)$ to be $(0, 0)$. If F is connected, then the *width* $w(F)$ is the pair, $(genus(F), |K \cap F|)$. If F is disconnected, then its *width* is the ordered set of the widths of its components, where we include repeated pairs and the ordering is non-increasing. Comparisons are made lexicographically at all levels.

Definition 2.2. Let F be an embedded surface in $M(\alpha)$ that is transverse to K. A *compressing disk* for $F \cap M$ is an embedded disk $D \subset M$ such that $D \cap F = \partial D$ is essential in $F \cap M$. Let B be the closure of a neighborhood of D in $M(\alpha) \setminus F$. Let F' be the surface obtained from F by removing $B \cap F$ and replacing it with the rest of ∂B. Then we say F' is obtained from F by *compressing* along D in M. A compressing disk D for $F \cap M$ is *dishonest* if ∂D bounds a disk D' in F. In this case, the disk D' is said to be the *witness* for D. If D is not dishonest we say it is *honest*. See Figure 1.

Lemma 2.3. *Suppose F' is obtained from a surface F in $M(\alpha)$ by compressing along some compressing disk for $F \cap M$, followed by removing any resulting components that lie in a ball in $M(\alpha)$. Then $w(F') < w(F)$.*

Proof. Let D be a compressing disk for $F \cap M$. Let F' denote the surface obtained from F by compressing along D, followed by removing any resulting components that lie in a ball in $M(\alpha)$. Suppose first that D is honest, so that ∂D is essential in F. If, furthermore, ∂D is non-separating, then the genus of F' is less than the genus of F, and hence the width is also less. If, on the other hand, ∂D is separating in F, then F' is disconnected, and both components have smaller genus. Hence, again width has decreased.

Now suppose D is dishonest, and D' is its witness. Thus, $D' \cap K \neq \emptyset$. The result of compressing along D produces a sphere in $M(\alpha)$ and a surface that has the same genus as F, but meets K fewer times, and thus has smaller width than F. This latter surface is precisely F'. □

Definition 2.4. A *compressing sequence* is a sequence $\{F_i\}$ of (possibly empty) 2-sided, embedded surfaces in $M(\alpha)$ such that for each i, either F_i or F_{i+1} is obtained from the other by compressing along a compressing disk in M, followed by discarding any resulting components that lie in a ball in $M(\alpha)$.

Note that by Lemma 2.3, the widths of consecutive elements of a compressing sequence must be different. This motivates the next definition.

Definition 2.5. Let $\{F_i\}$ be a compressing sequence. An element F_i is said to be a *thick level* of the sequence if $w(F_i) > w(F_{i\pm1})$.

Definition 2.6. Let F be a 2-sided, embedded surface in $M(\alpha)$. The *disk complex* of $F \cap M$ is the complex whose vertices correspond to isotopy classes of compressing disks for $F \cap M$. Two such vertices are connected by a 1-simplex if there are representatives of the corresponding isotopy classes that are disjoint.

Note that the endpoints of a 1-simplex of the disk complex of a surface may represent disks on the same side of the surface, or on opposite sides.

Definition 2.7. Let $\{F_i\}$ be a compressing sequence and F_i a thick level. Then F_{i-1} is obtained from F_i by compressing along a disk $D \subset M$, and F_{i+1} is obtained from F_i by compressing along a disk $E \subset M$. If D and E can be connected by a path in the disk complex of $F_i \cap M$, then we say the *angle* $\angle(F_i)$ is the minimal length of such a path. If D and E cannot be connected by such a path, then we say $\angle(F_i) = \infty$.

Definition 2.8. Let $\{F_i\}$ be a compressing sequence. Then the *size* of the entire sequence is the ordered set of pairs

$$\{(w(F_i), \angle(F_i)) \mid F_i \text{ is a thick level}\},$$

where repeated pairs are included, and the ordering is non-increasing. Two such sets are compared lexicographically.

Definition 2.9. A compressing sequence $\{F_i\}_{i=0}^n$ is *minimal* if its size is smallest among all sequences from F_0 to F_n.

Lemma 2.10. *Let $\{F_i\}$ be a minimal compressing sequence, and let F_i be a thick level. Then $\angle(F_i) = \infty$.*

Proof. The proof is by induction on $\angle(F_i)$. Let D and E be as in Definition 2.7. If $\angle(F_i) = 0$, then $D = E$. In this case $F_{i-1} = F_{i+1}$, and we can obtain a smaller compressing sequence by removing the subsequence $\{F_i, F_{i+1}\}$.

If $\angle(F_i) = 1$, then $D \cap E = \emptyset$. Since D and E are disjoint, we may form the surface F', obtained from F_i by simultaneously compressing along both D and E, followed by discarding any resulting components that lie in a ball in $M(\alpha)$. If F' is not isotopic to F_{i-1}, then F' will be obtainable from F_{i-1} by compressing along E and discarding components in a ball. Similarly, if F' is not isotopic to F_{i+1}, then F' will be obtainable from F_{i+1} by compressing along D and discarding components in a ball.

There are now three possible cases. If F' is isotopic to both F_{i-1} and F_{i+1}, then those two surfaces are isotopic to each other. In this case, we may discard the subsequence $\{F_i, F_{i+1}\}$ to obtain a smaller compressing sequence. If F' is isotopic to either F_{i-1} or F_{i+1} (but not both), then we may obtain a lower complexity compressing sequence by removing F_i. Finally, if F' is not isotopic to either F_{i-1} or F_{i+1}, then, by Lemma 2.3, we get a lower complexity compressing sequence by replacing F_i by F'.

If $\angle(F_i) > 1$ then let $\{C_i\}_{i=0}^n$ be a sequence of compressing disks for $F_i \cap M$ in M such that $\{D, C_0, \ldots, C_n, E\}$ is a minimal length path in its disk complex. Let F_- be the surface obtained from F_i by compressing along C_0. Now consider the compressing sequence $\{\ldots F_{i-1}, F_i, F_-, F_i, F_{i+1}, \ldots\}$. Both occurrences of F_i in this new sequence are thick levels with the same width as before, but in both cases we have reduced the angle. Hence we have produced a smaller compressing sequence. □

Corollary 2.11. *Let $\{F_i\}$ be a minimal compressing sequence and F_i a thick level. Let D and E be as in Definition 2.7. Then D and E are on opposite sides.*

Proof. By Lemma 2.10, $\angle(F_i) = \infty$. Hence, by definition there is no path in the disk complex of $F_i \cap M$ from D to E. But if D and E are on the same side X of F_i, then we can construct such a path by successively ∂-compressing D along subdisks of E in X. □

Corollary 2.12. *Let $\{F_i\}$ be a minimal compressing sequence and F_i a thick level. Then every pair of compressing disks for $F_i \cap M$ on opposite sides must intersect.*

Proof. Let D and E be as in Definition 2.7. By Corollary 2.11, D and E are on opposite sides of F_i. Let B and C be disjoint compressing disks for $F_i \cap M$ on opposite sides, where B and D are on the same side. As in the proof of Corollary 2.11, there must then be a path in the disk complex from D to B in the disk complex of $F_i \cap M$. Similarly, the disks C and E are on the same side, so there is a path between them in the disk complex. Finally, as B and C are disjoint, there is an edge between them in the disk complex. Putting the two paths together with this edge then forms a path from D to E in the disk complex. But this contradicts Lemma 2.10, which asserts that there is no such path. □

Definition 2.13. A properly embedded surface F in a 3-manifold M is *strongly irreducible* if there is at least one compressing disk on each side, and every pair of compressing disks on opposite sides intersects.

Corollary 2.14. *Let $\{F_i\}$ be a minimal compressing sequence and F_i a thick level. Then $F_i \cap M$ is strongly irreducible.*

Proof. By Corollary 2.11 there exist compressing disks on each side of F_i, and by Corollary 2.12 any such pair of disks on opposite sides must intersect. □

Definition 2.15. A *∂-compressing disk* for a properly embedded surface $F \subset M$ is a disk D such that $\partial D = \alpha \cup \beta$, where $\alpha = D \cap F$ is an essential arc on F and $D \cap \partial M = \beta$. We say F is *∂-compressible* if it has a ∂-compressing disk, and is *∂-incompressible* otherwise. We say F is *∂-strongly irreducible* if there is at least one compressing or ∂-compressing disk on each side of F, and any pair of compressing or ∂-compressing disks on opposite sides intersects.

The following terminology is due to Jesse Johnson.

Definition 2.16. Let F be a surface in $M(\alpha)$. Let D be an embedded disk such that $\partial D = \alpha \cup \beta$, $D \cap F = \alpha$, and $D \cap K = \beta$. Let B be the closure of a neighborhood of D in $M(\alpha) \setminus F$. Let F' be the surface obtained from F by removing $B \cap F$ and replacing it with the rest of ∂B. Then we say F' is obtained from F by *bridge compressing* along D. The disk D is a *bridge compressing disk* for F. The disk $\overline{D}$ that is the frontier of B in $M(\alpha) \setminus F$ is

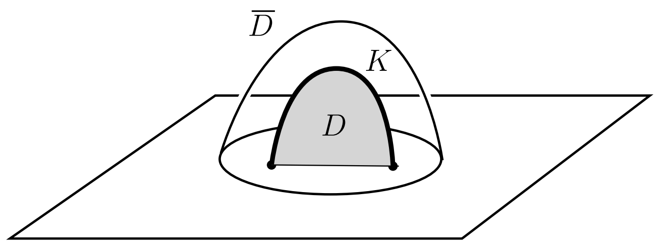

Figure 2: A bridge compressing disk and its associated compressing disk.

said to be the compressing disk for $F \cap M$ that is *associated* to the bridge compressing disk D. See Figure 2.

Note that a bridge compressing disk for $F \subset M(\alpha)$ meets M in a ∂-compressing disk for $F \cap M$. However, not every ∂-compressing disk for $F \cap M$ is the intersection of a bridge compressing disk with M.

Lemma 2.17. *Let $\{F_i\}$ be a minimal compressing sequence and F_i a thick level. Then there are two cases:*

(1) *$F_i \cap M$ is strongly irreducible and ∂-strongly irreducible.*

(2) *$\partial(F_i \cap M)$ intersects the slope bounding an incompressible and ∂-incompressible surface in M at most once.*

Furthermore, when $\partial(F_i \cap M)$ intersects the slope bounding an incompressible and ∂-incompressible surface in M exactly once, there are bridge compressing disks on opposite sides of F_i that meet in two points of K.

Proof. By Corollary 2.14, a thick level F_i meets M in a surface that is strongly irreducible. By Lemma 4.8 of [2], such a surface is either ∂-strongly irreducible, or has boundary that has intersection number at most one with the boundary of a surface S that is both incompressible and ∂-incompressible. The last paragraph in this proof implies that when the boundary of $F_i \cap M$ is at a distance of exactly one from S, there are ∂-compressions on opposite sides of $F_i \cap M$ that are the intersections of bridge compressing disks with M, on opposite sides of F_i, that meet in two points of K. □

3. Surfaces after Dehn filling

Lemma 3.1. *Suppose F and G are surfaces in $M(\alpha)$ that are transverse to K, isotopic in $M(\alpha)$, and (when restricted to M) not isotopic in M. Then there exists a compressing sequence $\{F_i\}_{i=0}^n$ such that*

(1) $F_0 = F$,

(2) $F_n = G$, *and*

(3) *for each i, either F_i or F_{i+1} is obtained from the other by a dishonest compression.*

We call a sequence given by the conclusion of the lemma a *dishonest compressing sequence.*

Proof. Let $\gamma\colon F \times I \to M(\alpha)$ be an isotopy from F to G which is in general position with respect to K. Let $F(t) = \gamma(F, t)$. Let $\{t'_i\}$ denote the values of t for which $F(t)$ is not transverse to K. For each i, choose $t_i \in (t'_{i-1}, t'_i)$. Then for each i, either $F(t_i)$ or $F(t_{i+1})$ is obtained from the other by bridge compressing along some disk D. It follows that one of these surfaces can be obtained from the other by compressing along the associated disk $\overline{D}$, and throwing away a 2-sphere in $M(\alpha)$. Note that the disk $\overline{D}$ is dishonest, and thus $\{F(t_i)\}$ is a dishonest compressing sequence from F to G. □

Lemma 3.2. *Suppose F is a closed, connected, essential, 2-sided surface in M, and G is a (possibly empty) surface in $M(\alpha)$ obtained by a compression of F followed by discarding sphere components in $M(\alpha)$. Then there exists a compressing sequence from F to G, any minimal such sequence has a thick level, and the first such thick level meets K.*

Proof. If F is compressible in $M(\alpha)$, then by shrinking the compressing disk off of K, it is isotopic in $M(\alpha)$ to a surface G' which can be compressed in M by an honest compressing disk. (See Figure 3.)

By Lemma 3.1 there exists a dishonest compressing sequence from F to G'. Note that the width of all surfaces in this sequence is of the form (g, x_i), where g is the genus of F. We may add one more element to this compressing sequence, namely the (possibly empty) surface G'' obtained from G' by compressing in M and possibly discarding a sphere in $M(\alpha)$, to obtain a compressing sequence from F to G''. By Lemma 3.1 there is now a dishonest compressing sequence from G'' to G. Putting this all together, we obtain a compressing sequence from F to G in which the genus of every element is at most g.

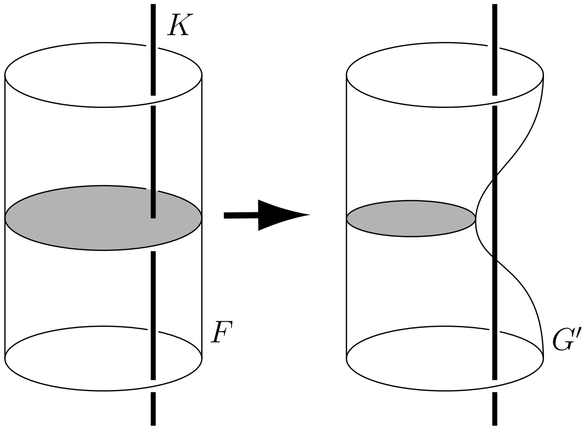

Figure 3: The surface G' obtained by shrinking F along a compressing disk in $M(\alpha)$.

Now let $\{F_i\}_{i=0}^n$ be a minimal compressing sequence from F to G, so that $F_0 = F$ and $F_n = G$. Since the size of this sequence is at most the size of the sequence constructed above, the genus of each component of F_i is at most g, for every i.

As $F \cap K = \emptyset$, $w(F) = (g, 0)$. Furthermore, as G is obtained from F by a compression, $w(F_n) = w(G) < w(F)$.

Since F is incompressible in M, it follows that $F_0 = F$ must be obtained from F_1 by a compression. Thus, $w(F_1) > w(F_0)$ and therefore $w(F_1) > w(F_n)$. It follows that for some i, $w(F_i) > w(F_{i\pm1})$, and thus F_i is a thick level.

Since the genus of each component of F_1 is at most g and F_0 is connected, it must be the case that the compression which results in the surface F_0 was dishonest. Thus, $F_1 \cap K \neq \emptyset$. It follows that the first thick level meets K. □

We are now prepared to prove Theorem 1.1. Recall the statement:

Theorem 1.1. *Let M be a compact, orientable, irreducible 3-manifold with a torus boundary component, T. Then there is a finite set of slopes Ω on T such that for any slope α on T and any closed, connected, 2-sided, essential surface F in M, at least one of the following holds:*

(1) *F is incompressible in $M(\alpha)$.*

(2) *α intersects some slope $\omega \in \Omega$ once.*

(3) *$\alpha \in \Omega$.*

Proof. Let Ω be the set of slopes bounding surfaces that are incompressible and ∂-incompressible or strongly irreducible and ∂-strongly irreducible. By [5] the set of slopes bounding incompressible and ∂-incompressible surfaces is finite. By [1], every strongly irreducible and ∂-strongly irreducible surface can be made almost normal with respect to a triangulation of M that has one vertex on T. By [6], the set of slopes bounding such surfaces is finite. Hence, Ω is a finite set.

If $M(\alpha)$ is reducible then there is an essential planar surface in M whose boundary slope on T is α, and thus $\alpha \in \Omega$. Thus we assume $M(\alpha)$ is irreducible.

Let F be a closed, connected, 2-sided, essential surface in M. Suppose G is a surface that is obtained from F by a compression in $M(\alpha)$ followed by discarding any resulting sphere components. It follows from Lemma 3.2 that there exists a minimal compressing sequence from F to G that has a thick level, and the first such thick level F_i meets K. Then by Lemma 2.17 there are two cases:

(1) $F_i \cap M$ is strongly irreducible and ∂-strongly irreducible, and hence $\partial(F_i \cap M) \in \Omega$.

(2) $\partial(F_i \cap M)$ intersects the slope bounding an incompressible and ∂-incompressible surface in M at most once. Hence, $\partial(F_i \cap M)$ intersects some slope in Ω at most once.

This yields conclusions (3) and (2) of the theorem, respectively. □

Lemma 3.3. *Suppose F and G are non-isotopic, closed, connected surfaces in M that are isotopic in $M(\alpha)$, and that F is essential in M. Then either $M(\alpha)$ is reducible, or there exists a compressing sequence from F to G and any minimal such compressing sequence is dishonest and has a thick level.*

Proof. By Lemma 3.1 there exists a dishonest sequence $\{F'_i\}$ from F to G. Note that for each i, $w(F'_i) = (g, x_i)$, where g is the genus of F. Now let $\{F_i\}$ denote a minimal compressing sequence from F to G. In particular, it follows that the size of this sequence is at most the size of the sequence $\{F'_i\}$. Thus, for all i, $w(F_i) \leq \max\{w(F'_j)\} = (g, x)$ for some x. In particular, for each i the genus of F_i is at most g.

We will now show that the entire compressing sequence $\{F_i\}$ is dishonest. To this end, assume n is the largest integer such that the subsequence $\{F_i\}_{i=0}^n$ of $\{F_i\}$ is dishonest. We will show that in fact, $\{F_i\}_{i=0}^n$ is the entire compressing sequence. If $M(\alpha)$ is irreducible, then every dishonest compression can be realized by an isotopy in $M(\alpha)$. We conclude F_n is isotopic in

$M(\alpha)$ to F. Thus F_n is incompressible in $M(\alpha)$. It follows that F_{n+1} cannot be obtained from F_n by an honest compression. Since F_n is connected and the genus of each component of F_{n+1} is at most g, we also conclude that F_n cannot be obtained from F_{n+1} by an honest compression. We conclude that either F_n or F_{n+1} is obtained from the other by a dishonest compression, contradicting the maximality of n.

What remains is to show that $\{F_i\}$ has a thick level. Note that as F and G miss K, $w(F) = w(G) = (g, 0)$. Since F is incompressible in M, it follows that $F_0 = F$ must be obtained from F_1 by a compression. Thus, $w(F_1)$ is larger than the widths of the first and last element of $\{F_i\}$. It follows that for some i, $w(F_i) > w(F_{i\pm1})$, and thus F_i is a thick level. □

We are now prepared to prove Theorem 1.2. Recall the statement:

Theorem 1.2. *Let M be a compact, orientable, irreducible 3-manifold with a torus boundary component, T. Then there is a finite set of slopes Ω on T such that for any slope α on T and any pair of non-isotopic, closed, connected, 2-sided surfaces F and G in M where F is essential in M, at least one of the following holds:*

(1) *F and G are not isotopic in $M(\alpha)$.*

(2) *α intersects some slope $\omega \in \Omega$ once and there is a level isotopy in $M(\alpha)$ between F and G.*

(3) *$\alpha \in \Omega$.*

Proof. Let Ω be the set of slopes bounding surfaces that are incompressible and ∂-incompressible or strongly irreducible and ∂-strongly irreducible. As in the proof of Theorem 1.1, the set Ω is finite. As before, we assume $M(\alpha)$ is irreducible.

Let F and G be non-isotopic, closed, connected, 2-sided surfaces in M, where F is essential in M, that are isotopic in $M(\alpha)$. By Lemma 3.3 there exists a compressing sequence from F to G, and any minimal such sequence $\{F_i\}$ is dishonest and has a thick level. It follows that all thick levels meet K.

By Lemma 2.17, either

(1) Some thick level meets M in a strongly irreducible and ∂-strongly irreducible surface, or

(2) Some thick level meets ∂M in a slope that intersects the slope bounding an incompressible, ∂-incompressible surface at most once.

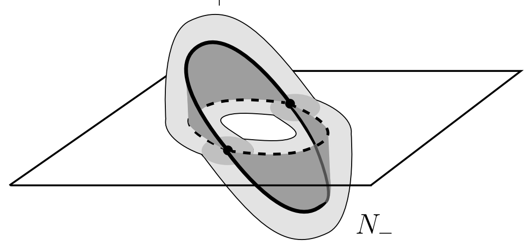

Figure 4: The solid torus $N_+ \cup N_-$

In the first case, $\alpha \in \Omega$. In the second case, either $\alpha \in \Omega$ or by Lemma 2.17, α meets some slope in Ω once and there are bridge compressing disks on opposite sides of each thick level that meet in two points of K. To complete the proof of Theorem 1.2, we must now show in the latter case that there is a level isotopy from F to G.

Recall that when $M(\alpha)$ is irreducible then a dishonest compressing sequence gives rise to an isotopy between F and G, with each F_i corresponding to an intermediate level of the isotopy. If F_i is a thick level, we will redefine this isotopy between F_{i-1} and F_{i+1} so that K lies on an intermediate surface, making the isotopy into a level isotopy. To do this, use the the bridge compressing disks B_+ and B_- for F_i given above so that $F_{i\pm1}$ is obtained from F_i by compressing along the associated compressing disk $\overline{B}_\pm$. Let N_+ and N_- be neighborhoods in $M(\alpha)$ of B_+ and B_-, respectively. Note that $N_+ \cup N_-$ is a solid torus whose core is K, and F_i cuts $\partial(N_+ \cup N_-)$ into two longitudinal annuli, A_+ and A_- (see Figure 4). Moreover, $F_{i\pm1}$ is isotopic to $(F_i - (N_+ \cup N_-)) \cup A_\pm$. We can now define an isotopy between F_{i-1} and F_{i+1} that keeps the surface fixed outside of $N_+ \cup N_-$, and inside $N_+ \cup N_-$ isotopes A_- to A_+ (rel ∂) such that K lies flat on an intermediate annulus. Doing this at each thick level thus yields a level isotopy from F to G. □

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