communications in
analysis and geometry
Volume 23, Number 2, 319–348, 2015

Curvatures and anisometry of maps

Benoît R. Kloeckner

We prove various inequalities measuring how far from an isometry a local map from a manifold of high curvature to a manifold of low curvature must be. We consider the cases of volume-preserving, conformal and quasiconformal maps. The proofs relate to a conjectural isoperimetric inequality for manifolds whose curvature is bounded above, and to a higher-dimensional generalization of the Schwarz–Ahlfors lemma.

1. Introduction

One of the basic facts of Riemannian geometry is that curvatures are isometry invariants: this explains, for example, why one cannot design a perfect map of a region on the earth. In this article, we shall be interested in quantifying this fact: how far from being an isometry a map from a region of a manifold to another manifold must be, when the source and target manifolds satisfy incompatible curvature bounds?

When the source manifold is the round 2-sphere and the target manifold is the Euclidean plane, this question is a cartography problem: a round sphere is a relatively good approximation of the shape of the Earth. It has been considered by Milnor [11] who described the best map when the source region is a spherical cap. Surprisingly, it seems like no other cases of the general question above have been considered.

1.1. Distortion and anisometry

To fill this gap, one has first to ask how we should measure the isometric defect of a map $\varphi : D \subset M \to N$ from a domain in a manifold M to a manifold N, assumed to be a diffeomorphism on its image. Milnor uses the distortion, defined as follows. Let $\sigma_1 = \sigma_1(\varphi)$ and $\sigma_2 = \sigma_2(\varphi)$ be the

Lipschitz constants of φ, i.e.,

$$\sigma_1 d(x,y) \leqslant d(\varphi(x),\varphi(y)) \leqslant \sigma_2 d(x,y) \quad \forall x,y \in D$$

and σ_1, σ_2 are, respectively, the greatest and least numbers satisfying such an inequality. Then the *distortion* of φ is the number $\operatorname{dist}(\varphi) = \log(\sigma_2/\sigma_1)$.

However, when the target manifold is not Euclidean, the distortion is ill-suited: it is zero for maps that are not isometries, but mere homotheties. More disturbing is the case when M is positively curved and N is negatively curved: to minimize distortion, one is inclined to take φ with a very small image, so that the curvature of N barely matters. To make this case more interesting, we propose the following definition of *anisometry*:

$$\operatorname{aniso}(\varphi) = |\log \sigma_1| + |\log \sigma_2|.$$

This quantity generalizes distortion in the sense that when $N = \mathbb{R}^n$,

$$\inf_\varphi \operatorname{aniso}(\varphi) = \inf_\varphi \operatorname{dist}(\varphi),$$

indeed one can in this case rescale the target to ensure $\sigma_1 \leqslant 1 \leqslant \sigma_2$. Note that another possible measure of the isometric defect would be the bi-Lipschitz constant $\max(1/\sigma_1, \sigma_2)$.

1.2. Azimuthal maps

To describe our results we will need to introduce a specific family of maps between model spaces. All considered manifolds will be of the same fixed dimension n; we set X_κ for the simply connected manifold of constant curvature κ (thus a sphere, the Euclidean space or a hyperbolic space).

Given a point $x \in X_\kappa$, we have polar coordinates (t,u) (t a positive real, u a unit tangent vector at x) given by the exponential map:

$$y = \exp_x(tu),$$

where t is less than the conjugate radius and y may be any point but the antipodal point to x (when $\kappa > 0$).

Definition. An *azimuthal* map is a map $\varphi : B \subset X_\rho \to X_\kappa$ where B is a geodesic ball, which reads in polar coordinates centered at x and $\varphi(x)$ as

$$\varphi(t,u) = (R(t), L(u)),$$

where L is a linear isometry from T_xX_ρ to $T_{\varphi(x)}X_\kappa$ and R is a differentiable function. In other words, we have

$$\varphi(\exp_x(tu)) = \exp_{\varphi(x)}(R(t)L(u)).$$

The function R is then called the *distance function* of φ.

As we consider only model spaces, L is irrelevant and the function R defines a unique azimuthal map up to isometries. The azimuthal map associated to each of the following distance functions bears a special name:

- $R(t) = t$: *equidistant* azimuthal map,
- $R(t) = \sigma t$ with $\sigma \in (0, 1)$: σ-*contracting* azimuthal map.

Moreover, given ρ and κ there exists exactly one family of conformal azimuthal maps and a unique volume-preserving azimuthal map $B \subset X_\rho \to X_\kappa$ (see below for details).

1.3. Description of the results

We shall not state our results in the greatest generality in this introduction, please see below for details.

Our main results have the following form: we assume M satisfies some kind of lower curvature bound associated with a parameter ρ, that N satisfies some kind of upper curvature bound (or more general geometric assumption) associated with a parameter $\kappa < \rho$, and that φ is a map (possibly satisfying extra assumptions) from a geodesic ball of center x and radius α in M to N.

Our methods provide half-local results, and we shall always assume that α is bounded above by some number. This bound shall be explicit most of the time and depends only on synthetic geometrical properties of M and N. In some cases (e.g., when the target is a Hadamard manifold) this bound will be completely harmless.

We then conclude that there is an azimuthal map $\bar{\varphi} : B_\rho(\alpha) \to X_\kappa$ (where $B_\rho(\alpha)$ is any geodesic closed ball of radius α in X_ρ) such that

$$\operatorname{aniso}(\varphi) \geqslant \operatorname{aniso}(\bar{\varphi})$$

with equality if and only if φ is conjugated to $\bar{\varphi}$ by isometries.

For simplicity, we shall write $\mathrm{Ric}_M \geqslant \rho$ to mean that the Ricci tensor and the metric tensor of M satisfy the usual bound

$$\mathrm{Ric}_x(u,u) \geqslant \rho \cdot (n-1) g_x(u,u) \quad \forall x, u.$$

Similarly, $K_N \leqslant \kappa$ means that the sectional curvature of N is not greater than κ at any tangent 2-plane.

We shall always assume implicitly that M (or more generally $B_M(x,\alpha)$) and N are complete; recall that they have the same dimension n.

Theorem 1.1 (General maps). *Assume $\mathrm{Ric}_M \geqslant \rho$, $K_N \leqslant \kappa$ where $\rho > \kappa$, and $\alpha \leqslant A_1(M,N)$ where $A_1(M,N)$ is an explicit positive constant.*

Then any map $\varphi : B(x,\alpha) \subset M \to N$ satisfies

$$\mathrm{aniso}(\varphi) \geqslant \mathrm{aniso}(\bar{\varphi})$$

where $\bar{\varphi}$ is:

- *the equidistant azimuthal map $B_\rho(\alpha) \to X_\kappa$ when $\kappa \geqslant 0$,*
- *the σ-contracting azimuthal map $B_\rho(\alpha) \to X_\kappa$ when $\kappa < 0$, where σ is such that the boundaries of $B_\rho(\alpha)$ and $B_\kappa(\sigma\alpha)$ have equal volume.*

Moreover in case of equalities φ and $\bar{\varphi}$ are conjugated by isometries (in particular, the source and image of φ have constant curvatures ρ and κ).

One can write $\mathrm{aniso}(\bar{\varphi})$ explicitly, see below. This theorem is proved using a rather direct generalization of Milnor's argument, which considers the constant curvature, two-dimensional case.

Remark. (1) It is interesting to see that the sign of κ has such an influence on the optimal map: when $\kappa > 0$ the best map is isometric along rays issued from the center, and increases distances in the orthogonal directions, while when $\kappa < 0$ the best map induces an isometry on the boundaries but contracts the radial rays. Of course, when $\kappa = 0$ all σ-contracting azimuthal maps are equivalent up to a homothety, and as long as $\sigma_1 \leqslant 1 \leqslant \sigma_2$ their anisometries are equal.

(2) The hypothesis on N can be relaxed thanks to the generalized Günther inequality proved by the author and Kuperberg [8]. In particular,

$K_N \leqslant \kappa$ can be replaced by mixed curvature bounds like

$$K_N \leqslant \rho \quad \text{and} \quad \mathrm{Ric}_N \leqslant (n-1)\kappa - n\rho$$

see Section 2.3 and Theorem 3.2 for the most general hypothesis and the above reference for various classical assumptions that imply this general hypothesis.

(3) The precise expression of A_1 is given in Section 3. In many cases one can adapt the result and its proof to larger α but we favored clarity over exhaustivity. For example, what happens for α close to $\frac{\pi}{\sqrt{\rho}}$ is that the boundary of $B_\rho(\alpha)$ becomes very small, and one can improve the equidistant azimuthal map by making it dilating along the rays.

We shall then consider maps satisfying special conditions. Two prominent examples are *volume-preserving* maps and *conformal* maps. In cartography, both make sense: area is obviously a relevant geographic information, and for many historical uses (e.g., navigation) measurement of angles on the map have been needed. Moreover, asking a map to be conformal means that zooming into the map will decrease arbitrarily the distortion of a smaller and smaller region. We therefore ask whether in general, asking φ to be volume-preserving or conformal increases the anisotropy lower bound by much.

In the theorems below, we shall make the assumption that N satisfies the best isoperimetric inequality holding on X_κ, meaning that for all smooth $\Omega \subset N$,

$$\mathrm{Vol}(\partial\Omega) \geqslant I_\kappa(\mathrm{Vol}(\Omega)),$$

where I_κ is the isoperimetric profile of X_κ defined by

$$I_\kappa(V) = \inf_{\Omega \subset X_\kappa} \{\mathrm{Vol}(\partial\Omega) \mid \mathrm{Vol}(\Omega) = V\}.$$

This assumption can be replaced by $K_N \leqslant \kappa$ in some cases.

One says that n is a *Hadamard manifold* if $K_N \leqslant 0$ and N is simply connected; it is conjectured that all Hadamard manifolds satisfy the isoperimetric inequality of X_κ whenever $K_N \leqslant \kappa$, but this conjecture has only been proved in a handful of cases: when $n = 2$ [1, 14], $n = 3$ [10], $(n = 4, \kappa = 0)$ [3] and $(n = 4, \kappa < 0)$ for small enough domains [9]. Moreover, the similar conjecture when $\kappa > 0$ holds in dimension $n = 4$ for uniquely geodesic domains [9]. When $n = 4$ the curvature assumption can generally be relaxed as for Theorem 1.1, see Section 2.3 below and [9].

This means that in most dimensions, our results below hold under a curvature assumption only conditionally to a strong conjecture; but note that even in the case when $N = X_\kappa$ these results are new.

Theorem 1.2 (Volume-preserving maps). *Assume* $\mathrm{Ric}_M \geqslant \rho$, *$N$ satisfies the best isoperimetric inequality holding on X_κ for some $\kappa < \rho$, and $\alpha \leqslant \mathrm{inj}(x)$ for a given $x \in M$.*

Then any volume-preserving map $\varphi : B(x, \alpha) \subset M \to N$ satisfies

$$\mathrm{aniso}(\varphi) \geqslant \mathrm{aniso}(\bar\varphi),$$

where $\bar\varphi$ is the unique volume-preserving azimuthal map $B_\rho(\alpha) \to X_\kappa$.

Assume further that the only domains in N satisfying the equality case in the isoperimetric inequality are balls isometric to geodesic balls in X_κ. Then whenever $\mathrm{aniso}(\varphi) = \mathrm{aniso}(\bar\varphi)$, *the domain of φ has constant curvature ρ and its range is isometric to a constant curvature ball $B_\kappa(R(\alpha))$. However, there are uncountably many different maps achieving equality.*

Remark. Here we have put little restriction on α (we only restrict it below the injectivity radius at x for simplicity), but in fact stronger restriction can appear when one wants to apply the result. Indeed, if one is only able to show that small enough domains of N satisfy the desired isoperimetric inequality, then one can still use Theorem 1.2 for small enough α: then a map $B(x, \alpha) \to N$ either has a small image, or a large σ_2.

Theorem 1.3 (Conformal maps). *Assume* $\mathrm{Ric}_M \geqslant \rho$, *$N$ satisfies the best isoperimetric inequality holding on X_κ for some $\kappa < \rho$, and $\alpha \leqslant A_3(M, N)$ where $A_3(M, N)$ is an explicit positive constant.*

Then any conformal map $\varphi : B(x, \alpha) \subset M \to N$ satisfies

$$\mathrm{aniso}(\varphi) \geqslant \mathrm{aniso}(\bar\varphi),$$

where $\bar\varphi$ is:

- *the conformal azimuthal map $B_\rho(\alpha) \to X_\kappa$ with $R'(0) = 1$ when $\kappa \geqslant 0$,*
- *the conformal azimuthal map $B_\rho(\alpha) \to X_\kappa$ that induces an isometry on the boundaries when $\kappa < 0$.*

Assume further that the only domains in N satisfying the equality case in the isoperimetric inequality are balls isometric to geodesic balls in X_κ. Then whenever $\mathrm{aniso}(\varphi) = \mathrm{aniso}(\bar\varphi)$, *the maps φ and $\bar\varphi$ are conjugated by isometries (in particular, the domain and range of φ have constant curvatures ρ*

and κ), except that when $\kappa = 0$ one can compose $\bar{\varphi}$ with any homothety such that we still have $\sigma_1 \leqslant 1 \leqslant \sigma_2$, and still get an optimal map.

Remark. We shall see that A_3 can, in fact, be chosen independently of N (but depending on κ). Moreover, when $\kappa \leqslant 0$ we can take $A_3 = \mathrm{inj}(x)$.

Conformal maps are rare in higher dimension, so we also tackle quasiconformal maps, whose angular distortion is controlled. Recall that a smooth map φ is said to be Q-quasiconformal if at each point x in its domain, we have $\mathrm{dist}(D\varphi_x) \leqslant Q$, i.e., its infinitesimal distortion is uniformly bounded; conformal maps are precisely the 1-quasiconformal maps.

Theorem 1.4. *Assume $\mathrm{Ric}_M \geqslant \rho$, N satisfies the best isoperimetric inequality holding on X_κ for some $\kappa < \rho$, let Q be a number greater than 1 and assume $\alpha \leqslant A_4(M, N, Q)$ where $A_4(M, N, Q)$ is some positive constant.*

Then any Q-quasiconformal map $\varphi : B(x, \alpha) \subset M \to N$ satisfies

$$\mathrm{aniso}(\varphi) \geqslant \mathrm{aniso}(\bar{\varphi}),$$

where $\bar{\varphi}$ is an explicit Q-conformal azimuthal map, which is C^1 but not C^2.

Assume further that the only domains in N satisfying the equality case in the isoperimetric inequality are balls isometric to geodesic balls in X_κ. Then whenever $\mathrm{aniso}(\varphi) = \mathrm{aniso}(\bar{\varphi})$, *the maps φ and $\bar{\varphi}$ are conjugated by isometries (in particular, the domain and range of φ have constant curvatures ρ and κ), except that when $\kappa = 0$ one can compose $\bar{\varphi}$ with any homothety such that we still have $\sigma_1 \leqslant 1 \leqslant \sigma_2$, and still get an optimal map.*

Remark. Here the constant A_4 is less explicit than in the other result, but it is still perfectly constructive. Moreover, we shall see that when $\kappa \leqslant 0$, we can take $A_4 = \mathrm{inj}(x)$.

It is also interesting to compare what we obtain from the above inequalities when α is small.

Corollary 1.5. *If $\mathrm{Ric}_M \geqslant \rho$ and $K_N \leqslant \kappa$, any map $\varphi : B(x, \alpha) \subset M \to N$ satisfies*

$$\mathrm{aniso}(\varphi) \geqslant \frac{1}{6}(\rho - \kappa)\alpha^2 + o(\alpha^2).$$

If φ is conformal, then

$$\operatorname{aniso}(\varphi) \geqslant \frac{1}{4}(\rho - \kappa)\alpha^2 + o(\alpha^2).$$

If φ is volume preserving, then

$$\operatorname{aniso}(\varphi) \geqslant \frac{n}{2(n+2)}(\rho - \kappa)\alpha^2 + o(\alpha^2).$$

Remark. (1) In this corollary, one can easily replace the curvature assumptions by scalar curvature bounds, since only small balls are considered. Note that the isoperimetric inequality needed in Theorems 1.2 and 1.3 has been proved to be true for small enough domains under the curvature assumption $K_N < \kappa$ (or even $K_N \leqslant \kappa$ in some cases) by Johnson and Morgan [12] and under $\mathrm{Scal}_N < \kappa$ by Druet [4]. To obtain a Taylor series, these strict assumptions are sufficient (but then the remainder term cannot be made explicit).

(2) In all our results, one considers maps from the higher-curvature manifold to the lower-curvature one. These results imply similar estimates for maps $\varphi : B(y, \alpha) \subset N \to M$, because either such a map contracts some distances by much (hence has large anisometry), or its image contains a ball of radius bounded below, allowing us to apply the results above to φ^{-1}. However, the estimates one gets that way are certainly not sharp, and we do not know whether $\bar{\varphi}^{-1}$ is optimal in any of the situation treated above; it seems that even the case of a map from a ball in the plane to a round 2-sphere is open. One might want to perturb the equidistant azimuthal map to enlarge the boundary of its image, so as to limit the distortion along the boundary. It is not clear whether this can be achieved without increasing distortion too much anywhere else.

Organization of the paper. Next section gives notations and some background. We prove our main results in the following three sections (general maps, then volume-preserving maps, then conformal and quasiconformal maps). The technique we use in the conformal and quasiconformal cases turns out to have been used by Gromov to generalize the Schwarz–Pick–Ahlfors lemma. In the final Section 5, we shall state and prove a result of this flavor that seems not to be in the literature (but certainly is in its topological closure).

2. Toolbox

2.1. Notations

Let X_κ be the model space of curvature κ and dimension n, i.e. a round sphere when $\kappa > 0$, the Euclidean space when $\kappa = 0$, and a hyperbolic space when $\kappa < 0$.

We denote by $B_M(x,t)$ (respectively $S_M(x,t)$) the geodesic closed ball (respectively sphere) of radius t and center x in M. When there is no ambiguity, we let $B(t) = B_M(x,t)$ and $S(t) = S_M(x,t)$. To simplify notation, we set $B_\kappa(t)$ (respectively $S_\kappa(t)$) for any geodesic closed ball (respectively sphere) of radius t in X_κ.

The volumes of manifolds, submanifolds and domains shall be denoted either by $\mathrm{Vol}(\cdot)$ or $|\cdot|$. We let $\omega_{n-1} = |S_0(1)|$ be the $(n-1)$-dimensional volume of the unit sphere in $X_0 = \mathbb{R}^n$.

When there is no ambiguity, σ_i shall denote $\sigma_i(\varphi)$.

When x is a point in a manifold and u a tangent vector at x, we let $\gamma_u(t) = \exp_x(tu)$ be the time t of the geodesic issued from x with velocity u.

We shall denote by T^1M the unit tangent bundle of a Riemannian manifold M, by $\mathrm{inj}(x)$ the injectivity radius at $x \in M$ and by $\mathrm{inj}(M)$ the injectivity radius of M.

2.2. Geometry of model spaces

The model spaces X_κ are well understood, let us recall a few facts about them.

2.2.1. Trigonometric functions. It will be convenient to use the functions $\sin_\kappa$ defined by

$$\sin_\kappa(a) = \begin{cases} \dfrac{\sin(\sqrt{\kappa}a)}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ a & \text{if } \kappa = 0, \\ \dfrac{\sinh(\sqrt{-\kappa}a)}{\sqrt{-\kappa}} & \text{if } \kappa < 0. \end{cases}$$

We then set

$$\cos_\kappa(a) := \sin_\kappa'(a) = \begin{cases} \cos(\sqrt{\kappa}a) & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh(\sqrt{-\kappa}a) & \text{if } \kappa < 0 \end{cases}$$

and

$$\tan_\kappa(a) := \frac{\sin_\kappa(a)}{\cos_\kappa(a)} = \begin{cases} \dfrac{\tan(\sqrt{\kappa}a)}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ a & \text{if } \kappa = 0, \\ \dfrac{\tanh(\sqrt{-\kappa}a)}{\sqrt{-\kappa}} & \text{if } \kappa < 0. \end{cases}$$

We shall also use occasionally

$$\arctan_\kappa(x) := \tan_\kappa^{-1}(x) = \begin{cases} \dfrac{\arctan(\sqrt{\kappa}x)}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ x & \text{if} \kappa = 0, \\ \dfrac{\operatorname{arctanh}(\sqrt{-\kappa}x)}{\sqrt{-\kappa}} & \text{if } \kappa < 0 \end{cases}$$

and we have the derivatives $\tan'_\kappa = 1 + \kappa \tan^2_\kappa$ and $\arctan'_\kappa(x) = \frac{1}{1+\kappa x^2}$.

A trigonometric formula that will prove useful is

$$\sin_\kappa(2 \arctan_\kappa x) = \frac{2x}{1+\kappa x^2}.$$

We shall need the following Taylor series:

$$\begin{aligned} \sin_\kappa(t) &= t - \frac{\kappa}{6}t^3 + \frac{\kappa^2}{24}t^5 + O(t^5), \\ \cos_\kappa(t) &= 1 - \frac{\kappa}{2}t^2 + \frac{\kappa^2}{24}t^4 + O(t^6), \\ \tan_\kappa(t) &= t + \frac{\kappa}{3}t^3 + \frac{2\kappa^2}{15}t^5 + O(t^5), \\ \arctan_\kappa(x) &= x - \frac{\kappa}{3}x^3 + \frac{\kappa^2}{5}x^5 + O(x^7). \end{aligned}$$

2.2.2. Volumes. Let x be a point on X_κ, t be a positive real and u be a unit tangent vector at x; then setting $y = \exp_x(tu)$ we can express the volume measure dy on X_k by the formula

$$dy = \sin_\kappa^{n-1}(t)\, dt\, du,$$

where dt is Lebesgue measure on $[0, +\infty)$ and du is the volume measure on the unit tangent sphere $T_x^1 X_\kappa$ naturally identified with the unit round sphere S^{n-1}.

In this volume formula, one can decompose the density into factors 1 (in the direction of the ray from the pole) and $\sin_\kappa(t)$ (in the $n-1$ orthogonal directions). This shows that up to isometry there exists exactly one azimuthal, conformal map φ from the ball $B_\rho(x,\alpha)$ to X_κ such that $D\varphi_x$ is a homothety of ratio σ (i.e., $R'(0)=\sigma$), whose distance function is driven by the following differential equation:

$$R'(t)=\frac{\sin_\kappa(R(t))}{\sin_\rho(t)}.$$

Moreover, the $(n-1)$-dimensional volume of a geodesic sphere $S_\kappa(t)$ of radius t is

$$A_\kappa(t):=|S_\kappa(t)|=\omega_{n-1}\sin_\kappa^{n-1}(t),$$

where ω_{n-1} is the volume of S^{n-1}; when $\kappa>0$ we only consider t below the conjugate radius $\pi/\sqrt{\kappa}$. We also name the volume of geodesic balls of X_κ:

$$V_\kappa(t):=|B_\kappa(t)|=\omega_{n-1}\int_0^t \sin_\kappa^{n-1}(s)\,ds.$$

Given ρ and κ, there is exactly one volume-preserving azimuthal map, defined by the distance function

$$R(t)=V_\kappa^{-1}(V_\rho(t)).$$

That R is as above is clearly necessary for an azimuthal map to be volume preserving, but the local volume formula shows that it is also sufficient.

It is known that in X_κ the least perimeter volume of given domains is balls, so that the isoperimetric profile of X_κ is given by

$$I_\kappa(V_\kappa(t))=A_\kappa(t).$$

Note that the lesser is κ, the greater is I_κ and the more stringent is the corresponding isoperimetric inequality.

Using the above Taylor series, we get:

$$\begin{aligned}
A_\kappa(t)&=\omega_{n-1}t^{n-1}\left(1-\frac{(n-1)\kappa}{6}t^2+O(t^4)\right),\\
V_\kappa(t)&=\frac{\omega_{n-1}}{n}t^n\left(1-\frac{n(n-1)\kappa}{6(n+2)}t^2+O(t^4)\right),\\
I_\kappa(v)&=n^{\frac{n-1}{n}}\omega_{n-1}^{\frac{1}{n}}v^{\frac{n-1}{n}}-\frac{(n-1)\kappa}{2(n+2)}\cdot\frac{n^{\frac{n+1}{n}}}{\omega_{n-1}^{\frac{1}{n}}}v^{\frac{n+1}{n}}+O(v^{\frac{n+3}{n}}).
\end{aligned}$$

2.3. Candle functions and comparison

To study anisometry of maps under curvature bounds of the domain and range, we will need some tools of comparison geometry, relating the geometry of M and N to the geometry of X_ρ and X_κ. We will notably rely on Bishop and Günther's inequality, which in their common phrasing compare volume of balls. It will be useful to discuss their more general form, which is about comparing Jacobians of exponential maps.

Given a point $x \in M$, a vector $u \in T^1_x M$ and a real number t, let $y = \exp_x(tu)$ and define the *candle function* $j_x(tu)$ as a normalized Jacobian of the exponential map by

$$dy = j_x(tu)\, dt\, du,$$

where dy denotes the Riemannian volume and du is the spherical measure on $T^1_x M$.

In the case of X_κ, this function does not depend on x nor on u and is equal to $\sin_\kappa^{n-1}(t)$.

Definition 2.1. The manifold M is said to satisfy the *candle* condition $\mathrm{Candle}(\kappa, \ell)$ if for all x, u and all $t \leqslant \ell$ it holds

$$j_x(tu) \geqslant \sin_\kappa^{n-1}(t).$$

The manifold M is said to satisfy the *logarithmic candle derivative* condition $\mathrm{LCD}(\kappa, \ell)$ if for all x, u and all $t \leqslant \ell$ it holds

$$\frac{j'(t)}{j(t)} \geqslant \frac{s'(t)}{s(t)}, \tag{1}$$

where $j(t) := j_x(tu)$ and $s(t) := \sin_\kappa^{n-1}(t)$.

The name "candle condition" is motivated by the fact that j_x describes the fade of the light of a candle (or of the gravitational fields generated by a punctual mass) in M.

By integration, $\mathrm{Candle}(\kappa, \ell)$ implies that spheres and balls of radius at most ℓ have volume at least as large as the volume of the spheres and balls of equal radius in X_κ.

The candle condition is an integrated version of the logarithmic candle derivative condition, which itself follows for $\ell = \mathrm{inj}(N)$ from the sectional curvature condition $K \leqslant \kappa$: this is known as Günther's theorem, see [5]. With Greg Kuperberg, we proved in [8] that it also follows from a weaker curvature bound, involving the “root-Ricci curvature”. In particular, we proved that manifolds satisfying a relaxed bound on K and a suitably strengthened bound on Ric still satisfy a LCD condition, and therefore a Candle one.

The strong form of Bishop's theorem is that the reversed inequality in (1) holds under the curvature lower bound $\mathrm{Ric} \geqslant \kappa$ (for ℓ the conjugate time of X_κ). The corresponding comparison on the volumes of spheres and balls follows and are also referred to as Bishop's inequality.

We shall establish Theorem 1.1 using the comparison of spheres; the assumption $K_N \leqslant \kappa$ can therefore be relaxed to $\mathrm{Candle}(\kappa, \ell)$ where ℓ can be taken to be, e.g., ∞ when N is a Hadamard manifold or chosen suitably otherwise, see the proof below.

2.4. Volume of ellipsoids and hyperplanes

A couple of our arguments will rely on a simple and classical lemma, which we state and prove for the sake of completeness.

Let q be a scalar product in Euclidean space of dimension n, endowed with the standard inner product $\langle\cdot,\cdot\rangle$. We shall denote by $\sigma_1(q)$ and $\sigma_2(q)$ the largest, respectively, smallest numbers such that

$$\sigma_1(q)\langle u,u\rangle \leqslant q(u,u) \leqslant \sigma_2(q)\langle u,u\rangle \quad \forall u \in \mathbb{R}^n$$

and say that q is at most Q-distorted if $\sigma_2/\sigma_1 \leqslant Q$. We shall also denote by $|q|$ the determinant of q, that is the ratio of the volume of its unit ball to the volume of Euclidean unit ball (both volumes computed with respect to the Lebesgue measure associated with $\langle\cdot,\cdot\rangle$).

Lemma 2.2. *Let q_0 be the restriction of q to any hyperplane. Then we have*

$$|q| \geqslant |q_0|\sigma_1(q)$$

and

$$|q| \geqslant \frac{1}{Q}|q_0|^{\frac{n}{n-1}}.$$

There is equality in this second inequality if and only if q has eigenvalues λ and $Q\lambda$, with respective multiplicities 1 and $n-1$, and the hyperplane defining q_0 is the $Q\lambda$ eigenspace of q.

Proof. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of q and $\mu_1 \geqslant \cdots \geqslant \mu_{n-1}$ be the eigenvalues of q_0. In particular, $\lambda_n = \sigma_1(q)$. Then Rayleigh quotients show that $\mu_i \leqslant \lambda_i$ for all $i < n$. It follows

$$\begin{aligned} |q| &= \lambda_1 \lambda_2 \dots \lambda_n \\ &\geqslant |q_0| \lambda_n = |q_0| \sigma_1(q). \end{aligned}$$

But by the distortion bound, we have

$$Q\sigma_1 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant \sigma_1$$

so that $(Q\sigma_1)^{n-1} \geqslant |q_0|$ and

$$\sigma_1 \geqslant \frac{|q_0|^{\frac{1}{n-1}}}{Q}$$

and the desired inequality follows. The equality case is straightforward. □

3. General maps

Assume that $\mathrm{Ric}_M \geqslant \rho$ and that N satisfies $\mathrm{Candle}(\kappa, \ell_0)$ for some ℓ_0 (on which we shall put some restriction later on).

Let $\varphi : B_M(x, \alpha) \to N$ be a diffeomorphism on its image, where $\alpha < \mathrm{inj}(x)$, the injectivity radius of M at x.

In what follows, we shall assume bounds involving $1/\sqrt{\kappa}$: our convention is that this number is $+\infty$ whenever $\kappa \leqslant 0$.

Lemma 3.1. *If $\sigma_2(\varphi)\alpha \leqslant \ell_0 \leqslant \frac{\pi}{2\sqrt{\kappa}}$, we have*

$$\sigma_2(\varphi) \geqslant \frac{\sin_\kappa(\sigma_1(\varphi)\alpha)}{\sin_\rho(\alpha)}.$$

The proof is a mere generalization of Milnor's argument in [11].

Proof. Denote by $S(\alpha)$ the geodesic sphere of center x and radius α. Bishop's inequality ensures that $|S(\alpha)| \geqslant \omega_{n-1} \sin_\rho^{n-1}(\alpha)$. On the other hand, $\varphi(S(\alpha))$ encloses the ball of N of radius $\sigma_1\alpha$ centered at $y = \varphi(x)$. Given a unit vector $u \in T_y N$, let $\ell(u)$ be the first time at which γ_u hits $\varphi(S(\alpha))$, and $\beta(u)$ be the angle between $\dot\gamma_u(\ell(u))$ and the outward normal to $\varphi(S(\alpha))$. We have $\ell(u) \geqslant \sigma_1\alpha$ and obviously $\cos(\beta(u)) \leqslant 1$. Moreover when $\kappa > 0$, we have

$$\ell(u) \leqslant \sigma_2\alpha \leqslant \frac{\pi}{2\sqrt{\kappa}}$$

so that the comparison candle function $\sin_\kappa^{n-1}$ is increasing on $[0, \ell(u)]$.

Then, letting j be the candle function of N at y and du be the usual measure on the unit Riemannian sphere, we get

$$\begin{aligned}|\varphi(S(\alpha))| &\geqslant \int_{T^1_y N} \frac{j_u(\ell(u))}{\cos\beta(u)}\,du \\ &\geqslant \int_{T^1_y N} \sin_\kappa^{n-1}(\ell(u))\,du \\ &\geqslant \omega_{n-1}\sin_\kappa^{n-1}(\sigma_1\alpha).\end{aligned}$$

There is at least one point on $S(\alpha)$ at which the Jacobian of the restriction of φ to $S(\alpha)$ is at least

$$\frac{|\varphi(S(\alpha))|}{|S(\alpha)|} \geqslant \frac{\sin_\kappa^{n-1}(\sigma_1(\varphi)\alpha)}{\sin_\rho^{n-1}(\alpha)}$$

and the lemma follows. □

Let us now define the α-bound A_1.

Definition. Let $A_1 = A_1(M,N)$ be the greatest number such that for all $\alpha \leqslant A_1$ we have $\alpha \leqslant \mathrm{inj}(M)$, if $\kappa > 0$:

$$\alpha\frac{\sin_\rho(\alpha)}{\sin_\kappa(\alpha)} \leqslant \min\left(\mathrm{inj}(N), \frac{\pi}{2\sqrt{\kappa}}\right) \tag{2}$$

and if $\kappa \leqslant 0$

$$\frac{\alpha^2}{\sin_\kappa^{-1}\circ\sin_\rho(\alpha)} \leqslant \mathrm{inj}(N). \tag{3}$$

Remark. (1) A_1 depends on M and N only through their curvature/candle bounds ρ and κ and their injectivity radii;

(2) if we do not insist on a uniform bound over possible centers, we can replace $\mathrm{inj}(M)$ by $\mathrm{inj}(x)$;

(3) if N is a Hadamard manifold, then $A_1 = \mathrm{inj}(M)$ (or $\mathrm{inj}(x)$);

(4) if $\mathrm{inj}(M)$ and $\mathrm{inj}(N)$ are large enough, when $\kappa > 0$ we have $A_1 \geqslant \frac{\pi}{2\sqrt{\rho}}$ and $A_1 \to \frac{\pi}{\sqrt{\rho}}$ when $\kappa \to 0$;

(5) in some cases (e.g., when one can apply Klingenberg's Theorems, see [2] Theorems 5.9 and 5.10), the curvature bound on N is sufficient to get an estimate on $\mathrm{inj}(N)$, and therefore to get a bound A_1' that does not depend on the injectivity radius of the range.

Theorem 3.2. *Assume* $\mathrm{Ric}_M \geqslant \rho$, *$N$ satisfies* $\mathrm{Candle}(\kappa, \mathrm{inj}(N))$ *for some $\kappa < \rho$ (e.g., $K_N \leqslant \kappa$) and $\alpha \leqslant A_1(M,N)$ defined above. Let $\varphi : B_M(x,\alpha) \to N$ be any smooth map.*

If $\kappa \geqslant 0$ then

$$\mathrm{aniso}(\varphi) \geqslant \log \frac{\sin_\kappa(\alpha)}{\sin_\rho(\alpha)}$$

and there is equality if and only if φ is conjugated via isometries to the equidistant azimuthal map from $B_\rho(\alpha)$ to $B_\kappa(\alpha)$ (in particular, $B_M(x,\alpha)$ and its image must have constant curvatures ρ and κ).

If $\kappa < 0$ then letting $\sigma_0 = \sigma_0(\kappa,\rho,\alpha)$ be the number in $(0,1)$ such that $\sin_\kappa(\sigma_0\alpha) = \sin_\rho(\alpha)$, we have

$$\mathrm{aniso}(\varphi) \geqslant \log \frac{1}{\sigma_0}$$

and there is equality if and only if φ is the σ_0-contracting azimuthal map from $B_\rho(\alpha)$ to $B_\kappa(\sigma_0\alpha)$ (in particular, $B_M(x,\alpha)$ and its image must have constant curvatures ρ and κ).

Notice that σ_0 is the dilation coefficient that makes the volumes of the spheres $S_\kappa(\sigma_0\alpha)$ and $S_\rho(\alpha)$ coincide; it makes the σ_0-contracting azimuthal map a non-dilating map, i.e., $\sigma_2(\bar\varphi) = 1$ when $\kappa < 0$.

Proof. We can assume $\sigma_2\alpha$ is small enough to apply Lemma 3.1, otherwise the way we designed A_1 ensures that σ_2 is so large that $\mathrm{aniso}(\varphi)$ is a least the claimed lower bound.

Let us start with the $\kappa \geqslant 0$ case. From Lemma 3.1 we have

$$\mathrm{aniso}(\varphi) \geqslant |\log \sigma_1| + \log \sin_\kappa(\sigma_1\alpha) - \log \sin_\rho(\alpha). \tag{4}$$

The derivative of the right-hand side with respect to σ_1 is

$$-\frac{1}{\sigma_1} + \frac{\alpha}{\tan_\kappa(\sigma_1\alpha)} < 0$$

when $\sigma_1 < 1$ and

$$\frac{1}{\sigma_1} + \frac{\alpha}{\tan_\kappa(\sigma_1\alpha)} > 0$$

when $\sigma_1 > 1$. This shows that the right-hand side of (4) achieves its minimum when $\sigma_1 = 1$, so that

$$\mathrm{aniso}(\varphi) \geqslant \log \sin_\kappa(\alpha) - \log \sin_\rho(\alpha).$$

In case of equality, one must have $\sigma_1 = 1$ and $\sigma_2 = \sin_\kappa(\alpha)/\sin_\rho(\alpha)$, therefore there is equality in Lemma 3.1. This forces $B_M(x, \alpha)$ and its image to have constant curvatures ρ and κ and $S(\alpha)$ must be mapped to the geodesic sphere of radius α and center $\varphi(x)$ in N. Since $\sigma_1 = 1$, φ must then map $S(a)$ to $S_\kappa(a)$ for all a. Each ray from $\varphi(x)$ to $S_\kappa(\alpha)$ must be mapped by φ^{-1} to a curve of length at most α that connects x to $S(\alpha)$, therefore unit rays are mapped to unit rays. The whole map φ then depends only on its derivative at x, which must preserve the norms. It follows that φ is azimuthal equidistant, up to isometries.

In the $\kappa < 0$ case, (4) also holds but is not optimal anymore. Indeed, the derivative of its right-hand side is positive both when $\sigma_1 > 1$ and when $\sigma_1 < 1$ since $\tan_\kappa(x) \leqslant x$. But when $\sigma_1 < \sigma_0$, the lower bound on σ_2 given by Lemma 3.1 is less than 1. It follows

$$\operatorname{aniso}(\varphi) \geqslant \log \frac{1}{\sigma_0},$$

which is achieved by $\bar{\varphi}$. The case of equality is treated as above. □

Corollary 3.3. *In the above setting,*

$$\operatorname{aniso}(\varphi) \geqslant \frac{\rho - \kappa}{6}\alpha^2 + O(\alpha^4),$$

where the implied constant in the remainder term only depends on the curvature bounds.

4. Area-preserving maps

Let us now prove Theorem 1.2 in the following form.

Theorem 4.1. *Assume* $\operatorname{Ric}_M \geqslant \rho$*,* N *satisfies the best isoperimetric inequality holding on* X_κ *and* $\alpha \leqslant \operatorname{inj}(x)$*. Then any volume-preserving map* $\varphi : B(x, \alpha) \subset M \to N$ *satisfies*

$$\operatorname{aniso}(\varphi) \geqslant \frac{n}{n-1} \log \frac{I_\kappa \circ V_\rho(\alpha)}{A_\rho(\alpha)}$$

and equality is achieved by the unique volume-preserving azimuthal map $\bar{\varphi} : B_\rho(\alpha) \to X_\kappa$*.*

Assume further that the only domains in N *satisfying the equality case in the isoperimetric inequality are balls isometric to geodesic balls in* X_κ*. Then* *whenever* $\operatorname{aniso}(\varphi) = \operatorname{aniso}(\bar{\varphi})$*, the domain of* φ *has constant curvature* ρ *and its range is isometric to a constant curvature ball* $B_\kappa(R(\alpha))$*. However, there are uncountably many different maps achieving equality.*

Proof. The key point is the following lemma, which is a direct adaptation of Theorem 3.5 in [12].

Lemma 4.2. *Under the assumptions* $\operatorname{Ric}_M \geqslant \rho$ *and* $\kappa \leqslant \rho$*, we have*

$$\frac{I_\kappa(|B(\alpha)|)}{|S(\alpha)|} \geqslant \frac{I_\kappa \circ V_\rho(\alpha)}{A_\rho(\alpha)}.$$

If there is equality, then $B(\alpha)$ *is isometric to* $B_\rho(\alpha)$*.*

Proof of Lemma. Setting $\delta_0 := \frac{|S(\alpha)|}{A_\rho(\alpha)}$, the strong form of Bishop's inequality yields for all $t \leqslant \alpha$

$$\delta_0 \leqslant \frac{|S(t)|}{A_\rho(t)} \leqslant 1.$$

By integration, it becomes

$$\delta_0 \leqslant \delta_1 := \frac{|B(\alpha)|}{V_\rho(\alpha)} \leqslant 1.$$

Since I_κ is concave, we have $I_\kappa(\delta_1 V_\rho(\alpha)) \geqslant \delta_1 I_\kappa(V_\rho(\alpha))$; therefore

$$I_\kappa(|B(\alpha)|) \geqslant \delta_0 I_\kappa(V_\rho(\alpha)) = |S(\alpha)| \frac{I_\kappa(V_\rho(\alpha))}{A_\rho(\alpha)}.$$

In case of equality, we must have $\delta_0 = \delta_1$, which implies $\delta_0 = \delta_1 = 1$. The equality case in Bishop's inequality then implies that $B(\alpha)$ is isometric to $B_\rho(\alpha)$. □

Now, since φ is volume preserving and N satisfies the isoperimetric inequality, we have

$$|\varphi(S(\alpha))| \geqslant I_\kappa(|B(\alpha)|) \geqslant |S(\alpha)| \frac{I_\kappa(V_\rho(\alpha))}{A_\rho(\alpha)}.$$

Then, there must be a point x on $S(\alpha)$ such that the Jacobian of $\varphi_{|S(\alpha)}$ is at least $\frac{I_\kappa(V_\rho(\alpha))}{A_\rho(\alpha)}$ so that

$$\sigma_2 \geqslant \left(\frac{I_\kappa(V_\rho(\alpha))}{A_\rho(\alpha)}\right)^{\frac{1}{n-1}},$$

but also, since $\operatorname{jac}\varphi_x = 1$, Lemma 2.2 shows that a direction transverse to the boundary must be contracted by φ and

$$\sigma_1 \leqslant \left(\frac{I_\kappa(V_\rho(\alpha))}{A_\rho(\alpha)}\right)^{-1}.$$

These two bounds combined imply the desired inequality on $\operatorname{aniso}(\varphi)$.

It is straightforward to see that the unique volume-preserving azimuthal map $\bar\varphi : B_\rho(\alpha) \to X_\kappa$ realizes equality. Moreover, if there is equality then there must be equality in the lemma, so that $B(\alpha)$ is isometric to $B_\rho(\alpha)$, and there must be equality in the isoperimetric inequality on N.

However, $\bar\varphi$ is far from being the only optimal map: both σ_1 and σ_2 are realized on the boundary, and for all $t < \alpha$,

$$\sigma_1(\bar\varphi_{|B_\rho(t)}) > \sigma_1(\bar\varphi) \quad \text{and} \quad \sigma_2(\bar\varphi_{|B_\rho(t)}) < \sigma_2(\bar\varphi).$$

If we compose $\bar\varphi$ with any diffeomorphism of $B_\rho(\alpha)$ close to identity and supported on some $B_\rho(t)$, we get another optimal map. $\square$

5. Conformal and quasiconformal maps

The following result is the heart of our results for quasiconformal maps; its formulation has been chosen to avoid repetition of arguments while keeping as much flexibility as we shall need, and it is therefore rather technical.

Theorem 5.1 (Main quasiconformal inequality). *Assume φ is a Q-quasiconformal maps from $B_M(x,\alpha)$ to N, where $\operatorname{Ric}_M \geqslant \rho$ and N satisfies the isoperimetric inequality of X_κ.*

Let G_κ be the function defined by

$$G_\kappa(x) = \sin_\kappa(2\arctan_\kappa(x)) = \frac{2x}{1+\kappa x^2}.$$

If $\kappa > 0$, assume further that the volume of the image of φ is not greater than the volume $\frac{1}{2}|X_\kappa|$ of an hemisphere of curvature κ.

Then for all $\beta < \alpha$ we have

$$\sigma_2(\varphi) \geqslant \frac{G_\kappa\left(\tan_\kappa(\frac{r(\beta)}{2}) \cdot \left(\frac{\tan_\rho(\frac{\alpha}{2})}{\tan_\rho(\frac{\beta}{2})}\right)^{\frac{1}{Q}}\right)}{\sin_\rho(\alpha)},$$

where $r(\beta)$ is the radius of a ball in X_κ that has the same volume as the image of $B_M(x,\beta)$.

The proof of this inequality follows a simple idea: at each time t, the isoperimetric inequality forces the image of the sphere of radius t to have large volume, and the quasiconformality then translates this into a large increase in the volume of the image of the ball. These two effects therefore amplify one another. At $t=\alpha$, we get a lower bound on $V(\alpha)$, and using the isoperimetric inequality again we bound from below the perimeter of the image of the α-ball. Comparing with the perimeter of the ball, we get a lower bound on σ_2.

Proof of the main quasiconformal inequality. For convenience, for all $t \in (0,\alpha)$ set $V(t) = |\varphi(B(t))|$. In particular, $r(t) = V_\kappa^{-1}(V(t))$.

Using Hölder's inequality we get

$$\begin{aligned} V'(t) &= \int_{S(t)} |\operatorname{jac}\varphi|(y)\,dy \\ &\geqslant \frac{\left(\int_{S(t)} |\operatorname{jac}\varphi|^{\frac{n-1}{n}}(y)\,dy\right)^{\frac{n}{n-1}}}{|S(t)|^{\frac{1}{n-1}}}, \end{aligned}$$

where $\operatorname{jac}\varphi(y)$ is the Jacobian of φ at y. Let φ_0 be the restriction of φ along $S(t)$: using Lemma 2.2, Bishop's inequality and the isoperimetric inequality on N it follows that

$$\begin{aligned} V'(t) &\geqslant \frac{\frac{1}{Q}\left(\int_{S(t)} |\operatorname{jac}\varphi_0|(y)\,dy\right)^{\frac{n}{n-1}}}{|S(t)|^{\frac{1}{n-1}}} \\ &\geqslant \frac{|\varphi(S(t))|^{\frac{n}{n-1}}}{Q|S_\rho(t)|^{\frac{1}{n-1}}}, \end{aligned}$$

$$V'(t) \geqslant \frac{I_\kappa(V(t))^{\frac{n}{n-1}}}{Q\omega_{n-1}^{\frac{1}{n-1}}\sin_\rho(t)}. \tag{5}$$

Let $F = F_{\kappa,Q}$ be defined by

$$F \circ V_\kappa(t) = Q \log \tan_\kappa(t/2)$$

and let us compute F':

$$\begin{aligned}
\frac{d}{dx}(F\circ V_\kappa(x)) &= \frac{d}{dx}(Q\log\tan_\kappa(x/2)),\\
V_\kappa'(x)F'(V_\kappa(x)) &= \frac{Q}{\sin_\kappa(x)},\\
F'(V_\kappa(x)) &= \frac{Q}{A_\kappa(x)\sin_\kappa(x)}\\
&= \frac{Q\omega_{n-1}^{\frac{1}{n-1}}}{A_\kappa(x)^{1+\frac{1}{n-1}}}\\
&= \frac{Q\omega_{n-1}^{\frac{1}{n-1}}}{(I_\kappa\circ V_\kappa(x))^{\frac{n}{n-1}}},\\
F' &= \frac{Q\omega_{n-1}^{\frac{1}{n-1}}}{I_\kappa^{\frac{n}{n-1}}}. \qquad (6)
\end{aligned}$$

From (5) and (6) it becomes

$$F'(V(t))V'(t) \geqslant \frac{1}{\sin_\rho(t)}.$$

As above, $\log(\tan_\rho(t/2))$ defines an antiderivative of $1/\sin_\rho(t)$ and integrating we conclude

$$F(V(\alpha)) - F(V(\beta)) \geqslant \log\frac{\tan_\rho(\alpha/2)}{\tan_\rho(\beta/2)} \qquad (7)$$

since F' is a positive function, F is increasing and invertible, so that the above inequality gives a lower bound on $V(\alpha)$; using $|\varphi(S(\alpha))| \geqslant I_\kappa(V(\alpha))$ and proceeding as in the proof of Theorem 3.2, we get

$$\begin{aligned}
\sigma_2(\varphi) &\geqslant \left(\frac{I_\kappa(V(\alpha))}{|S(\alpha)|}\right)^{\frac{1}{n-1}}\\
&\geqslant \frac{(I_\kappa\circ F^{-1})^{\frac{1}{n-1}}\left(F(V(\beta))+\log(\frac{\tan_\rho(\alpha/2)}{\tan_\rho(\beta/2)})\right)}{\omega_{n-1}^{\frac{1}{n-1}}\sin_\rho(\alpha)}
\end{aligned}$$

(beware that exponent $\frac{1}{n-1}$ is a multiplicative power while exponent -1 stands for inverse function).

Now

$$\begin{aligned}(I_\kappa \circ F^{-1})^{\frac{1}{n-1}} &= (I_\kappa \circ V_\kappa \circ (F \circ V_\kappa)^{-1})^{\frac{1}{n-1}} \\ &= (A_\kappa \circ (F \circ V_\kappa)^{-1})^{\frac{1}{n-1}} \\ &= \omega_{n-1}^{\frac{1}{n-1}} \sin_\kappa \circ (F \circ V_\kappa)^{-1}\end{aligned}$$

and the desired inequality follows from the identity

$$\sin_\kappa(2\arctan_\kappa x) = \frac{2x}{1+\kappa x^2}.$$

□

Remark. In the above proof, two small difficulties are hidden.

(1) When $\kappa > 0$, I_κ is decreasing beyond the volume of an hemisphere; this is why we assumed an upper bound on $V(\alpha)$.

(2) When $\kappa < 0$, F has bounded image so that F^{-1} is not defined on the whole positive axis. Our proof shows that any quasiconformal map must map small balls to domains of relatively small volume (bounded in terms of α, κ, ρ and the radius of the considered ball), for otherwise the differential inequality on $V(t)$ would blow up in time less than α and the map would not have compact image. This is the base to a generalization of the Schwarz–Pick–Ahlfors lemma by Gromov, see the appendix.

Definition. Let $A_3 = A_3(M, \kappa, \rho, n)$ be defined as the greatest real number such that for all $\alpha \leqslant A_3$ it holds

- $\alpha \leqslant \mathrm{inj}(M)$,
- if $\kappa > 0$,

$$\left(\frac{|X_\kappa|}{2V_\rho(\alpha)}\right)^{\frac{1}{n}} \geqslant 1 + (\rho - \kappa)\frac{\tan_\rho^2(\alpha/2)}{1+\kappa\tan_\rho^2(\alpha/2)}.$$

Let us now prove Theorem 1.3 which we restate as follows.

Theorem 5.2. *Assume* $\mathrm{Ric}_M \geqslant \rho$, *$N$ satisfies the best isoperimetric inequality holding on X_κ and $\alpha \leqslant A_3(M, \kappa, \rho, n)$. For the equality case below, assume further that any domain $\Omega \subset N$ such that $|\partial\Omega| = I_\kappa(|\Omega|)$ is isometric to a geodesic ball in X_κ. Let $\varphi : B(x, \alpha) \subset M \to N$ be a conformal map.*

If $\kappa > 0$, then

$$\operatorname{aniso}(\varphi) \geqslant \log\left(1 + (\rho-\kappa)\frac{\tan_\rho^2(\alpha/2)}{1+\kappa\tan_\rho^2(\alpha/2)}\right)$$

with equality when φ is conjugated by isometries to the conformal azimuthal map $B_\rho(\alpha) \to X_\kappa$ with $R'(0) = 1$.

If $\kappa = 0$, then

$$\operatorname{aniso}(\varphi) \geqslant \log\left(1 + \rho\tan_\rho^2(\alpha/2)\right)$$

with equality when φ is conjugated by isometries to a conformal azimuthal map $B_\rho(\alpha) \to X_0 = \mathbb{R}^n$ with $R'(0) \leqslant 1$ and $\sigma_2 \geqslant 1$ (e.g., $R'(0) = 1$).

If $\kappa < 0$, then

$$\operatorname{aniso}(\varphi) \geqslant \log\frac{-2\kappa\sin_\rho^2(\frac{\alpha}{2})}{\sqrt{1-\kappa\sin_\rho\alpha}-1}$$

with equality when φ is conjugated by isometries to the conformal azimuthal map $B_\rho(\alpha) \to X_\kappa$ that induces an isometry on the boundaries (or, equivalently, that preserves volumes along the boundary).

Remark. In the case $\kappa = 0$ it is easy to compare the bound for general maps and conformal ones. When $\alpha \to 0$, this will be done more generally below; when $\alpha \to \pi$, both lower bounds go to infinity, but in the conformal case it does so twice as fast (after taking logs!) in the sense that

$$\frac{\operatorname{aniso}(\bar\varphi_{\mathrm{c}})}{\operatorname{aniso}(\bar\varphi)} \to 2,$$

where $\bar\varphi$ and $\bar\varphi_{\mathrm{c}}$ denote the optimal azimuthal maps for radius α in the general and conformal cases, respectively. Conformality thus appears to have a significant effect on anisometry.

Proof. The bound A_3 has been designed so that either $B(\alpha)$ is mapped to a domain so large that at some point y, $D\varphi_y$ itself must have anisometry at least equal to the claimed bound, or the volume of $\varphi(B(\alpha))$ is at most $\frac{1}{2}|X_\kappa|$ and we can use the main quasiconformal inequality with $Q = 1$ and $\beta \to 0$.

Since $r(\beta) \geqslant \sigma_1\beta + o(\beta)$, when $\beta \to 0$ we have

$$\frac{\tan_\kappa(\frac{r(\beta)}{2})}{\tan_\rho(\beta/2)} \geqslant \sigma_1 + o_\beta(1)$$

and we obtain

$$\sigma_2(\varphi) \geqslant \bar\sigma_2 := \frac{G_\kappa(\sigma_1 \tan_\rho(\alpha/2))}{\sin_\rho(\alpha)}.$$

We would like to optimize in σ_1 the corresponding bound

$$f(\sigma_1) := |\log\sigma_1| + |\log\bar\sigma_2(\sigma_1)|$$

on $\operatorname{aniso}(\varphi)$.

For this, we observe that for all positive σ, the number $f(\sigma)$ is the anisometry of a conformal map with co-Lipschitz coefficient equal to σ. For this, let

$$\Phi_{\sigma,\kappa} : B_\rho(\alpha) \to X_\kappa$$

be the unique conformal azimuthal map such that $\sigma_1(\Phi_{\sigma,\kappa}) = \sigma$ (i.e., its distance function satisfies $R'_{\sigma,\kappa}(0) = \sigma$). Then following the proof of the main quasiconformal inequality with $Q = 1$, we see that all inequalities are equalities so that indeed $f(\sigma) = \operatorname{aniso}(\Phi_{\sigma,\kappa})$.

Moreover, if $\operatorname{aniso}(\varphi) = f(\sigma_1)$ (recall that σ_1 stands for $\sigma_1(\varphi)$) then we must have equality in all inequalities in the proof of the main quasiconformal inequality, and this implies that φ and $\Phi_{\sigma_1,\kappa}$ are conjugated by isometries.

Observe that $\operatorname{aniso}(\Phi_{\sigma,\kappa})$ is decreasing with κ, and increasing with σ whenever

$$\sigma \leqslant 1 \leqslant \sigma_2(\Phi_{\sigma,\kappa}). \tag{8}$$

It is clear that the minimum of $f(\sigma)$ occurs in this range. Observe further that

$$\operatorname{aniso}(\Phi_{\sigma,\kappa}) = \operatorname{aniso}(\Phi_{\sigma/\lambda,\lambda^2\kappa})$$

whenever σ and σ/λ both are in the range (8), since $\Phi_{\sigma/\lambda,\lambda^2\kappa}$ is the composition of $\Phi_{\sigma,\kappa}$ with a homothety of ratio λ.

When $\kappa > 0$, if $\sigma < 1$ is in the above range then we get

$$\operatorname{aniso}(\Phi_{\sigma,\kappa}) = \operatorname{aniso}(\Phi_{1,\sigma^2\kappa}) > \operatorname{aniso}(\Phi_{1,\kappa})$$

so that $f(\sigma_1) \geqslant f(1)$ with equality if and only if $\sigma = 1$.

When $\kappa < 0$, if $1 < \bar{\sigma}_2(\sigma) = \sigma_2(\Phi_{\sigma,\kappa})$, then

$$\operatorname{aniso}(\Phi_{\sigma,\kappa}) = \operatorname{aniso}(\Phi_{\sigma/\bar{\sigma}_2,\bar{\sigma}_2^2\kappa}) > \operatorname{aniso}(\Phi_{\sigma/\bar{\sigma}_2,\kappa})$$

so that $f(\sigma)$ reaches its unique minimum for the value of σ such that $\sigma_2(\Phi_{\sigma,\kappa}) = 1$.

When $\kappa = 0$, f is constant on the range (8).

Note that we could also have proceeded via calculus: setting $x = \sigma_1 \tan_\rho (\alpha/2)$ we then have

$$\begin{aligned}\frac{d}{d\sigma_1} e^{f(\sigma_1)} &= \frac{1}{\sigma_1^2}\left(xG'_\kappa(x) - G_\kappa(x)\right)\\ &= \frac{-4\kappa x^3}{\sigma_1^2(1+\kappa x^2)^2}.\end{aligned}$$

Therefore if $\kappa > 0$ then f has its only minimum when $\sigma_1 = 1$, and if $\kappa < 0$ then f has its only maximum when $\bar{\sigma}_2(\sigma_1) = 1$. When $\kappa = 0$, any value of σ_1 between this two cases yields the same result.

We only have left to compute $\min f$. When $\kappa \geqslant 0$, we get

$$\operatorname{aniso}(\varphi) \geqslant \log \frac{G_\kappa(\tan_\rho(\frac{\alpha}{2}))}{\sin_\rho(\alpha)}.$$

Then, using

$$\sin_\rho(\alpha) = \frac{2\tan_\rho(\frac{\alpha}{2})}{1+\rho\tan_\rho^2(\frac{\alpha}{2})}$$

we easily get the claimed inequality.

When $\kappa < 0$, the minimum of f is attained when $\bar{\sigma}_2 = 1$ and, therefore σ_1 is such that $G_\kappa(\sigma_1 \tan_\rho(\frac{\alpha}{2})) = \sin_\rho(\alpha)$. Since at this point we have $f(\sigma_1) = -\log(\sigma_1)$, we only have to invert G_κ to get the desired inequality. □

Corollary 5.3. *If φ is conformal, then*

$$\operatorname{aniso}(\varphi) \geqslant \frac{\rho-\kappa}{4}\alpha^2 + o(\alpha^2),$$

where the remainder depends on the curvature bounds and N.

As mentioned above, we can even relax the assumption on M and N to be $\mathrm{Scal}_M \geqslant \rho$ and $\mathrm{Scal}_N \leqslant \kappa$: the infinitesimal Bishop inequality holds true under a scalar curvature bound, and so does the isoperimetric inequality as proved by Druet [4].

Theorem 5.4. *Assume* $\mathrm{Ric}_M \geqslant \rho$ *and* N *satisfies the best isoperimetric inequality holding on* X_κ*. For the equality case below, assume further that any domain* $\Omega \subset N$ *such that* $|\partial\Omega| = I_\kappa(|\Omega|)$ *is isometric to a geodesic ball in* X_κ*. Let* $\varphi : B(x,\alpha) \subset M \to N$ *be a* Q*-conformal map.*

If $\kappa \geqslant 0$*, let* $\bar\varphi : B_\rho(\alpha) \to X_\kappa$ *be the azimuthal map whose distance function satisfies*

$$R(t) = t \text{ when } t \leqslant \beta, \quad R'(t) = \frac{\sin_\kappa(R(t))}{Q\sin_\rho(t)} \text{ when } t \geqslant \beta,$$

where $\beta > 0$ *is such that*

$$\frac{\sin_\kappa(\beta)}{Q\sin_\rho(\beta)} = 1.$$

There is a positive number $A_4 = A_4(M,N,Q)$ *such that if* $\alpha \leqslant A_4$ *then*

$$\mathrm{aniso}(\varphi) \geqslant \mathrm{aniso}(\bar\varphi)$$

and there is equality if and only if φ *and* $\bar\varphi$ *are conjugated by isometries (except in the case* $\kappa = 0$ *where the conjugating map on the range can be a homothety).*

If $\kappa < 0$*, let* $\bar\varphi : B_\rho(\alpha) \to X_\kappa$ *be the azimuthal map whose distance function satisfies*

$$R(t) = \sigma t \text{ when } t \leqslant \beta, \quad R'(t) = \frac{\sin_\kappa(R(t))}{Q\sin_\rho(t)} \text{ when } t \geqslant \beta,$$

where $\beta > 0$ *is such that*

$$\frac{\sin_\kappa(\sigma\beta)}{Q\sin_\rho(\beta)} = \sigma$$

and σ *is such that* $\sigma_2(\bar\varphi) = 1$ *(in particular,* $\bar\varphi$ *induces an isometry on the boundary). Then whenever* $\alpha \leqslant \mathrm{inj}(x)$ *we have*

$$\mathrm{aniso}(\varphi) \geqslant \mathrm{aniso}(\bar\varphi)$$

and there is equality if and only if φ *and* $\bar\varphi$ *are conjugated by isometries.*

Proof. The proof follows exactly the same lines as the proof of Theorem 5.2, using the quasiconformal inequality with the chosen β and Q. Fixing ρ and

α, for all given σ and κ we construct a comparison map

$$\bar{\varphi}_{\sigma,\kappa} : B_\rho(\alpha) \to X_\kappa$$

as in the conclusion of the theorem. We remark that, restricted to $B_\rho(\beta)$, $\bar{\varphi}$ is both the least anisometrical map and Q-conformal. Then for larger radii, A_4 is designed so that either the anisometry bound is true or the main quasiconformal inequality shows that $\bar{\varphi}_{\sigma_1(\varphi),\kappa}$ has lesser anisometry than φ.

To get the desired conclusion, we only have left to optimize the anisometry of these comparison maps in σ. This does not differ from the conformal case.

We do not give explicit values for the lower anisometry bounds, but they can be obtained explicitly from the above computations (though probably not in closed form). □

Remark 5.5. It can be checked that in Theorem 5.4 the optimal azimuthal map is C^1 but not C^2 when $Q > 1$.

Appendix A. The generalized Schwarz–Pick–Ahlfors lemma

The method used to prove the main quasiconformal inequality was already used by Gromov [6, 7] and Pansu [13] in relation with generalizations of Ahlfors lemma.

The classical Schwarz lemma says that a holomorphic map $f : \Delta \to \Delta$ from the unit disc to itself, such that $f(0) = 0$ must satisfy $|f'(0)| \leqslant 1$ and, in case of equality, f must be a rotation. Pick reinterpreted this result by endowing the disc with its hyperbolic metric: the lemma then amounts to say that any conformal map from $\mathbb{H}^2$ to $\mathbb{H}^2$ must be non-dilating in the hyperbolic metric, and if at any point its Jacobian has modulus 1 then the map must be a hyperbolic isometry. Then, Ahlfors extended this result to conformal maps from a surface with curvature bounded below by -1 to a surface with curvature bounded above by -1. This had a lasting impact on several fields of mathematics. Among possible generalization to higher dimensions, one that fits particularly well with the content of the present article is the following.

Theorem A.1. *Let M and N be complete manifolds of the same dimension (at least 2) with $\mathrm{Ric}_M \geqslant -1$ and $K_N \leqslant -1$, and let $\varphi : M \to N$ be a smooth conformal map. If the Cartan–Hadamard conjecture holds, then $|\operatorname{jac}\varphi(x)| \leqslant 1$ for all $x \in M$, and if there is equality at any one point, then φ lifts to an* *isometry of the universal coverings $\tilde\varphi : \tilde M \to \tilde N$ (in particular, M and N have constant curvature -1).*

The current knowledge gives us the conclusion unconditionally when N is the real hyperbolic space, and when the dimension is 2 or 3.

The above result can hardly be considered new, but we could not find a written proof; we therefore provide one.

Proof. Let $\tilde\varphi : \tilde M \to \tilde N$ be the lift to $\tilde N$ of the composition of the universal covering map $\pi : \tilde M \to M$ with φ. Then $\tilde\varphi$ is a smooth conformal map with the same local behavior as φ.

We apply to $\tilde\varphi$ inequality (7) from the proof of the main quasiconformal inequality. As in the beginning of the proof of Theorem 5.2, with $Q = 1$ and $\beta \to 0$ we get for all $x \in \tilde M$ and all $\alpha > 0$:

$$\tanh\left(\frac{r(\alpha)}{2}\right) \geqslant \sigma_0 \tanh\left(\frac{\alpha}{2}\right)$$

where $r(\alpha)$ is the radius of a ball in hyperbolic space whose volume equals $|\tilde\varphi(B_{\tilde M}(x,\alpha))|$, and σ_0 is the conformal dilation factor at x (i.e., $\sigma_0^n = |\operatorname{jac}\tilde\varphi(x)|$).

If we had $\sigma_0 > 1$, then for large enough α the above inequality would yield $r(\alpha) \geqslant \infty$, a contradiction. Therefore $\sigma_0 \leqslant 1$ independently of x. Together with the conformality of $\tilde\varphi$, this implies that $\tilde\varphi$ is distance non-increasing.

If $\sigma_0 = 1$ (for one given x), then we have from the above inequality $r(\alpha) \geqslant \alpha$ for all α, so that $\tilde\varphi$ maps balls of volume at most $V_{-1}(r)$ to balls of volume at least $V_{-1}(r)$, while not increasing distances. This implies that we have equalities in the Bishop and Günther inequalities, so that $\tilde M$ and $\tilde N$ both have constant curvature -1 and $\tilde\varphi$ is an isometry. □

Remark.
(1) The above result may seem weak in the sense that it asks for a conformal map, which may not exist for given M and N. However, the hypothesis cannot be weakened to quasiconformal as there are local Q-quasiconformal diffeomorphisms of arbitrarily high supremum of the Jacobian. Using the above method one can only get bounds on averaged Jacobians, i.e., volume of balls.

(2) Theorem A.1 can be interpreted as follows: given a manifold M, if one can find in the same conformal class two complete metrics g and σg such that $\operatorname{Ric}_g \geqslant -1$ and $K_{\sigma g} \leqslant -1$, then σ is uniformly bounded above by 1, and if there is a point at which $\sigma(x) = 1$ then $\sigma \equiv 1$.

Acknowledgment

It is a pleasure to thank Charles Frances, Étienne Ghys and Pierre Pansu for interesting discussions related to the content of the present article.

References

[1] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Diff. Geom. **11**(4) (1976), 573–598.

[2] J. Cheeger and D.G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Mathematical Library, **9**, North-Holland Publishing Co., Amsterdam, 1975.

[3] C.B. Croke, *A sharp four-dimensional isoperimetric inequality*, Comment. Math. Helv. **59**(2) (1984), p. 187–192.

[4] O. Druet, *Sharp local isoperimetric inequalities involving the scalar curvature*, Proc. Amer. Math. Soc. **130**(8) (2002), 2351–2361 (electronic).

[5] S. Gallot, D. Hulin and J. Lafontaine, *Riemannian geometry*, 2nd ed., Universitext, Springer-Verlag, Berlin, 1990.

[6] M. Gromov, *Structures métriques pour les variétés riemanniennes*, in Textes Mathématiques, **1**, ed. J. Lafontaine and P. Pansu, CEDIC, Paris, 1981.

[7] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82**(2) (1985), 307–347.

[8] B.R. Kloeckner and G. Kuperberg, A refinement of Günther's candle inequality, 2012, `arXiv:1204.3943`, *Asian J. Math.*, to appear.

[9] B.R. Kloeckner and G. Kuperberg, The generalized Cartan-Hadamard conjecture and the problem of the Little Prince, 2013, preprint, `arXiv:math.1303.3115`.

[10] B. Kleiner, *An isoperimetric comparison theorem*, Invent. Math. **108**(1) (1992), 37–47.

[11] J. Milnor, *A problem in cartography*, Amer. Math. Monthly **76** (1969), 1101–1112.

[12] F. Morgan and D.L. Johnson, *Some sharp isoperimetric theorems for Riemannian manifolds*, Indiana Univ. Math. J. **49**(3) (2000), 1017–1041.

[13] P. Pansu, *Quasiconformal mappings and manifolds of negative curvature*, Curvature and Topology of Riemannian Manifolds (Katata, 1985), Lecture Notes in Math. **1201**, Springer, Berlin, 1986, 212–229.

[14] A. Weil, *Sur les surfaces à courbure négative*, C.R. Acad. Sci. Paris **182** (1926), 1069–1071.

Université de Grenoble I
Institut Fourier
CNRS UMR 5582
BP 74, 38402 Saint Martin d'Hères cedex
France
E-mail address: benoit.kloeckner@ens-lyon.org

Received March 27, 2014