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Cox rings of rational surfaces and flag varieties of ADE-types

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The Cox rings of del Pezzo surfaces are closely related to the Lie groups E_n. In this paper, we generalize the definition of Cox rings to G-surfaces defined by us earlier, where the Lie groups $G = A_n, D_n$ or E_n. We show that the Cox ring of a G-surface S is closely related to an irreducible representation V of G, and is generated by degree one elements. The Proj of the Cox ring of S is a sub-variety of the orbit of the highest weight vector in V, and both are closed sub-varieties of $\mathbb{P}(V)$ defined by quadratic equations. The GIT quotient of the Spec of such a Cox ring by a natural torus action is considered.

1. Introduction

This is a continuation of our studies in which flat G-bundles over an elliptic curve are related to rational surfaces S of type G, where G is a Lie group of simply laced type in [11] and non-simply laced type in [12]. The affine E_n case is considered in [13]. These studies generalize a classical result of Looijenga [16, 17], Friedman *et al.* [6], Donagi [5] and so on, about the case of $G = E_n$ and del Pezzo surfaces.

For instance, an E_n-surface S is simply a blowup of a *del Pezzo* surface X_n of degree $9-n$ at a general point, where X_n is a blowup of $\mathbb{P}^2$ at n points in general position. The del Pezzo surface X_n is well known to be closely linked to E_n [3, 18]. For example, the orthogonal complement of the canonical class K_{X_n} in $H^2(X_n, \mathbb{Z})$, equipped with the natural intersection product, is the root lattice of E_n [3, 18], where we extend the exceptional E_n-series to $0 \leq n \leq 8$ by setting $E_0 = 0, E_1 = \mathbb{C}, E_2 = A_1 \times \mathbb{C}$, $E_3 = A_2 \times A_1, E_4 = A_4$ and $E_5 = D_5$. Recall that for the del Pezzo surface X_n, a curve l is called a *line* if $l^2 = l \cdot K_{X_n} = -1$ (which is really of degree 1 under the anti-canonical morphism for $n \leq 7$). In [11], we use these root lattices and lines to construct an adjoint principal E_n-bundle $\mathcal{E}_n$ over X_n and its representation bundle (that is, an associated principal E_n-bundle) $\mathcal{L}_{E_n}$ over X_n (corresponding to the left-end node in the Dynkin diagram, see figure 1).

In Section 2.1, we describe a D_n-surface (resp. an A_n-surface) S as a rational surface with a fixed ruling $S \to \mathbb{P}^1$ (resp. a fixed birational morphism $S \to \mathbb{P}^2$). Note that the description of A_n-surfaces is slightly different from the description in [11], where it is more indirect. Here we use a more direct description to obtain the same root lattice. The results about A_n-surfaces cited from [11] are all about lattice structures and hence keep true. Similar to the E_n-surface case, there is an adjoint principal D_n-bundle $\mathcal{D}_n$ (resp. an adjoint principal A_n-bundle $\mathcal{A}_n$) over a D_n-surface (resp. an A_n-surface) and an associated bundle $\mathcal{L}_{D_n}$ (resp. $\mathcal{L}_{A_n}$) determined by the lines on this surface. For simplicity, we also use $\mathcal{L}_G$ to denote the bundle $\mathcal{L}_{E_n}$, $\mathcal{L}_{D_n}$ or $\mathcal{L}_{A_n}$, in the context.

Moreover, both the vector space $V = H^0(S, \mathcal{L}_G)$ and any fiber of the bundle $\mathcal{L}_G$ are representations of G. The vector space V, or a subspace of it (denoted still by V), is just the corresponding fundamental representation of G determined by the left-end node $\alpha_L = \alpha_n$. Thus we have $G/P \subset \mathbb{P}(V)$, where P is the maximal parabolic subgroup of G associated with α_n.

In the classical $G = E_n$ case, the representations and the flag varieties G/P are related to the *Cox rings* of the del Pezzo surfaces X_n.

The notion of Cox rings is introduced by Cox [2] and formulated by Hu and Keel [7]. Let X be an algebraic variety. Assume that the Picard group $\mathrm{Pic}(X)$ is freely generated by the classes of divisors $D_0, D_1, \ldots, D_r$. Then the *total homogeneous coordinate ring*, or the *Cox ring* of X with respect to this basis is given by

$$\mathrm{Cox}(X) := \bigoplus_{(m_0,\ldots,m_r)\in\mathbb{Z}^{r+1}} H^0(X, \mathcal{O}_X(m_0D_0 + \cdots + m_rD_r))$$

with multiplication induced by the multiplication of functions in the function field of X. Different choices of bases yield (non-canonically) isomorphic Cox rings.

The Cox ring of X is naturally graded by $\mathrm{Pic}(X)$. Moreover, in the two-dimensional case, it is also graded by $\deg(D) := (-K_X)D$, where $-K_X$ is the anti-canonical class of X.

In [2], it is shown that for a toric variety X, $\mathrm{Cox}(X)$ is a polynomial ring with generators t_E, where E runs over the irreducible components of the boundary $X \setminus U$ and U is the open torus orbit. For a smooth del Pezzo surface X_n of degree at most 6, $\mathrm{Cox}(X_n)$ is finitely generated by sections of degree one elements (which are sections of -1 curves for $n \leq 7$; and in the X_8 case, sections of -1 curves and two linearly independent sections of $-K_{X_8}$), and these generators satisfy a collection of quadratic relations (see [1, 4, 10, 21] etc). Thus in particular, a smooth del Pezzo surface is a Mori Dream Space in the sense of Hu and Keel [7], and as a result, the GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(X_n))$ by the action of the Néron–Severi torus T_{NS} of X_n is isomorphic to X_n [7].

The Cox rings of del Pezzo surfaces are closely related to universal torsors and homogeneous varieties (see, for example, [4, 8, 19, 20] etc.). For the Lie group $G = E_n$ with $4 \leq n \leq 8$, it is shown that there are the following two successive embeddings

$$\mathrm{Proj}(\mathrm{Cox}(X_n)) \hookrightarrow G/P \hookrightarrow \mathbb{P}(V),$$

where V is the fundamental representation associated with the left-end node α_L in the Dynkin diagrams (see figures 1 to 3), and the Proj is considered with respect to the anti-canonical grading.

Motivated from above, we want to give a geometric description of above results in terms of the representation bundle $\mathcal{L}_G$ and also generalize these results to all ADE cases. In this paper, we show how the Lie groups, the representations and the flag varieties are tied together with the rational surfaces.

For this, let S be a *G-surface* (Definition 3) with G a simple Lie group of simply laced type. Let $\mathcal{L}_G$ be the fundamental representation bundle over S determined by *lines*. Let $\mathcal{W}$ be the fundamental representation bundles determined by *rulings* (see Section 2.2). Let $\mathrm{Sym}^2 \mathcal{L}_G$ be the second symmetric power of $\mathcal{L}_G$. Let P be the maximal parabolic subgroup of G associated with $\mathcal{L}_G$.

Our main results are the following:

Theorem 1 (Theorem 9). *Let S, G, $\mathcal{L}_G$ and $\mathcal{W}$ be as above. There is a canonical fiberwise quadratic form $\mathcal{Q}$ on $\mathcal{L}_G$,*

$$\mathcal{Q} : \mathcal{L}_G \to \mathrm{Sym}^2\mathcal{L}_G \to \mathcal{W},$$

such that $\ker(\mathcal{Q}) \subset \mathbb{P}(\mathcal{L}_G)$ is a fiber bundle over S with fiber being the homogeneous variety G/P, where $\ker(\mathcal{Q})$ is the subscheme of $\mathbb{P}(\mathcal{L}_G)$ defined by $x \in \mathbb{P}(\mathcal{L}_G)$, such that $\mathcal{Q}(x) = 0$.

Moreover, by taking global sections, we realize G/P as a subvariety of $\mathbb{P}(H^0(S, \mathcal{L}_G))$ cut out by quadratic equations, for $G \neq E_8$. For $G = E_8$, we should replace $H^0(S, \mathcal{L}_G)$ by a subspace V of dimension 248.

We have a uniform definition for an ADE-surface in [11] (see also Section 2). Using this definition, we can give a uniform definition of the Cox ring of a G-surface S (Definition 11), where G is the ADE Lie group. For $G = E_n$, it turns out that the Cox ring of an E_n-surface S is the same as the Cox ring of a del Pezzo surface X_n of degree $9-n$. Let $\mathrm{Cox}(S,G)$ be the Cox ring of a G-surface S. Let $T_G \subseteq P$ be the maximal subtorus of G, and $T_{S,G}$ be the torus defined in Section 3.3.

Theorem 2 (Theorems 14 to 16 and Propositions 18 and 19).

(1) *The Cox ring of an ADE-surface S is generated by degree 1 elements, and the ideal of relations between the degree 1 generators is generated by quadrics.*

(2) *We have $\mathbb{C}^* \times T_G$-equivariant embeddings:*

$$\mathrm{Spec}(\mathrm{Cox}(S,G)) \hookrightarrow C(G/P) \hookrightarrow H^0(S, \mathcal{L}_G).$$

Taking the Proj*, we have T_G-equivariant embeddings:*

$$\mathrm{Proj}(\mathrm{Cox}(S,G)) \hookrightarrow G/P \hookrightarrow \mathbb{P}(H^0(S, \mathcal{L}_G)).$$

Both of the first two spaces are embedded into the last space as subvarieties defined by quadratic equations.

(3) *The GIT quotient of* $\mathrm{Spec}(\mathrm{Cox}(S,G))$ *by the action of the torus $T_{S,G}$ is, respectively, X_n for $G = E_n$, $\mathbb{P}^1$ for $G = D_n$ and a point for $G = A_n$.*

Thus, we have a uniform description for Cox rings of ADE-surfaces and their relations to configurations of curves, representation theory and flag varieties, as is the purpose of this paper.

Note that in the E_6 and E_7 cases, the proof of the embedding $\mathrm{Proj}(\mathrm{Cox}(S,G)) \hookrightarrow G/P$ was achieved by Derenthal [4] with the help of a computer program. Trying to simplify this proof is also a very interesting question. For $G = E_8$, the embedding was proved by Serganova and Skorobogatov [20]. These results about E_n $(4 \leq n \leq 8)$ answer a conjecture of Batyrev and Popov [1]. Here we just cite their results without new proofs.

2. ADE-surfaces and associated principal G-bundles

Let $G = A_n, D_n$ or E_n be a complex (semi-)simple Lie groups. In this section, we first briefly recall the definitions and constructions of G-surfaces and associated principal G-bundles from [11]. After that, we study the quadratic forms defined fiberwise over these associated principal G-bundles.

2.1. ADE-surfaces

The definition of ADE-surfaces is motivated from the classical del Pezzo surfaces [11]. According to the results of [11, 18], over a del Pezzo surface X_n $(0 \leq n \leq 8)$ of degree $9-n$, there is a root lattice structure of the Lie group E_n, and the lines and the rulings in X_n can be related to the fundamental representations associated with the endpoints of the Dynkin diagram, via a natural way. Inspired by these, we can consider general G-surfaces, where $G = A_n, D_n$ or E_n.

When the simply laced Lie group G is simple, that is, $G = E_n$ for $4 \leq n \leq 8$, A_n for $n \geq 1$ or D_n for $n \geq 3$, we gave a uniform definition of ADE-surfaces in [11], using the pair (S, C). It turns out that when $G = E_n$, after blowing down an exceptional curve, we obtain the classical del Pezzo surfaces X_n.

Notations. Let h be the (divisor, the same below) class of a line in $\mathbb{P}^2$. Fix a ruled surface structure of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ over $\mathbb{P}^1$, and let f, s be the classes of a fiber and a section in the natural projection from $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ to $\mathbb{P}^1$. If S is a blowup of one of these surfaces, then we use the same notations to denote the pullback class of h, f, s and use l_i to denote the exceptional class corresponding to the blowup at a point x_i. Let K_S be the canonical class of S. Since for S the Picard group and the divisor class group are isomorphic, we use $\mathrm{Pic}(S)$ to denote the divisor class group of S. The Picard group $\mathrm{Pic}(S)$ is generated by $h, l_1, \ldots, l_n$ or by $f, s, l_1, \ldots, l_n$, respectively.

Definition 3. Let (S, C) be a pair consisting of a smooth rational surface S and a smooth rational curve $C \subset S$ with $C^2 \neq 4$. The pair (S, C) is called an ADE-surface, or a G-surface for the Lie group $G = A_n, D_n$ or E_n if it satisfies the following two conditions:

(i) any rational curve on S has a self-intersection number at least -1;

(ii) the sub-lattice $\langle K_S, C\rangle^\perp$ of $\mathrm{Pic}(S)$ is an irreducible root lattice of rank equal to $r-2$, where r is the rank of $\mathrm{Pic}(S)$.

The following proposition shows that such surfaces can be classified into three types, and the curve C in fact sits in the negative part of the Mori cone.

Proposition 4 ([11], Proposition 2.6). *Let (S, C) be an ADE-surface. Let $n = \mathrm{rank}(\mathrm{Pic}(S)) - 2$. Then $C^2 \in \{-1, 0, 1\}$ and*

(i) *when $C^2 = -1$, $\langle K_S, C\rangle^\perp$ is of E_n-type, where $4 \le n \le 8$;*

(ii) *when $C^2 = 0$, $\langle K_S, C\rangle^\perp$ is of D_n-type, where $n \ge 3$;*

(iii) *when $C^2 = 1$, $\langle K_S, C\rangle^\perp$ is of A_n-type.*

In the following corollary, n points on $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ are said to be *in general position*, if the surface obtained by blowing up these points contains no irreducible rational curves with self-intersection number less than or equal to -2.

Corollary 5. *Let (S, C) be an ADE-surface.*

(i) *In the E_n case, blowing down the (-1) curve C of S, we obtain a del Pezzo surface X_n of degree $9 - n$.*

(ii) *In the D_n case, S is just a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at n points in general position with C as the natural ruling.*

(iii) *In the A_n case, the linear system $|C|$ defines a birational map $\varphi_{|C|} : S \to \mathbb{P}^2$. Therefore S is just the blowup of $\mathbb{P}^2$ at $n + 1$ points in general position, and C is a smooth curve which represents the class determined by lines in $\mathbb{P}^2$.*

Corollary 6. *Let (S, C) be an ADE-surface, and G be the corresponding simple Lie group. The lattice $\mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S)$ is the corresponding weight lattice. Hence its dual $\mathrm{Hom}(\mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S), \mathbb{C}^*)$ is a maximal torus of G.*

Proof. The intersection pairing

$$\langle C, K_S\rangle^\perp \times \mathrm{Pic}(S) \to \mathbb{Z}$$

induces a perfect non-degenerate pairing

$$\langle C, K_S\rangle^\perp \times \mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S) \to \mathbb{Z}.$$

Since $\langle C, K_S\rangle^\perp$ is the (simply laced) root lattice of G, $\mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S)$ is the weight lattice of G. And the last statement follows since G is simply connected. □

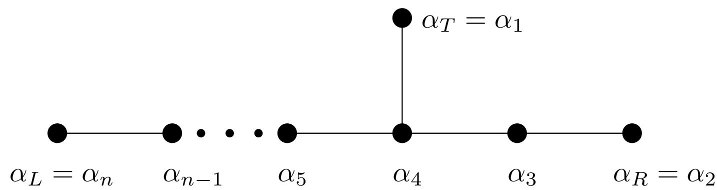

Figure 1: The root system $E_n : \alpha_1 = -h + l_1 + l_2 + l_3$, $\alpha_i = l_i - l_{i-1}$, $2 \leq i \leq n$.

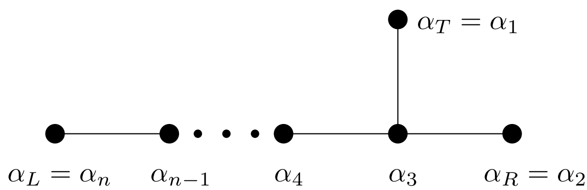

Figure 2: The root system $D_n : \alpha_1 = -f + l_1$, $\alpha_i = l_i - l_{i-1}$, $2 \leq i \leq n$.

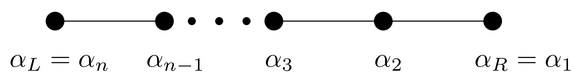

Figure 3: The root system $A_n : \alpha_i = l_{i+1} - l_i$, $1 \leq i \leq n$.

For convenience, we draw the Dynkin diagrams of the root lattices $\langle K_S, C\rangle^{\perp}$ for the given ADE-surfaces (S, C) as figures 1 to 3.

In these Dynkin diagrams, we specify three special nodes: the top node α_{T}, the right-end node α_{R} and the left-end node α_{L}, if any. These special nodes determine three fundamental representation bundles.

Definition 7. Let (S, C) be an ADE-surface.

(1) A class $l \in \mathrm{Pic}(S)$ is called a line if $l^2 = lK_S = -1$ and $lC = 0$.

(2) A class $r \in \mathrm{Pic}(S)$ is called a ruling if $r^2 = 0$, $rK_S = -2$ and $rC = 0$.

(3) A section $s_D \in H^0(S, \mathcal{O}_S(D))$ is called of degree d if $D(-K_S) = d$.

We denote the root system of the root lattice in Proposition 4 (respectively, the set of lines, the set of rulings) by $R(S, C)$ (respectively, $I(S, C)$, $J(S, C)$).

Note that there is a $\mathbb{Z}$-basis for $\mathrm{Pic}(S)$, such that all these sets and the curve C can be written down concretely (see [11] for details). The adjoint principal G-bundle (where G is of rank n) is

$$\mathcal{G} := \mathcal{O}_S^{\oplus\, n} \bigoplus \bigoplus_{\alpha \in R(S,C)} \mathcal{O}_S(\alpha).$$

The fundamental representation bundles determined by α_{L}, denoted by $\mathcal{L}_G$, are the following (see [11] for details):

For $G = E_n$ with $3 \leq n \leq 7$,

$$\mathcal{L}_{E_n} := \bigoplus_{l \in I(S,C)} \mathcal{O}_S(l);$$

and for $G = E_8$,

$$\mathcal{L}_{E_8} := \mathcal{O}_S(-K_S)^{\oplus\, 8} \bigoplus \bigoplus_{l \in I(S,C)} \mathcal{O}_S(l) \cong \mathcal{E}_8 \otimes \mathcal{O}_S(-K_S).$$

For $G = D_n$ and A_n,

$$\mathcal{L}_G := \bigoplus_{l \in I(S,C)} \mathcal{O}_S(l).$$

For $G = E_n$, $3 \leq n \leq 7$, the rulings in corresponding surfaces are used to construct the fundamental representation bundles $\mathcal{R}_{E_n}$ determined by α_{R} (see [11] for details):

For $G = E_n$ with $3 \leq n \leq 6$,

$$\mathcal{R}_{E_n} := \bigoplus_{D \in J(S,C)} \mathcal{O}_S(D).$$

For $G = E_7$,

$$\mathcal{R}_{E_7} := \mathcal{O}_S(-K_S)^{\oplus\, 7} \bigoplus \bigoplus_{D \in J(S,C)} \mathcal{O}_S(D) \cong \mathcal{E}_7 \otimes \mathcal{O}_S(-K_S).$$

We summarize some facts from [11] about these representation bundles in the following lemma.

Lemma 8. *For any irreducible representation V_λ of G with the highest weight λ, denote by $\Pi(\lambda)$ or $\Pi(V_\lambda)$ the set of all weights of V_λ.*

(i) *For $G = A_{n-1}, D_n$, or E_n, the exceptional class l_n represents the highest weight associated with α_{L}. Therefore $\Pi(l_n) = I(S,C)$ for $G \neq E_8$; and $\Pi(l_8) = I(S,C) \cup \{-K_S\}$ for $G = E_8$.*

(ii) *For* $G = E_n$*, the class* $h - l_1$ *represents the highest weight associated with* α_{R}*. Therefore* $\Pi(h - l_1) = J(S, C)$ *for* $3 \le n \le 6$*;* $\Pi(h - l_1) = J(S, C) \cup \{-K_S\}$ *for* $n = 7$*; and* $J(S, C) \subsetneq \Pi(h - l_1)$ *for* $n = 8$*.*

Proof. (i) According to figures 1 to 3, by the definition of the pairing between weights and roots in page 759 of [11], we see that $l_n(\alpha_{\mathrm{L}}) = -l_n \cdot \alpha_{\mathrm{L}} = 1$, while $l_n(\alpha_i) = -l_n \cdot \alpha_i = 0$, if $\alpha_i \neq \alpha_{\mathrm{L}}$. Thus l_n represents the highest weight associated with α_{L}.

For $G \neq E_8$, l_n is minuscule (that is, $W(G)$ acts on $\Pi(l_n)$ transitively), and by Leung and Zhang [11], $W(G)$ acts on $I(S, C)$ transitively. Therefore $\Pi(l_n) = I(S, C)$.

For $G = E_8$, $-K_S \in \Pi(l_8)$ because $-K_S = l_8 - (-3h + l_1 + \cdots + l_7 + 2l_8)$ and $-3h + l_1 + \cdots + l_7 + 2l_8$ is a positive root of E_8. In fact, $-K_S$ is the zero weight in $\Pi(l_8)$ (that is, $W(E_8)$ acts on $-K_S$ trivially). Now l_n is quasi-minuscule (that is, $W(G)$ acts on non-zero weights of $\Pi(l_n)$ transitively), and by Manin [18] or Leung and Zhang [11], $W(G)$ acts on $I(S, C)$ transitively. Therefore $\Pi(l_8) = I(S, C) \cup \{-K_S\}$.

(ii) The proof is similar. □

2.2. Quadratic forms over associated bundles

Let V_λ be a fundamental representation of a semisimple Lie group G with the fundamental weight λ. Let $\mathrm{Sym}^2 V_\lambda$ be the second symmetric product of V. Since 2λ is the highest weight in the weight set of $\mathrm{Sym}^2 V_\lambda$, $V_{2\lambda}$ is a summand of the representation $\mathrm{Sym}^2 V_\lambda$, where $V_{2\lambda}$ is the fundamental representation associated with the highest weight 2λ. Therefore there is another representation W such that $\mathrm{Sym}^2 V_\lambda = W \bigoplus V_{2\lambda}$.

With the help of the program LiE [15], we list the decomposition of $\mathrm{Sym}^2 V_\lambda$ for simply laced Lie group G with λ the fundamental weight associated with $\alpha_{\mathrm{L}} = \alpha_n$ (see figures 1 to 3).

In the $G = E_n$ case, for $4 \le n \le 6$, W is a non-trivial irreducible G-module of the least dimension, which is a minuscule representation of G. If $r = 7$, then W is the adjoint representation, which is quasi-minuscule (that is, all the non-zero weights have multiplicity 1 and form one orbit of the Weyl group $W(E_7)$ of E_7). If $r = 8$, then $W = W_1 \bigoplus \mathbb{C}$, where W_1 is the irreducible representation associated with the node α_{R} (of dimension 3875), and $\mathbb{C}$ is the trivial representation.

In the $G = D_n$ case, $W = \mathbb{C}$ is the trivial representation.

In the $G = A_n$ case, $W = \{0\}$, that is, $\mathrm{Sym}^2 V_\lambda = V_{2\lambda}$.

Let P be the maximal parabolic subgroup of G corresponding to the fundamental representation V_λ. Then we have a homogeneous variety G/P. It is well known that $G/P \hookrightarrow \mathbb{P}(V_\lambda)$ is a subvariety defined by quadratic relations [14]. A way to write explicitly the quadratic relations is the following. Let $C(G/P)$ be the affine cone over G/P. Let pr be the natural projection $\mathrm{Sym}^2 V_\lambda \to W$, and $\mathrm{Ver} : V_\lambda \to \mathrm{Sym}^2 V_\lambda$ be the Veronese map $x \mapsto x^2$, then it is well known that $C(G/P)$ is the fiber $(\mathrm{pr} \circ \mathrm{Ver})^{-1}(0)$ (as a scheme, see [1] Proposition 4.2 and references therein). Thus the homogeneous variety G/P is defined by the quadratic form

$$Q : V_\lambda \to \mathrm{Sym}^2 V_\lambda \to W.$$

In fact, we can show that the quadratic form could be globally defined over fundamental representation bundles

$$\mathcal{Q} : \mathcal{L}_G \to \mathrm{Sym}^2 \mathcal{L}_G \to \mathcal{W},$$

such that G/P is fiberwise defined by $\mathcal{Q}$.

Let $\mathcal{L}_G$ be the fundamental representation bundle defined as in the end of Section 2.1 by lines on an ADE-surface S. That is, $\mathcal{L}_G$ corresponds to the left-end node α_{L} (or equivalently, associated with the fundamental weight l_n corresponding to α_{L} for $G = A_{n-1}, D_n$ or E_n, by Lemma 8). For a quadratic form over a vector bundle $\mathcal{L}_G$, we denote $\mathcal{Q}^{-1}(0)$ the subscheme of $\mathbb{P}(\mathcal{L}_G)$ defined by $x \in \mathbb{P}(\mathcal{L}_G)$, such that $\mathcal{Q}(x) = 0$.

By Lemma 8,

$$\mathcal{L}_G = \bigoplus_{\mu \in \Pi(l_n) \subseteq \mathrm{Pic}(S)} \mathcal{O}_S(\mu)^{\oplus k_\mu},$$

where the multiplicity $k_\mu = 8$ if $\mu = -K_S$ and $G = E_8$; otherwise, $k_\mu = 1$. Let $\Pi(\mathrm{Sym}^2\mathcal{L}_G)$ be the set of weights of $\mathrm{Sym}^2 \mathcal{L}_G$ which is saturated (see Section 13.4 of [9]). Then

$$\Pi(\mathrm{Sym}^2\mathcal{L}_G) = \{\lambda_1 + \lambda_2 | \lambda_1, \lambda_2 \in \Pi(l_n)\} \subseteq \mathrm{Pic}(S)$$

and

$$\mathrm{Sym}^2 \mathcal{L}_G = \bigoplus_{\mu \in \Pi(\mathrm{Sym}^2\mathcal{L}_G) \subseteq \mathrm{Pic}(S)} \mathcal{O}_S(\mu)^{\oplus m_\mu},$$

where m_μ is the multiplicity uniquely determined by μ and $\mathrm{Sym}^2\mathcal{L}_G$. Since $2l_n$ occurs with multiplicity one, by the saturatedness, $\Pi(2l_n) \subseteq \Pi(\mathrm{Sym}^2\mathcal{L}_G)$.

Therefore $\mathrm{Sym}^2\mathcal{L}_G$ contains a summand $\mathcal{V}_{2l_n}$ which is an irreducible representation bundle associated with the highest weight $2l_n$. We write $\mathcal{V}_{2l_n}$ as

$$\mathcal{V}_{2l_n} = \bigoplus_{\mu\in\Pi(\mathrm{Sym}^2\mathcal{L}_G)\subseteq\mathrm{Pic}(S)} \mathcal{O}_S(\mu)^{\oplus\, n_\mu},$$

where $n_\mu = 0$ if $\mu \notin \Pi(2l_n)$ and $1 \le n_\mu \le m_\mu$ if $\mu \in \Pi(2l_n)$.

The other summand $\mathcal{W}$ of $\mathrm{Sym}^2\mathcal{L}_G$ is automatically a representation bundle:

$$\mathcal{W} = \bigoplus_{\mu\in\Pi(\mathrm{Sym}^2\mathcal{L}_G)\subseteq\mathrm{Pic}(S)} \mathcal{O}_S(\mu)^{\oplus\,(m_\mu - n_\mu)}.$$

We are mainly interested in the representation bundle $\mathcal{W}$, which we discuss case by case according to $G = E_n$, D_n or A_{n-1}.

(i) For $G = E_n$, $h - l_1 \in \Pi(\mathrm{Sym}^2\mathcal{L}_G)$. By LiE [15], $\mathcal{W}$ contains a weight space with the weight $h - l_1$. Thus the set $J(S,C)$ of rullings on S are contained in the set $\Pi(\mathcal{W})$ of weights of $\mathcal{W}$. Therefore, as a vector bundle, $\mathcal{W}$ contains $\bigoplus_{\mu\in J(S,C)} \mathcal{O}_S(\mu)$ as summands. By counting the rank of $\mathcal{W}$ [15] and the number of the elements of $J(S,C)$, we find that for $4 \le n \le 7$,

$$\mathcal{W} = \bigoplus_{\mu\in J(S,C)} \mathcal{O}_S(\mu) = \mathcal{R}_{E_n}$$

is the irreducible representation bundle associated with α_{R} (Lemma 8).

Similarly, for $G = E_8$, by LiE [15], $\mathcal{W}$ is a direct sum of $\mathcal{R}_{E_8}$ and a line bundle which is a trivial representation. Note that among the weights of $\mathrm{Sym}^2\,\mathcal{L}_G$, only $-2K_S$ appears as a zero weight (Lemma 8). Thus the line bundle considered here is nothing but $\mathcal{O}_S(-2K_S)$. Therefore

$$\mathcal{W} = \mathcal{R}_{E_n} \bigoplus \mathcal{O}_S(-2K_S).$$

(ii) For $G = D_n$, $f \in \Pi(\mathrm{Sym}^2\mathcal{L}_G)$. By LiE [15], $\mathcal{W}$ is a line bundle which is a trivial representation bundle. Note that the only zero weight of $\mathrm{Sym}^2\mathcal{L}_G$ is f. Therefore $\mathcal{W} \cong \mathcal{O}_S(f)$.

(iii) For $G = A_{n-1}$, by a dimension counting, $\mathrm{Sym}^2\mathcal{L}_G \cong \mathcal{V}_{2l_n}$. Therefore $\mathcal{W} = 0$.

Thus we achieved the first statement of the following theorem.

Theorem 9. *The notations are as above.*

(1) We have a decomposition of representation bundles:

$$\operatorname{Sym}^2 \mathcal{L}_G = \mathcal{W} \bigoplus \mathcal{V}_{2l_n}.$$

Here $\mathcal{W} = \mathcal{R}_{E_n}$ *for* $G = E_n$ *with* $4 \leq n \leq 7$*;* $\mathcal{W} = \mathcal{R}_{E_8} \bigoplus \mathcal{O}_S(-2K_S)$ *for* $G = E_8$*;* $\mathcal{W} = \mathcal{O}_S(f)$ *for* $G = D_n$*; and* $\mathcal{W} = 0$ *for* $G = A_{n-1}$*.*

(2) The projection to the first summand defines a quadratic form on $\mathcal{L}_G$

$$\mathcal{Q} : \mathcal{L}_G \to \operatorname{Sym}^2 \mathcal{L}_G \to \mathcal{W},$$

such that the homogeneous variety G/P *is the fiber of the subscheme (considered as a scheme defined over* S*)* $\mathbb{P}(\mathcal{Q}^{-1}(0)) \subseteq \mathbb{P}(\mathcal{L}_G)$*.*

(3) By taking global sections, for $G \neq E_8$*, we realize* G/P *as a subvariety of* $\mathbb{P}(H^0(S, \mathcal{L}_G))$*, cut out by quadratic equations. For* $G = E_8$*, we replace* $H^0(S, \mathcal{L}_G)$ *by a subspace* V *of dimension* 248*, where* $V = \mathbb{C}\langle s_K \rangle^{\oplus 8} \bigoplus \bigoplus_{\mu \in I(S,G)} H^0(S, \mathcal{O}_S(\mu))$ *with* s_K *a fixed non-zero global section of* $\mathcal{O}_S(-K_S)$*.*

Proof. It remains to verify (2) and (3), which are essentially consequences of (1).

(2) Note that fiberwise, the map

$$\mathcal{Q} : \mathcal{L}_G \to \operatorname{Sym}^2 \mathcal{L}_G = \mathcal{W} \bigoplus \mathcal{V}_{2l_n} \to \mathcal{W}$$

is exactly the map (Lemma 8)

$$Q : V_{l_n} \to \operatorname{Sym}^2 V_{l_n} \cong W \bigoplus V_{2l_n} \to W,$$

where V_{l_n}, W and V_{2l_n} are as in the beginning of Section 2.2.

By Lichtenstein [14], $Q^{-1}(0)$ is the cone over G/P in V_{l_n}, that is $\mathbb{P}(Q^{-1}(0)) = G/P \subseteq \mathbb{P}(V_{l_n})$.

(3) First by Leung and Zhang [11], every element $\mu \in I(S,G)$ is represented by a unique irreducible curve in an ADE-surface S and hence $\dim H^0(S, \mathcal{O}_S(\mu)) = 1$. For $G \neq E_8$, recall that $\mathcal{L}_G = \bigoplus_{\mu \in I(S,G)} \mathcal{O}_S(\mu)$. Therefore, we can choose a unique global section for each summand of $\mathcal{L}_G$ up to a constant.

By Lichtenstein [14], $C(G/P) \subseteq V_{l_n}$ is defined by finitely many quadratic polynomials. Let $f(x_\mu|_{\mu \in I(S,G)})$'s be such polynomials. Let s_μ be the global section of $\mathcal{O}_S(\mu), \mu \in I(S,G)$. Then $H^0(S, \mathcal{L}_G) = \{\sum_{\mu \in I(S,G)} x_\mu s_\mu | x_\mu \in \mathbb{C}\}$, and the same polynomials $f(x_\mu|_{\mu \in I(S,G)})$'s define G/P.

For $G = E_8$, since $H^0(S, \mathcal{O}_S(-K_S))$ is of dimension two, we should fix any one non-zero global section s_K of $\mathcal{O}_S(-K_S)$. Similarly by Lichtenstein [14], $C(G/P) \subseteq V_{l_8}$ is defined by finitely many quadratic polynomials. Let $f(x_\mu|_{\mu\in\Pi(l_8)})$'s be such polynomials. Thus, we take a subspace of $H^0(S, \mathcal{L}_G)$ of dimension 248 as follows: $V = \mathbb{C}\langle s_K\rangle^{\oplus\, 8} \bigoplus \bigoplus_{\mu\in I(S,G)} H^0(S, \mathcal{O}_S(\mu))$. As a vector space $V = \{x_1 s_{K,1} + \cdots + x_8 s_{K,8} + \sum x_\mu s_\mu | x_i, x_\mu \in \mathbb{C}\}$ where $s_{K,i} = s_K$ is the basis of the i-th $\mathbb{C}\langle s_K\rangle$, and the same polynomials f's define G/P. □

Remark 10. The bundle $\mathcal{W}$ appearing in Theorem 9 can be called the *representation bundle determined by rulings*, since in the $G = D_n$ and E_n cases, it is constructed by using the rulings.

3. Cox rings of ADE-surfaces and flag varieties

3.1. Cox rings of ADE-surfaces

The notion of Cox rings is defined by Cox [2] for toric varieties and he shows that for a toric variety, its Cox ring is precisely its total coordinate ring. Hu and Keel [7] give a general definition of Cox rings for $\mathbb{Q}$-factorial projective varieties X with $\mathrm{Pic}(X)_{\mathbb{Q}} \cong N^1(X)$, and show that it is related to GIT and Mori Dream Spaces. Batyrev and Popov [1], followed by Derenthal and so on [4], make a deep study on the Cox rings of del Pezzo surfaces. From their studies, for the del Pezzo surface X_n ($n \le 7$), its Cox ring is closely linked to the fundamental representation V with the highest weight α_L in the Dynkin diagram of E_n. More precisely, the projective variety defined by this ring is embedded into the flag variety G/P, where G is the complex Lie group E_n, and P is the maximal parabolic subgroup determined by the node α_L. Both G/P and this variety are subvarieties of $\mathbb{P}(V)$ defined by quadrics.

Motivated from these, we can define (generalized) Cox rings for G-surfaces as follows.

Definition 11. Let (S, C) be a G-surface with $G = A_n, D_n$ or E_n, and a $\mathbb{Z}$-basis of $\mathrm{Pic}(S)$ be chosen as in Section 2.1. Then we define the Cox ring of (S, C) as

$$\mathrm{Cox}(S, G) := \bigoplus_{D\in \mathrm{Pic}(S), DC=0} H^0(S, \mathcal{O}_S(D))$$

with a well-defined multiplication (see Section 1).

Notice that $\mathrm{Cox}(S, G)$ is naturally graded by the degree defined in Definition 7.

Remark 12. As usual, let X_n be a del Pezzo surface of degree $9-n$ with $4 \leq n \leq 8$, let $S \to X_n$ be a blowup at a general point, and C be the corresponding exceptional curve. Then for the E_n-surface (S, C), we have

$$\mathrm{Cox}(S, E_n) \cong \bigoplus_{D \in \mathrm{Pic}(X_n)} H^0(X_n, \mathcal{O}_{X_n}(D)) = \mathrm{Cox}(X_n).$$

Thus the definition of Cox rings of E_n-surfaces is the same as the classical definition of Cox rings for del Pezzo surfaces X_n. The reason for the displayed isomorphism is that the contraction morphism $\pi : S \to X_n$ induces an isomorphism $\pi^* : \mathrm{Pic}(X_n) \to C^{\perp} \subseteq \mathrm{Pic}(S)$ such that the pull-back of rational functions $H^0(X_n, \mathcal{O}_{X_n}(D)) \to H^0(S, \mathcal{O}_S(\pi^* D))$ is an isomorphism for any divisor D of X_n.

Corollary 13. (1) *For the D_n-surface (S, C), $C \equiv f$ is a smooth fiber. Then we have*

$$\mathrm{Cox}(S, D_n) = \bigoplus_{D \in \mathrm{Pic}(S), Df = 0} H^0(S, \mathcal{O}_S(D)).$$

(2) *For the A_n-surface (S, C), $C \equiv h$ (linear equivalence) is a twisted cubic. Then we have*

$$\mathrm{Cox}(S, A_n) = \bigoplus_{D \in \mathrm{Pic}(S), Dh = 0} H^0(S, \mathcal{O}_S(D)).$$

Theorem 14. *Let $G = A_n, D_n\,(n \geq 3)$ or $E_n\,(4 \leq n \leq 8)$. The Cox ring $\mathrm{Cox}(S, G)$ is finitely generated, and generated by degree 1 elements. For $G \neq E_8$, the generators of $\mathrm{Cox}(S, G)$ are global sections of invertible sheaves defined by lines on S. For $G = E_8$, we should add to the above set of generators two linearly independent global sections of the anti-canonical sheaf on X_8.*

Proof. Let f, s, h, l_i's be as in Section 2.1 (Notations).

(1) For the $G = E_n$ case, see [1, 4, 10, 19].

(2) For the $G = D_n$ case, let $D \in \mathrm{Pic}(S)$ and $DF = 0$. Assume that D is effective. Then we can write $D \equiv \sum a_i D_i$ (here "$\equiv$" means the linear equivalence) with D_i irreducible curves and $a_i \geq 0$. Choose a smooth fiber

F. Then $D_iF \geq 0$. Thus $DF = 0$ implies $a_i = 0$ or $D_iF = 0$ for all i. By the Hodge index theorem, $D_iF = 0$ implies that $D_i \equiv F$ or $D_i = l_j$ or $D_i = f - l_k$, for some j, k. Thus $D \equiv a_0F + \sum_i a_il_i + \sum_j b_j(f - l_j)$ with $a_0, a_i, b_j \geq 0$. Moreover, we can assume that $\{i \mid a_i \neq 0\} \cap \{j \mid b_j \neq 0\} = \emptyset$.

Let x_i (resp. y_i) be a non-zero global section of $\mathcal{O}_S(l_i)$ (resp. $\mathcal{O}_S(f - l_i)$). Thus, by induction, we can show that

$$\dim H^0(S, \mathcal{O}_S(D)) = \dim H^0(S, \mathcal{O}_S(a_0F)) = a_0 + 1.$$

The proof goes as follows. We have a short exact sequence

$$0 \to \mathcal{O}_S(a_0F + (a_i - 1)l_i) \to \mathcal{O}_S(a_0F + a_il_i) \to \mathcal{O}_{l_i}(a_0F + a_il_i) \to 0.$$

Note that $\mathcal{O}_{l_i}(a_0F + a_il_i) \cong \mathcal{O}_{\mathbb{P}^1}(-a_i)$, since $l_i \cong \mathbb{P}^1$ and $l_i(a_0F + a_il_i) = -a_i$. Thus we have a long exact sequence

$$\begin{aligned} 0 &\to H^0(S, \mathcal{O}_S(a_0F + (a_i - 1)l_i)) \to H^0(S, \mathcal{O}_S(a_0F + a_il_i)) \\ &\to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a_i)) \to H^1(S, \mathcal{O}_S(a_0F + (a_i - 1)l_i)) \to \cdots . \end{aligned}$$

When $a_i \geq 1$, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a_i)) = 0$, and therefore

$$H^0(S, \mathcal{O}_S(a_0F + (a_i - 1)l_i)) \cong H^0(S, \mathcal{O}_S(a_0F + a_il_i)).$$

Hence by induction we have

$$H^0(S, \mathcal{O}_S(a_0F)) \cong H^0(S, \mathcal{O}_S(a_0F + a_il_i)).$$

By repeating this process, we have

$$H^0(S, \mathcal{O}_S(D)) \cong H^0(S, \mathcal{O}_S(a_0F)).$$

It remains to prove $\dim H^0(S, \mathcal{O}_S(a_0F))) = a_0 + 1$. Also this comes from the following short exact sequence:

$$0 \to \mathcal{O}_S((a_0 - 1)F) \to \mathcal{O}_S(a_0F) \to \mathcal{O}_F(a_0F) \to 0.$$

Here $\mathcal{O}_F(a_0F) \cong \mathcal{O}_{\mathbb{P}^1}$, since $F \cong \mathbb{P}^1$ and $(a_0F)F = 0$. Taking the long exact sequence, we have

$$\begin{aligned} 0 &\to H^0(S, \mathcal{O}_S((a_0 - 1)F)) \to H^0(S, \mathcal{O}_S(a_0F)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \\ &\to H^1(S, \mathcal{O}_S((a_0 - 1)F)) \to H^1(S, \mathcal{O}_S(a_0F)) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to \cdots . \end{aligned}$$

Since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$, we shall have $H^1(S, \mathcal{O}_S(a_0F)) = 0$ if $H^1(S, \mathcal{O}_S((a_0-1)F)) = 0$. For $a_0 = 1$, we have $H^1(S, \mathcal{O}_S((a_0-1)F)) = H^1(S, \mathcal{O}_S) = 0$, since S is a rational surface. Thus by induction, we have for all $a_0 \geq 0$, $H^1(S, \mathcal{O}_S(a_0F)) = 0$. Then from the last long exact sequence we have

$$\begin{aligned}\dim H^0(S, \mathcal{O}_S(a_0F)) &= \dim H^0(S, \mathcal{O}_S((a_0-1)F)) + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \\ &= \dim H^0(S, \mathcal{O}_S((a_0-1)F)) + 1.\end{aligned}$$

Therefore by induction, we have

$$\dim H^0(S, \mathcal{O}_S(a_0F)) = a_0 + 1.$$

Let $H^0(S, \mathcal{O}_S(F)) = \mathbb{C}\langle v_1, v_2\rangle$, where v_1, v_2 are two linearly independent global sections of $\mathcal{O}_S(F)$. Then the linearly independent generators of $H^0(S, \mathcal{O}_S(D))$ can be taken as $u_k(\Pi_i x_i^{a_i})(\Pi_j y_j^{b_j})$, where $u_k = v_1^k v_2^{a_0-k}, k = 0, \ldots, a_0$.

Let $n \geq 2$. Thus we have at least two different singular fibers: $l_1 + (f - l_1)$ and $l_2 + (f - l_2)$. Then x_1y_1 and x_2y_2 are linearly independent elements in $H^0(S, \mathcal{O}_S(F))$. Thus we can take $v_1 = x_1y_1, v_2 = x_2y_2$.

Therefore, the Cox ring is generated by global sections of the invertible sheaves defined by lines (when $n \geq 2$).

In fact, if $(x) = F'$ is a smooth fiber, then we must have

$$x = a(x_1y_1) + b(x_2y_2)$$

with $a \neq 0$ and $b \neq 0$.

(3) For the $G = A_n$ case, let $D \in \mathrm{Pic}(S)$, such that $Dh = 0$. Then obviously, $D \equiv a_1l_1 + \cdots + a_{n+1}l_{n+1}$. $D \geq 0$ if and only if $a_i \geq 0$. Let $x_i \neq 0$ be a global section of $\mathcal{O}_S(l_i), 1 \leq i \leq n+1$. Note that

$$\dim H^0(S, \mathcal{O}_S(a_1l_1 + \cdots + a_{n+1}l_{n+1})) = 1,$$

and $x_1^{a_1} \cdots x_{n+1}^{a_{n+1}}$ generates the one-dimensional vector space $H^0(S, \mathcal{O}_S(a_1l_1 + \cdots + a_{n+1}l_{n+1}))$. By Definition 7,

$$\deg(x_1^{a_1} \cdots x_{n+1}^{a_{n+1}}) := D(-K_S) = a_0 + \cdots + a_{n+1}.$$

Thus, the Cox ring is in fact a polynomial ring with $n+1$ variables:

$$\mathrm{Cox}(S, A_n) = k[x_1, \ldots, x_{n+1}]. \qquad \square$$

By this theorem, the Cox ring $\mathrm{Cox}(S, G)$ of a G-surface S is a quotient of the polynomial ring $P(S, G) = k[x_1, \ldots, x_{N_G}]$ by an ideal $\mathcal{I}(S, G)$:

$$\mathrm{Cox}(S, G) = k[x_1, \ldots, x_{N_G}]/\mathcal{I}(S, G),$$

where N_G is the number of lines (Definition 7) in the G-surface S for $G \neq E_8$; for $G = E_8$, N_G is the number of lines plus 8.

Theorem 15. *For any ADE-surface S, the ideal $\mathcal{I}(S, G)$ is generated by quadrics.*

Proof. (1) For $G = A_n$, the ideal $\mathcal{I}(S, G) = 0$.

(2) For $G = E_n$, see [4] for $4 \leq n \leq 7$ and [20, 21] for $n = 8$.

(3) For $G = D_n$, let x_i, y_i be as in the proof of Theorem 14. We want to show that

$$\mathrm{Cox}(S, D_n) = k[x_1, y_1, \ldots, x_n, y_n]/\mathcal{I}(S, D_n),$$

where

$$\mathcal{I}(S, D_n) = (a_{31}x_1y_1 + a_{32}x_2y_2 + a_{33}x_3y_3, \ldots, a_{n1}x_1y_1 + a_{n2}x_2y_2 + a_{n3}x_ny_n),$$

and all $a_{ij} \neq 0$.

By the proof of Theorem 14, we see that all the generating relations come from the ruling f. The vector space $H^0(S, \mathcal{O}_S(f))$ is a two-dimensional space. Moreover, any two singular fibers are different. Therefore, when $n \geq 3$, any two elements of $\{x_1y_1, \ldots, x_ny_n\}$ are linearly independent, and any three elements are linearly dependent. Thus, the ideal $\mathcal{I}(S, D_n)$ is of desired form. $\square$

3.2. Cox rings and flag varieties

Let G be a complex simple Lie groups, and λ be a fundamental weight. Let P be the corresponding maximal parabolic subgroup and V_λ be the highest weight module. It is well known that the homogeneous space G/P (the orbit of the highest weight vector of V_λ) could be embedded into the projective space $\mathbb{P}(V_\lambda)$ with quadratic relations as generating relations. It is showed in [4] that for the del Pezzo surfaces X_n with $n = 6, 7$,

$$\mathrm{Spec}(\mathrm{Cox}(X_n)) \hookrightarrow C(E_n/P).$$

Here P is the maximal parabolic subgroup determined by the left-end node α_L in the Dynkin diagram (figure 1), and $C(E_n/P)$ is the affine cone over the homogeneous space E_n/P.

Given an ADE-surface S, we let $\mathcal{L}_G$ be the representation bundle determined by lines on S. The vector space of global sections $H^0(S, \mathcal{L}_G)$ is the fundamental representation of G associated with the node α_L (for $G = E_8$ we should replace it by a subspace V).

The following result relates the Cox ring of a G-surface with the homogeneous variety G/P. Thus we obtain a uniform description of the relation between the Cox rings $\mathrm{Cox}(S, G)$, the homogeneous space G/P, and fundamental representation bundles defined by lines in S, for any Lie group $G = A_n, D_n, E_n$.

Theorem 16. *Let $G = A_n, D_n$ or E_n. Let S, $\mathcal{L}_G$ and P be as above. We have an embedding:* $\mathrm{Proj}(\mathrm{Cox}(S, G)) \hookrightarrow G/P$.

Proof. For $G = E_n$ and $4 \leq n \leq 7$, the result is known, see [1, 4]. For $G = E_8$, see [20].

For $G = A_n$, in fact we have an isomorphism: $\mathrm{Proj}(\mathrm{Cox}(S, G)) \cong A_n/P \cong \mathbb{P}^n$. For $G = D_n$, by the proof of Theorem 15,

$$\mathrm{Cox}(S, D_n) = k[x_1, y_1, \ldots, x_n, y_n]/\mathcal{I}(S, D_n),$$

where

$$\mathcal{I}(S, D_n) = (a_{31}x_1y_1 + a_{32}x_2y_2 + a_{33}x_3y_3, \ldots, a_{n1}x_1y_1 + a_{n2}x_2y_2 + a_{n3}x_ny_n),$$

and all $a_{ij} \neq 0$. By Lemma 17, the affine coordinate ring of $C(D_n/P)$ is

$$k[x_1, y_1, \ldots, x_n, y_n]/(x_1y_1 + \cdots + x_ny_n).$$

There exist non-zero b_i, $3 \leq i \leq n$, such that the coefficient of the term x_iy_i in the sum $\sum_{3\leq i\leq n} b_i(a_{i1}x_1y_1 + a_{i2}x_2y_2 + a_{i3}x_iy_i)$ is non-zero, by dimension counting.

Therefore, we have a surjective homomorphism from the affine coordinate ring of $C(G/P)$ to that of $\mathrm{Spec}(\mathrm{Cox}(S, G))$, which defines a closed embedding

$$\mathrm{Spec}(\mathrm{Cox}(S, G)) \hookrightarrow C(G/P).$$

□

The following result is well known. But since we cannot find an appropriate reference, we include its proof here.

Lemma 17. *Let* $V_{l_n} = \bigoplus_{\mu \in I(S,G)} V_{(\mu)}$ *be the irreducible representation of* $D_n = \mathrm{SO}(2n, \mathbb{C})$ *associated with the highest weight* l_n*. Let* P *be the maximal hyperbolic subgroup of* D_n *associated with the highest weight* l_n*.*

Then there exists a basis $\{u_i, v_i | i = 1, \ldots, n\}$ *for* V_{l_n} *such that* $D_n/P \subseteq \mathbb{P}(V_{l_n})$ *is defined by the quadratic equation* $Q(x_1, y_1, \ldots, x_n, y_n) = \sum_{i=1}^{n} x_i y_i = 0$.

Proof. By Leung and Zhang [11], $\Pi(l_n) = I(S, G) = \{l_i, f - l_i | i = 1, \ldots, n\}$. Hence we can take a basis for V_{l_n} as $\{u_i, v_i | i = 1, \ldots, n\}$, where for $1 \leq i \leq n$, u_i (resp. v_i) is the basis for the one-dimensional weight space with the weight l_i (resp. $f - l_i$).

As in the beginning of Section 2.2, let Q be the composite map

$$Q : V_{l_n} \to \mathrm{Sym}^2 V_{l_n} = V_{2l_n} \bigoplus W \to W,$$

where $W \cong \mathbb{C}$ is the trivial representation. By Lichtenstein [14], $D_n/P \subseteq \mathbb{P}(V_{l_n})$ is defined by the equation $Q = 0$. We only need to write down Q explicitly.

The following map

$$Q' : V_{l_n} = \mathbb{C}^{2n}\langle u_i, v_i | 1 \leq i \leq n \rangle \to \mathbb{C},$$
$$\sum (x_i u_i + y_i v_i) \mapsto \sum x_i y_i$$

defines a non-degenerate symmetric quadratic form which is D_n-invariant. (In fact, $D_n = \mathrm{SO}(2n, \mathbb{C})$ is the Lie group preserving this non-degenerate symmetric quadratic form with determinant one.) Therefore by the Complete Reducibility Theorem for semisimple Lie groups, $\mathbb{C}$ is a summand of the D_n-module $\mathrm{Sym}^2 V_{l_n}$. Hence $W \cong \mathbb{C}$ and $Q = Q'$. □

3.3. Cox rings and the GIT quotients

Let (S, C) be a G-surface. The subset $C^{\perp}$ of $\mathrm{Pic}(S)$ is a free abelian group of rank equal to $\mathrm{rank}(\mathrm{Pic}(S)) - 1$. We have the following short exact sequence:

$$0 \to \mathbb{Z}C \to \mathrm{Pic}(S) \to \mathrm{Pic}(S)/\mathbb{Z}C \to 0.$$

Taking the dual, we have

$$1 \to \mathrm{Hom}(\mathrm{Pic}(S)/\mathbb{Z}C, \mathbb{C}^*) \to T_{\mathrm{NS}} \to \mathbb{C}^* \to 1.$$

We denote the torus $\mathrm{Hom}(\mathrm{Pic}(S)/\mathbb{Z}C, \mathbb{C}^*)$ by $T_{S,G}$. Note that, for $G = E_n$, $T_{S,G}$ is exactly the Néron–Severi torus T_{NS} of the del Pezzo surface X_n obtained by blowing down C from S.

The torus $T_{S,G}$ is an extension of $\mathbb{C}^*$ by a maximal torus T_G of G. One can see that the lattice $\langle C, K_S\rangle^\perp$ is a sublattice of $C^\perp$ of rank equal to $\mathrm{rank}(C^\perp) - 1$. In fact, we have a short exact sequence:

$$0 \to \mathbb{Z}K_S \to \mathrm{Pic}(S)/\mathbb{Z}C \to \mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S) \to 0.$$

Since the character group $\chi(T_G)$ of T_G is isomorphic to the weight lattice $\mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S)$ (if we take G to be the simply connected one), we have $T_G \cong \mathrm{Hom}(\mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S), \mathbb{C}^*)$, by Corollary 6. Therefore the following sequence is exact:

$$1 \to T_G \to T_{S,G} \to \mathbb{C}^* \to 1.$$

The torus $T_{S,G}$ acts on $\mathrm{Cox}(S, G)$ (and therefore acts on $\mathrm{Spec}(\mathrm{Cox}(S, G))$) naturally.

Proposition 18. *The embeddings*

$$\mathrm{Proj}(\mathrm{Cox}(S, G)) \hookrightarrow G/P \hookrightarrow \mathbb{P}(V_{l_n})$$

arising in Theorem 16 are T_G-equivariant.

Proof. This is known for $G = E_n$ by Batyrev and Popov [1], Derenthal [4] and Serganova and Skorobogatov [19].

For $G = D_n$, since our coordinate system $\{x_i, y_i | i = 1, \ldots, n\}$ is chosen by the weight vectors (see Lemma 17), T_G acts on these spaces as scalars on each coordinate. According to Lemma 17 and the proof of Theorem 16, these embeddings are T_G-equivariant.

For $G = A_{n-1}$, it is trivial, since $\mathrm{Proj}(\mathrm{Cox}(S, G)) \cong G/P \cong \mathbb{P}(V_{l_n})$ and T_G acts on these spaces as scalars on each coordinate. □

In the E_n case, by Hu and Keel [7], the GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(X_n))$ by T_{NS} is exactly the surface X_n. In general, we have

Proposition 19. *Let X_n be the del Pezzo surface obtained from an E_n-surface S by blowing down C. The GIT quotient of* $\mathrm{Spec}(\mathrm{Cox}(S, G))$ *by the action of $T_{S,G}$ is, respectively, X_n for $G = E_n$, $\mathbb{P}^1$ for $G = D_n$ and a point for $G = A_n$.*

Proof. For the case $G = E_n$, we can apply the result of Hu and Keel (Proposition 2.9 in [7]), since $\mathrm{Cox}(S, E_n) \cong \mathrm{Cox}(X_n)$ is finitely generated by Batyrev and Popov [1], and since $T_{\mathrm{NS}}(X_n) \cong T_{S,G}$.

It remains to prove the cases $G = D_n$ and $G = A_n$.

For $G = D_n$, we apply a linearization argument as in Hu and Keel [7]. In this case $\mathrm{Pic}(S) = \mathbb{Z}\langle f, s, l_1, \dots, l_n\rangle$ and $C \equiv f$. Let $R = \mathrm{Cox}(S, D_n)$. Note that R is naturally graded by the lattice $\mathrm{Pic}(S)/\mathbb{Z}f$. For example, $a_0 f + \sum_{i=1}^n a_i l_i \in f^\perp$ with $a_i \in \mathbb{Z}$ is graded by $a_0 s + \sum_{i=1}^n a_i l_i \in \mathrm{Pic}(S)/\mathbb{Z}f$. Then $R = \bigoplus_{v\in \mathrm{Pic}(S)/\mathbb{Z}f} R_v$. According to Theorem 14, R is finitely generated. Note that $T_{S,G}$ acts naturally on R. So $R = \bigoplus_{v\in \mathrm{Pic}(S)/\mathbb{Z}f=\chi(T_{S,G})} R_v$ is the eigenspace decomposition for this action. Thus

$$H^0(\mathrm{Spec}(\mathrm{Cox}(S,G)), L_v)^{T_{S,G}} = R_v,$$

where L_v is the line bundle determined by the linearization $v \in \mathrm{Pic}(S)$. And the ring of invariants is

$$R(\mathrm{Spec}(\mathrm{Cox}(S,G)), L_v)^{T_{S,G}} = R(S, \mathcal{O}_S(v')),$$

where $v' \in f^\perp$ is graded by v, and $R(S, \mathcal{O}_S(v'))$ (similar for $R(\mathrm{Spec}(\mathrm{Cox}(S,G)), L_v)$) denotes the graded ring $\bigoplus_{n\geq 0} H^0(S, \mathcal{O}_S(nv'))$. (This notation is taken from [7].) Thus $\mathbb{P}^1 \cong \mathrm{Proj}(R(S, \mathcal{O}_S(f)))$ is the GIT quotient for the linearization $v = s \in \chi(T_{S,G})_{\mathbb{Q}}$.

The proof for $G = A_n$ is similar. □

Appendix A. Two non-simple but semisimple cases

Note that $G = E_3 = A_2 \times A_1$ and $G = D_2 = A_1 \times A_1$ are not simple, but semisimple. For completeness, in these two cases, we define G-surfaces (S, C) and the Cox rings $\mathrm{Cox}(S, G)$ similarly as in Corollary 5 and Definition 11, and we compute briefly the coordinate rings of the G/P and the Cox rings explicitly. It turns out that there is no embedding of $\mathrm{Spec}(\mathrm{Cox}(S,G))$ into $C(G/P)$.

A.1. The case $G = E_3$

Let (S, C) be an E_3-surface, that is, S is a blowup of a del Pezzo surface X_3 of degree 6 at a general point, and C is the exceptional curve. Note that X_3 is a blowup of $\mathbb{P}^2$ at three points in general position. The representation bundle $\mathcal{L}_{E_3}$ is the tensor product of the standard representation bundles of A_2 and A_1.

Precisely, $\mathcal{L}_{E_3} = \mathcal{V}_2 \otimes \mathcal{V}_1$ where $\mathcal{V}_i (i = 1, 2)$ is the standard representation of A_i. By checking the highest weights, it is easy to see that

(i) $\mathcal{L}_{E_3}$ is determined by the set of -1 curves

$$\{l_1, l_2, l_3, h - l_1 - l_2, h - l_1 - l_3, h - l_2 - l_3\};$$

(ii) $\mathcal{V}_2^*$ is determined by $\{h - l_1, h - l_2, h - l_3\}$ and $\mathcal{V}_2$ is determined by $\{-(h - l_1), -(h - l_2), -(h - l_3)\}$;

(iii) $\mathcal{V}_1$ is determined by $\{h, 2h - l_1 - l_2 - l_3\}$.

Note that $\mathcal{O}_S(h) \bigoplus \mathcal{O}_S(2h - l_1 - l_2 - l_3)$ is a standard representation bundle of the adjoint principal bundle $\mathscr{A}_1 := \mathcal{O}_S \bigoplus \mathcal{O}_S(\alpha_1) \bigoplus \mathcal{O}_S(-\alpha_1)$ (recall that $\alpha_1 = -h + l_1 + l_2 + l_3$).

The $\mathrm{Cox}(S, E_3)$ is defined as in Definition 11. Then $\mathrm{Cox}(S, E_3) \cong \mathrm{Cox}(X_3)$, and it is well known that $\mathrm{Cox}(X_3) \cong \mathbb{C}[y_1, \ldots, y_6]$, since X_3 is toric [2]. Therefore,

$$\mathrm{Proj}(\mathrm{Cox}(S, E_3)) \cong \mathbb{P}(H^0(S, \mathcal{L}_{E_3})) = \mathbb{P}^5.$$

And $E_3/P = \mathbb{P}^2 \times \mathbb{P}^1$. Denote $V = H^0(S, \mathcal{L}_{E_3})$.

Then we have

$$\mathrm{Proj}(\mathrm{Cox}(S, E_3)) = \mathbb{P}(V) \ (\cong \mathbb{P}^5), \text{ and } E_3/P = \mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}(V),$$

where the embedding $G/P \hookrightarrow \mathbb{P}(H^0(S, \mathcal{L}_{E_3})) = \mathbb{P}^5$ corresponds to the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$.

A.2. The case $G = D_2$

Let (S, C) be a D_2-surface, that is, S is a blowup of the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at two points in general position, and C is a smooth fiber. The rank 4 representation bundle $\mathcal{L}_{D_2}$ is the tensor product of the standard representation bundles of A_1 (recall that $D_2 = A_1 \times A_1$).

Note that $\mathcal{L}_{D_2} = \mathcal{O}_S(l_1) \bigoplus \mathcal{O}_S(l_2) \bigoplus \mathcal{O}_S(f - l_1) \bigoplus \mathcal{O}_S(f - l_2)$. Let $\mathcal{V}_1 := \mathcal{O}_S(l_1 - s) \bigoplus \mathcal{O}_S(l_2 - s)$ and $\mathcal{V}_2 := \mathcal{O}_S(s) \bigoplus \mathcal{O}_S(s + f - l_1 - l_2)$. Then we find that $\mathcal{L}_{D_2} = \mathcal{V}_1 \otimes \mathcal{V}_2$. By checking the highest weights, we see that $\mathcal{V}_1, \mathcal{V}_2$ are the corresponding standard representations.

Thus we have $G/P \cong \mathbb{P}^1 \times \mathbb{P}^1$. And the embedding

$$G/P \hookrightarrow \mathbb{P}(H^0(S, \mathcal{L}_{D_2})) \cong \mathbb{P}^3$$

corresponds to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. On the other hand, the Cox ring $\mathrm{Cox}(S, D_2)$, defined as in Definition 11, is a sub-ring of $\mathrm{Cox}(X_3)$ generated by degree 1 elements in $\mathrm{Cox}(X_3)$, since S is also a del Pezzo surface X_3. That $\mathrm{Cox}(S, D_2)$ is a sub-ring of $\mathrm{Cox}(X_3)$ follows directly from their definitions. Therefore, we have $\mathrm{Cox}(S, D_2) = \mathbb{C}[x_1, \ldots, x_4]$, and hence,

$$\mathrm{Proj}(\mathrm{Cox}(S, D_2)) \cong \mathbb{P}(H^0(S, \mathcal{L}_{D_2})) \cong \mathbb{P}^3.$$

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