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The positive mass theorem and Penrose inequality for graphical manifolds

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We give, via elementary methods, explicit formulas for the ADM mass which allow us to conclude the positive mass theorem and Penrose inequality for a class of graphical manifolds which includes, for instance, those with flat normal bundle.

1. Introduction

A smooth connected n-dimensional Riemannian manifold (M^n, g), with $n \geq 3$, is said to be *asymptotically flat* if there exists a compact subset K of M and a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus \{|x| \leq 1\}$ such that in this coordinate chart the metric $g(x) = g_{ij}(x)dx_i \otimes dx_j$, with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \{|x| \leq 1\}$, satisfies

$$\begin{aligned} g_{ij} - \delta_{ij} &= O(|x|^{-p}), \quad & g_{ijk} &= O(|x|^{-p-1}), \\ g_{ijkl} &= O(|x|^{-p-2}), \quad & S &= O(|x|^{-q}), \end{aligned}$$

at infinity, where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and g_{ijk}, g_{ijkl} denote the partial derivatives of g_{ij},

$$g_{ijk} = \frac{\partial g_{ij}}{\partial x^k} \quad \text{and} \quad g_{ijkl} = \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}, \tag{1}$$

for all $1 \leq i, j, k, l \leq n$. Here S is the scalar curvature, δ_{ij} is the Kronecker delta, and $p > \frac{n-2}{2}$ and $q > n$ are constants.

Definition 1.1. The ADM mass of an asymptotically flat manifold (M^n, g) is the limit

$$m_{\mathrm{ADM}} = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{iji} - g_{iij})\nu_j \, d\mu, \tag{2}$$

where $S_r = \{x \in \mathbb{R}^n \mid |x| = r\}$ is the coordinate sphere of radius r, $d\mu$ is the area element of S_r in the coordinate chart, ω_{n-1} is the volume of the unit sphere S_1 and $\nu = (\nu_1, \ldots, \nu_n) = r^{-1}x$ is the outward unit normal to S_r.

It is worthwhile to reminder that Definition 1.1 was given by the physicists Arnowitt, Deser and Misner [1], who defined it for the three-dimensional case, and Bartnik [2] proved that the limit (2) exists and independs of the choice of an asymptotically flat chart Φ, hence the ADM mass is a geometric invariant of (M^n, g). The positivity of the ADM mass in all dimensions is a long-standing question and a pillar of the mathematical relativity. In a seminal work, Schoen and Yau [14] gave an affirmative answer for the three-dimensional case and, in the follow-up paper [15], gave affirmative answer for dimensions $3 \leq n \leq 7$. For manifolds that are conformally flat or spin affirmative answers were given by Schoen and Yau [16] and Witten [19], respectively. The Riemannian positive mass theorem can be stated as

Theorem A ([14–16, 19]). *Let M^n be an asymptotically flat manifold with nonnegative scalar curvature. Assume that M is spin, or $3 \leq n \leq 7$, or M is conformally flat. Then the ADM mass is positive unless M^n is isometric to the Euclidean space $\mathbb{R}^n$.*

The Riemannian Penrose conjecture asserts that any asymptotically flat manifold M^n with nonnegative scalar curvature containing an outermost minimal hypersurface Σ (possibly disconnected) of area A has ADM mass satisfying

$$m_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}. \tag{3}$$

Furthermore, the equality in (3) implies that M^n is isometric to the Riemannian Schwarzschild manifold $(\mathbb{R}^n \setminus \{0\}, (1 + \frac{m}{2|x|^{n-2}})^{\frac{4}{n-2}}\delta)$, where δ denotes the Euclidean metric of $\mathbb{R}^n$ and $m = m_{\mathrm{ADM}}$. This inequality was first proved in the three-dimensional case by Huisken and Ilmanen [11] under the additional hypothesis that the horizon Σ is connected. Bray [3] proved this conjecture, still in dimension three, without connectedness assumption on Σ. For $3 \leq n \leq 7$, this conjecture was proved by Bray and Lee [4], with the extra requirement that M be spin for the rigidity statement. The Riemannian Penrose inequality can be stated as

Theorem B ([3, 4, 11]). *Let M^n be an asymptotically flat manifold with nonnegative scalar curvature. Assume that $3 \leq n \leq 7$ and there exists an* *outermost minimal hypersurface* $\Sigma^{n-1} \subset M^n$ *with area* A*. Then it holds*

$$m_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Moreover, under the hypothesis that M *is spin then the equality occurs if and only if* M *is isometric to the Riemannian Schwarzschild manifold of mass* $m = m_{\mathrm{ADM}}$.

In arbitrary dimension, Lam [13] obtained an elementary and straightforward proof for the positive mass theorem and Penrose inequality for codimension one graphical manifolds, which was extended in some sense to hypersurfaces by Huang and Wu in [8–10], and to more general kind of codimension one graphs by de Lima and Girão in [5, 6]. The present paper deals with graphical manifolds with arbitrary codimension. We will give here, via elementary methods, explicit formulas for the ADM mass which allow us to conclude the positive mass theorem and Penrose inequality for a class of graphical manifolds which includes, for instance, those with flat normal bundle. Example 1.1 below shows that there exist examples of asymptotically flat graphical manifolds of arbitrary codimension and with flat normal bundle. We bring to the fore that graphical manifolds with flat normal bundle are subject of study in several recent works, see for example [12, 17, 18] and references therein.

In order to enunciate our theorems, we will start with some notations and definitions.

Definition 1.2. A C^2 map $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$, with $n \geq 3$, where $\Omega \subset \mathbb{R}^n$ is a bounded subset, is said to be *asymptotically flat* if the scalar curvature S of the graph of f endowed with its natural metric is an integrable function over $\mathbb{R}^n$ and moreover the partial derivatives $f_i^\alpha = \frac{\partial f^\alpha}{\partial x_i}$ and $f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x_i \partial x_j}$ satisfy

$$|f_i^\alpha(x)| = O(|x|^{-\frac{p}{2}}) \text{ and } |f_{ij}^\alpha(x)| = O(|x|^{-\frac{p}{2}-1}),$$

at infinity, for all $\alpha = 1, \ldots, m$ and $i, j = 1, \ldots, n$, where $p > (n-2)/2$.

Let $M = \mathrm{Gr}(f) = \{(x, f(x)) \mid x \in \mathbb{R}^n\}$ be the graph of an asymptotically flat map $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$. The metric on M induced by the ambient space $\mathbb{R}^n \times \mathbb{R}^m$, that will denote by natural metric on M, is given by $g = g_{ij} dx^i \otimes$

dx^j of M, where

$$g_{ij} = \delta_{ij} + f_i^\alpha f_j^\alpha, \tag{4}$$

hence $g_{ij} = O(|x|^{-p})$ and $g_{ijk} = O(|x|^{-p-1})$. The vectors $\partial_i = (e_i, f_i^\alpha e_\alpha)$ form the coordinate vector fields, and the vectors $\eta^\alpha = (-Df^\alpha, e_\alpha)$, where $Df^\alpha = f_i^\alpha e_i$ denotes the gradient vector field of f^α, form a basis for the normal bundle $T^\perp M$ of M. Here e_i and e_α denotes the canonical vectors of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

By abuse of notation, let us consider the functions f^α also defined on M by identifying $f^\alpha = f^\alpha \circ \pi$, where $\pi : M \to \mathbb{R}^n$ is the natural projection $\pi(x, f(x)) = x$. The gradient vector field of $f^\alpha : M \to \mathbb{R}$ satisfies

$$\nabla f^\alpha = g^{jk} f_k^\alpha \partial_j, \tag{5}$$

where (g^{ij}) denotes the inverse matrix $(g_{ij})^{-1}$.

Let $S : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ be the scalar curvature of (M, g) and $S^\perp : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ the function given by

$$S^\perp = \langle R^\perp(\nabla f^\alpha, \nabla f^\beta)\eta^\beta, \eta^\alpha\rangle, \tag{6}$$

where $R^\perp$ denotes the normal curvature tensor of the submanifold $M \subset \mathbb{R}^{n+m}$.

It is natural to ask about the abundance of asymptotically flat graphical manifolds with flat normal bundle. Of course, codimension one graphs have flat normal bundle since they are hypersurfaces. Example 1.1 below exhibits a class of asymptotically flat graphical manifolds of arbitrary codimension and with flat normal bundle.

Example 1.1. Let $f^\alpha : \mathbb{R}^{n_\alpha} \setminus \Omega_\alpha \to \mathbb{R}$, with $\alpha = 1, \ldots, k$, be asymptotically flat functions, where Ω_α are bounded open subsets. Given m_α, with $\alpha = 1, \ldots, k$, positive integer numbers, consider the map $F^\alpha = (f^\alpha, \ldots, f^\alpha) : \mathbb{R}^n \setminus \Omega_\alpha \to \mathbb{R}^{m_\alpha}$. Write $\mathbb{R}^n \setminus \Omega = (\mathbb{R}^{n_1} \setminus \Omega_1) \times \cdots (\mathbb{R}^{n_k} \setminus \Omega_k)$, for some bounded open set $\Omega \subset \mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$. Consider the map

$$F = (F^1, \ldots, F^k) : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k}.$$

It is simple to see that F is an asymptotically flat map. In Remark 2.1, we observe that the graph of F has flat normal bundle.

In the theorem below, we will state an explicit formula for the ADM mass for graphs of asymptotically flat maps $f : \mathbb{R}^n \to \mathbb{R}^m$.

Theorem 1.2. *Let M^n be a graph of an asymptotically flat map $f : \mathbb{R}^n \to \mathbb{R}^m$ endowed with its natural metric $g = g_{ij}dx^i \otimes dx^j$. Then the ADM mass of M satisfy*

$$m_{\mathrm{ADM}} = \frac{1}{2(n-1)\omega_{n-1}} \int_M (S + S^\perp) \frac{1}{\sqrt{G}}\, dM,$$

where $G = \det(g_{ij})$ is the determinant of the metric coefficient matrix (g_{ij}).

As a consequence, we will derive the positiveness of the ADM mass for entire graphs with flat normal bundle.

Corollary 1.3. *Let M^n be the graph of an asymptotically flat map $f : \mathbb{R}^n \to \mathbb{R}^m$ endowed with its natural metric. Assume that M has nonnegative scalar curvature and flat normal fiber bundle. Then the ADM mass of M is nonnegative.*

In the next two theorems, we obtain explicit formulas for the ADM mass for asymptotically flat graphs with boundary. These will allow us to conclude the Penrose inequality in our setting.

Theorem 1.4. *Let $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$ be an asymptotically flat map, where Ω is a bounded open set with Lipschitz boundary. Assume that f is constant along each connected component of $\Sigma = \partial\Omega$. Let M^n be the graph of f endowed with its natural metric. Then,*

$$m_{\mathrm{ADM}} = \frac{1}{2(n-1)\omega_{n-1}} \left(\int_M (S + S^\perp) \frac{1}{\sqrt{G}} dM + \int_\Sigma \frac{|Df|^2}{1+|Df|^2} H^\Sigma d\Sigma \right),$$

where $G = \det(g_{ij})$, and H^Σ, $\mathcal{H}^\Sigma$ are the mean curvatures of Σ seen as a hypersurface in $\mathbb{R}^n$ and in M, in the direction to the unit vectors pointing outward to Ω and M, respectively.

Under the assumption that f is constant along each connected component of Σ, the mean curvatures H^Σ, $\mathcal{H}^\Sigma$ of Σ, seen as hypersurfaces in $\mathbb{R}^n$ and in M, in the directions of their corresponding unit normal vectors pointing outward Ω and M, respectively, satisfy

$$\mathcal{H}^\Sigma = -\frac{1}{\sqrt{1+|Df|^2}} H^\Sigma, \tag{7}$$

where $|Df|^2 = |Df^1|^2 + \cdots + |Df^m|^2$ (see Remark 4.1).

Theorem 1.5 below will be proved from Theorem 1.4 and an approximation argument.

Theorem 1.5. *Let $f : \mathbb{R}^n \setminus \Omega$ be a continuous map, where $\Omega \subset \mathbb{R}^n$ is a bounded open set, that is asymptotically flat in $\mathbb{R}^n \setminus \bar{\Omega}$ and constant along each connected component of $\Sigma = \partial\Omega$. Assume that the graph $M = \mathrm{Gr}(f)$ extends C^2 up its boundary ∂M and that, along each connected component Σ' of Σ, the manifold M is tangent to a cylinder $\Sigma' \times \ell$, where ℓ is a straight line of $\mathbb{R}^m$. Assume further that $S^\perp$ is bounded in neighborhood of Σ. Then,*

$$m_{\mathrm{ADM}} = \frac{1}{2(n-1)\omega_{n-1}} \left(\int_M (S + S^\perp) \frac{1}{\sqrt{G}} dM + \int_\Sigma H^\Sigma \, d\Sigma \right),$$

where H^Σ is the mean curvature of Σ, seen as a hypersurface in $\mathbb{R}^n$, in the direction of the unit vector ν pointing outward to Ω.

Under hypothesis of Theorem 1.5, we will prove in Section 5 that $\lim_{x\to\partial\Omega} |Df|^2 = +\infty$. This implies, together with (7) and an approximation argument, that the boundary ∂M is a minimal hypersurface of M.

Now we will state the Penrose inequality for graphical manifolds of arbitrary codimension. Following [10] closely, we can use the following Alexandrov–Fenchel inequality due to Guan and Li [7] and an elementary lemma.

Proposition 1.6 ([7]). *Let $\Omega \subset \mathbb{R}^n$ be a mean-convex star-shaped bounded domain with boundary $\partial\Omega = \Sigma$. Consider $|\Sigma| = \mathcal{H}^{n-1}(\Sigma)$ the total volume of Σ. Then,*

$$\frac{1}{2(n-1)\omega_{n-1}} \int_\Sigma H_\Sigma \, d\Sigma \geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \tag{8}$$

where H^Σ is the mean curvature of Σ, seen as a hypersurface in $\mathbb{R}^n$, in the direction of the unit vector field ν pointing inward to Ω. Furthermore, the equality in (8) occurs if and only if Σ is a sphere.

Lemma 1.7 ([10]). *Let $a_1, \ldots, a_k$ be nonnegative real numbers and $0 \leq \beta \leq 1$. Then,*

$$\sum_{i=1}^k a_i^\beta \geq \left(\sum_{i=1}^k a_i \right)^\beta.$$

If $0 \leq \beta < 1$, the equality holds if and only if at most one of a_i is non-zero.

By Theorem 1.5, Proposition 1.6 and Lemma 1.7, we can conclude our main result this paper.

Theorem 1.8. *Under hypothesis of Theorem 1.5, we assume that M has non-negative scalar curvature and flat normal bundle. Assume further that each connected component of Ω is mean-convex star-shaped. Then,*

$$m_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{9}$$

where $|\Sigma|$ denotes the total volume of Σ. Furthermore, the equality in (9) implies that the scalar curvature S is identically zero and Σ is a sphere.

2. Preliminaries

We assume all the notations given in the previous section. Let $(U_{\alpha\beta})$ be the non-singular matrix given by

$$U_{\alpha\beta} = \langle \eta^\alpha, \eta^\beta \rangle = \delta_{\alpha\beta} + \langle Df^\alpha, Df^\beta \rangle \tag{10}$$

and let $(U^{\alpha\beta})$ be the inverse matrix of $(U_{\alpha\beta})$. Using that $\partial_i = (e_i, f_i^\alpha e_\alpha)$ and $\eta^\alpha = (-Df^\alpha, e_\alpha)$, we obtain $\bar{\nabla}_{\partial_i}\eta^\alpha = \eta_i^\alpha = (-Df_i^\alpha, 0)$, hence $\langle \bar{\nabla}_{\partial_i}\eta^\alpha, \partial_j \rangle = -f_{ij}^\alpha$ and $\langle \bar{\nabla}_{\partial_i}\eta^\alpha, \eta^\beta \rangle = \langle Df_i^\alpha, Df^\beta \rangle$. Thus, the shape operator A^α with respect to the normal vector η^α and the second fundamental form B of the graph $M \subset \mathbb{R}^n \times \mathbb{R}^m$ are given by

$$\begin{aligned} A^\alpha \partial_i &= -(\bar{\nabla}_{\partial_i}\eta^\alpha)^T = f_{ik}^\alpha g^{kj}\partial_j, \\ B(\partial_i, \partial_j) &= f_{ij}^\alpha U^{\alpha\beta}\eta^\beta. \end{aligned} \tag{11}$$

Remark 2.1. Let $F : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$ be the map as given in Example 1.1. We have

$$\begin{aligned} F = (f^{11}, \dots, f^{1\,m_1}, \dots, f^{k1}, \dots, f^{k\,m_k}) :& \\ (\mathbb{R}^{n_1} \setminus \Omega_1) \times \cdots \times (\mathbb{R}^{n_k} \setminus \Omega_k) &\to \mathbb{R}^m. \end{aligned}$$

Write $x = (x_{11}, \dots, x_{1\,n_1}, \dots, x_{k1}, \dots, x_{k\,n_k}) \in \mathbb{R}^n \setminus \Omega$. Then, for all $i, j = 1, \dots, k$, the partial derivatives $\frac{\partial f^{jl}}{\partial x_{ir}} = 0$ if $i \neq j$, for all $l = 1, \dots, m_i$ and $r = 1, \dots, n_i$. This implies that $A^{jl}\partial_{ir} = 0$, if $i \neq j$. Furthermore, since $f^{i1} = \cdots = f^{i\,m_i}$, we have $A^{i1}\partial_{ir} = \cdots = A^{i\,m_i}\partial_{ir}$. Thus, by the Ricci equation, we obtain that the normal curvature $R^\perp$ of M vanishes identically.

By the Gauss Equation, the curvature tensor R of M satisfies

$$\begin{aligned} R_{ilkj} &= \langle R(\partial_i,\partial_l)\partial_k,\partial_j\rangle = \langle B(\partial_i,\partial_j),B(\partial_l,\partial_k)\rangle - \langle B(\partial_i,\partial_k),B(\partial_l,\partial_j)\rangle \\ &= f^\gamma_{ij}U^{\gamma\alpha}f^\mu_{kl}U^{\mu\beta}U_{\alpha\beta} - f^\gamma_{ik}U^{\gamma\alpha}f^\mu_{jl}U^{\mu\beta}U_{\alpha\beta} = (f^\gamma_{ij}f^\alpha_{kl} - f^\gamma_{ik}f^\alpha_{jl})U^{\gamma\alpha}. \end{aligned}$$

Thus, the scalar curvature S of M is given by

$$S = g^{ij}g^{kl}R_{ilkj} = g^{ij}g^{kl}U^{\alpha\beta}(f^\beta_{ij}f^\alpha_{kl} - f^\beta_{ik}f^\alpha_{jl}). \tag{12}$$

We will prove the following:

Proposition 2.2. *The scalar curvature S of the graph M and the function $S^\perp = \langle R^\perp(\nabla f^\alpha,\nabla f^\beta)\eta^\beta,\eta^\alpha\rangle$ as given in (6) satisfy*

$$S + S^\perp = \mathrm{div}_{\mathbb{R}^n} X,$$

where $X : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field given by

$$X = \left(U^{\alpha\beta}(f^\beta_i f^\alpha_{kk} - f^\beta_k f^\alpha_{ik}) + U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df^\mu_k\rangle(f^\alpha_i f^\beta_k - f^\alpha_k f^\beta_i)\right) e_i. \tag{13}$$

In order to prove Proposition 2.2 we will need some preliminaries. The first one is the following:

Lemma 2.3. *The following items hold:*

1) $g^{ij} = \delta_{ij} - U^{\alpha\beta}f^\alpha_i f^\beta_j$;
2) $U^{\alpha\beta} = \delta_{\alpha\beta} - g(\nabla f^\alpha,\nabla f^\beta)$;
3) $g(\nabla f^\alpha,\nabla f^\beta) = \langle Df^\alpha, Df^\gamma\rangle U^{\gamma\beta}$.

Proof. By (4), we have $f^\beta_j g_{ij} = f^\beta_j(\delta_{ij} + f^\alpha_i f^\alpha_j) = f^\beta_i + f^\alpha_i\langle Df^\beta, Df^\alpha\rangle = f^\beta_i + f^\alpha_i(U_{\alpha\beta} - \delta_{\alpha\beta}) = f^\alpha_i U_{\alpha\beta}$. Hence, multiplying both sides by $g^{ik}U^{\beta\mu}$, we obtain

$$f^\beta_k U^{\beta\mu} = f^\mu_i g^{ik}. \tag{14}$$

Again by (4), we have $\delta_{ij} = g_{ik}g^{kj} = (\delta_{ik} + f^\alpha_i f^\alpha_j)g^{kj} = g^{ij} + f^\alpha_i f^\alpha_j g^{kj}$. Using (14), we obtain $g^{ij} = \delta_{ij} - f^\alpha_i f^\alpha_k g^{kj} = \delta_{ij} - f^\alpha_i f^\beta_j U^{\beta\alpha}$, which proves Item 1.

Now, using (5) and Item 1, we obtain

$$\begin{aligned} g(\nabla f^\alpha, \nabla f^\beta) &= g^{ij} f_i^\alpha f_j^\beta = (\delta_{ij} - U^{\gamma\mu} f_i^\gamma f_j^\mu) f_i^\alpha f_j^\beta \\ &= \langle Df^\alpha, Df^\beta \rangle - U^{\gamma\mu} \langle Df^\gamma, Df^\alpha \rangle \langle Df^\mu, Df^\beta \rangle \\ &= \langle Df^\alpha, Df^\gamma \rangle (\delta_{\gamma\beta} - U^{\gamma\mu}(U_{\mu\beta} - \delta_{\mu\beta})) \\ &= \langle Df^\alpha, Df^\gamma \rangle U^{\gamma\beta} = (U_{\alpha\gamma} - \delta_{\alpha\gamma}) U^{\gamma\beta} \\ &= \delta_{\alpha\beta} - U^{\alpha\beta}, \end{aligned}$$

which proves Items 2 and 3. Lemma 2.3 is proved. □

The result below will be useful to prove Proposition 2.2. For our purposes, it is convenient to write $M_{ij} = \delta_{ij} - g^{ij}$. By (12), we can write

$$S = (\delta_{ij} - M_{ij})(\delta_{kl} - M_{kl}) R_{ilkj} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \tag{15}$$

where

1) $\mathrm{I} = \delta_{ij}\delta_{kl} U^{\alpha\beta}(f_{ij}^\beta f_{kl}^\alpha - f_{ik}^\beta f_{jl}^\alpha)$;
2) $\mathrm{II} = \delta_{ij} M_{kl} U^{\alpha\beta}(f_{ij}^\beta f_{kl}^\alpha - f_{ik}^\beta f_{jl}^\alpha)$;
3) $\mathrm{III} = \delta_{kl} M_{ij} U^{\alpha\beta}(f_{ij}^\beta f_{kl}^\alpha - f_{ik}^\beta f_{jl}^\alpha)$;
4) $\mathrm{IV} = M_{ji} M_{kl} U^{\alpha\beta}(f_{ij}^\beta f_{kl}^\alpha - f_{ik}^\beta f_{lj}^\alpha)$.

Lemma 2.4. *The scalar curvature* $S = \mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV}$, *as written in (15), satisfies*

1) $\mathrm{I} = U^{\alpha\beta}(f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)_i$;
2) $\mathrm{II} = \mathrm{III} = -\frac{1}{2}\left(U_i^{\alpha\beta}(f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha) + U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle F_{ik,k}^{\beta\alpha}\right)$
3) $\mathrm{IV} = U^{\alpha\nu}U^{\beta\mu}U^{\theta\gamma}\langle Df^\mu, Df_i^\gamma\rangle\langle Df^\nu, Df_k^\theta\rangle F_{ik}^{\beta\alpha}$;

where $F_{ik}^{\beta\alpha} = f_i^\beta f_k^\alpha - f_k^\beta f_i^\alpha$ *and* $F_{ik,l}^{\beta\alpha}$ *denotes the partial derivative* $\frac{\partial}{\partial x_l} F_{ik}^{\beta\alpha}$.

Proof. Item 1 follows from the fact that $\mathrm{I} = f_{ii}^\beta f_{kk}^\alpha - f_{ik}^\beta f_{ik}^\alpha = (f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)_i$.

Since $U^{\alpha\beta} = U^{\beta\alpha}$ we have

$$\begin{aligned} \delta_{kl} M_{ij} U^{\alpha\beta}(f_{ij}^\beta f_{kl}^\alpha - f_{ik}^\beta f_{jl}^\alpha) &= \delta_{ij} M_{kl} U^{\alpha\beta}(f_{kl}^\beta f_{ij}^\alpha - f_{ki}^\beta f_{lj}^\alpha) \\ &= \delta_{ij} M_{kl}(U^{\beta\alpha} f_{kl}^\alpha f_{ij}^\beta - U^{\alpha\beta} f_{jl}^\alpha f_{ik}^\beta) \\ &= \delta_{ij} M_{kl} U^{\alpha\beta}(f_{ij}^\beta f_{kl}^\alpha - f_{ik}^\beta f_{jl}^\alpha), \end{aligned}$$

which proves that II = III. By Item 1 of Lemma 2.3, $M_{kl} = f_k^\gamma f_l^\mu U^{\gamma\mu}$. This implies

$$\begin{aligned}\delta_{ij} M_{kl} U^{\alpha\beta}(f_{ij}^\beta f_{kl}^\alpha - f_{ik}^\beta f_{jl}^\alpha) &= f_k^\gamma f_l^\mu U^{\gamma\mu} U^{\alpha\beta}(f_{ii}^\beta f_{kl}^\alpha - f_{ik}^\beta f_{il}^\alpha)\\ &= U^{\alpha\beta}U^{\gamma\mu}(f_{ii}^\beta f_k^\gamma f_{kl}^\alpha f_l^\mu - f_{ki}^\beta f_i^\gamma f_{kl}^\alpha f_l^\mu)\\ &= U^{\gamma\beta}U^{\alpha\mu} f_{kl}^\gamma f_l^\mu (f_{ii}^\beta f_k^\alpha - f_{ik}^\beta f_i^\alpha)\\ &= U^{\beta\gamma}U^{\alpha\mu} f_{ij}^\gamma f_j^\mu (f_{kk}^\beta f_i^\alpha - f_{ki}^\beta f_k^\alpha)\\ &= U^{\alpha\gamma}U^{\beta\mu}\langle Df_i^\gamma, Df^\mu\rangle (f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha).\end{aligned} \tag{16}$$

Since $U^{\alpha\gamma}U_{\gamma\mu} = \delta_{\alpha\mu}$ it follows that $U_i^{\alpha\gamma} = -U^{\alpha\gamma}U^{\beta\mu}U_{\gamma\mu,i}$, hence

$$U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle = -U_i^{\alpha\beta} - U^{\alpha\gamma}U^{\beta\mu}\langle Df_i^\gamma, Df^\mu\rangle. \tag{17}$$

It is easy to see that $F_{ik,k}^{\beta\alpha} = (f_i^\beta f_k^\alpha - f_k^\beta f_i^\alpha)_k = (f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha) - (f_i^\alpha f_{kk}^\beta - f_k^\alpha f_{ik}^\beta)$. Thus we obtain

$$\begin{aligned}&U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle (f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)\\ &\quad= U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle F_{ik,k}^{\beta\alpha}\\ &\qquad+ U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle (f_i^\alpha f_{kk}^\beta - f_k^\alpha f_{ik}^\beta)\\ &\quad= U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle F_{ik,k}^{\beta\alpha}\\ &\qquad+ U^{\beta\mu}U^{\alpha\gamma}\langle Df^\mu, Df_i^\gamma\rangle (f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha).\end{aligned} \tag{18}$$

Using (17) and (18) we obtain

$$\begin{aligned}&U^{\alpha\gamma}U^{\beta\mu}\langle Df_i^\gamma, Df^\mu\rangle (f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)\\ &\quad= -U_i^{\alpha\beta}(f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)\\ &\qquad- U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle (f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)\\ &\quad= -U_i^{\alpha\beta}(f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)\\ &\qquad- U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle F_{ik,k}^{\beta\alpha}\\ &\qquad- U^{\beta\mu}U^{\alpha\gamma}\langle Df^\mu, Df_i^\gamma\rangle (f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha).\end{aligned} \tag{19}$$

Using (16) and (19) we obtain

$$\begin{aligned}2\delta_{ij} M_{kl} U^{\alpha\beta}(f_{ij}^\beta f_{kl}^\alpha - f_{ik}^\beta f_{jl}^\alpha) &= -U_i^{\alpha\beta}(f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)\\ &\qquad - U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle F_{ik,k}^{\beta\alpha},\end{aligned}$$

which concludes Item 2.

Using Item 1 of Lemma 2.3, we have

$$\begin{aligned}
M_{ij}M_{kl}U^{\alpha\beta}(f^\beta_{ij}f^\alpha_{kl} - f^\beta_{ik}f^\alpha_{lj}) &= f^\gamma_i f^\mu_j U^{\gamma\mu} f^\theta_k f^\nu_l U^{\theta\nu} U^{\alpha\beta}(f^\beta_{ij}f^\alpha_{kl} - f^\beta_{ik}f^\alpha_{lj}) \\
&= U^{\gamma\mu}U^{\theta\nu}U^{\alpha\beta}\langle Df^\mu, Df^\beta_i\rangle\langle Df^\nu, Df^\alpha_k\rangle f^\gamma_i f^\theta_k \\
&\quad - U^{\gamma\mu}U^{\theta\nu}U^{\alpha\beta}\langle Df^\mu, Df^\alpha_l\rangle\langle Df^\theta, Df^\beta_i\rangle f^\gamma_i f^\nu_l \\
&= U^{\gamma\mu}U^{\theta\nu}U^{\alpha\beta}\langle Df^\mu, Df^\beta_i\rangle\langle Df^\nu, Df^\alpha_k\rangle f^\gamma_i f^\theta_k \\
&\quad - U^{\gamma\mu}U^{\nu\theta}U^{\beta\alpha}\langle Df^\mu, Df^\beta_k\rangle\langle Df^\nu, Df^\alpha_i\rangle f^\gamma_i f^\theta_k \\
&= U^{\gamma\mu}U^{\theta\nu}U^{\alpha\beta}\langle Df^\mu, Df^\beta_i\rangle\langle Df^\nu, Df^\alpha_k\rangle F^{\gamma\theta}_{ik} \\
&= U^{\beta\mu}U^{\alpha\nu}U^{\theta\gamma}\langle Df^\mu, Df^\gamma_i\rangle\langle Df^\nu, Df^\theta_k\rangle F^{\beta\alpha}_{ik}.
\end{aligned}$$

We conclude Item 3. Lemma 2.4 is proved. □

Finally, we will prove Proposition 2.2. Using (15) and Lemma 2.4, we have

$$\begin{aligned}
S &= U^{\alpha\beta}(f^\beta_i f^\alpha_{kk} - f^\beta_k f^\alpha_{ik})_i \\
&\quad + U^{\alpha\beta}_i(f^\beta_i f^\alpha_{kk} - f^\beta_k f^\alpha_{ik}) + U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df^\mu_i\rangle F^{\beta\alpha}_{ik,k} \\
&\quad + U^{\alpha\nu}U^{\beta\mu}U^{\theta\gamma}\langle Df^\mu, Df^\gamma_i\rangle\langle Df^\nu, Df^\theta_k\rangle F^{\beta\alpha}_{ik} \\
&= (U^{\alpha\beta}(f^\beta_i f^\alpha_{kk} - f^\beta_k f^\alpha_{ik}) + U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df^\mu_k\rangle F^{\beta\alpha}_{ki})_i + V^{\alpha\beta}_{ik}F^{\beta\alpha}_{ik},
\end{aligned} \tag{20}$$

where

$$V^{\alpha\beta}_{ik} = U^{\alpha\nu}U^{\beta\mu}U^{\theta\gamma}\langle Df^\mu, Df^\gamma_i\rangle\langle Df^\nu, Df^\theta_k\rangle - (U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df^\mu_i\rangle)_k. \tag{21}$$

Claim 2.1. $V^{\alpha\beta}_{ik}F^{\beta\alpha}_{ik} = \langle R^\perp(\nabla f^\gamma, \nabla f^\mu)\eta^\mu, \eta^\gamma\rangle$.

In fact, using (11) and Item 1 of Lemma 2.3, it follows:

$$\begin{aligned}
g(A^\mu\partial_i, A^\gamma\partial_k) &= f^\mu_{il}g^{lr}f^\gamma_{kr} = f^\mu_{il}f^\gamma_{kr}(\delta_{lr} - U^{\theta\nu}f^\nu_l f^\theta_r) \\
&= \langle Df^\mu_i, Df^\gamma_k\rangle - U^{\theta\nu}\langle Df^\mu_i, Df^\nu\rangle\langle Df^\gamma_k, Df^\theta\rangle.
\end{aligned} \tag{22}$$

Now, by (21) and using that $U_r^{\alpha\beta} = -U^{\alpha\gamma}U^{\beta\mu}U_{\gamma\mu,r}$, we obtain

$$\begin{aligned} V_{ik}^{\alpha\beta} &= U^{\alpha\nu}U^{\beta\mu}U^{\theta\gamma}(U_{\mu\gamma,i} - \langle Df_i^\mu, Df^\gamma\rangle)\langle Df^\nu, Df_k^\theta\rangle \\ &\quad - U_k^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle - U^{\alpha\gamma}U_k^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle \\ &\quad - U^{\alpha\gamma}U^{\beta\mu}\langle Df_k^\gamma, Df_i^\mu\rangle - U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_{ik}^\mu\rangle \\ &= -U^{\alpha\nu}U_i^{\beta\theta}\langle Df^\nu, Df_k^\theta\rangle - U^{\alpha\nu}U^{\beta\mu}U^{\theta\gamma}\langle Df_i^\mu, Df^\gamma\rangle(U_{\nu\theta,k} \\ &\quad - \langle Df_k^\nu, Df^\theta\rangle) - U_k^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle - U^{\alpha\gamma}U_k^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle \\ &\quad - U^{\alpha\gamma}U^{\beta\mu}\langle Df_k^\gamma, Df_i^\mu\rangle - U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_{ik}^\mu\rangle \\ &= -C_{ik}^{\alpha\beta} + U_k^{\alpha\gamma}U^{\beta\mu}\langle Df_i^\mu, Df^\gamma\rangle + U^{\alpha\nu}U^{\beta\mu}U^{\theta\gamma}\langle Df_i^\mu, Df^\gamma\rangle\langle Df_k^\nu, Df^\theta\rangle \\ &\quad - U_k^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle - U^{\alpha\gamma}U^{\beta\mu}\langle Df_k^\gamma, Df_i^\mu\rangle \\ &= -C_{ik}^{\alpha\beta} + U^{\alpha\nu}U^{\beta\mu}U^{\theta\gamma}\langle Df_i^\mu, Df^\gamma\rangle\langle Df_k^\nu, Df^\theta\rangle - U^{\alpha\gamma}U^{\beta\mu}\langle Df_k^\gamma, Df_i^\mu\rangle, \end{aligned}$$

where $C_{ik}^{\alpha\beta}$ is given by

$$C_{ik}^{\alpha\beta} = U^{\alpha\nu}U_i^{\beta\theta}\langle Df^\nu, Df_k^\theta\rangle + U^{\alpha\gamma}U_k^{\beta\mu}\langle Df^\gamma, Df_i^\mu\rangle + U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_{ik}^\mu\rangle.$$

Note that $C_{ik}^{\alpha\beta} = C_{ki}^{\alpha\beta}$. Since $F_{ik}^{\beta\alpha} = -F_{ki}^{\beta\alpha}$ we obtain that $C_{ik}^{\alpha\beta}F_{ik}^{\beta\alpha} = 0$. Thus, using that $\nabla f^\alpha = U^{\alpha\gamma}f_i^\gamma\partial_i$, it follows from (22) and from the Ricci equation that

$$\begin{aligned} V_{ik}^{\alpha\beta}F_{ik}^{\beta\alpha} &= -(f_i^\beta f_k^\alpha - f_k^\beta f_i^\alpha)U^{\alpha\gamma}U^{\beta\mu}g(A^\mu\partial_i, A^\gamma\partial_k) \\ &= -(g(A^\mu(\nabla f^\mu), A^\gamma(\nabla f^\gamma)) - g(A^\mu(\nabla f^\gamma), A^\gamma(\nabla f^\mu))) \\ &= -\langle R^\perp(\nabla f^\gamma, \nabla f^\mu)\eta^\mu, \eta^\gamma\rangle, \end{aligned} \tag{23}$$

which together with (20) concludes the proof of Proposition 2.2.

3. Proof of Theorem 1.2.

Since $f = (f^1, \dots, f^m) : \mathbb{R}^n \to \mathbb{R}^m$ is an asymptotically flat map, we have that $f_i^\alpha = O(|x|^{-p/2})$ and $f_{ik}^\alpha = O(|x|^{-p/2-1})$, for all $i, k = 1, \dots, n$ and $\alpha = 1, \dots, m$. In particular, $U^{\alpha\gamma}$ tends to $\delta_{\alpha\gamma}$ when $|x| \to \infty$. Moreover, using Items 2 and 3 of Lemma 2.3, we have $U^{\alpha\beta} - \delta_{\alpha\beta} = -g(\nabla f^\alpha, f^\beta) = U^{\alpha\gamma}\langle Df^\gamma, Df^\beta\rangle = O(|x|^{-p})$. This implies

$$(U^{\alpha\beta} - \delta_{\alpha\beta})(f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha) = O(|x|^{-2p-1}) \tag{24}$$

and

$$U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_k^\mu\rangle f_i^\alpha f_k^\beta = O(|x|^{-2p-1}). \tag{25}$$

We have $2p+1 > n-1 = \dim S_r$, since $p > (n-2)/2$. Thus, by (24) and (25), we obtain

$$\lim_{r\to\infty}\int_{S_r} U^{\alpha\beta}(f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)\frac{x^i}{|x|} = \lim_{r\to\infty}\int_{S_r}(f_i^\alpha f_{kk}^\alpha - f_k^\alpha f_{ik}^\alpha)\frac{x^i}{|x|} \tag{26}$$

and

$$\lim_{r\to\infty}\int_{S_r} U^{\alpha\gamma}U^{\beta\mu}\langle Df^\gamma, Df_k^\mu\rangle(f_i^\alpha f_k^\beta - f_k^\alpha f_i^\beta) = 0. \tag{27}$$

Furthermore the function $S^\perp = \langle R^\perp(\nabla f^\alpha, \nabla f^\beta)\eta^\beta, \eta^\alpha\rangle \in O(|x|^{-2p-2})$ since, by (22) and (23), it can be expressed as

$$S^\perp = U^{\alpha\gamma}U^{\beta\mu}(\langle Df_k^\gamma, Df_i^\mu\rangle + U^{\theta\nu}\langle Df_i^\mu, Df^\nu\rangle\langle Df_k^\gamma, Df^\theta\rangle)(f_i^\alpha f_k^\beta - f_i^\beta f_k^\alpha). \tag{28}$$

Since $2p+2 > n$ it follows that $S^\perp : \mathbb{R}^n \to \mathbb{R}$ is integrable.

By hypothesis, $S : \mathbb{R}^n \to \mathbb{R}$ is integrable. Using that $g_{kik} - g_{kki} = f_i^\alpha f_{kk}^\alpha - f_k^\alpha f_{ik}^\alpha$, from Proposition 2.2 together with (26) and (27) and the divergence theorem, we obtain

$$\begin{aligned}\int_M (S+S^\perp)\frac{1}{\sqrt{G}}dM &= \int_{R^n} S+S^\perp \\ &= \lim_{r\to\infty}\int_{S_r}\left\langle X, \frac{x}{|x|}\right\rangle = \lim_{r\to\infty}\int_{S_r}(f_i^\alpha f_{kk}^\alpha - f_k^\alpha f_{ik}^\alpha)\frac{x^i}{|x|} \\ &= 2(n-1)\omega_{n-1}m_{\text{ADM}},\end{aligned}$$

where $G = \det(g_{ij})$. Theorem 1.2 is proved.

4. Proof of Theorem 1.4

Let ν be the unit vector field orthogonal to $\partial\Omega$ pointing outward to Ω and let $H^\Sigma = -\mathrm{div}_{\mathbb{R}^n}\nu$ be the mean curvature of $\Sigma = \partial\Omega$ seen as a hypersurface in $\mathbb{R}^n$.

Since each connected component of Σ is a level set of f^α, for all α, it follows that the gradient vector field Df^α is normal to Σ, hence

$$Df^\alpha = \langle Df^\alpha, \nu\rangle\,\nu \quad \text{in } \Sigma. \tag{29}$$

In particular, Df^α and Df^β are linearly dependent which implies that

$$f_i^\alpha f_k^\beta - f_k^\alpha f_i^\beta = \langle (Df^\beta \wedge Df^\alpha)e_i, e_k\rangle = 0 \quad \text{in } \Sigma, \tag{30}$$

for all $\alpha, \beta = 1, \ldots, n$. Here, "$\wedge$" : $\mathbb{R}^n \times \mathbb{R}^n \to (\mathbb{R}^n)^*$ is the skew-symmetric tensor given by $(u \wedge v)w = \langle v, w\rangle u - \langle u, w\rangle v$, for all $u, v, w \in \mathbb{R}^n$.

Using (13), (29) and (30) we obtain

$$\begin{aligned}\langle X, \nu\rangle &= U^{\alpha\beta}(f_i^\beta f_{kk}^\alpha - f_k^\beta f_{ik}^\alpha)\nu^i \\ &= U^{\alpha\beta}(\Delta f^\alpha \langle Df^\beta, \nu\rangle - \mathrm{Hes}_{f^\alpha}(Df^\beta, \nu)).\end{aligned} \tag{31}$$

By a simple computation, we have that

$$\Delta f^\alpha = \Delta_\Sigma f^\alpha + \mathrm{Hes}_{f^\alpha}(\nu, \nu) - H^\Sigma \langle \nu, Df^\alpha\rangle \tag{32}$$

Using that f^α is constant along Σ it follows that $\Delta_\Sigma f^\alpha = 0$ and $Df^\beta = \langle Df^\beta, \nu\rangle \nu$ in Σ. Thus, by (31) and (32), we obtain

(33)
$$\begin{aligned}\langle X, \nu\rangle &= U^{\alpha\beta}(\mathrm{Hes}_{f^\alpha}(\nu, \nu)\langle Df^\beta, \nu\rangle - \mathrm{Hes}_{f^\alpha}(Df^\beta, \nu) - H^\Sigma\langle \nu, Df^\alpha\rangle\langle Df^\beta, \nu\rangle) \\ &= -U^{\alpha\beta} H^\Sigma \langle \nu, Df^\alpha\rangle \langle Df^\beta, \nu\rangle \\ &= -U^{\alpha\beta} H^\Sigma \langle Df^\alpha, Df^\beta\rangle\end{aligned}$$

Using (29), we have $U_{\alpha\beta} = \delta_{\alpha\beta} + \lambda^\alpha\lambda^\beta$, in Σ, for all α, β, where $\lambda^\alpha = \langle Df^\alpha, \nu\rangle$. This implies that $U^{\alpha\beta} = \delta_{\alpha\beta} - \lambda^\alpha\lambda^\beta/(1 + |\lambda|^2)$, in Σ, where $|\lambda|^2 = |Df|^2 = (\lambda^1)^2 + \cdots + (\lambda^m)^2$. Thus, in Σ, it holds

(34)
$$U^{\alpha\gamma}\langle Df^\gamma, Df^\alpha\rangle = \quad U^{\alpha\gamma}\lambda^\alpha\lambda^\gamma = \left(\delta_{\alpha\beta} - \frac{\lambda^\alpha\lambda^\beta}{1 + |\lambda|^2}\right)\lambda^\alpha\lambda^\beta = \frac{|Df|^2}{1 + |Df|^2}.$$

As in the proof of Theorem 1.2, using that f is an asymptotically flat map we have that $\lim_{r\to\infty} \int_{S_r} \langle X, \frac{x}{|x|}\rangle = 2(n-1)\omega_{n-1} m_{\mathrm{ADM}}$. By Proposition 2.2 and the divergence theorem, we obtain from (33) and (34) that

$$\begin{aligned}\int_{\mathbb{R}^n - \Omega} S + S^\perp &= \lim_{r\to\infty} \int_{S_r} \left\langle X, \frac{x}{|x|}\right\rangle + \int_\Sigma \langle X, \nu\rangle \\ &= 2(n-1)\omega_{n-1} m_{\mathrm{ADM}} - \int_\Sigma \frac{|Df|^2}{1 + |Df|^2} H^\Sigma.\end{aligned} \tag{35}$$

Theorem 1.4 is proved.

Remark 4.1. We would like to state here the equality (7) referred in the Introduction. In general, let M_1 and M_2 be n-dimensional submanifolds of $\mathbb{R}^{n+m}$ with smooth boundaries $\Sigma = \partial M_1 = \partial M_2$. Let ν^1 and ν^2 be the unit conormal vectors of Σ with respect to M_1 and M_2, pointing outward, respectively. Let us denote by B^Σ, B_1, B_2, $A_1^\Sigma \nu_1$ and $A_2^\Sigma \nu_2$, the second fundamental forms of Σ into $\mathbb{R}^{n+m}$, M_1 into $\mathbb{R}^{n+m}$, M_2 into $\mathbb{R}^{n+m}$, Σ into M_1 and Σ into M_2, respectively. It is clear that

$$B_1|_{T\Sigma} + A_1^\Sigma \nu_1 = B^\Sigma = B_2|_{T\Sigma} + A_2^\Sigma \nu_2. \tag{36}$$

Let H_1, H_2, $H_1^\Sigma \nu_1$ and $H_2^\Sigma \nu_2$ be the mean curvature vectors of M_1 into $\mathbb{R}^{n+m}$, M_2 into $\mathbb{R}^{n+m}$, Σ into M_1 and Σ into M_2, respectively. Taking the traces in (36), we obtain

$$H_1 - B_1(\nu_1, \nu_1) + H_1^\Sigma \nu_1 = H_2 - B_2(\nu_2, \nu_2) + H_2^\Sigma \nu_2. \tag{37}$$

Now, backing down to our setting, consider $M_1 = M$ the graph of a smooth map $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$ that is constant along each connected component of $\partial\Omega$. As in the Introduction, we denote by $\mathcal{H}^\Sigma = H_1^\Sigma$. The boundary $\Sigma = \partial M$ is a hypersurface of a totally geodesic submanifold M_2 in $\mathbb{R}^{n+m}$, given by disjoint union of parts of n-dimensional planes of the form $\mathbb{R}^n \times \{v\}$, for some $v \in \mathbb{R}^m$. We also denote by $H^\Sigma = H_2^\Sigma$. By (37) and using that $\langle B_1(\cdot,\cdot), \nu_1\rangle = 0$, we obtain

$$\mathcal{H}^\Sigma = H^\Sigma \langle \nu_1, \nu_2\rangle. \tag{38}$$

Fix a point $p \in \Sigma$ and identify $\mathbb{R}^n = T_pM_2 = \mathbb{R}^n \times \{0\}$. By a change of variables in $\mathbb{R}^n$, assume that $\nu_2(p) = e_1 = (e_1, 0)$. We obtain $f_i^\alpha(p) = \lambda^\alpha \langle e_1, e_i\rangle = \lambda^\alpha \delta_{1i}$, where $\lambda^\alpha = \langle Df^\alpha(p), \nu_2(p)\rangle$. Hence, $g_{ij}(p) = \delta_{ij} + |Df|^2$ $\delta_{i1}\delta_{j1}$. In particular, $g^{11}(p) = \frac{1}{1+|Df|^2}$ and $g^{ij}(p) = \delta_{ij}$, for all $i, j = 2, \ldots, n$. Note also that, at p, it holds $\partial_1 = (e_1, f_1^\alpha e_\alpha)$, and $\partial_i = (e_i, 0)$, for all $i = 2, \ldots, n$. Hence, ∂_1 is a multiple of ν_1. Since ∂_1 points inward M_1 it follows that $\nu_1 = -(g_{11})^{-\frac{1}{2}} \partial_1 = -(1 + |Df|^2)^{-\frac{1}{2}} (\nu_2 + (0, \lambda^\alpha e_\alpha))$, hence $\langle \nu_2(p), \nu_1(p)\rangle = -\frac{1}{\sqrt{1+|Df|^2}}$. This, together with (38), imply that

$$\mathcal{H}^\Sigma = -\frac{1}{\sqrt{1 + |Df|^2}} H^\Sigma. \tag{39}$$

5. Proof of Theorem 1.5

Before we prove Theorem 1.5, we will need of the following result.

Lemma 5.1. *Under hypothesis of Theorem 1.5, we have* $\lim_{x\to\Sigma} |Df|^2 = +\infty$.

In fact, fix $x_0 \in \Sigma$ and let Σ' be the connected component of Σ that contains x_0. At the point $(x_0, f(x_0)) \in \partial M$, M is tangent to the cylinder $\Sigma' \times \ell$, where ℓ is a straight line of $\mathbb{R}^m$. Let $A = A_{\Sigma'} : \mathbb{R}^m \to \mathbb{R}^m$ be an isometry that transforms ℓ into the vertical line $A(\ell) = \{(z, 0, \dots, 0) \mid z \in \mathbb{R}\}$. Consider the map $\bar{f} = A \circ f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$ and $M_A = \mathrm{Gr}(\bar{f})$ with its natural metric. Since $\bar{f}(x) = \bar{f}^\alpha(x) e_\alpha = f^\alpha(x)\bar{e}_\alpha$, where $\bar{e}_\alpha = Ae_\alpha = A_\alpha^\beta e_\beta$, we have that M_A is isometric to M. Hence, they have the same ADM-mass and scalar curvatures. Moreover, $|Df|^2 = |D\bar{f}|^2$, and using (28), $S_A^\perp = S^\perp$, everywhere in $\mathbb{R}^n \setminus \Omega$. So, without loss of generality, we can assume that, at the point $(x_0, f(x_0))$, the boundary ∂M is tangent to the cylinder $\Sigma' \times \{(z, 0, \dots, 0) \mid z \in \mathbb{R}\}$.

Claim 5.1. $\lim_{x\to x_0} \nabla f^\alpha(x) = \pm\delta_{\alpha 1}(0, e_1)$. In particular, $\lim_{x\to x_0} U^{\alpha\beta} = \delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta}$, for all α, β.

In fact, first we assume, by contradiction, that $\lim_{x\to x_0} \nabla f^\alpha = 0$, for all α. By Item 2 of Lemma 2.3, $U^{\alpha\beta} = \delta_{\alpha\beta} - g(\nabla f^\alpha, \nabla f^\beta)$. Thus,

$$\lim_{x\to x_0} U^{\alpha\beta} = \delta_{\alpha\beta}, \tag{40}$$

for all α, β. Using that $\partial_i = (e_i, f_i^\beta e_\beta)$, $\nabla f^\gamma = U^{\gamma\alpha} f_i^\alpha \partial_i$ and $g(\nabla f^\gamma, \nabla f^\beta) = U^{\gamma\alpha}\langle Df^\alpha, Df^\beta\rangle$ we have

$$\begin{aligned} \nabla f^\gamma &= U^{\gamma\alpha} f_i^\alpha \partial_i = U^{\gamma\alpha}(Df^\alpha, \langle Df^\alpha, Df^\beta\rangle e_\beta) \\ &= (U^{\gamma\alpha} Df^\alpha,\, g(\nabla f^\gamma, \nabla f^\beta) e_\beta). \end{aligned} \tag{41}$$

Consider $\pi^1, \pi^2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ the orthogonal projections $\pi^1(x, y) = x$ and $\pi^2(x, y) = y$. By (40) and (41) we have

$$0 = \lim_{x\to x_0} \pi_*^1(\nabla f^\alpha) = \lim_{x\to x_0} U^{\alpha\beta} Df^\beta = \lim_{x\to x_0} Df^\alpha,$$

for all α. Thus, M is tangent to the plane $\mathbb{R}^n \times \{0\}$ at the point $(x_0, f(x_0))$, which is a contradiction. Thus, we take $1 \le \gamma \le m$ so that $\limsup_{x\to x_0} |\nabla f^\gamma| > 0$.

Using that M is tangent to the cylinder $\Sigma' \times \{(z, 0, \dots, 0) \mid z \in \mathbb{R}\})$, at $(x_0, f(x_0))$, the vector $\eta = (0, e_1)$ is tangent to M and normal to ∂M at $(x_0, f(x_0))$. Since f^α is constant along each connected component of Σ

and M extends C^2 up to its boundary, it follows that, for all α, either $\lim_{x\to x_0} \nabla f^\alpha = 0$ or $\lim_{x\to x_0} \nabla f^\alpha/|\nabla f^\alpha| = \pm\eta$. In particular, $\lim_{x\to x_0} \nabla f^\gamma/|\nabla f^\gamma| = \pm\eta$. Thus, by (41),

$$\pm e_1 = \pm\,\pi^2(\eta) = \lim_{x\to x_0} \pi^2(\nabla f^\gamma/|\nabla f^\gamma|) = \lim_{x\to x_0} g(\nabla f^\gamma/|\nabla f^\gamma|, \nabla f^\beta)e_\beta. \tag{42}$$

This implies that

$$\begin{aligned} \lim\nolimits_{x\to x_0} g(\nabla f^\gamma/|\nabla f^\gamma|, \nabla f^\beta) = 0,\ \text{for all } \beta \neq 1; \\ \lim\nolimits_{x\to x_0} g(\nabla f^\gamma/|\nabla f^\gamma|, \nabla f^1) = \pm 1. \end{aligned} \tag{43}$$

If we assume that $\limsup_{x\to x_0} |\nabla f^\beta| > 0$, for some $\beta \neq 1$ then, by (43), we obtain

$$\begin{aligned} 0 &= \lim_{x\to x_0} g(\nabla f^\gamma/|\nabla f^\gamma|, \nabla f^\beta) = \limsup_{x\to x_0} g\big(\nabla f^\gamma/|\nabla f^\gamma|, \nabla f^\beta/|\nabla f^\beta|\big)|\nabla f^\beta| \\ &= \pm g(\eta,\eta) \limsup_{x\to x_0} |\nabla f^\beta| = \pm \limsup_{x\to x_0} |\nabla f^\beta|, \end{aligned}$$

which is a contradiction. Thus, it holds

$$\lim_{x\to x_0} \nabla f^\beta = 0,\ \text{for all } \beta \neq 1, \tag{44}$$

We conclude that $\gamma = 1$, which implies that $\pm\eta = \lim_{x\to x_0} \nabla f^1/|\nabla f^1|$. Moreover, again using (43), we obtain that $\lim_{x\to\Sigma} |\nabla f^1| = \lim_{x\to\Sigma} g(\nabla f^1/|\nabla f^1|, \nabla f^1) = 1$. This implies that $\lim_{x\to x_0} \nabla f^1 = \pm(0, e_1)$. Hence, $\lim_{x\to x_0} \nabla f^\gamma(x) = \pm\,\delta_{1\gamma}(0, e_1)$. In particular, since $U^{\alpha\beta} = \delta_{\alpha\beta} - g(\nabla f^\alpha, \nabla f^\beta)$, we obtain $\lim_{x\to x_0} U^{\alpha\beta} = \delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta}$, for all α, β. Claim 5.1 is proved. The claim below concludes the proof of Lemma 5.1.

Claim 5.2. $\lim_{x\to x_0} Df^\alpha = 0$, for all $\alpha \neq 1$, and $\lim_{x\to x_0} |Df^1| = +\infty$. In particular, $\lim_{x\to x_0} |Df|^2 = +\infty$.

In fact, by (41) and Claim 5.1, $\lim_{x\to x_0} U^{\gamma\alpha} Df^\alpha = 0$, for all γ. Thus, using that $1 = \lim_{x\to x_0} g(\nabla f^1, \nabla f^1) = \lim_{x\to x_0} U^{1\alpha}\langle Df^\alpha, Df^1\rangle$, we have $\lim_{x\to x_0} |Df^1| = +\infty$. Claim 5.2 is proved.

Now, we will finish the proof of Theorem 1.5. Let $F^k = (f^{1;k}, f^{2;k}, \dots, f^{m;k}) : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$, with $k = 1, 2, \dots$, be a sequence of smooth maps satisfying:

(i) F^k coincides with f outside a compact subset containing Σ;

(ii) $F^k = f$ everywhere in Σ;

(iii) if M_k is the graph of f_k with its natural metric then the closure $\bar{M}_k$ converges to $\bar{M}$ with respect to the C^2-topology.

Note that Theorem 1.4 applies for $F^k : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$. By using 5, the ADM-mass of M_k coincides with the ADM-mass of M. Using 5 and (12), for all $x \in \mathbb{R}^n \setminus \Omega$, the scalar curvature $S_k : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ of the graph $\bar{M}_k$ satisfies $\lim_{k\to\infty} S_k = S$ uniformly. Using (28), $S_k^\perp(x)$ converges to $S^\perp(x)$, for all $x \in \mathbb{R}^n \setminus \bar{\Omega}$. Since $F^k|_\Sigma = f|_\Sigma$ is constant along each connected component of Σ, $S_k^\perp = 0$ along Σ, hence $S_k^\perp$ is bounded in a neighborhood of Σ. Furthermore, by Item 5, $\lim_{k\to\infty} |DF^k|^2 = +\infty$, everywhere in Σ. Thus, by the Dominated Convergence Theorem and Theorem 1.4, we have

$$\begin{aligned} m_{\mathrm{ADM}} &= \frac{1}{2(n-1)\omega_{n-1}} \lim_{k\to\infty} \left(\int_{\mathbb{R}^n\setminus\Omega} (S_k + S_k^\perp) + \int_\Sigma \frac{|DF^k|^2}{1+|DF^k|^2} H^\Sigma \right) \\ &= \frac{1}{2(n-1)\omega_{n-1}} \left(\int_{\mathbb{R}^n\setminus\Omega} (S + S^\perp) + \int_\Sigma H^\Sigma \right) \end{aligned}$$

Theorem 1.5 is proved.

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