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On geometrically constrained variational problems of the Willmore functional I. The Lagrangian–Willmore problem

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In this paper, we study a kind of geometrically constrained variational problem of the Willmore functional. A surface $l : \Sigma \to \mathbb{C}^2$ is called a Lagrangian–Willmore surface (in short, a LW surface) or a Hamiltonian–Willmore surface (in short, a HW surface) if it is a critical point of the Willmore functional under Lagrangian deformations or Hamiltonian deformations, respectively. We extend the L^∞ estimates of the second fundamental form of Willmore surfaces to both HW and LW surfaces and thus get a gap theorem for both HW and LW surfaces. To investigate the existence of HW surfaces we introduce a sixth-order flow which is called by us the Hamiltonian–Willmore flow (in short, the HW flow) decreasing the Willmore energy and we prove that this flow is well posed.

1. Introduction

Let Σ be a surface (possibly with boundary) and f an immersion from Σ to $\mathbb{R}^n$. The Willmore functional is defined as

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |H|^2 d\mu_f,$$

where H is the mean curvature vector of f defined by

$$H = \operatorname{tr} A,$$

and $d\mu_f$ is the area element on Σ induced by f.

For a smooth compactly supported variation $f : \Sigma \times I \to \mathbb{R}^n$ and $V = \partial_t f$ we have [39]

$$\frac{d}{dt}\mathcal{W}(f) = \frac{1}{2}\int_\Sigma \langle W(f), V\rangle d\mu_f, \tag{1.1}$$

where

$$W(f) = \Delta^\perp H + g^{ik}g^{jl}A^\circ_{ij}\langle A^\circ_{kl}, H\rangle,$$

$\Delta^\perp$ is the Laplace–Beltrami operator along the normal vector bundle of f, i.e., $\Delta^\perp = \nabla^{\perp *}\nabla^\perp$, and A° is the trace-free part of the second fundamental form A. Here, $\nabla^\perp$ is the normal connection.

A smooth immersion $f : \Sigma \to \mathbb{R}^n$ is called a *Willmore immersion*, if it is a critical point of W under compactly supported variations, i.e., f is a Willmore immersion if and only if

$$\Delta^\perp H + Q(A^\circ)H = 0, \tag{1.2}$$

where $Q(A^\circ)H = g^{ik}g^{jl}A^\circ_{ij}\langle A^\circ_{kl}, H\rangle$.

An intensively studied problem is the estimate of the Willmore energy for closed immersed surfaces, inspired by Willmore. It is proved by Willmore [42] that for any closed oriented surface $f : \Sigma \to \mathbb{R}^3$ we have $\mathcal{W}(f) \geq 4\pi$, with equality only for the round spheres (and this is also true for closed surfaces in $\mathbb{R}^n$ by Simon's [35] monotonicity formula). Furthermore Willmore conjectured that the Willmore energy of a torus in $\mathbb{R}^3$ is at least $2\pi^2$ and the Clifford torus, which is the torus of revolution whose generating circle centers at $\sqrt{2}$ from the axis and has radius 1 defined by

$$(u, v) \to ((\sqrt{2} + \cos u)\cos v, (\sqrt{2} + \cos u)\sin v, \sin u),$$

is the unique minimizer up to conformal transformations of the ambient space [40, 42]. Later it was conjectured that this statement is also true for immersed tori in $\mathbb{R}^n$ and this is also called the Willmore conjecture. There are a lot of partial answers to this conjecture. See, for example, [1, 6, 17, 18, 21, 28, 30–32, 37, 38, 41]. This long standing conjecture was proved by Marques and Neves in a recent paper [24], when the ambient manifold is $\mathbb{R}^3$.

In another direction, the variational aspect of the Willmore functional was also intensively studied. The existence of a torus in $\mathbb{R}^n$, which minimizes the Willmore functional, was established by Simon [35] and he established a condition for the existence of higher genus Willmore surfaces with minimal Willmore energy, which was later verified by Bauer and Kuwert [4].

Minicozzi initiated the study of the geometrically constrained variational problem of the Willmore functional. He proved the existence of smooth minimizers of the Willmore functional among closed Lagrangian tori in $\mathbb{C}^2$ [25] by using techniques developed in Simon's paper [35]. His proof used a direct variational method: for a given minimizing sequence of Lagrangian tori one can extract a subsequence which converges in the sense of measure to a limit and then proved regularity. In his paper, Minicozzi proposed a weaker conjecture that the Willmore energy of a Lagrangian torus in $\mathbb{C}^2$ is at least $2\pi^2$ and the Clifford torus, which is a Lagrangian torus in $\mathbb{R}^4$, minimizes the Willmore energy among Lagrangian tori. Furthermore, he conjectured the Clifford torus minimizes the Willmore energy in its Hamiltonian isotopy class. The last conjecture is the weakest one but it has a very interesting relation with Oh's conjecture (see [23, 25], Theorem 1). For recent development of this conjecture see the work of Ilmanen [11] and Anciaux [2].

Inspired by the work of Minicozzi, we want to study critical points of the Willmore functional under Lagrangian or Hamiltonian deformations. A critical point $f : \Sigma \to \mathbb{C}^2$ of the Willmore functional under Lagrangian, Hamiltonian deformations, resp. is a Lagrangian immersion which satisfies the following Euler–Lagrange equation:

$$\Pi W(f)^\sharp = 0, \tag{1.3}$$

$$\operatorname{div} JW(f) = 0, \tag{1.4}$$

respectively, where Π is the L^2 projection from the space of 1-forms on $f(\Sigma)$ to the space of closed 1-forms with compact support on Σ, J is the standard complex structure of $\mathbb{C}^2$ and for a vector field V on $f(\Sigma)$, $V^\sharp$ is defined by

$$V^\sharp := V \lrcorner \omega,$$

where ω is the standard symplectic form in $\mathbb{C}^2$.

A Lagrangian immersion $f : \Sigma \to \mathbb{C}^2$ satisfying (1.3) ((1.4), resp.) is called a *Lagrangian–Willmore surface* (in short a LW surface), a *Hamiltonian–Willmore surface* (in short a HW surface), respectively. A Lagrangian surface, which is a Willmore surface, i.e., it satisfies (1.2), is called a Willmore–Lagrangian surface. By definition, a Willmore–Lagrangian surface is a LW surface.

In [15], Kuwert and Schätzle proved the following theorem for surfaces.

Theorem A. *There exists a constant* $\epsilon_0 = \epsilon_0(n) > 0$*, such that if* $f : \Sigma \to B_\rho = B_\rho(0)$ *is properly immersed with*

$$\|A\|_{L^2(B_\rho)} < \epsilon_0,$$

where $B_\rho(0) \subseteq \mathbb{R}^n$ *is the ball of radius* ρ *centered at* 0*, then we have the estimate*

$$\|A\|_{L^\infty(B_{\frac{\rho}{2}})} \leq C(\rho\|W\|_{L^2(B_\rho)} + \frac{1}{\rho}\|A\|_{L^2(B_\rho)}),$$

where C *is a constant depends only on the dimension* n*. Here we abbreviate* $\|A\|_{L^2(B_\rho)} := \|A\|_{L^2_g(f^{-1}(B_\rho))}$.

This result provides a decay estimate of the L^∞ norm of A of Willmore surfaces under a smallness condition of its L^2 norm, and this decay estimate, in turn, immediately implies that a properly immersed Willmore surface in $\mathbb{R}^n$ with A small in L^2 must be a plane, i.e, a gap theorem.

In this paper, we first use curvature estimate to obtain a gap theorem for HW surface. In the following, we will consider compact (or non-compact) surfaces without boundary.

Theorem 4.3. *There exists a constant* $\epsilon_0 > 0$*, such that if* $l : \Sigma \to \mathbb{C}^2$ *is a properly immersed HW surface with* $\|A\|_{L^2(\Sigma)} < \epsilon_0$*, then* l *is a Lagrangian plane.*

In our case, we need to estimate $\|W\|_{L^2(B_\rho)}$, which is zero for Willmore surfaces. To do this we choose a proper variational vector field and successfully bound $\|W\|_{L^2(B_\rho)}$ by lower order terms. This estimate implies Theorem 4.3 for HW surfaces. Since a LW surface is automatically a HW surface by definition, both this curvature estimate and the gap theorem hold for LW surfaces too. This variational vector field is chosen with the help of the crucial

Observation. *If* $l : \Sigma \to \mathbb{C}^2$ *is Lagrangian, then* $dW^\sharp(l) = dK \wedge H^\sharp + d(Q(A^\circ)H)^\sharp$ *has at most third-order derivatives.*

For a general surface in $\mathbb{R}^4$, $dW^\sharp$ has fifth-order derivatives.

In [12], Kuwert and Schätzle got an L^∞ estimate for A° of Willmore surfaces under the smallness assumption of the L^2 norm of A° and the finiteness of the L^2 norm of A. An alternative proof of this gap theorem for Willmore spheres was given by Bernard and Rivière [5]. See also Mondino and Nguyen [27]. Our argument cannot yet be directly used to extend these results to either HW surfaces or LW surfaces and we leave it as a question to be considered in the future.

A flow method was introduced to investigate the Willmore functional. In [13], Kuwert and Schätzle considered the negative gradient flow of the Willmore functional and proved its short time existence. Furthermore, they proved that the curvature concentrates if a singularity develops. Later, in [12], a blow-up analysis was carried out around singularities and they proved that a suitable blow-up converges to a compact or non-compact non-umbilic Willmore surface. Together with an L^∞ estimate of the trace-free part of the second fundamental form A° for Willmore surfaces under the smallness assumption of the L^2 norm they obtained the long time existence and convergence of this flow, i.e., there is an $\varepsilon_0(n) > 0$ such that if the L^2 norm of A° is smaller than $\varepsilon_0(n)$, then the Willmore flow exists smoothly for all time and converges to a round sphere. In [14] a singularity removable result in codimension one case was proved and as an application they improved the longtime convergence result for a sphere to $\varepsilon_0 = 8\pi$ when $n = 3$. In [29], Rivière found a divergence form of the Willmore equation and used his formulation to prove a new singularity removable result. By using his result about removable singularity and a theorem of Montiel [26] which states that any non-umbilic Willmore sphere in $\mathbb{R}^4$ has Willmore energy larger than 8π he extended the longtime convergence result to $n = 4$ with $\varepsilon_0 = 8\pi$.

Motivated by the Willmore flow introduced by Kuwert and Schätzle, in this paper, we introduce a flow method to study the existence of HW surfaces and to estimate the Willmore energy of Lagrangian surfaces in $\mathbb{C}^2$. This is a sixth-order flow which decreases the Willmore energy. As a first step in this direction we prove the well posedness of this flow among Lagrangian immersions. Since this flow decreases the Willmore functional, the flow preserves the embeddedness of surfaces, if the initial surface has Willmore energy less than 8π. This follows directly from a classical result of Li and Yau [21].

Theorem 5.3. *Let l_0 be a closed Lagrangian surface in $\mathbb{C}^2$. The HW flow*

$$\begin{cases} \dfrac{\partial}{\partial t} l = -J\nabla \operatorname{div}(JW(l)), \\ l(\cdot, 0) = l_0(\cdot), \end{cases} \tag{1.5}$$

is well posed. Namely, there exists a $T_0 > 0$ and a family of Lagrangian surfaces $l(t)$, $t \in [0, T_0)$ satisfying (1.5).

Equation (1.5) is equivalent to

$$\frac{\partial}{\partial t} l_t \lrcorner \omega = V^\sharp = -dd^* W(l_t)^\sharp.$$

In this paper, d^* is the adjoint operator of the exterior differential d with respect to a Riemannian metric. If one considers the flow (1.5) for a general surface (i.e., a surface may be not Lagrangian), then the flow is not parabolic, for the operator dd^* in flow (1.5) is only the "half" Hodge–Laplacian. However, on the space of Lagrangian surfaces, with the crucial observation mentioned above one can rewrite the operator as

$$-(dd^* + d^*d)W(l)^\sharp + d^*\left(dK \wedge H^\sharp + d(Q(A^\circ)H)^\sharp\right), \tag{1.6}$$

which has a form $\Delta W(l)^\sharp$ with low-order (fourth-order) terms. The well posedness for a flow with velocity given in (1.6) could be proved by using the result of Huisken–Polden [10]. In order to go back to flow (1.5), one needs to show that the flow with velocity (1.6) preserves the Lagrangian condition. This could be a quite difficult problem, since the flow here considered is a higher order flow. The preservation of the Lagrangian condition under the mean curvature flow was proved by Smoczyk [36].

Our idea of proof of Theorem 5.3 is to use the Weinstein tubular neighborhood theorem to reduce the flow to a sixth-order parabolic scalar flow, together with the method developed in the paper of Huisken and Polden [10] (see also [16]). The methods presented in the proof of Theorem 5.3 can be used to provide a (new) proof of the flows considered in [19, 33].

The rest of this paper is organized as follows: in Section 2, we recall some basic notations and results from symplectic geometry, which will be the geometric framework of this paper. In Section 3, we define the LW surfaces and HW surfaces and derive the Euler–Lagrange equations. And then, we prove the crucial observation in this section. In Section 4, we get curvature estimates for such surfaces, which can be considered as generalizations of corresponding results for Willmore surfaces. The gap result, Theorem 4.3, follows from this curvature estimate. In Section 5, we introduce the Hamiltonian–Willmore flow (in short, the HW flow) and prove the well posedness.

2. Lagrangian surfaces

In this section, we recall some basic material from symplectic geometry. For more information, we refer to [34].

Let $\mathbb{C}^2 = \mathbb{R}^4$ be the two-dimensional complex plane with complex coordinates (z_1, z_2), where $z_i = x_i + \sqrt{-1}y_i$, and the standard metric,

$$ds^2 = \sum_{i=1,2} dx_i^2 + dy_i^2,$$

and the symplectic structure

$$\omega = \sum_{i=1,2} dx_i \wedge dy_i.$$

Given two vectors, u and v, we will write $\langle u, v\rangle$ for the inner product of u and v. Notice that $\omega(u, v) = \langle Ju, v\rangle$, where J is the standard complex structure on $\mathbb{C}^2$. J satisfies that $\nabla J = 0$, where ∇ is the derivative w.r.t the standard metric ds^2. We can write

$$\omega = d\alpha,$$

where $\alpha = \frac{1}{2}\sum_i (x_i dy_i - y_i dx_i)$.

A Lagrangian surface of $\mathbb{C}^2$ is defined in

Definition 2.1. Let Σ be a surface in $\mathbb{C}^2$, with tangent and normal bundles, $T\Sigma$ and $N\Sigma$, respectively. Then Σ is Lagrangian if and only if one of the following equivalent conditions holds:

(1) ω restricted to Σ is zero;

(2) $JT\Sigma = N\Sigma$; and

(3) α restricted to Σ is closed.

From (2), we know that if Σ is a Lagrangian surface in $\mathbb{C}^2$, then the complex structure induces an isomorphism from the tangent bundle of Σ to the normal bundle of Σ

$$J : T\Sigma \to N\Sigma.$$

Definition 2.2. An immersion l from a surface Σ into $\mathbb{C}^2$ is called a Lagrangian immersion if $l^*\omega = 0$.

Let $L := l(\Sigma)$. In the paper, we will not distinguish $L = l(\Sigma)$ and the immersion $l : \Sigma \to \mathbb{C}^2$, if there is no confusion.

We define some special deformations of Lagrangian immersions.

Definition 2.3. (1) A smooth vector field V along a Lagrangian surface $l : \Sigma \to \mathbb{C}^2$ is called a Lagrangian (resp., Hamiltonian) vector field or a

Lagrangian (resp., Hamiltonian) variation if the associated one-form

$$V^\sharp := l^*(V \lrcorner \omega)$$

is closed (resp., exact)[1].

(2) A smooth family of maps $l : I \times \Sigma \to \mathbb{C}^2$, $I \subseteq R$, $l_t(p) = l(t,p)$ is called a Lagrangian (resp., Hamiltonian) variation if its derivatives w.r.t. $t \in I$ is Lagrangian (resp., Hamiltonian).

Let $V^\perp$ denote the normal component of V and $V^{\mathrm{T}} = V - V^\perp$ the tangential component of V. Since l is Lagrangian, one can easily check that

$$l^*(V \lrcorner \omega) = l^*(V^\perp \lrcorner \omega). \tag{2.1}$$

Using the Weinstein's tubular neighborhood theorem, recalled in Theorem 5.5 below, one can show that the space of immersed Lagrangian surfaces is a smooth infinite-dimensional manifolds and the tangential space at an immersed Lagrangian surface consists of the Lagrangian variations.

The following lemma is well known.

Lemma 2.4. *Let V be a variation along a Lagrangian surface Σ of $\mathbb{C}^2$. Then V is an Hamiltonian vector field, i.e., $V^\sharp = df$ for some function f if and only if*

$$V = -J\nabla f, \tag{2.2}$$

where ∇ is the derivative with respect to the metric on Σ.

Proof. For convenience of the reader, we provide the proof. If V is an Hamiltonian vector field, i.e., $V^\sharp = df$ for some function f, we have for any vector field X on Σ

$$V^\sharp(X) = df(X) = \langle \nabla f, X \rangle.$$

By the definition of $V^\sharp$ we have

$$V^\sharp(X) = (V \lrcorner \Omega)(X) = \omega(V, X) = \langle JV, X \rangle.$$

Hence we have

$$\nabla f = JV,$$

[1]For the convenience of notations, we will omit l^* in the sequel, if there is no confusion.

and hence $V = -J\nabla f$. The proof of another direction follows from the same computation. □

It is also easy to check

$$\langle V, U\rangle = \langle V^\sharp, U^\sharp\rangle,$$

where on the left-hand side $\langle\cdot,\cdot\rangle$ is the inner product for vector fields with respect to the pull-back metric and on the right-hand side $\langle\cdot,\cdot\rangle$ is the inner product for one-forms.

Examples: Now we give some examples of Lagrangian surfaces.

(a) A plane P in $\mathbb{C}^2$ is Lagrangian if and only if $P^\perp = \sqrt{-1}P$. For example, $P = \{(x_1, 0, x_2, 0)\}$ is a Lagrangian plane.

(b) By a classical result of Gromov, it is well known that a Lagrangian immersion ϕ from a genus zero compact surface will not be embedded. The Whitney sphere is a (simplest) Lagrangian immersed sphere, which is defined by

$$\begin{aligned} \phi : \mathbb{S}^2 &\to \mathbb{C}^2 \\ (x, y, z) &\mapsto \frac{1}{1+z^2}(x(1+\mathrm{i}z), y(1+\mathrm{i}z)), \end{aligned}$$

where $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$.

(c) $\{(\gamma_1(t), \gamma_2(s)) \in \mathbb{C}^2 \,|\, t, s \in \mathbb{S}^1\}$ is a Lagrangian surface, for any two curves $\gamma_1, \gamma_2 : \mathbb{S}^1 \to \mathbb{R}^2$. The Clifford torus $\mathbb{S}^1(\sqrt{2}) \times \mathbb{S}^1(\sqrt{2})$ is a Lagrangian surface, which was conjectured by Minicozzi [25] that it has the smallest Willmore energy in the class of Lagrangian tori. There are many examples of Lagrangian tori, which have many good properties. See, for example, [7, 20].

(d) Lagrangian graphs. A graph $\{(x_1, x_2, h_1(x_1, x_2), h_2(x_1, x_2)) \,|\, (x_1, x_2) \in \mathbb{R}^2\}$ is Lagrangian if and only if there exists a function $h : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla h = (h_1, h_2)$.

(e) Cotangent bundle. Let Σ be a manifold and $T^*\Sigma$ its cotangent bundle. Let $(U, x_1, \ldots, x_n)$ a local coordinate chart of Σ and $(T^*U, x_1, x_2, \ldots, x_n, \xi_1, \ldots, \xi_n)$ the associated coordinate chart for $T^*\Sigma$. The cotangent bundle carries a canonical symplectic form ω_0, which is given by $\sum dx_i \wedge d\xi_i$ in the associated local coordinate chart. This symplectic form is an exact form. Namely, there is a (tautological) one-form λ on

$T^*\Sigma$ with

$$\omega_0 = -d\lambda.$$

Locally, $\lambda = \sum x_i d\xi_i$. Both λ and ω are globally defined, namely, they are independent of the choice local coordinates. By a tautological one-form we mean that

$$\alpha^*\lambda = \alpha, \qquad \text{for all one-form } \alpha : \Sigma \to T^*\Sigma. \tag{2.3}$$

Σ can be embedded into $T^*\Sigma$ by using the zero section, i.e, the zero 1-form, $i_0 : x \mapsto (x, 0)$. This embedding is clearly Lagrangian. A one-form α on Σ can be viewed as a submanifold in $T^*\Sigma$, i.e, $L_\alpha = \{(x, \alpha(x))\ x \in \Sigma\}$, which is a generalization of the graphs given in (d). L_α is Lagrangian if and only if α is closed. This follows from (2.3), for

$$\alpha^*(\omega_0) = -\alpha^*(d\lambda) = -d(\alpha^*\lambda) = -d\alpha.$$

If α is exact, i.e., $\alpha = df$ for certain function $f : \Sigma \to \mathbb{R}$, the corresponding Lagrangian surface could be obtained from the zero section through an Hamiltonian variation. The reverse is also true.

Let $f_t : \Sigma \to \mathbb{R}$ be a smooth family of functions and $V = \frac{d}{dt}df$ the Hamiltonian vector field along Lagrangian surface L_{α_t} with $\alpha_t = df_t$. It is easy to check that

$$V^\sharp = V \lrcorner \omega_0 = d\left(\frac{d}{dt}f\right).$$

3. LW surfaces and HW surfaces

In this section, we introduce two kinds of geometrically constrained variational problems of the Willmore functional, i.e., the critical points of the Willmore functional under Lagrangian deformations or under Hamiltonian deformations. We compute the Euler–Lagrange equations of these variations and study properties of such critical points.

Let Σ be a surface and $\mathbb{C}^2$ be the standard two-dimensional complex plane with the standard symplectic form ω, the almost complex structure J and the standard metric, which are compatible with each other, as introduced in the last section.

Definition 3.1. Let $l : \Sigma \to \mathbb{C}^2$ be a smooth Lagrangian immersion.

(1) l is called a Lagrangian stationary Willmore surface, or Lagrangian–Willmore surface (in short, a LW surface) if it is a critical point of the Willmore functional under compactly supported Lagrangian variations

(2) l is called a Hamiltonian stationary Willmore surface, or Hamiltonian–Willmore surface (in short, a HW surface) if it is a critical point of the Willmore functional under compactly supported Hamiltonian variations.

Let l be a Lagrangian immersion. By definition l is a LW surface if and only if

$$\int_\Sigma \langle W(l), V\rangle d\mu = \int_\Sigma \langle W(l)^\sharp, V^\sharp\rangle d\mu = 0, \tag{3.1}$$

for any compactly supported Lagrangian variation vector fields V.

Let $\Omega^1(\Sigma)$ denote the set of 1-forms and $Z^1(\Sigma)$ denote the set of closed one-forms with compact supports and

$$\Pi : \Omega^1(\Sigma) \to Z^1(\Sigma)$$

be the L^2 projection. Then

Proposition 3.2. *Let $l : \Sigma \to \mathbb{C}^2$ be a LW surface, then*

$$\Pi(W(l)^\sharp) = 0. \tag{3.2}$$

Proof. This is a direct consequence of (3.1) and Definition 2.3. □

Proposition 3.3. *A Lagrangian immersion $l : \Sigma \to \mathbb{C}^2$ is a HW-surface if and only if*

$$d^* W(l)^\sharp = 0, \tag{3.3}$$

or equivalently,

$$\operatorname{div}(JW(l)) = 0. \tag{3.4}$$

Proof. For any function f on Σ, $V^\sharp := df$ is an Hamiltonian variation. Since l is a HW surface, we have

$$\begin{aligned} 0 &= \int_\Sigma \langle W(l), V\rangle d\mu = \int_\Sigma \langle W(l)^\sharp, df\rangle d\mu \\ &= \int_\Sigma \langle d^* W(l)^\sharp, f\rangle d\mu. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} 0 &= \int_\Sigma \langle W(l), V\rangle d\mu = -\int_\Sigma \langle W(l), J\nabla f\rangle d\mu \\ &= \int_\Sigma \langle JW(l), \nabla f\rangle d\mu = \int_\Sigma \langle \operatorname{div}(JW(l)), f\rangle d\mu. \end{aligned}$$

□

From the proof, it follows that for any normal vector field V

$$d^* V^\sharp = \operatorname{div}(JV). \tag{3.5}$$

In order to calculate $dW(l)^\sharp$, let us recall a theorem proved by Dazord [9]. See also [3].

Lemma 3.4. (Dazord) *Let L be a Lagrangian submanifold of a Kähler manifold M and H be the mean curvature vector of L. Then*

$$dH^\sharp = i^*\rho,$$

where $i : L \to M$ is the isometric embedding and ρ is the Ricci form of M. In particular if M is Einstein–Kähler, i.e., $\rho = c\omega$ for some constant c, then $H^\sharp$ is closed.

The Weitzenböck formula will be used in our computations:

Lemma 3.5. *Let M be an n-dimensional oriented Riemannian manifold with the Levi–Civita connection D. If $\{e_k\}$ is a local orthonormal basis and $\{\omega^k\}$ is its dual, then*

$$\Delta^h = -\sum_k D^2_{e_k,e_k} + \sum_{kj} \omega^k \wedge i(e_j) R_{e_k e_j},$$

where $D^2_{XY} \equiv D_X D_Y - D_{D_X Y}$ are the second derivatives, $\Delta^h = dd^* + d^*d$ is the Hodge–Laplacian and $R_{XY} = -D_X D_Y + D_Y D_X + D_{[X,Y]}$ is the curvature tensor.

We have

Lemma 3.6. *Let $l : \Sigma \to \mathbb{C}^2$ be a Lagrangian immersion, then*

$$dd^* H^\sharp = -(W(l) - KH - Q(A^\circ)H)^\sharp. \tag{3.6}$$

It follows that

$$d(W(l) - Q(A^\circ)H - KH)^\sharp = 0, \tag{3.7}$$

or equivalently,

$$dW(l)^\sharp = dK \wedge H^\sharp + d(Q(A^\circ)H)^\sharp. \tag{3.8}$$

Here K is the Gauss curvature of the immersion l.

Proof. We prove it by a direct computation. By Lemma 3.4, we have $d^* dH^\sharp = 0$, and

$$\begin{aligned} dd^* H^\sharp &= (dd^* + d^* d)H^\sharp = \Delta^h H^\sharp = -\Delta H^\sharp + Ric(H^\sharp) \\ &= -(\Delta^\perp H)^\sharp + KH^\sharp = -(\Delta^\perp H - KH)^\sharp \\ &= -(W(l) - KH - Q(A^\circ)H)^\sharp, \end{aligned}$$

where in the third equality we used the Weitzenböck formula, in the fourth equality we used the relation between the normal Laplacian $\Delta^\perp$ and the usual Laplacian $\Delta = \nabla^*\nabla$ (w.r.t. the induced metric on Σ) acting on forms:

$$\Delta^\perp V \lrcorner \omega = \Delta(V \lrcorner \omega),$$

which is proved by Oh (see [22], Lemma 3.3). □

Hence, we have

Proposition 3.7. *Let $l : \Sigma \to \mathbb{C}^2$ be a Lagrangian immersion. Then if it is a HW immersion,*

$$\Delta^h W(l)^\sharp - d^*(dK \wedge H^\sharp) - d^* d(Q(A^\circ)H)^\sharp = 0, \tag{3.9}$$

where by Δ^h we denote the Hodge–Laplacian $\Delta^h := dd^* + d^* d$.

Proof. Summarizing the results from Theorem 3.3 and Lemma 3.6 we have

$$\Delta^h W(l)^\sharp = d^* dW(l)^\sharp = d^*(dK \wedge H^\sharp) + d^* d(Q(A^\circ)H)^\sharp. \qquad \square$$

Examples: Now we give some examples of LW surfaces and HW surfaces. By definition, all Willmore–Lagrangian surfaces are LW surfaces.

(a) The Lagrangian planes are simplest examples.

(b) The Whitney sphere has $W(\phi) = 8\pi$. As mentioned in the previous section that a genus zero compact surface will not be embedded as Lagrangian surface, by Li–Yau's inequality [21], we have $W(\phi) \geq 8\pi$ for any compact surface with genus zero . Hence, the Whitney sphere is a LW surface. In fact, it is Willmore surface.

(c) The existence of a minimizer of the Willmore functional among Lagrangian tori was proved by Minicozzi [25]. It is clearly a LW surface. The Clifford torus, which is a Willmore–Lagrangian surface (and hence a LW surface), has $W(\phi) = 2\pi^2$.

It is interesting to ask when a HW surface is a Willmore surface.

Proposition 3.8. *Let Σ be a closed surface and $l : \Sigma \to \mathbb{C}^2$ a HW immersion. If $W(l)$ is Hamiltonian variation, i.e., there exists a function f on $l(\Sigma)$ such that*

$$W(l) = J\nabla f,$$

then l is a Willmore immersion.

Proof. Since $W(l)$ is a Hamiltonian variation along l, l is a HW immersion implies that

$$\int_\Sigma |W(l)|^2 = 0.$$

Therefore $W(l) = 0$, i.e., l is a Willmore immersion.

Or, one can show is as follows:

$$W(l) = J\nabla f$$

and l is a HW surface, we get

$$0 = \operatorname{div} JW(l) = -\Delta f,$$

which implies that $f = c$ for some constant c. Therefore $W(l) = 0$, i.e., l is a Willmore immersion. □

$W(l) = J\nabla f$ for some function f means it defines a Hamiltonian variation on l. This does not hold in general. But in the following we will see that modulo some lower order terms it does define a Hamiltonian variation vector field (cf. Lemma 3.6 in the last subsection). This observation helps us to find curvature estimates for HW surfaces.

Definition 3.9. Let $l : \Sigma \to \mathbb{C}^2$ be a Lagrangian and H its mean curvature. There exists a $\mathbb{S}^1$-valued function θ satisfying

$$H = J\nabla\theta.$$

This local function is called *Lagrangian angle*. Its derivative is globally defined.

One can also compute the Lagrangian angle as follows. Let $\mathbb{C}^2$ be the complex plane with complex coordinates (z_1, z_2) and $\omega = dz_1 \wedge dz_2$. Then we have

$$l^*(\omega) = e^{i\theta}\mathrm{vol}(l),$$

where $\mathrm{vol}(l)$ denotes the area form of the immersion l

Now we consider Lagrangian graphs. Assume that l is a Lagrangian graph, that is there is a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$l(x, y) = (x, y, \partial_x\varphi, \partial_y\varphi).$$

If l is a LW graph, then φ satisfies a sixth-order quasi-linear elliptic equation, which we show in the following:

From above we know

$$J\nabla\theta = H, \tag{3.10}$$

and hence

$$\begin{aligned} JW(l) &= J\Delta^{\perp}H + JQ(A^\circ)H \\ &= \Delta JH + JQ(A^\circ)H \\ &= -\Delta\nabla\theta + JQ(A^\circ)H. \end{aligned}$$

The Lagrangian angle, now it is globally defined, can be computed as follows.

$$\mathrm{e}^{i\theta} = (dz_1 \wedge dz_2)\mid_{l(\mathbb{R}^2)}, \tag{3.11}$$

where $*$ is the Hodge star operator with respect to the metric g_l. Thus we have

$$\begin{aligned}
\mathrm{e}^{i\theta} &= *(dx + \mathrm{i}d\varphi_x) \wedge (dy + \mathrm{i}d\varphi_y) \\
&= *(1 - \det(D_0^2\varphi) + \mathrm{i}\Delta_0\varphi)dx \wedge dy, \\
&= \frac{1}{\sqrt{\det(g_l)}}(1 - \det(D_0^2\varphi) + \mathrm{i}\Delta_0\varphi) \\
&= \frac{1}{\sqrt{\det(g_l)}} \det(I_2 + \mathrm{i}D_0^2\varphi),
\end{aligned}$$

where $\Delta_0 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the standard Laplacian on $\mathbb{R}^2$, D_0^2 is the Hessian and I_2 is 2×2 identity matrix. A direct computation gives

$$\det(g_l) = (1 - \det(D_0^2\varphi))^2 + (\Delta_0\varphi)^2 = \det(I_2 + (D_0^2\varphi)^2).$$

Therefore we have

$$\theta = -\mathrm{i}\log\frac{1}{\sqrt{\det(g_l)}}(1 - \det(D_0^2\varphi) + \mathrm{i}\Delta_0\varphi) = -\mathrm{i}\log\frac{\det(I_2 + \mathrm{i}D_0^2\varphi)}{\det(I_2 + (D_0^2\varphi)^2)} \tag{3.12}$$

and

$$\begin{aligned}
\operatorname{div}_{g_l}\Delta JH &= -\operatorname{div}_{g_l}\Delta\nabla\theta + \operatorname{div}_{g_l}(JQ(A^\circ)H) \\
&= -\Delta^2\theta - 2K\Delta\theta - g_l(\nabla\theta, \nabla K) + \operatorname{div}_{g_l}(JQ(A^\circ)H),
\end{aligned}$$

where K is the Gauss curvature of $l(\mathbb{R}^2)$. The representation form of the Lagrangian angle θ for graph is well known.

Proposition 3.10. *Let $l : \mathbb{R}^2 \to \mathbb{C}^2$ be a LW immersion. If l is a graph, i.e., $l = (x, y, \partial_x\varphi, \partial_y\varphi)$ for some function φ on $\mathbb{R}^2$. Then φ satisfies the following sixth-order quasi-linear elliptic equation*

$$\operatorname{div}_{g_l}(JW(l)) = -\Delta^2\theta - 2K\Delta\theta - g_l(\nabla\theta, \nabla K) + \operatorname{div}_{g_l}(JQ(A^\circ)H) = 0,$$

where θ is given by (3.12), Δ is the Laplace–Beltrami operator with respect to the metric g_l.

Proof. First notice that $-2K\Delta\theta - g_l(\nabla\theta, \nabla K) + \operatorname{div}_{g_l}(JQ(A^\circ)H)$ contains at highest the fourth-order terms of φ. From (3.12) we claim that

$$\Delta\theta = \Delta^2\varphi + lot, \tag{3.13}$$

where lot has at highest the third-order derivatives. To show this claim, we first remark that

$$\Delta\theta = g^{ik}\theta_{ij} + lot,$$

where (g^{jk}) is the inverse metric of g_l. Using (3.12) one can compute it directly to show that

$$\theta_k := \frac{\partial}{\partial x_k}\theta = g^{ij}\varphi_{ijk}.$$

One can find its proof, for example, in [8]. For convenience of the reader, we provide it here. Let $A = (A_{ij}) = I_2 + \mathrm{i}D_0^2\varphi$, $B = I_2 + (D_0^2\varphi)^2$, $(A^{ij}) = A^{-1}$ and $(B^{ij}) = B^{-1}$. The matrices A and g_l have the following relation:

$$B = (I_2 - \mathrm{i}D_0^2\varphi)A, \quad A^{-1} = B^{-1}(I_2 - \mathrm{i}D_0^2\varphi).$$

Using the above relation, we have

$$\begin{aligned}\theta_k &= -\mathrm{i}\sum\left(A^{ij}A_{ij,k} - \frac{1}{2}B^{ij}B_{ij,k}\right)\\ &= -\mathrm{i}\sum\left(g^{il}(d^*_{lj} - i\varphi_{lj})\cdot i\varphi_{ijk} - \frac{1}{2}B^{ij}\cdot 2\varphi_{il}\varphi_{ljk}\right)\\ &= \sum B^{ij}\varphi_{ijk} = g^{ij}\varphi_{ijk}.\end{aligned}$$

It follows that

$$\Delta\theta = g^{im}g^{jk}\varphi_{imjk} + lot = \Delta^2\varphi + lot.$$

□

We remark that in general the Lagrangian angle θ is not globally defined. However, $\nabla\theta$ and hence, $\Delta\theta$ are globally defined.

With the same computations one can show that

Proposition 3.11. *Let Σ be a manifold and $T^*\Sigma$ its cotangent bundle with the canonical symplectic form ω_0 mentioned in the previous section. Let $\beta = df$ for certain function $f : \Sigma \to \mathbb{R}$. If the corresponding Lagrangian surface* *is a HW surface, then f satisfies a sixth-order elliptic equation with a form*

$$\Delta^3 f + lot = 0,$$

where Δ is the Laplacian with respect to the metric of Σ.

4. Curvature estimates for the LW and HW surfaces

In this section, we extend the L^∞ estimates of the second fundamental form of Willmore surfaces ([15], Theorem 2.2) to HW surfaces in $\mathbb{C}^2$.

Theorem 4.1. *Let $l : \Sigma \to \mathbb{C}^2$ be a properly immersed HW surface. There is a constant $\epsilon_0 > 0$, such that if*

$$\int_{B_\rho(0)} |A|^2 d\mu < \epsilon_0^2, \tag{4.1}$$

where $B_\rho(0)$ is the ball of radius ρ centered at 0, then we have

$$\|A\|_{L^\infty(B_{\frac{\rho}{2}})} \leq \frac{C}{\rho} \|A\|_{L^2(B_\rho)}, \tag{4.2}$$

where C is a constant and here we abbreviate $\|A\|_{L^2(B_\rho)} := \int_{l^{-1}(B_\rho)} |A|^2 d\mu$.

Remark 4.2. Furthermore in [13] they got the decay estimates for A°, under assumptions of the smallness of A° in L^2 and the finiteness of A in L^2. We could not extend yet this result to our case by using the following observations and we leave it as a question to be considered later .

By this estimate, we get immediately the following gap theorem.

Theorem 4.3. *If $l : \Sigma \to \mathbb{C}^2$ is a properly immersed HW surface with $\|A\|_{L^2(\Sigma)} < \epsilon_0$, where ϵ_0 is the constant of the above theorem, then l is a Lagrangian plane.*

Since by definition a LW surface is automatically a HW surface, these results also hold for LW surfaces.

To prove Theorem 4.1 we need several lemmas. The main observation which helps us to find this result is equality (3.6) in Lemma 3.6, which says the Willmore operator $W(l)$ defines a Hamiltonian deformation on l, modulo some second-order terms.

Equality (3.6) shows that $W(l) - KH - Q(A^\circ)H$ defines a Hamiltonian variation on l. This fact can be used to derive the following integral inequality.

Lemma 4.4. *Let $l : \Sigma \to \mathbb{C}^2$ be a HW surface. Assume that B_ρ is the standard ball with radius ρ in $\mathbb{C}^2$ and $\tilde{\gamma} : \mathbb{C}^2 \to R^+$ is a cut-off function belonging to $C_c^1(B_\rho)$ satisfying $|D\tilde{\gamma}| \le \frac{C}{\rho}$. Then for $\gamma = \tilde{\gamma} \circ l$ we have*

$$\int_\Sigma |W(l)|^2 \gamma^4 d\mu \le C \int_\Sigma |A|^6 \gamma^4 d\mu + \frac{C}{\rho^2} \int_\Sigma |\nabla A|^2 \gamma^2 d\mu, \tag{4.3}$$

where C is a constant.

Proof. Because $(W(l) - KH - Q(A^\circ)H)^\sharp = -dd^* H^\sharp$, we have $W(l) - KH - Q(A^\circ)H = -J\nabla d^* H^\sharp$, and by the definition of HW surfaces, we get

$$\begin{aligned} 0 &= \int_\Sigma \langle W(l), J\nabla(\gamma^4 d^* H^\sharp)\rangle d\mu \\ &= \int_\Sigma \langle W(l), -\gamma^4 (W(l) - KH - Q(A^\circ)H d\mu + \int_\Sigma \langle W(l), 4\gamma^3 J\nabla\gamma d^* H^\sharp\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \int_\Sigma |W(l)|^2 \gamma^4 d\mu &= -\int_\Sigma \langle W(l), KH + Q(A^\circ)H\rangle \gamma^4 d\mu \\ &\quad + \int_\Sigma \langle W(l), 4\gamma^3 J\nabla\gamma d^* H^\sharp\rangle \\ &\le \frac{1}{2}\int_\Sigma |W(l)|^2 \gamma^4 d\mu + C\int_\Sigma |A|^6 \gamma^4 d\mu + \frac{C}{\rho^2}\int_\Sigma |d^* H^\sharp|^2 \gamma^2 d\mu. \end{aligned}$$

Noting that $H^\sharp(.) = g(JH, .)$ we have

$$|d^* H^\sharp|^2 \le C|\nabla A|^2,$$

which finish the proof of this lemma. □

To proceed, we need the following lemma (Lemma 2.3 in [15]).

Lemma 4.5. *Let $l : \Sigma \to B_\rho$ be properly immersed, where B_ρ is the standard ball with radius ρ in $\mathbb{R}^n$, and let $\tilde{\gamma} \in C_c^1(B_\rho)$ satisfy $|D\tilde{\gamma}| \leq \frac{C}{\rho}$. Then for $\gamma = \tilde{\gamma} \circ l$ we have*

$$\begin{aligned}\int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu \leq C \left(\int_\Sigma |W(l)|^2 \gamma^4 d\mu + \frac{1}{\rho^4} \int_{\{\gamma>0\}} |A|^2 d\mu \right) \\ + C \int_\Sigma (|A|^2 |\nabla A|^2 + |A|^6) \gamma^4 d\mu),\end{aligned} \tag{4.4}$$

where C is a constant.

Combing the estimates of the above two lemmas we get

Lemma 4.6. *Let $l : \Sigma \to \mathbb{C}^2$ be a HW surface. Assume that B_ρ is the standard ball with radius ρ in $\mathbb{C}^2$ and $\tilde{\gamma} : \mathbb{C}^2 \to R^+$ is a cut-off function belongs to $C_c^1(B_\rho)$ satisfying $|D\tilde{\gamma}| \leq \frac{C}{\rho}$. Then for $\gamma = \tilde{\gamma} \circ l$ we have*

$$\begin{aligned}\int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu \leq C \int_\Sigma (|A|^2 |\nabla A|^2 + |A|^6) \gamma^4 d\mu + \frac{C}{\rho^2} \int_\Sigma |\nabla A|^2 \gamma^2 d\mu \\ + \frac{C}{\rho^4} \int_{\{\gamma>0\}} |A|^2 d\mu,\end{aligned} \tag{4.5}$$

where C is a constant.

The next issue is to estimate the right-hand side of the inequality in Lemma 4.6. Comparing to the left-hand side, we know that the right-hand side contains terms with lower order (in derivatives). Hence, we need a Sobolev-type inequality to control the right-hand side by the left-hand side.

Theorem 4.7. (Michael Simon Sobolev inequality). *Let $l : \Sigma \to \mathbb{R}^n$ be an immersed surface. Then for any $\varphi \in C_c^1(\Sigma)$, we have*

$$\left(\int_\Sigma \varphi^2 d\mu \right)^{\frac{1}{2}} \leq C \int_\Sigma (|\nabla \varphi| + |H||\varphi|) d\mu,$$

where C is a constant depending only on n.

Using this inequality and integral estimates, we have

Proposition 4.8. *There is a constant $\epsilon_0 > 0$, such that if $l : \Sigma \to \mathbb{C}^2$ is a properly immersed HW surface with*

$$\int_\Sigma |A|^2 d\mu < \epsilon_0^2,$$

then it satisfies the inequality

$$\int_\Sigma (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^6)\gamma^4 d\mu \le \frac{C}{\rho^4}\int_{B_\rho} |A|^2 d\mu,$$

where C is a constant.

Proof. Choose $\varphi = |A||\nabla A|\gamma^2$ in the Sobolev inequality, using $|\nabla|\varphi|| \le |\nabla\varphi|$ we get

$$\begin{aligned}\int_\Sigma |A|^2|\nabla A|^2\gamma^4 d\mu &\le C\left(\int_\Sigma |A||\nabla^2 A|\gamma^2 d\mu\right)^2 + C\left(\int_\Sigma |A||\nabla A|\gamma|\nabla\gamma| d\mu\right)^2 \\ &\quad + C\left(\int_\Sigma |\nabla A|^2\gamma^2 d\mu\right)^2 + C\left(\int_\Sigma |A|^2|\nabla A|\gamma^2 d\mu\right)^2 \\ &\le C\left(\int_{\{\gamma>0\}} |A|^2 d\mu\right)\left(\int_\Sigma (|\nabla^2 A|^2 + |A|^2|\nabla A|^2)\gamma^4 d\mu\right) \\ &\quad + C\left(\int_\Sigma |\nabla A|^2\gamma^2 d\mu\right)^2 + \frac{C}{\rho^4}\int_{\{\gamma>0\}} |A|^2 d\mu. \end{aligned} \tag{4.6}$$

On the other hand, we have

$$\begin{aligned}\int_\Sigma |\nabla A|^2\gamma^2 d\mu &\le \int_\Sigma |A||\nabla^2 A|\gamma^2 d\mu + \frac{C}{\rho}\int_\Sigma |A||\nabla A|\gamma d\mu \\ &\le C\left(\int_{\{\gamma>0\}} |A|^2 d\mu\right)^{\frac{1}{2}}\left(\int_\Sigma |\nabla^2 A|^2\gamma^4 d\mu\right)^{\frac{1}{2}} \\ &\quad + \frac{1}{2}\int_\Sigma |\nabla A|^2\gamma^2 d\mu + \frac{C}{\rho^2}\int_{\{\gamma>0\}} |A|^2 d\mu, \end{aligned} \tag{4.7}$$

or

$$
\begin{aligned}
\int_\Sigma |\nabla A|^2\gamma^2 d\mu &\le \int_\Sigma |A||\nabla^2 A|\gamma^2 d\mu + \frac{C}{\rho}\int_\Sigma |A||\nabla A|\gamma d\mu \\
&\le \frac{C}{\epsilon\rho^2}\int_{\{\gamma>0\}} |A|^2 d\mu + C\rho^2\epsilon\int_\Sigma |\nabla^2 A|^2\gamma^4 d\mu \\
&\quad + \frac{1}{2}\int_\Sigma |\nabla A|^2\gamma^2 d\mu + \frac{C}{\rho^2}\int_{\{\gamma>0\}} |A|^2 d\mu, \qquad (4.8)
\end{aligned}
$$

where ϵ is a positive constant to be determined later.

From above, we conclude

$$
\begin{aligned}
\int_\Sigma |A|^2|\nabla A|^2\gamma^4 d\mu &\le C\left(\int_{\{\gamma>0\}} |A|^2 d\mu\right)\left(\int_\Sigma (|\nabla^2 A|^2 + |A|^2|\nabla A|^2)\gamma^4 d\mu\right) \\
&\quad + \frac{C}{\rho^4}\int_{\{\gamma>0\}} |A|^2 d\mu. \qquad (4.9)
\end{aligned}
$$

Secondly, we take $\varphi = |A|^3\gamma^2$ in the Sobolev inequality to obtain

$$
\begin{aligned}
\int_\Sigma |A|^6\gamma^4 d\mu &\le C\left(\int_\Sigma |A|^2|\nabla A|\gamma^2 d\mu\right)^2 + C\left(\int_\Sigma |A|^3\gamma|\nabla\gamma| d\mu\right)^2 \\
&\quad + C\left(\int_\Sigma |A|^4\gamma^2 d\mu\right)^2 \\
&\le C\left(\int_\Sigma |\nabla A|^2\gamma^2 d\mu\right)^2 + C\left(\int_\Sigma |A|^4\gamma^2 d\mu\right)^2 \\
&\quad + \frac{C}{\rho^4}\int_{\{\gamma>0\}} |A|^2 d\mu \\
&\le C\left(\int_\Sigma |\nabla A|^2\gamma^2 d\mu\right)^2 + C\int_{\{\gamma>0\}} |A|^2 d\mu\left(\int_\Sigma |A|^6\gamma^4 d\mu\right) \\
&\quad + \frac{C}{\rho^4}\int_{\{\gamma>0\}} |A|^2 d\mu. \qquad (4.10)
\end{aligned}
$$

Combing (4.9) and (4.10) one gets

$$
\begin{aligned}
&\int_\Sigma (|A|^2|\nabla A|^2 + |A|^6)\gamma^4 d\mu \\
&\quad \le C\left(\int_{\{\gamma>0\}} |A|^2 d\mu\right)\int_\Sigma (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^6)\gamma^4 d\mu \\
&\quad\quad + \frac{C}{\rho^4}\int_{\{\gamma>0\}} |A|^2 d\mu. \qquad (4.11)
\end{aligned}
$$

Combing the estimates (4.5), (4.8) and (4.11) we get

$$\begin{aligned}&\int_\Sigma (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^6)\gamma^4 d\mu\\&\leq C\left(\int_{\{\gamma>0\}} |A|^2 d\mu) \int_\Sigma (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^6\right)\gamma^4 d\mu\\&\quad + C\epsilon \int_\Sigma |\nabla^2 A|^2\gamma^4 d\mu + \frac{C}{\rho^4}\left(1+\frac{1}{\epsilon}\right)\int_{\{\gamma>0\}} |A|^2 d\mu.\end{aligned}$$

We can choose ϵ such that $C\epsilon < 1$ and when ϵ_0 is small enough, we get the inequality in the lemma. □

We need the following interpolation inequality.

Lemma 4.9. *Let $l : \Sigma \to B_\rho$ be properly immersed, where B_ρ is the standard ball with radius ρ in $\mathbb{R}^n$, and let $\tilde{\gamma} \in C^1_c(B_\rho)$ satisfy $|D\tilde{\gamma}| \leq \frac{C}{\rho}$ and $\gamma = \tilde{\gamma} \circ l$. Then for any normal-valued form ϕ, we have*

$$\|\gamma^2\phi\|^4_{L^\infty} \leq C\|\gamma^2\phi\|^2_{L^2}\left(\int_\Sigma (|\nabla^2\phi|^2 + |H|^4|\phi|^2)\gamma^4 d\mu + \frac{1}{\rho^4}\int_{\{\gamma>0\}} |\phi|^2 d\mu\right).$$

Proof. For a proof of this lemma we refer to [15], lemma 2.4. □

*Proof of Theorem 4.1 (**Continuation**).* Using Lemma 4.9 with $\phi = A$ and combing with Proposition 4.8, when ρ is big enough, we get

$$\begin{aligned}\|\gamma^2 A\|^4_{L^\infty} &\leq C\|\gamma^2 A\|^2_{L^2}\left(\int_\Sigma (|\nabla^2 A|^2 + |A|^6|)\gamma^4 d\mu + \frac{1}{\rho^4}\int_{\{\gamma>0\}} |A|^2 d\mu\right)\\&\leq \frac{C}{\rho^4}\|A\|^2_{L^2(\{\gamma>0\})}\int_{\{\gamma>0\}} |A|^2 d\mu\\&\leq \frac{C}{\rho^4}\left(\int_{\{\gamma>0\}} |A|^2 d\mu\right)^2. \qquad (4.12)\end{aligned}$$

This completes the proof. □

5. The HW-flow

In this section, we introduce a sixth-order flow, which could be used to study the existence problem of the HW surfaces.

Let L be a closed surface and $l : L \times I \to \mathbb{C}^2$, $I = (0, T)$ for some $T \in (0, \infty]$ be a family of surfaces in $\mathbb{C}^2$ and V_t be a normal vector field along l_t defined by

$$V_t^\sharp := V_t \lrcorner \omega = -dd^* W(l_t)^\sharp.$$

One can check that

$$V_t = -J\nabla \mathrm{div}\,(JW(l_t)). \tag{5.1}$$

In fact, from Lemma 2.4 we have

$$V_t = -J\nabla(d^* W(l_t)^\sharp).$$

Notice that $W(l_t)^\sharp$ is a one-form and $d^* W(l_t)^\sharp$ is a function. To show (5.1), we need to check that for any normal vector field U on Σ

$$d^* U^\sharp = \mathrm{div}(JU).$$

This follows from

$$\begin{aligned} \int f d^* U^\sharp &= \int \langle df, U^\sharp \rangle = -\int \langle J\nabla f, U\rangle \\ &= \int \langle \nabla f, JU \rangle = \int f \mathrm{div}\,(\nabla JU), \end{aligned}$$

for any compactly supported function f. Notice the variation V_t is an Hamiltonian variation.

Now we define a flow to deform Lagrangian immersions in $\mathbb{C}^2$.

Definition 5.1. We call $l_t(.) := l(., t)$ a solution of the Hamiltonian–Willmore flow (in short, the HW-flow) if there holds

$$\left\{ \begin{array}{rcl} \dfrac{\partial}{\partial t} l_t &=& V_t, \quad \forall t \in I, \\ l(., 0) &=& l_0(.), \end{array} \right. \tag{5.2}$$

where l_0 is an initial Lagrangian surface.

Flow (5.2) is equivalent to

$$\frac{\partial}{\partial t} l_t \lrcorner \omega = V^\sharp = -dd^* W(l_t)^\sharp, \tag{5.3}$$

up to diffeomorphisms of Σ. More precisely, (5.3) is equivalent to

$$\left\{ \left(\frac{\partial}{\partial t} l_t\right)^\perp = V_t, \quad \forall t \in I, \right. \tag{5.4}$$

The HW-flow is an "H^1-gradient flow" of the Willmore energy.

Proposition 5.2. *Let l_t, $t \in I$ be a solution of the HW-flow, then the Willmore energy $\mathcal{W}(l_t)$ is non-increasing in t. $\frac{d}{dt}\mathcal{W}(l_t) = 0$ at t if and only if l_t is a HW surface.*

Proof. We can calculate directly to get

$$\begin{aligned}
\frac{d}{dt}\mathcal{W}(l_t) &= \frac{1}{2}\int_{l_t(\Sigma)} \langle W(l_t), V_t\rangle \\
&= \frac{1}{2}\int_{l_t(\Sigma)} \langle W(l_t)^\sharp, V_t^\sharp\rangle \\
&= -\frac{1}{2}\int_{l_t(\Sigma)} \langle d^*W(l_t)^\sharp, d^*W(l_t)^\sharp\rangle \\
&\le 0,
\end{aligned}$$

which implies the conclusion. □

The rest of this section is devoted to the proof of the following result.

Theorem 5.3. *There exists a time $T > 0$, such that there exists a unique smooth solution for the HW-flow on the time interval $[0, T)$.*

By using Lemma 3.6, we have

$$\begin{aligned}
V_t^\sharp &= -dd^*W(l_t)^\sharp \\
&= -\Delta^h W(l_t)^\sharp - d^*dW(l_t)^\sharp \\
&= -\Delta^h W(l_t)^\sharp - d^*d(Q(A^\circ)H)^\sharp - d^*(dK \wedge H^\sharp).
\end{aligned}$$

Hence the flow looks like

$$\frac{\partial}{\partial t} l_t \lrcorner \omega = -\Delta^h W(l_t)^\sharp + lot.$$

In order to have some feelings about this flow, we first consider the case of Lagrangian graphs. Let φ_t be a family of smooth functions, whose graphs satisfy (5.3). We have computed $d^*W(l)^\sharp = \mathrm{div}\,(JW(l))$ in Section 3, which has form in terms of φ

$$d^*W(l)^\sharp = \mathrm{div}(JW(l)) = \Delta^3\varphi + lot.$$

In the graphical case, $l_t(x_1, x_2) = (x_1, x_2, \partial_1\varphi_t, \partial_2\varphi_t)$. Hence,

$$\frac{d}{dt}l_t = \left(0, 0, \partial_1\frac{d}{dt}\varphi_t, \partial_2\frac{d}{dt}\varphi_t\right) = \partial_1\frac{d}{dt}\varphi_t\frac{\partial}{\partial y_1} + \partial_2\frac{d}{dt}\varphi_t\frac{\partial}{\partial y_2}.$$

It follows that

$$\frac{d}{dt}l_t \lrcorner \omega = -\left(\partial_1\frac{d}{dt}\varphi_t dx_1 + \partial_2\frac{d}{dt}\varphi_t dx_2\right) = -d\left(\frac{d}{dt}\varphi_t\right). \tag{5.5}$$

Hence the HW flow for l_t is

$$d\left(\frac{d}{dt}\varphi_t\right) = dd^*W(l)^\sharp. \tag{5.6}$$

If we can solve

$$\frac{d}{dt}\varphi_t = d^*W(l)^\sharp = \Delta^3\varphi_t + lot, \tag{5.7}$$

then we have a solution for the HW flow (5.6). Flow (5.7) is a sixth parabolic flow. Here we have used a simple fact that the operator d does not depend on t, though all other operators do depend on t.

For general Lagrangian surfaces, we use Weinstein's tubular neighborhood theorem to view nearby Lagrangian surfaces as generalized graphs over the initial surface.

Let Σ be a Lagrangian surface in $\mathbb{C}^2$ and U a tubular neighborhood of Σ. Recall that a theorem of Weinstein says that there is a symplectomorphism mapping U to V, where V is a neighborhood of Σ in $T^*\Sigma$, the cotangent bundle of Σ.

Before stating this theorem, we recall some facts about $T^*\Sigma$.

Lemma 5.4. *With a canonical symplectic form, the cotangent bundle $T^*\Sigma$ is a symplectic manifold and Σ, viewed as the zero section of $T^*\Sigma$, is a Lagrangian submanifold of $T^*\Sigma$. Let f be a one-form, i.e., a section of $T^*\Sigma$. Then $\{(p, f(p)\,|\,p \in \Sigma\} \subset T^*\Sigma$ is a Lagrangian submanifold of $T^*\Sigma$ if and only if f is a closed one-form on Σ.*

We have

Theorem 5.5 Weinstein tubular neighborhood theorem. *Let (M, ω) be a symplectic manifold and X be a compact Lagrangian submanifold, ω_0 the canonical symplectic form on T^*X. Assume that $i_0 : X \to T^*X$ is the Lagrangian embedding as the zero section, and $i : X \to M$ is the Lagrangian embedding given by inclusion. There are neighborhoods V of X in T^*X and U of X in M and a symplectodiffeomorphism $\phi : U \to V$, such that $\phi^*\omega_0 = \omega$ and $\phi \circ i = i_0$.*

With the help of this theorem we can write the HW-flow equation in the cotangent bundle of the initial surface.

First, notice that the above results hold for immersions. Let $l_0 : \Sigma \to \mathbb{C}^2$ be an immersion. Apply the above Theorem, we have symplectomorphism ϕ from U to V, where U is a neighborhood of $L := l_0(\Sigma) \subset \mathbb{C}^2$ and V a neighborhood of the zero section $\Sigma \to T^*\Sigma$. Denote $L_t = l_t(\Sigma)$ and assume that L_t belong to U for small t. Let

$$\widetilde{l_t} := \phi \circ l_t.$$

Theorem 5.5 implies that $\widetilde{l_t} : \Sigma \to T^*\Sigma$ are Lagrangian.

From the HW flow for l_t, we want to compute a flow to $\widetilde{l_t}$, which is a family of immersions $\tilde{l}_t : \Sigma \to T^*\Sigma$, where $\tilde{l}_t := \phi \circ l_t$. By (5.3) it is easy to check that $\tilde{l}_l$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{l}_t \lrcorner \omega_0 &= \left(\frac{\partial}{\partial t}\phi \circ l_t\right) \lrcorner (\phi^{-1})^*\omega \\ &= (\phi^{-1})^* \left(\left(\frac{\partial}{\partial t} l_t\right) \lrcorner \omega\right) \\ &= -(\phi^{-1})^*(dd^*W(l_t)^\sharp) \\ &= -dd^*_{\tilde{g}_t} W(\tilde{l}_t)^\sharp, \end{aligned} \tag{5.8}$$

where $W(\tilde{l}_t)$ is the Willmore operator for the immersion $\tilde{l}_t : \Sigma \to V \subset T^*\Sigma$ with the metric $\tilde{g}_t := (\tilde{l}_t)^*\tilde{g}$, $d^*_{\tilde{g}_t}$ is the adjoint operator of d w.r.t $\tilde{g}_t$ and $\sharp$ is defined with respect to the symplectic form ω_0. As mentioned above, any nearby Lagrangian surface in $T^*\Sigma$ can be expressed by a closed one-form α_t, up to a diffeomorphisms of Σ. We may assume that $\widetilde{l_t} = \alpha_t$ for a family of one-forms α_t. Since $dd^*_{\tilde{g}_t} W(\tilde{l}_t)$ is an Hamiltonian vector field, it is well known that $\tilde{l}_t$ can be actually represented as a family of exact one-forms df_t,

$f_t : \Sigma \to \mathbb{R}$, i.e., $\alpha_t = df_t$. Similar to the proof of (5.5), we have

$$\frac{\partial}{\partial t}\tilde{l}_t \lrcorner \omega_0 = -d\left(\frac{d}{dt}f_t\right). \tag{5.9}$$

For the convenience of the reader we provide a proof of (5.9) in Appendix A. Hence, the HW-flow equation is equivalent to the following flow equation, possibly in a smaller interval I,

$$\frac{d}{dt}f_t = d^*_{\tilde{g}_t}W(\tilde{l}_t)^\sharp + h_t. \tag{5.10}$$

where $h_t : \Sigma \to \mathbb{R}$. Now like the Lagrangian graphs (see Proposition 3.11), one can show that

$$d^*W(\tilde{l}_t)^\sharp = \Delta^3_{\tilde{g}_t} f + lot.$$

Now we can apply the following general existence result for quasi-linear parabolic equations on compact manifolds without boundary, due to Polden and Huisken.

Theorem 5.6 ([10], Theorem 7.15). *Suppose that for a smooth initial data u_0 the operator of $2p$ order*

$$A(u) = A^{i_1j_1\ldots i_pj_p}(x, u, \nabla u, \ldots, \nabla^{2p-1}u)D_{i_1j_1\ldots i_pj_p}$$

is smooth and strongly elliptic in a neighborhood of u_0. Then the evolution equation

$$D_t u = -A(u)u + b,$$

where $b = b(x, u, \nabla u, \ldots, \nabla^{2p-1}u)$ is smooth, has a unique smooth solution on the interval $[0, T)$, for some $T \in (0, \infty]$.

For a proof of this theorem we also refer to Lamm's diploma thesis [16].

Remark 5.7. Here $A^{i_1j_1\ldots i_pj_p}$ is strongly elliptic means that it can be decomposed as

$$A^{i_1j_1\ldots i_pj_p} = (-1)^p E^{i_1j_1}E^{i_2j_2}\ldots E^{i_pj_p},$$

where the two-form E is strictly positive: $E \geq \lambda g$ for some $\lambda > 0$.

Our operator is actually strongly elliptic in a neighborhood of the initial surface, and hence by applying Theorem 5.6. we get the short time existence and uniqueness of

$$\frac{d}{dt} f_t = d^*_{\tilde{g}_t} W(\tilde{l}_t)^\sharp \tag{5.11}$$

with the initial condition $f(0) = 0$. Let $\tilde{l}_t$ be the Lagrangian surfaces in $T^*\Sigma$ corresponding to closed form df_t. Set $l_t = \phi^{-1}\tilde{l}_t : \Sigma \to \mathbb{C}^2$. l_t is a solution of the HW flow. This proves that existence part of Theorem 5.3. The uniqueness part follows similarly.

6. Appendix A. Proof of (5.9)

Let α_t : be a family of closed one-forms on Σ and $L_{\alpha_t} = \{(x, \alpha_t(x))\}$ the Lagrangian surfaces. L_{α_t} is the image of $\tilde{l}_t := \alpha_t : \Sigma \to T^*\Sigma$ We want to compute the Lagrangian variation $\frac{d}{dt}\tilde{l}_t$. The computation is similar to (5.5) for the Lagrangian graphs. Let $\alpha_t = \sum_i \alpha_i dx_i$ in the associated local coordinate, see the Example (e) in Section 2. We denote its variation $\frac{d}{dt}\alpha_t = \sum_i a_i(x)dx_i$ locally. The variation $\frac{d}{dt}\tilde{l}_t$ of $\tilde{l}_t$ is

$$T_{\tilde{l}_t}(T^*\Sigma) \ni \left(0, \frac{d}{dt}\alpha_t(x)\right) = \sum a_i \frac{\partial}{\partial \xi_i}(x, \alpha_t(x)).$$

Recall that the canonical symplectic form on $T^*\Sigma$ is given by $\omega_0 = \sum_i dx_i \wedge d\xi_i$. Hence we have

$$\frac{d}{dt}\tilde{l}_t \lrcorner \omega_0 = -\sum a_i(x)dx_i = -\frac{d}{dt}\alpha_t.$$

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