# Linearly degenerate partial differential equations and quadratic line complexes

### E.V. Ferapontov and J. Moss

A quadratic line complex is a three-parameter family of lines in projective space $\mathbb{P}^3$ specified by a single quadratic relation in the Plücker coordinates. Fixing a point $\mathbf{p}$ in $\mathbb{P}^3$ and taking all lines of the complex passing through $\mathbf{p}$ we obtain a quadratic cone with vertex at $\mathbf{p}$. This family of cones supplies $\mathbb{P}^3$ with a conformal structure, which can be represented in the form $f_{ij}(\mathbf{p})\, dp^i dp^j$ in a system of affine coordinates $\mathbf{p} = (p^1, p^2, p^3)$. With this conformal structure we associate a three-dimensional second-order quasilinear wave equation

$$\sum_{i,j} f_{ij}(u_{x_1}, u_{x_2}, u_{x_3}) u_{x_i x_j} = 0,$$

whose coefficients can be obtained from $f_{ij}(\mathbf{p})$ by setting $p^1 = u_{x_1}$, $p^2 = u_{x_2}$, $p^3 = u_{x_3}$. We show that any partial differential equations (PDE) arising in this way is linearly degenerate, furthermore, any linearly degenerate PDE can be obtained by this construction. This provides a classification of linearly degenerate wave equations into 11 types, labelled by Segre symbols of the associated quadratic complexes. We classify Segre types for which the structure $f_{ij}(\mathbf{p})\, dp^i dp^j$ is conformally flat, as well as Segre types for which the corresponding PDE is integrable.

## 1. Introduction

We study second-order quasilinear equations of the form
(1.1)
$$f_{11} u_{x_1 x_1} + f_{22} u_{x_2 x_2} + f_{33} u_{x_3 x_3} + 2 f_{12} u_{x_1 x_2} + 2 f_{13} u_{x_1 x_3} + 2 f_{23} u_{x_2 x_3} = 0,$$

where $u(x_1, x_2, x_3)$ is a function of three independent variables, and the coefficients $f_{ij}$ depend on the first-order derivatives $u_{x_1}, u_{x_2}, u_{x_3}$ only. Throughout the paper, we assume the nondegeneracy condition $\det f_{ij} \neq 0$. Partial differential equations (PDEs) of this type, which can be called quasilinear wave equations, arise in a wide range of applications in mechanics, general

relativity, differential geometry and the theory of integrable systems. The class of Equation (1.1) is invariant under the group $\mathbf{SL}(4)$ of linear transformations of the dependent and independent variables $x_1, x_2, x_3$ and $u$ that constitutes the natural equivalence group of the problem. Transformations from the equivalence group act projectively on the space $\mathbb{P}^3$ of first-order derivatives $p^i = u_{x_i}$, and preserve conformal class of the quadratic form

$$(1.2) \qquad\qquad\qquad f_{ij}(\mathbf{p})\, dp^i dp^j.$$

This correspondence between quasilinear wave equations and conformal structures in projective space was proposed and thoroughly investigated in [7].

In the present paper, we concentrate on the particular subclass of *linearly degenerate* equations (the concept of linear degeneracy is discussed in Section 2). Linearly degenerate PDEs are known to be quite exceptional from the point of view of solvability of the Cauchy problem: the gradient catastrophe, which is typical for genuinely nonlinear systems, does not occur, so that one has global existence of classical solutions with sufficiently small initial data.

Remarkably, linearly degenerate PDEs of the form (1.1) are naturally associated with quadratic complexes of lines in $\mathbb{P}^3$. In invariant terms, the construction can be summarized as follows. Consider a *line complex*, that is, a three-parameter family of lines in $\mathbb{P}^3$. Fixing a point $\mathbf{p}$ in $\mathbb{P}^3$ and taking all lines of the complex that pass through $\mathbf{p}$, we obtain a one-parameter family of lines which form a cone with vertex at $\mathbf{p}$. A complex is said to be *quadratic* if it is specified by a single quadratic equation in the Plücker coordinates or, equivalently, if the above cones are quadratic for all $\mathbf{p}$. This family of cones supplies $\mathbb{P}^3$ with a conformal structure which can be represented in the form (1.2) in any system of affine coordinates. Since large part of this paper is based on explicit calculations with quadratic complexes, we will need the coordinate version of the above construction.

Recall that the Plücker coordinates of a line in $\mathbb{P}^3$ through the points $\mathbf{p} = (p^1 : p^2 : p^3 : p^4)$ and $\mathbf{q} = (q^1 : q^2 : q^3 : q^4)$ are defined as $p^{ij} = p^i q^j - p^j q^i$. They satisfy the quadratic Plücker relation, $\Omega = p^{23}p^{14} + p^{31}p^{24} + p^{12}p^{34} = 0$, which defines a four-dimensional quadric in $\mathbb{P}^5$. A quadratic line complex is specified by an additional homogeneous quadratic relation among the Plücker coordinates,

$$Q(p^{ij}) = 0.$$

From algebro-geometric point of view, a quadratic line complex is a complete intersection of two quadrics in $\mathbb{P}^5$. Projective classification of quadratic complexes in $\mathbb{P}^3$ is therefore equivalent to the theory of normal forms of pencils of $6 \times 6$ symmetric matrices corresponding to the equation of the complex and the Plücker quadric, respectively. The associated conformal structure, also known as the Monge equation of the line complex ([18], p. 328), can be obtained by taking $\mathbf{q}$ "infinitesimally close" to $\mathbf{p}$, setting $q^i = p^i + dp^i$ and passing to a system of affine coordinates, say $p^4 = 1$, $dp^4 = 0$. Expressions for the Plücker coordinates take the form $p^{4i} = dp^i$, $p^{ij} = p^i dp^j - p^j dp^i$, $i, j = 1, 2, 3$, and the equation of the complex takes the so-called Monge form,

$$Q(dp^i, \ p^i dp^j - p^j dp^i) = f_{ij}(\mathbf{p}) \, dp^i dp^j = 0.$$

This provides the required conformal structure (1.2) and the associated Equation (1.1). Note that the coefficients $f_{ij}$ will automatically be quadratic in $\mathbf{p}$. We prove that this geometric construction gives *all* linearly degenerate PDEs of the form (1.1). Recall that the singular surface of the complex is a locus in $\mathbb{P}^3$, where conformal structure (1.2) degenerates, $\det f_{ij} = 0$. This surface is known to be Kummer's quartic with 16 double points, see [15], p. 762 for the modern treatment of this topic. It can be viewed as the locus where Equation (1.1) changes type.

Quadratic line complexes[1] have been extensively investigated in the classical works by Plücker [40], Kummer [24], Klein [22] and many other prominent geometers of 19th and 20th centuries, see also [4, 36] and references therein for more recent developments. Lie [25] studied certain classes of PDEs associated with line complexes. These included first-order PDEs governing surfaces which are tangential to the cones of the associated conformal structure, and second-order PDEs for surfaces whose asymptotic tangents belong to a given line complex (as well as surfaces conjugate to a given complex). Large part of this theory has nowadays become textbook material [15, 17, 18, 46]. Our first main result (Theorem 1 in Section 2.3) gives a characterization of PDEs (1.1) associated with quadratic complexes.

**Theorem 1.** *The following conditions are equivalent:*

(a) *Equation (1.1)/conformal structure (1.2) is associated with a quadratic line complex.*

---

[1]This terminology is due to the classical sources such as [18], as well as modern textbooks [5, 9]. The term "quadratic complex" is also adopted in most of the recent journal publications on the subject; note that [15] use 'quadric complex".

(b) *Equation (1.1) is linearly degenerate.*

(c) *Conformal structure (1.2) satisfies the condition*

(1.3) $$\partial_{(k}f_{ij)} = \varphi_{(k}f_{ij)}.$$

> *Here $\partial_k = \partial_{p^k}$, $\varphi = \varphi_k(\mathbf{p})\,dp^k$ is a covector and brackets denote complete symmetrization in $i, j, k \in \{1, 2, 3\}$. The above equivalence holds in any dimension $\geq 3$.*

Based on the projective classification of quadratic complexes by their Segre types (we follow Jessop [18], p. 206, which builds on the previous work by Klein [23], Segre [43] and Weiler [47]), we obtain a complete list of 11 normal forms of linearly degenerate PDEs (Theorem 2 of Section 3). Recall that the Segre type of a quadratic complex is uniquely determined by the Jordan normal form of the $6 \times 6$ matrix $Q\Omega^{-1}$ where $Q$ and $\Omega$ are the symmetric matrices of the complex and the Plücker quadric, see Section 3 for more details. For instance, the Segre symbol [111111] indicates that the operator $Q\Omega^{-1}$ has purely diagonal Jordan normal form, the Segre symbol [222] indicates that $Q\Omega^{-1}$ has three $2 \times 2$ Jordan blocks, etc. We will also use the "refined" Segre symbol which takes into account possible coincidences among the eigenvalues of $Q\Omega^{-1}$. Thus, the symbol [(11)(11)(11)] denotes the subcase of [111111] with three pairs of coinciding eigenvalues, the symbol [(111)(111)] denotes the subcase with two triples of coinciding eigenvalues, etc. Segre symbols are the key projective invariants of quadratic complexes. Note that complexes with the same Segre symbol do not need to be projectively equivalent: they may have additional continuous moduli (eigenvalues of $Q\Omega^{-1}$).

The most generic linearly degenerate PDE corresponds to the Segre symbol [111111]:

$$(a_1 + a_2 u_{x_3}^2 + a_3 u_{x_2}^2)u_{x_1 x_1} + (a_2 + a_1 u_{x_3}^2 + a_3 u_{x_1}^2)u_{x_2 x_2}$$
$$+ (a_3 + a_1 u_{x_2}^2 + a_2 u_{x_1}^2)u_{x_3 x_3} 2(\alpha u_{x_3} - a_3 u_{x_1} u_{x_2})u_{x_1 x_2}$$
$$+ 2(\beta u_{x_2} - a_2 u_{x_1} u_{x_3})u_{x_1 x_3} + 2(\gamma u_{x_1} - a_1 u_{x_2} u_{x_3})u_{x_2 x_3} = 0,$$

here $a_i, \alpha, \beta$ and $\gamma$ are constants such that $\alpha + \beta + \gamma = 0$. We point out that for generic values of constants this PDE is not integrable. The particular choice $\alpha = \beta = \gamma = 0$, $a_1 = a_2 = a_3 = 1$, leads to the (nonintegrable)

equation for minimal hypersurfaces in the Euclidean space $\mathbb{E}^4$,

$$(1 + u_{x_3}^2 + u_{x_2}^2)u_{x_1 x_1} + (1 + u_{x_3}^2 + u_{x_1}^2)u_{x_2 x_2} + (1 + u_{x_2}^2 + u_{x_1}^2)u_{x_3 x_3}$$
$$- 2u_{x_1}u_{x_2}u_{x_1 x_2} - 2u_{x_1}u_{x_3}u_{x_1 x_3} - 2u_{x_2}u_{x_3}u_{x_2 x_3} = 0,$$

while the choice $a_1 = a_2 = a_3 = 0$ results in the (integrable) nonlinear wave equation,

$$\alpha u_{x_3}u_{x_1 x_2} + \beta u_{x_2}u_{x_1 x_3} + \gamma u_{x_1}u_{x_2 x_3} = 0,$$

$\alpha + \beta + \gamma = 0$, which appeared in the context of Veronese webs in three dimensions [48], as well as in the theory of Einstein–Weyl geometries of hyper-CR type [11].

Theorem 3 of Section 3 gives a complete list of complexes with the flat conformal structure (1.2) (characterized by the vanishing of the Cotton tensor, which is responsible for conformal flatness in three dimensions): this result can be viewed as our contribution to the classical theory of quadratic complexes in $\mathbb{P}^3$.

**Theorem 3.** *A quadratic complex defines flat conformal structure if and only if its Segre symbol is one of the following:*

$$[111(111)]^*, \ [(111)(111)], \ [(11)(11)(11)],$$
$$[(11)(112)], \ [(11)(22)], \ [(114)], \ [(123)], \ [(222)], \ [(24)], \ [(33)].$$

*Here the asterisk denotes a particular subcase of* $[111(111)]$*, where the matrix* $Q\Omega^{-1}$ *has eigenvalues* $(1, \epsilon, \epsilon^2, 0, 0, 0)$*,* $\epsilon^3 = 1$*.*

Theorem 4 of Section 3 gives a complete list of normal forms of linearly degenerate *integrable* equations of the form (1.1). In general, the integrability aspects of quasilinear wave Equations (1.1) (not necessarily linearly degenerate) were investigated in [7], based on the method of hydrodynamic reductions [13]. It was shown that the moduli space of integrable equations is 20-dimensional. The recent result of [14], stating that the integrability of a PDE of the form (1.1) is equivalent to the Einstein–Weyl property of the symbol of its formal linearization, provides an efficient geometric test of integrability. It was demonstrated in [37] that the coefficients of "generic" integrable equations (1.1) can be parametrized by generalized hypergeometric functions. For linearly degenerate PDEs, the integrability is also equivalent

to the existence of a linear scalar Lax pair of the form

$$\psi_{x_2} = f(u_{x_1}, u_{x_2}, u_{x_3}, \lambda)\psi_{x_1}, \quad \psi_{x_3} = g(u_{x_1}, u_{x_2}, u_{x_3}, \lambda)\psi_{x_1},$$

where $\lambda$ is an auxiliary spectral parameter, so that (1.1) follows from the compatibility condition $\psi_{x_2x_3} = \psi_{x_3x_2}$. It was pointed out in [7] that flatness of the conformal structure (1.2) is the necessary condition for integrability (this simplifies the classification of integrable equations, indeed, the corresponding complexes must be contained in the list of Theorem 3). The main result reads as follows.

**Theorem 4.** *A quadratic complex corresponds to an integrable PDE if and only if its Segre symbol is one of the following:*

$$[(11)(11)(11)], \ [(11)(112)], \ [(11)(22)], \ [(123)], \ [(222)], \ [(33)].$$

*Modulo equivalence transformations (which are allowed to be complex-valued) this leads to a complete list of normal forms of linearly degenerate integrable PDEs:*
**Segre symbol** $[(11)(11)(11)]$

$$\alpha u_{x_3} u_{x_1x_2} + \beta u_{x_2} u_{x_1x_3} + \gamma u_{x_1} u_{x_2x_3} = 0, \ \alpha + \beta + \gamma = 0;$$

**Segre symbol** $[(11)(112)]$

$$u_{x_1x_1} + u_{x_1} u_{x_2x_3} - u_{x_2} u_{x_1x_3} = 0;$$

**Segre symbol** $[(11)(22)]$

$$u_{x_1x_2} + u_{x_2} u_{x_1x_3} - u_{x_1} u_{x_2x_3} = 0;$$

**Segre symbol** $[(123)]$

$$u_{x_2x_2} + u_{x_1x_3} + u_{x_2} u_{x_3x_3} - u_{x_3} u_{x_2x_3} = 0;$$

**Segre symbol** $[(222)]$

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = 0;$$

**Segre symbol** $[(33)]$

$$u_{x_1x_3} + u_{x_1} u_{x_2x_2} - u_{x_2} u_{x_1x_2} = 0.$$

In different contexts, the canonical forms of Theorem 4 have appeared in [1, 10, 30, 31, 34, 37–39, 48]. In particular, the same normal forms appeared in [37] in the alternative approach to linear degeneracy based on the requirement of "nonsingular" structure of generalized Gibbons–Tsarev systems which governs hydrodynamic reductions of PDEs in question. It was demonstrated recently in [39] that all (nonlinear) equations from Theorem 4 are members of the same integrable hierarchy.

Section 4 contains remarks about the Cauchy problem for linearly degenerate PDEs. We observe that for some linearly degenerate PDEs (1.1), the coefficients $f_{ij}$ can be represented in the form $f_{ij} = \eta_{ij} + \varphi_{ij}$ where $\eta$ is a constant-coefficient matrix with diagonal entries $1, -1, -1$, while $\varphi_{ij}$ vanish at the "origin" $u_{x_1} = u_{x_2} = u_{x_3} = 0$. PDEs of this type can be viewed as nonlinear perturbations of the linear wave equation. Under the so-called "null conditions" of Klainerman and Alinhac, the papers [3, 8, 19, 21] establish global existence of smooth solutions with small initial data for multi-dimensional nonlinear wave equations. It remains to point out that both null conditions are automatically satisfied for linearly degenerate PDEs: they follow from condition (1.3) satisfied in the vicinity of the origin. Our numerical simulations clearly demonstrate that solutions with small initial data do not break down, and behave essentially like solutions to the linear wave equation.

## 2. Linearly degenerate PDEs

In this section, we discuss the concept of linear degeneracy for multi-dimensional second-order PDEs. After recalling the definition of linear degeneracy for first-order quasilinear systems (Section 2.1), we extend it to second-order quasilinear PDEs in two dimensions (Section 2.2). In higher dimensions, the property of linear degeneracy is defined by the requirement of linear degeneracy of all travelling wave reductions of a given PDE to two dimensions. This leads to the constraint (1.3) which characterizes conformal structures coming from quadratic line complexes (Theorem 1 of Section 2.3).

### 2.1. Linearly degenerate first-order quasilinear systems

Let us consider a quasilinear system

$$\mathbf{v}_t + A(\mathbf{v})\mathbf{v}_x = 0,$$

where $\mathbf{v} = (v^1, \ldots, v^n)$ is the vector of dependent variables, $A$ is an $n \times n$ matrix, and $t, x$ are independent variables. Recall that $A$ is said to be

*linearly degenerate* if its eigenvalues, assumed real and distinct, are constant in the direction of the corresponding eigenvectors. Explicitly, $L_{r^i}\lambda^i = 0$, no summation, where $L_{r^i}$ is Lie derivative of the eigenvalue $\lambda^i$ in the direction of the corresponding eigenvector $r^i$. Linearly degenerate systems are quite exceptional from the point of view of solvability of the initial value problem, and have been thoroughly investigated in literature, see e.g., [27, 41, 42, 44]. There exists a simple invariant criterion of linear degeneracy which does not appeal to eigenvalues/eigenvectors. Let us introduce the characteristic polynomial of $A$,

$$\det(\lambda E - A(\mathbf{v})) = \lambda^n + f_1(\mathbf{v})\lambda^{n-1} + f_2(\mathbf{v})\lambda^{n-2} + \cdots + f_n(\mathbf{v}).$$

The condition of linear degeneracy can be represented in the form [12]

$$\nabla f_1 \; A^{n-1} + \nabla f_2 \; A^{n-2} + \cdots + \nabla f_n = 0,$$

where $\nabla$ is the gradient, $\nabla f = (\frac{\partial f}{\partial v^1}, \ldots, \frac{\partial f}{\partial v^n})$ and $A^k$ denotes $k$th power of the matrix $A$. In the $2 \times 2$ case this condition simplifies to

$$(2.1) \qquad\qquad \nabla(tr A) \; A = \nabla(\det A).$$

## 2.2. Linearly degenerate second-order PDEs in two dimensions

Here we consider second-order equations of the form

$$(2.2) \qquad f_{11}(u_t, u_x)u_{tt} + 2f_{12}(u_t, u_x)u_{tx} + f_{22}(u_t, u_x)u_{xx} = 0.$$

Setting $u_t = p^1$ and $u_x = p^2$, we obtain the equivalent first-order quasilinear representation

$$(2.3) \qquad p_x^1 = p_t^2, \; f_{11}(p^1, p^2)p_t^1 + 2f_{12}(p^1, p^2)p_x^1 + f_{22}(p^1, p^2)p_x^2 = 0.$$

We will call PDE (2.2) *linearly degenerate* if this is the case for the corresponding quasilinear system (2.3). With $\mathbf{v} = (p^1, p^2)$, the constraint (2.1) leads to the conditions of linear degeneracy in the form

$$(2.4) \qquad 2\partial_1\left(\frac{f_{12}}{f_{11}}\right) + \partial_2\left(\ln\frac{f_{11}}{f_{22}}\right) = 0, \quad 2\partial_2\left(\frac{f_{12}}{f_{22}}\right) + \partial_1\left(\ln\frac{f_{22}}{f_{11}}\right) = 0,$$

$\partial_k = \partial_{p^k}$. In equivalent form, these conditions appeared in [32], where they were solved implicitly leading to the following result (see also [16, 33, 35] for related work).

**Proposition 1.** *A generic linearly degenerate PDE of the form (2.2) can be represented in the form*

$$(2.5) \qquad\qquad u_{tt} - (v + w)u_{tx} + vwu_{xx} = 0,$$

*where the coefficients $v(u_t, u_x)$ and $w(u_t, u_x)$ are defined by the implicit relations*

$$(2.6) \qquad\qquad f(v) = vu_x - u_t, \quad g(w) = wu_x - u_t,$$

*here $f, g$ are two arbitrary functions. Furthermore, by virtue of (2.5), coefficients $v(u_t, u_x)$ and $w(u_t, u_x)$ satisfy the equations*

$$(2.7) \qquad\qquad v_t = wv_x, \quad w_t = vw_x.$$

*Formulae (2.6) establish a Bäcklund transformation between the second-order PDE (2.5) and the linearly degenerate system (2.7).*

*Proof.* Setting in (2.4) $f_{11} = 1$, $f_{12} = -(v + w)/2$, $f_{22} = vw$, one obtains a pair of uncoupled Hopf equations for $v$ and $w$: $\partial_2 v + v\partial_1 v = 0$, $\partial_2 w + w\partial_1 w = 0$. Their implicit solution leads to the formula (2.6). Finally, differentiating (2.6) by $t$ and $x$ one arrives at (2.7). Notice that relations (2.6) can be rewritten in the form

$$u_t = \frac{wf(v) - vg(w)}{v - w}, \quad u_x = \frac{f(v) - g(w)}{v - w},$$

where the consistency condition,

$$\left(\frac{f(v) - g(w)}{v - w}\right)_t = \left(\frac{wf(v) - vg(w)}{v - w}\right)_x,$$

constitutes the general conservation law of the linearly degenerate system (2.7). Thus, (2.5) can be interpreted as the equation for the corresponding potential variable $u$, so that any two equations of the form (2.5) are equivalent to each other. This finishes the proof. $\square$

**Proposition 2.** *A PDE of the form (2.2) is linearly degenerate if and only if the null curves of the corresponding conformal structure $f_{11}(p^1, p^2)(dp^1)^2 + 2f_{12}(p^1, p^2)\, dp^1 dp^2 + f_{22}(p^1, p^2)(dp^2)^2$ are straight lines (in projective space $\mathbb{P}^2$ with affine coordinates $p^1, p^2$).*

*Proof.* Setting $f_{11} = 1$, $f_{12} = -(v+w)/2$, $f_{22} = vw$ (see the proof of Proposition 1), we can rewrite the conformal structure in factorized form

$$(dp^1 - vdp^2)(dp^1 - wdp^2).$$

Thus, the null curves are defined by the equations $dp^1 - vdp^2 = 0$ and $dp^1 - wdp^2 = 0$, respectively. It remains to note that the equations $\partial_2 v + v\partial_1 v = 0$, $\partial_2 w + w\partial_1 w = 0$ are equivalent to the condition that these curves are straight lines: indeed, they can be written as $dv \wedge (dp^1 - vdp^2) = 0$, $dw \wedge (dp^1 - wdp^2) = 0$. This finishes the proof.          $\square$

**Example.** For $f(v) = \sqrt{1-v^2}$, $g(w) = \sqrt{1-w^2}$, relations (2.6) reduce to one and the same quadratic equation for $v$ and $w$. Taking two different roots of this equation results in the so-called Born–Infeld equation,

$$(1 + u_x^2)u_{tt} - 2u_t u_x u_{tx} + (u_t^2 - 1)u_{xx} = 0,$$

which is the Euler–Lagrange equation for the area functional $\int \sqrt{1 + u_x^2 - u_t^2}\, dx\, dt$ governing minimal surfaces in Minkowski space.

**Remark.** Conditions (2.4) can be represented in tensorial form

$$\partial_{(k} f_{ij)} = \varphi_{(k} f_{ij)}.$$

Here $\varphi = \varphi_k dp^k$ is a covector, and brackets denote complete symmetrization in the indices $i, j, k \in \{1, 2\}$. Explicitly, this gives

(2.8)
$$\partial_1 f_{11} = \varphi_1 f_{11}, \quad \partial_2 f_{22} = \varphi_2 f_{22},$$
$$\partial_2 f_{11} + 2\partial_1 f_{12} = \varphi_2 f_{11} + 2\varphi_1 f_{12}, \quad \partial_1 f_{22} + 2\partial_2 f_{12} = \varphi_1 f_{22} + 2\varphi_2 f_{12},$$

and the elimination of $\varphi_1, \varphi_2$ from the first two relations results in (2.4).

### 2.3. Linearly degenerate second-order PDEs in three dimensions and quadratic line complexes

A three-dimensional (3D) PDE of the form (1.1) is said to be *linearly degenerate* if all its travelling wave reductions to two dimensions are linearly degenerate in the sense of Section 2.2. More precisely, setting $u(x_1, x_2, x_3) = v(\xi, \eta) + \zeta$, where $\xi, \eta, \zeta$ are arbitrary linear forms in the variables $x_i$, we

obtain a two-dimensional equation of the form (2.2) for $v(\xi, \eta)$. The requirement of linear degeneracy of *all* such reductions imposes strong constraints on the coefficients $f_{ij}$:

**Theorem 1.**  *The following conditions are equivalent:*

(a) *Equation (1.1)/conformal structure (1.2) is associated with a quadratic line complex.*

(b) *Equation (1.1) is linearly degenerate.*

(c) *Conformal structure (1.2) satisfies the condition (1.3),*

$$\partial_{(k}f_{ij)} = \varphi_{(k}f_{ij)}.$$

*Here $\partial_k = \partial_{p^k}$, $\varphi = \varphi_k(\mathbf{p})\,dp^k$ is a covector and brackets denote complete symmetrization in $i, j, k \in \{1, 2, 3\}$. The above equivalence holds in any dimension $\geq 3$.*

*Proof.* The equivalence of (b) and (c) can be seen as follows. Let us seek travelling wave reductions in the form $u(x_1, x_2, x_3) = v(\xi, \eta) + \alpha x_1 + \beta x_2 + \gamma x_3$, where $\xi = x_1 + \lambda x_3$, $\eta = x_2 + \mu x_3$ and $\alpha, \beta, \gamma, \lambda, \mu$ are arbitrary constants. We have

$$u_{x_1} = v_\xi + \alpha, \quad u_{x_2} = v_\eta + \beta, \quad u_{x_3} = \lambda v_\xi + \mu v_\eta + \gamma,$$

as well as

$$u_{x_1 x_1} = v_{\xi\xi}, \quad u_{x_1 x_2} = v_{\xi\eta}, \quad u_{x_2 x_2} = v_{\eta\eta},$$
$$u_{x_1 x_3} = \lambda v_{\xi\xi} + \mu v_{\xi\eta}, \; u_{x_2 x_3} = \lambda v_{\xi\eta} + \mu v_{\eta\eta}, \; u_{x_3 x_3} = \lambda^2 v_{\xi\xi} + 2\lambda\mu v_{\xi\eta} + \mu^2 v_{\eta\eta}.$$

The reduced Equation (1.1) takes the form

$$a v_{\xi\xi} + 2b v_{\xi\eta} + c v_{\eta\eta} = 0,$$

where

$$a = f_{11} + 2\lambda f_{13} + \lambda^2 f_{33}, \quad b = f_{12} + \lambda f_{23} + \mu f_{13} + \lambda\mu f_{33},$$
$$c = f_{22} + 2\mu f_{23} + \mu^2 f_{33},$$

we point out that the coefficients $a, b, c$ are now viewed as functions of $v_\xi$ and $v_\eta$. For the reduced equation, conditions of linear degeneracy (2.8) take

the form

$$\partial_{v_\xi} a = \varphi_1 a, \quad \partial_{v_\eta} c = \varphi_2 c, \quad \partial_{v_\eta} a + 2\partial_{v_\xi} b = \varphi_2 a + 2\varphi_1 b,$$
$$\partial_{v_\xi} c + 2\partial_{v_\eta} b = \varphi_1 c + 2\varphi_2 b.$$

Let us take the first condition, $\partial_{v_\xi} a = \varphi_1 a$. The calculation of $\partial_{v_\xi} a$ gives

$$\partial_{u_\xi} a = \partial_1 f_{11} + \lambda\partial_3 f_{11} + 2\lambda(\partial_1 f_{13} + \lambda\partial_3 f_{13}) + \lambda^2(\partial_1 f_{33} + \lambda\partial_3 f_{33}),$$

which is polynomial in $\lambda$ of degree three. We point out that, due to the presence of arbitrary constants $\alpha, \beta, \gamma$ in the expressions for $u_{x_1}, u_{x_2}, u_{x_3}$ and the coefficients of this polynomial can be viewed as independent of $\lambda, \mu$. Thus, $\varphi_1$ must be linear in $\lambda$, so that we can set $\varphi_1 \to \varphi_1 + \lambda\varphi_3$ (keeping the same notation $\varphi_1$ for the first term). Ultimately, the relation $\partial_{v_\xi} a = \varphi_1 a$ takes the form

$$\partial_1 f_{11} + \lambda\partial_3 f_{11} + 2\lambda(\partial_1 f_{13} + \lambda\partial_3 f_{13}) + \lambda^2(\partial_1 f_{33} + \lambda\partial_3 f_{33})$$
$$= (\varphi_1 + \lambda\varphi_3)(f_{11} + 2\lambda f_{13} + \lambda^2 f_{33}).$$

Equating terms at different powers of $\lambda$ we obtain four relations,

$$\partial_1 f_{11} = \varphi_1 f_{11}, \quad \partial_3 f_{33} = \varphi_3 f_{33},$$
$$\partial_3 f_{11} + 2\partial_1 f_{13} = \varphi_3 f_{11} + 2\varphi_1 f_{13}, \quad \partial_1 f_{33} + 2\partial_3 f_{13} = \varphi_1 f_{33} + 2\varphi_3 f_{13}.$$

Similar analysis of the three remaining conditions of linear degeneracy of the reduced equation (where one should set $\varphi_2 \to \varphi_2 + \mu\varphi_3$) leads to the full set (1.3) of conditions of linear degeneracy in 3D:

$$\partial_1 f_{11} = \varphi_1 f_{11}, \quad \partial_2 f_{22} = \varphi_2 f_{22}, \quad \partial_3 f_{33} = \varphi_3 f_{33},$$
$$\partial_2 f_{11} + 2\partial_1 f_{12} = \varphi_2 f_{11} + 2\varphi_1 f_{12}, \quad \partial_1 f_{22} + 2\partial_2 f_{12} = \varphi_1 f_{22} + 2\varphi_2 f_{12},$$
$$\partial_3 f_{11} + 2\partial_1 f_{13} = \varphi_3 f_{11} + 2\varphi_1 f_{13}, \quad \partial_1 f_{33} + 2\partial_3 f_{13} = \varphi_1 f_{33} + 2\varphi_3 f_{13},$$
$$\partial_2 f_{33} + 2\partial_3 f_{23} = \varphi_2 f_{33} + 2\varphi_3 f_{23}, \quad \partial_3 f_{22} + 2\partial_2 f_{23} = \varphi_3 f_{22} + 2\varphi_2 f_{23},$$
$$\partial_1 f_{23} + \partial_2 f_{13} + \partial_3 f_{12} = \varphi_1 f_{23} + \varphi_2 f_{13} + \varphi_3 f_{12}.$$

On elimination of $\varphi$'s, these conditions give rise to seven first-order differential constraints for $f_{ij}$.

The equivalence of (a) and (c) follows from [2, 45]. Indeed, (1.3) means that the conformal structure possesses a quadratic complex of null lines. This can be most easily seen as follows. Take a projective space $\mathbb{P}^3$ with the conformal structure corresponding to a linearly degenerate PDE (1.1).

A travelling wave reduction of (1.1) corresponds to a plane, $\mathbb{P}^2 \subset \mathbb{P}^3$, with induced conformal structure. Since the reduced PDE is linearly degenerate, each $\mathbb{P}^2$ will carry two one-parameter families of null lines (Proposition 2 of Section 2.2), which will automatically be null lines of the ambient conformal structure in $\mathbb{P}^3$. Varying a travelling wave reduction (which corresponds to varying $\mathbb{P}^2 \subset \mathbb{P}^3$), we will get a three-parameter family of null lines of the ambient conformal structure. Thus, we obtain a quadratic complex of null lines. This finishes the proof of Theorem 1.                      $\square$

The general solution of conditions of linear degeneracy (1.3) can be obtained as follows. In $\mathbb{P}^3$, the Plücker coordinates of a line through the points $\mathbf{p} = (p^1 : p^2 : p^3 : p^4)$ and $\mathbf{q} = (q^1 : q^2 : q^3 : q^4)$ are six $2 \times 2$ minors of the matrix

$$\begin{pmatrix} p^1 & p^2 & p^3 & p^4 \\ q^1 & q^2 & q^3 & q^4 \end{pmatrix},$$

explicitly, $p^{ij} = p^i q^j - p^j q^i$. These coordinates are known to satisfy the quadratic Plücker relation, $\Omega = p^{23} p^{14} + p^{31} p^{24} + p^{12} p^{34} = 0$. A quadratic complex is a three-parameter family of lines in $\mathbb{P}^3$ specified by an additional homogeneous quadratic relation among the Plücker coordinates,

$$Q(p^{ij}) = 0.$$

Fixing a point $\mathbf{p}$ in $\mathbb{P}^3$ and taking the lines of the complex which pass through $\mathbf{p}$ one obtains a quadratic cone with vertex at $\mathbf{p}$. The family of these cones supplies $\mathbb{P}^3$ with a conformal structure. Its equation can be obtained by setting $q^i = p^i + dp^i$ and passing to a system of affine coordinates, say, $p^4 = 1$, $dp^4 = 0$. The expressions for the Plücker coordinates take the form $p^{4i} = dp^i$, $p^{ij} = p^i dp^j - p^j dp^i$, $i, j = 1, 2, 3$, and the equation of the complex takes the so-called Monge form,

$$Q(dp^i, \ p^i dp^j - p^j dp^i) = f_{ij}(\mathbf{p}) \, dp^i dp^j = 0.$$

This provides the required conformal structure, which we always assume to be nondegenerate (that is, $\det f_{ij}$ is not identically zero). Since a homogeneous quadratic form $Q$ in six variables depends, modulo a constant factor and a multiple of $\Omega$, on 19 arbitrary parameters, we obtain the 19-parameter generic solution of relations (1.3), that is, the 19-parameter family of linearly degenerate PDEs (1.1). The singular surface of the complex is a locus in $\mathbb{P}^3$ where the conformal structure degenerates, $\det f_{ij} = 0$. This is known to be a quartic with 16 ordinary double points (Kummer's quartic). It can be viewed as the locus where PDE (1.1) changes its type.

**Example 1.** The so-called tetrahedral complex, see [18], Chapter VII, is defined by the equation

$$Q = b_1 p^{41} p^{23} + b_2 p^{42} p^{31} + b_3 p^{43} p^{12} = 0.$$

Its Monge form is

$$b_1 dp^1 (p^2 dp^3 - p^3 dp^2) + b_2 dp^2 (p^3 dp^1 - p^1 dp^3) + b_3 dp^3 (p^1 dp^2 - p^2 dp^1) = 0,$$

or, equivalently,

$$(b_3 - b_2) p^1 dp^2 dp^3 + (b_1 - b_3) p^2 dp^1 dp^3 + (b_2 - b_1) p^3 dp^1 dp^2 = 0.$$

It corresponds to the "nonlinear wave equation",

$$(b_3 - b_2) u_{x_1} u_{x_2 x_3} + (b_1 - b_3) u_{x_2} u_{x_1 x_3} + (b_2 - b_1) u_{x_3} u_{x_1 x_2} = 0,$$

which probably first appeared in [48] in the context of Veronese webs in 3D. The associated Kummer surface degenerates into four planes: $p^1 = 0$, $p^2 = 0$, $p^3 = 0$, plus the plane at infinity. The lines forming tetrahedral complex are characterized by the property that their four points of intersection with the above planes have constant cross-ratio (determined by $b_1, b_2, b_3$). Introducing the 1-form

$$\omega = (\lambda - b_2)(\lambda - b_3) u_{x_1} dx^1 + (\lambda - b_1)(\lambda - b_3) u_{x_2} dx^2 \\ + (\lambda - b_1)(\lambda - b_2) u_{x_3} dx^3,$$

which depends quadratically on the auxiliary "spectral parameter" $\lambda$, one can verify that the above PDE is equivalent to the condition $d\omega \wedge \omega = 0$. Thus, the foliation $\omega = 0$ is integrable for any value of $\lambda$, and defines the structure known as a 3D Veronese web.

**Example 2.** The so-called special complex, see [18], Chapter VII, is defined by the equation

$$Q = (p^{12})^2 + (p^{13})^2 + (p^{23})^2 - (p^{14})^2 - (p^{24})^2 - (p^{34})^2 = 0.$$

Its Monge form is

$$(p^1 dp^2 - p^2 dp^1)^2 + (p^1 dp^3 - p^3 dp^1)^2 + (p^2 dp^3 - p^3 dp^2)^2 - (dp^1)^2 - (dp^2)^2 \\ - (dp^3)^2 = 0,$$

or, equivalently,

$$(p^{2^2} + p^{3^2} - 1)dp^{1^2} + (p^{1^2} + p^{3^2} - 1)dp^{2^2} + (p^{1^2} + p^{2^2} - 1)dp^{3^2}$$
$$- 2p^1 p^2 dp^1 dp^2 - 2p^1 p^3 dp^1 dp^3 - 2p^2 p^3 dp^2 dp^3 = 0.$$

It corresponds to the equation

$$(u_{x_2}^2 + u_{x_3}^2 - 1)u_{x_1 x_1} + (u_{x_1}^2 + u_{x_3}^2 - 1)u_{x_2 x_2} + (u_{x_1}^2 + u_{x_2}^2 - 1)u_{x_3 x_3}$$
$$- 2u_{x_1} u_{x_2} u_{x_1 x_2} - 2u_{x_1} u_{x_3} u_{x_1 x_3} - 2u_{x_2} u_{x_3} u_{x_2 x_3} = 0,$$

which comes from the Lagrangian $\int \sqrt{u_{x_1}^2 + u_{x_2}^2 + u_{x_3}^2 - 1}\, dx$ governing minimal hypersurfaces $x_4 = u(x_1, x_2, x_3)$ in Minkowski space with the metric $-dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$. The associated Kummer surface is the double sphere $p^{1^2} + p^{2^2} + p^{3^2} = 1$. The external part of the sphere is the domain of hyperbolicity of our equation: quadratic cones of the complex are tangential to the sphere. We point out that the equation for minimal hypersurfaces is not integrable in dimensions higher than two.

## 3. Normal forms of quadratic line complexes and linearly degenerate PDEs

In this section, we utilize the projective classification of quadratic line complexes following [18]. Complexes are characterized by the so-called Segre symbols which govern normal forms of pairs of the associated quadratic forms. To be more precise, let $\Omega$ and $Q$ be the $6 \times 6$ symmetric matrices of quadratic forms specifying the Plücker quadric and the complex. Then, for instance, the Segre symbol [111111] means that the operator $Q\Omega^{-1}$ has purely diagonal Jordan normal form, the Segre symbol [222] means that the operator $Q\Omega^{-1}$ has three $2 \times 2$ Jordan blocks, etc. Since $Q$ is defined up to transformations of the form $Q \to \alpha Q + \beta \Omega$, we can always assume $Q\Omega^{-1}$ to be traceless. The associated conformal structures and the corresponding linearly degenerate PDEs result from the equation of the complex upon setting $p^{ij} = p^i dp^j - p^j dp^i$, $p^4 = 1$, $dp^4 = 0$, as explained in Section 2.3 (in some cases it will be more convenient to use different affine projections, say, $p^1 = 1$, $dp^1 = 0$: this will be indicated explicitly where appropriate). Here is the summary of our results. Theorem 2 gives a complete list of normal forms of linearly degenerate PDEs based on the classification of quadratic complexes (for simplicity, we use the notation $u_{x_i} = u_i$, $u_{x_i x_j} = u_{ij}$, etc). Theorem 3 provides a classification of complexes with the flat conformal structure $f_{ij} dp^i dp^j$, and Theorem 4 characterizes complexes corresponding

to integrable PDEs. Theorems 2 to 4 will be proved simultaneously by going through the list of normal forms of quadratic complexes.

**Theorem 2.** *Any linearly degenerate PDE of the form (1.1) can be brought by an equivalence transformation to one of the 11 canonical forms, labelled by Segre symbols of the associated quadratic complexes.*

*Case 1:* **Segre symbol** $[111111]$

$$(a_1 + a_2 u_3^2 + a_3 u_2^2)u_{11} + (a_2 + a_1 u_3^2 + a_3 u_1^2)u_{22} + (a_3 + a_1 u_2^2 + a_2 u_1^2)u_{33}$$
$$+ 2(\alpha u_3 - a_3 u_1 u_2)u_{12} + 2(\beta u_2 - a_2 u_1 u_3)u_{13} + 2(\gamma u_1 - a_1 u_2 u_3)u_{23}$$
$$= 0,$$

$\alpha + \beta + \gamma = 0.$

*Case 2:* **Segre symbol** $[11112]$

$$(\lambda u_2^2 + \mu u_3^2 + 1)u_{11} + (\lambda u_1^2 + \mu)u_{22} + (\mu u_1^2 + \lambda)u_{33}$$
$$+ 2(\alpha u_3 - \lambda u_1 u_2)u_{12} + 2(\beta u_2 - \mu u_1 u_3)u_{13} + 2\gamma u_1 u_{23} = 0,$$

$\alpha + \beta + \gamma = 0.$

*Case 3:* **Segre symbol** $[1113]$

$$(\lambda u_2^2 + \mu u_3^2 + 2u_3)u_{11} + (\lambda u_1^2 + \mu)u_{22} + (\mu u_1^2 + \lambda)u_{33}$$
$$+ 2(\mu u_3 - \lambda u_1 u_2 - 1)u_{12} + 2(\beta u_2 - \mu u_1 u_3 - u_1)u_{13} + 2\gamma u_1 u_{23} = 0,$$

$\mu + \beta + \gamma = 0.$

*Case 4:* **Segre symbol** $[1122]$

$$(\lambda u_2^2 + 1)u_{11} + (\lambda u_1^2 + 4)u_{22} + \lambda u_{33} + 2(\alpha u_3 - \lambda u_1 u_2)u_{12} + 2\beta u_2 u_{13}$$
$$+ 2\gamma u_1 u_{23} = 0,$$

$\alpha + \beta + \gamma = 0.$

*Case 5:* **Segre symbol** $[114]$

$$\lambda u_{11} + (\lambda u_3^2 + 4)u_{22} + (\lambda u_2^2 - 2u_1)u_{33} + 2\alpha u_3 u_{12} + 2(u_3 - \alpha u_2)u_{13}$$
$$- 2\lambda u_2 u_3 u_{23} = 0.$$

*Case 6:* **Segre symbol** [123]

$$\lambda u_{11} + (\lambda u_3^2 + 4)u_{22} + [\lambda u_2^2 + 2u_2]u_{33} + 2\alpha u_3 u_{12} + 2(1 - \lambda u_2)u_{13}$$
$$+ 2(\gamma u_1 - \lambda u_2 u_3 - u_3)u_{23} = 0,$$

$\alpha - \lambda + \gamma = 0.$

*Case 7:* **Segre symbol** [222]
*Subcase 1:*

$$u_{11} + u_{22} + u_{33} + 2\alpha u_3 u_{12} + 2\beta u_2 u_{13} + 2\gamma u_1 u_{23} = 0,$$

*Subcase 2:*

$$(u_2^2 + u_3^2)u_{11} + (u_1^2 + u_3^2)u_{22} + (u_1^2 + u_2^2)u_{33} + 2(\alpha u_3 - u_1 u_2)u_{12}$$
$$+ 2(\beta u_2 - u_1 u_3)u_{13} + 2(\gamma u_1 - u_2 u_3)u_{23} = 0,$$

$\alpha + \beta + \gamma = 0.$

*Case 8:* **Segre symbol** [15]

$$\lambda u_{11} + (\lambda u_3^2 - 2u_3)u_{22} + (\lambda u_2^2 - 4u_1)u_{33} + 2(\lambda u_3 + 1)u_{12} + 2(2u_3 - \lambda u_2)u_{13}$$
$$+ 2(u_2 - \lambda u_2 u_3)u_{23} = 0.$$

*Case 9:* **Segre symbol** [24]
*Subcase 1:*

$$u_{11} + u_{22} - 2u_1 u_{33} + 2\lambda u_3 u_{12} + 2(u_3 - \lambda u_2)u_{13} = 0.$$

*Subcase 2:*

$$u_3^2 u_{22} + (1 + u_2^2)u_{33} + 2u_{12} + 2\lambda u_2 u_{13} - 2(\lambda u_1 + u_2 u_3)u_{23} = 0.$$

*Case 10:* **Segre symbol** [33]

$$\lambda u_{11} + (\lambda u_3^2 - 2u_3)u_{22} + (\lambda u_2^2 - 2u_2)u_{33} + 2(\lambda u_3 + 1)u_{12} + 2(\lambda u_2 + 1)u_{13}$$
$$- 2(2\lambda u_1 + \lambda u_2 u_3 - u_2 - u_3)u_{23} = 0.$$

*Case 11:* **Segre symbol** [6]
*Subcase 1:*

$$2u_3u_{11} + u_{22} + 2u_2u_{33} - 2u_1u_{13} - 2u_3u_{23} = 0.$$

*Subcase 2:*

$$(u_3^2 - 2u_2)u_{11} - 2u_3u_{22} + u_1^2u_{33} + 2u_1u_{12} - 2u_1u_3u_{13} + 2u_2u_{23} = 0.$$

Calculating the Cotton tensor (whose vanishing is responsible for conformal flatness in three dimensions) we obtain a complete list of quadratic complexes with the flat conformal structure. Recall that the flatness of $f_{ij}dp^idp^j$ is a necessary condition for integrability of the corresponding PDE [7]. We observe that the requirement of conformal flatness imposes further constraints on the parameters appearing in cases 1 to 11 of Theorem 2, which are characterized by certain coincidences among eigenvalues of the corresponding Jordan normal forms of $Q\Omega^{-1}$ (some Segre types do not possess conformally flat specializations at all). In what follows we label conformally flat subcases by their "refined" Segre symbols, e.g., the symbol $[(11)(11)(11)]$ denotes the subcase of $[111111]$ with three pairs of coinciding eigenvalues, the symbol $[(111)(111)]$ denotes the subcase with two triples of coinciding eigenvalues, etc. see [18]. Although the subject is classical, the following result is new.

**Theorem 3.** *A quadratic complex defines flat conformal structure if and only if its Segre symbol is one of the following:*

$$[111(111)]^*, \quad [(111)(111)], \quad [(11)(11)(11)],$$
$$[(11)(112)], [(11)(22)], [(114)], [(123)], [(222)], [(24)], [(33)].$$

*Here the asterisk denotes a particular subcase of* $[111(111)]$ *where the matrix* $Q\Omega^{-1}$ *has eigenvalues* $(1, \epsilon, \epsilon^2, 0, 0, 0)$, $\epsilon^3 = 1$. *Modulo equivalence transformations this gives the following list of normal forms of the associated PDEs:*

**Segre symbol** $[111(111)]^*$

$$(1 - 2u_2u_3)u_{11} + (1 - 2u_1u_3)u_{22} + 2(u_1 - u_2)u_{33} + 2(1 + u_1u_3 + u_2u_3)u_{12}$$
$$+ 2(u_1u_2 - u_3 - u_2^2)u_{13} + 2(u_1u_2 + u_3 - u_1^2)u_{23} = 0,$$

**Segre symbol** $[(111)(111)]$

$$(u_2^2 + u_3^2 - 1)u_{11} + (u_1^2 + u_3^2 - 1)u_{22} + (u_1^2 + u_2^2 - 1)u_{33} - 2u_1 u_2 u_{12}$$
$$- 2u_1 u_3 u_{13} - 2u_2 u_3 u_{23} = 0,$$

**Segre symbol** $[(11)(11)(11)]$

$$\alpha u_3 u_{12} + \beta u_2 u_{13} + \gamma u_1 u_{23} = 0, \ \alpha + \beta + \gamma = 0,$$

**Segre symbol** $[(11)(112)]$

$$u_{11} + u_1 u_{23} - u_2 u_{13} = 0,$$

**Segre symbol** $[(11)(22)]$

$$u_{12} + u_2 u_{13} - u_1 u_{23} = 0,$$

**Segre symbol** $[(114)]$

$$u_{22} + u_1 u_{33} - u_3 u_{13} = 0,$$

**Segre symbol** $[(123)]$

$$u_{22} + u_{13} + u_2 u_{33} - u_3 u_{23} = 0,$$

**Segre symbol** $[(222)]$

$$u_{11} + u_{22} + u_{33} = 0,$$

**Segre symbol** $[(24)]$

$$u_3^2 u_{22} + (1 + u_2^2)u_{33} + 2u_{12} - 2u_2 u_3 u_{23} = 0,$$

**Segre symbol** $[(33)]$

$$u_{13} + u_1 u_{22} - u_2 u_{12} = 0.$$

Since conformal flatness is the necessary condition for integrability, a complete list of linearly degenerate integrable PDEs can be obtained by

going through the list of Theorem 3 and either calculating the integrability conditions as derived in [7], or verifying the existence of a Lax pair. Another possibility is to utilize the recent result of [14] according to which the integrability of a second-order quasilinear PDE is equivalent to the Einstein–Weyl property of the symbol of its formal linearization. A direct computation shows that the requirement of integrability eliminates Segre types $[111(111)]^*, [(111)(111)], [(114)], [(24)]$, leading to the following result:

**Theorem 4.** *A linearly degenerate PDE is integrable if and only if the corresponding complex has one of the following Segre types:*

$$[(11)(11)(11)], [(11)(112)], [(11)(22)], [(123)], [(222)], [(33)].$$

*Modulo equivalence transformations, this leads to the five canonical forms of linearly degenerate integrable PDEs (we exclude the linearizable case with Segre symbol $[(222)]$). For each integrable equation we present its Lax pair in the form $[X, Y] = 0$, where $X$ and $Y$ are parameter-dependent vector fields which commute modulo the corresponding equation:*

**Segre symbol** $[(11)(11)(11)]$

$$\alpha u_3 u_{12} + \beta u_2 u_{13} + \gamma u_1 u_{23} = 0,$$

$\alpha + \beta + \gamma = 0$. *Setting $\alpha = a - b$, $\beta = b - c$, $\gamma = c - a$, we obtain the Lax pair:* $X = \partial_{x^3} - \frac{\lambda - b}{\lambda - c} \frac{u_3}{u_1} \partial_{x^1}$, $Y = \partial_{x^2} - \frac{\lambda - b}{\lambda - a} \frac{u_2}{u_1} \partial_{x^1}$.

**Segre symbol** $[(11)(112)]$

$$u_{11} + u_1 u_{23} - u_2 u_{13} = 0,$$

*Lax pair:* $X = \partial_{x^1} - \lambda u_1 \partial_{x^3}$, $Y = \partial_{x^2} + (\lambda^2 u_1 - \lambda u_2) \partial_{x^3}$.

**Segre symbol** $[(11)(22)]$

$$u_{12} + u_2 u_{13} - u_1 u_{23} = 0,$$

*Lax pair:* $X = \lambda \partial_{x^1} - u_1 \partial_{x^3}$, $Y = (\lambda - 1) \partial_{x^2} - u_2 \partial_{x^3}$.

**Segre symbol** $[(123)]$

$$u_{22} + u_{13} + u_2 u_{33} - u_3 u_{23} = 0,$$

*Lax pair:* $X = \partial_{x^2} + (\lambda - u_3) \partial_{x^3}$, $Y = \partial_{x^1} + (\lambda^2 - \lambda u_3 + u_2) \partial_{x^3}$.

**Segre symbol** [(33)]

$$u_{13} + u_1 u_{22} - u_2 u_{12} = 0,$$

*Lax pair:* $X = \lambda \partial_{x^1} - u_1 \partial_{x^2}, \ Y = \partial_{x^3} + (\lambda - u_2)\partial_{x^2}.$

**Remark 1.** In different contexts, the five canonical forms of Theorem 4 have appeared in [1, 10, 30, 31, 34, 37–39, 48]. The nonequivalence of the above PDEs can also be seen by calculating the Kummer surfaces of the corresponding line complexes. In all cases the Kummer surfaces degenerate into a collection of planes:
– Case 1: four planes in general position, one of them at infinity.
– Case 2: two double planes, one of them at infinity.
– Case 3: three planes, one of them double, with the double plane at infinity.
– Case 4: one quadruple plane at infinity.
– Case 5: two planes, one of them triple, with the triple plane at infinity.

**Remark 2.** Although all equations from Theorem 3 are not related via the equivalence group $\mathbf{SL}(4)$, there may exist more complicated Bäcklund-type links between them. Thus, let $\alpha, \beta, \gamma$ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ be two triplets of numbers such that $\alpha + \beta + \gamma = 0$ and $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = 0$. Consider the system of two first-order relations for the functions $u$ and $v$,

$$\alpha\tilde{\gamma}v_1 u_3 - \gamma\tilde{\alpha}v_3 u_1 = 0, \ \ \alpha\tilde{\beta}v_2 u_3 - \beta\tilde{\alpha}v_3 u_2 = 0.$$

Eliminating $v$ (that is, solving the above relations for $v_1$ and $v_2$ and imposing the compatibility condition $v_{12} = v_{21}$), we obtain the second-order equation $\alpha u_3 u_{12} + \beta u_2 u_{13} + \gamma u_1 u_{23} = 0$. Similarly, eliminating $u$ we obtain the analogous equation for $v$, $\tilde{\alpha}v_3 v_{12} + \tilde{\beta}v_2 v_{13} + \tilde{\gamma}v_1 v_{23} = 0$. This construction first appeared in [48] in the context of Veronese webs in 3D. It shows that any two integrable equations of the Segre type [(11)(11)(11)] are related by a Bäcklund transformation. Similarly, the relations

$$(\lambda - 1)v_2 - u_2 v_3 = 0, \ \ \lambda v_1 - u_1 v_3 = 0$$

provide a Bäcklund transformation between the equation for $u$, $u_{12} + u_2 u_{13} - u_1 u_{23} = 0$ and the equation for $v$, $v_3 v_{12} + (\lambda - 1)v_2 v_{13} - \lambda v_1 v_{23} = 0$, thus establishing the equivalence of integrable equations of the types [(11)(22)] and [(11)(11)(11)].

### Proof of Theorems 2 to 4:

We follow the classification of quadratic complexes as presented in [18], p. 206–232. This constitutes 11 canonical forms which are analysed case-by-case below. In each case, we calculate the conditions of vanishing of the Cotton tensor (responsible for conformal flatness in three dimensions), as well as the integrability conditions as derived in [7], see also [14] for their geometric reformulation. Recall that conformal flatness is a necessary condition for integrability: this requirement already leads to a compact list of conformally flat subcases which can be checked for integrability by calculating the Lax pair. Our results are summarized as follows.

*Case 1 (generic):* **Segre symbol** [111111]**.** The equation of the complex is

$$\lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 + \lambda_3(p^{13} + p^{42})^2 - \lambda_4(p^{13} - p^{42})^2$$
$$+ \lambda_5(p^{14} + p^{23})^2 - \lambda_6(p^{14} - p^{23})^2 = 0.$$

Here $\lambda_i$ are the eigenvalues of $Q\Omega^{-1}$. Its Monge form is

$$[a_1 + a_2(p^3)^2 + a_3(p^2)^2](dp^1)^2 + [a_2 + a_1(p^3)^2 + a_3(p^1)^2](dp^2)^2$$
$$+ [a_3 + a_1(p^2)^2 + a_2(p^1)^2](dp^3)^2 + 2[\alpha p^3 - a_3 p^1 p^2]dp^1 dp^2$$
$$+ 2[\beta p^2 - a_2 p^1 p^3]dp^1 dp^3 + 2[\gamma p^1 - a_1 p^2 p^3]dp^2 dp^3 = 0,$$

where $a_1 = \lambda_5 - \lambda_6$, $a_2 = \lambda_3 - \lambda_4$, $a_3 = \lambda_1 - \lambda_2$, $\alpha = \lambda_5 + \lambda_6 - \lambda_3 - \lambda_4$, $\beta = \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6$ and $\gamma = \lambda_3 + \lambda_4 - \lambda_1 - \lambda_2$, notice that $\alpha + \beta + \gamma = 0$. The corresponding PDE takes the form

$$(a_1 + a_2 u_3^2 + a_3 u_2^2)u_{11} + (a_2 + a_1 u_3^2 + a_3 u_1^2)u_{22} + (a_3 + a_1 u_2^2 + a_2 u_1^2)u_{33}$$
$$+ 2(\alpha u_3 - a_3 u_1 u_2)u_{12} + 2(\beta u_2 - a_2 u_1 u_3)u_{13} + 2(\gamma u_1 - a_1 u_2 u_3)u_{23} = 0,$$

which is the first case of Theorem 2. The analysis of integrability/conformal flatness leads to the four subcases, depending on how many $a$'s equal zero.

*Subcase 1:* $a_1 = a_2 = a_3 = 0$. This subcase, which corresponds to the so-called tetrahedral complex, is integrable and conformally flat, leading to the nonlinear wave equation [48],

$$\alpha u_3 u_{12} + \beta u_2 u_{13} + \gamma u_1 u_{23} = 0.$$

The Kummer surface of this complex consists of four planes in $\mathbb{P}^3$ in general position. The lines of the complex intersect these planes at four points

with constant cross-ratio (depending on $\alpha, \beta$ and $\gamma$). The corresponding affinor $Q\Omega^{-1}$ has three pairs of coinciding eigenvalues. The notation for such complexes is $[(11)(11)(11)]$, see Example 1 of Section 2.3.

*Subcase 2*: $a_1 = a_2 = 0$. This subcase possesses no nondegenerate integrable specializations. The conditions of conformal flatness imply $\alpha = -2\beta$, $a_3 = \pm\beta$. For any choice of the sign the corresponding affinor $Q\Omega^{-1}$ has two triples of coinciding eigenvalues. Complexes of this type are denoted $[(111)(111)]$, and are known as "special": they consist of tangent lines to a nondegenerate quadric surface in $\mathbb{P}^3$. Particular example of this type is the PDE for minimal surfaces in Minkowski space, see Example 2 in Section 2.3.

*Subcase 3*: $a_1 = 0$. The further analysis splits into two essentially different branches. The first branch corresponds to $\gamma = 0$, $a_3 = \pm a_2$, in this case we have both conformal flatness and integrability. The corresponding complexes are the same as in subcase 1, with Segre symbols $[(11)(11)(11)]$. The second branch corresponds to $\beta = \alpha$, $a_2 = \pm\alpha$, $a_3^2 + 3\alpha^2 = 0$ or $\beta = \alpha$, $a_3 = \pm\alpha$, $a_2^2 + 3\alpha^2 = 0$. All these subcases are conformally flat, but not integrable. They are projectively equivalent to each other, with the same Segre symbol $[111(111)]^*$, where the asterisk indicates that the eigenvalues of the (traceless) operator $Q\Omega^{-1}$ are proportional to $(1, \epsilon, \epsilon^2, 0, 0, 0)$, here $\epsilon$ is a cubic root of unity, $\epsilon^3 = 1$. There exists an equivalent real normal form of complexes of this type, the simplest one we found is

$$(p^{24} + p^{14})^2 + 2(p^{12} + p^{34})(p^{23} + p^{31}) = 0.$$

The corresponding Monge form is

$$[1 - 2p^2p^3](dp^1)^2 + [1 - 2p^1p^3](dp^2)^2 + 2(p^1 - p^2)(dp^3)^2$$
$$+ 2[1 + p^1p^3 + p^2p^3]dp^1dp^2 + 2[p^1p^2 - p^3 - (p^2)^2]dp^1dp^3$$
$$+ 2[p^1p^2 + p^3 - (p^1)^2]dp^2dp^3 = 0,$$

with the associated PDE

$$(1 - 2u_2u_3)u_{11} + (1 - 2u_1u_3)u_{22} + 2(u_1 - u_2)u_{33} + 2(1 + u_1u_3 + u_2u_3)u_{12}$$
$$+ 2(u_1u_2 - u_3 - u_2^2)u_{13} + 2(u_1u_2 + u_3 - u_1^2)u_{23} = 0,$$

which is not integrable, although the corresponding conformal structure is flat. The associated Kummer surface is a double quadric, $2p^3 + (p^1)^2 - (p^2)^2$. This is the first case of Theorem 3.

*Subcase 4*: All $a's$ are nonzero. Here we have three essentially different branches which however, gives no new examples. Thus, the first branch corresponds to $a_1 = \epsilon_1\gamma$, $a_2 = \epsilon_2\beta$, $a_3 = \epsilon_3\alpha$ and $\epsilon_i = \pm1$; in all these cases, we have both conformal flatness and integrability. The corresponding complexes are the same as in subcase 1, with Segre symbols $[(11)(11)(11)]$. The second branch is $\alpha = \beta = \gamma = 0$, $a_2 = \epsilon_2 a_1$, $a_3 = \epsilon_3 a_1$ and $\epsilon_i = \pm1$. This coincides with subcase 2, with Segre symbol $[(111)(111)]$. The third branch is $a_1 = \epsilon_1\frac{\gamma^2}{\alpha-\beta}$, $a_2 = \epsilon_2\frac{\beta^2}{\gamma-\alpha}$, $a_3 = \epsilon_3\frac{\alpha^2}{\beta-\gamma}$ and $\epsilon_i = \pm1$, where $\alpha,\beta,\gamma \in \{1, \epsilon, \epsilon^2\}$ are three distinct cubic roots of unity. This is the same as subcase 3, with Segre symbol $[111(111)]^*$.

*Case 2:* **Segre symbol** $[11112]$**.** The equation of the complex is

$$\lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 + \lambda_3(p^{13} + p^{42})^2 - \lambda_4(p^{13} - p^{42})^2$$
$$+ 4\lambda_5 p^{14} p^{23} + (p^{14})^2 = 0.$$

Its Monge form is

$$[\lambda(p^2)^2 + \mu(p^3)^2 + 1](dp^1)^2 + [\lambda(p^1)^2 + \mu](dp^2)^2 + [\mu(p^1)^2 + \lambda](dp^3)^2$$
$$+ 2[\alpha p^3 - \lambda p^1 p^2]dp^1 dp^2 + 2[\beta p^2 - \mu p^1 p^3]dp^1 dp^3 + 2\gamma p^1 dp^2 dp^3 = 0,$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_3 - \lambda_4$, $\alpha = -\lambda_3 - \lambda_4 + 2\lambda_5$, $\beta = \lambda_1 + \lambda_2 - 2\lambda_5$, $\gamma = -\alpha - \beta$, so that the corresponding PDE is

$$(\lambda u_2^2 + \mu u_3^2 + 1)u_{11} + (\lambda u_1^2 + \mu)u_{22} + (\mu u_1^2 + \lambda)u_{33}$$
$$+ 2(\alpha u_3 - \lambda u_1 u_2)u_{12} + 2(\beta u_2 - \mu u_1 u_3)u_{13} + 2\gamma u_1 u_{23} = 0.$$

This is the second case of Theorem 2. We verified that in this case conditions of integrability are equivalent to conformal flatness, leading to the following subcases.
*Subcase 1*: $\lambda = \mu = 0$, $\alpha = 0$ (the possibility $\lambda = \mu = 0$, $\beta = 0$ is equivalent to $\alpha = 0$ via the interchange of indices 2 and 3), which simplifies to

$$u_{11} + 2\beta(u_2 u_{13} - u_1 u_{23}) = 0.$$

Modulo a rescaling this gives the corresponding subcases of Theorems 3 and 4.
*Subcase 2*: $\beta = -\alpha$, $\lambda = \epsilon_1\alpha$, $\mu = \epsilon_2\alpha$ and $\epsilon_i = \pm1$. One can show that subcase 2 is equivalent to subcase 1: all such complexes have the same Segre type $[(11)(112)]$.

*Case 3:* **Segre symbol** [1113]**.** The equation of the complex is

$$\lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 - \lambda_3(p^{13} - p^{42})^2 + \lambda_4(p^{13} + p^{42})^2$$
$$+ 4\lambda_4 p^{14}p^{23} + 2p^{14}(p^{13} + p^{42}) = 0.$$

Its Monge form is

$$[\lambda(p^2)^2 + \mu(p^3)^2 + 2p^3](dp^1)^2 + [\lambda(p^1)^2 + \mu](dp^2)^2 + [\mu(p^1)^2 + \lambda](dp^3)^2$$
$$+ 2[\mu p^3 - \lambda p^1 p^2 - 1]dp^1 dp^2 + 2[\beta p^2 - \mu p^1 p^3 - p^1]dp^1 dp^3 + 2\gamma p^1 dp^2 dp^3$$
$$= 0,$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_4 - \lambda_3$, $\beta = \lambda_1 + \lambda_2 - 2\lambda_4$, $\gamma = -\mu - \beta$, so that the corresponding PDE is

$$(\lambda u_2^2 + \mu u_3^2 + 2u_3)u_{11} + (\lambda u_1^2 + \mu)u_{22} + (\mu u_1^2 + \lambda)u_{33}$$
$$+ 2(\mu u_3 - \lambda u_1 u_2 - 1)u_{12} + 2(\beta u_2 - \mu u_1 u_3 - u_1)u_{13} + 2\gamma u_1 u_{23} = 0.$$

This is the third case of Theorem 2. One can show that it possesses no nondegenerate integrable/conformally flat subcases.

*Case 4:* **Segre symbol** [1122]**.** The equation of the complex is

$$\lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 + 4\lambda_3 p^{13}p^{42} + 4\lambda_4 p^{14}p^{23}$$
$$+ (p^{13})^2 + 4(p^{23})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^3 = 1$, $dp^3 = 0$ we obtain the associated Monge equation,

$$[\lambda(p^2)^2 + 1](dp^1)^2 + [\lambda(p^1)^2 + 4](dp^2)^2 + \lambda(dp^4)^2 + 2[\alpha p^4 - \lambda p^1 p^2]dp^1 dp^2$$
$$+ 2\beta p^2 dp^1 dp^4 + 2\gamma p^1 dp^2 dp^4,$$

where $\lambda = \lambda_1 - \lambda_2$, $\alpha = 2\lambda_4 - 2\lambda_3$, $\beta = 2\lambda_3 - \lambda_1 - \lambda_2$ and $\gamma = -\alpha - \beta$, so that the corresponding PDE is

$$(\lambda u_2^2 + 1)u_{11} + (\lambda u_1^2 + 4)u_{22} + \lambda u_{44} + 2(\alpha u_4 - \lambda u_1 u_2)u_{12} + 2\beta u_2 u_{14}$$
$$+ 2\gamma u_1 u_{24} = 0.$$

Relabelling independent variables gives the fourth case of Theorem 2. In this case, conditions of conformal flatness are equivalent to the integrability,

leading to $\lambda = \alpha = 0$,

$$u_{11} + 4u_{22} + 2\beta(u_2 u_{14} - u_1 u_{24}) = 0.$$

Modulo elementary changes of variables this gives the corresponding sub-cases of Theorems 3 and 4, with Segre symbol $[(11)(22)]$.

*Case 5:* **Segre symbol** $[114]$. The equation of the complex is

$$\lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 + 4\lambda_3(p^{14}p^{23} + p^{42}p^{13}) + 2p^{14}p^{42} + 4(p^{13})^2$$
$$= 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1$, $dp^1 = 0$ we obtain the associated Monge equation,

$$\lambda(dp^2)^2 + [\lambda(p^4)^2 + 4](dp^3)^2 + [\lambda(p^3)^2 - 2p^2](dp^4)^2$$
$$+ 2\alpha p^4 dp^2 dp^3 + 2[p^4 - \alpha p^3]dp^2 dp^4 - 2\lambda p^3 p^4 dp^3 dp^4 = 0,$$

where $\lambda = \lambda_1 - \lambda_2$, $\alpha = 2\lambda_3 - \lambda_1 - \lambda_2$, so that the corresponding PDE is

$$\lambda u_{22} + (\lambda u_4^2 + 4)u_{33} + (\lambda u_3^2 - 2u_2)u_{44} + 2\alpha u_4 u_{23} + 2(u_4 - \alpha u_3)u_{24}$$
$$- 2\lambda u_3 u_4 u_{34} = 0.$$

Relabelling independent variables gives the fifth case of Theorem 2. One can show that this equation is not integrable. The condition of conformal flatness gives $\lambda = \alpha = 0$,

$$4u_{33} - 2u_2 u_{44} + 2u_4 u_{24} = 0.$$

Such complexes are denoted $[(114)]$. Modulo elementary changes of variables this gives the corresponding subcase of Theorem 3.

*Case 6:* **Segre symbol** $[123]$. The equation of the complex is

$$- \lambda_1(p^{12} - p^{34})^2 + 4\lambda_2 p^{13}p^{42} + 4(p^{13})^2 + \lambda_3(4p^{14}p^{23} + (p^{12} + p^{34})^2)$$
$$+ 2p^{14}(p^{12} + p^{34}) = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1$, $dp^1 = 0$ we obtain the associated Monge equation,

$$\lambda(dp^2)^2 + [\lambda(p^4)^2 + 4](dp^3)^2 + [\lambda(p^3)^2 + 2p^3](dp^4)^2$$
$$+ 2\alpha p^4 dp^2 dp^3 + 2[1 - \lambda p^3]dp^2 dp^4 + 2[\gamma p^2 - \lambda p^3 p^4 - p^4]dp^3 dp^4 = 0,$$

where $\lambda = \lambda_3 - \lambda_1$, $\alpha = 2\lambda_2 - \lambda_1 - \lambda_3$, $\gamma = \lambda - \alpha$, so that the corresponding PDE is

$$\lambda u_{22} + (\lambda u_4^2 + 4)u_{33} + (\lambda u_3^2 + 2u_3)u_{44} + 2\alpha u_4 u_{23} + 2(1 - \lambda u_3)u_{24}$$
$$+ 2(\gamma u_2 - \lambda u_3 u_4 - u_4)u_{34} = 0.$$

Relabelling independent variables gives the sixth case of Theorem 2. In this case, conditions of conformal flatness are equivalent to the integrability. One can show that both require $\lambda = \alpha = \gamma = 0$, which gives

$$2u_{33} + u_{24} + u_3 u_{44} - u_4 u_{34} = 0.$$

Appropriate relabellings and rescalings give the corresponding subcases of Theorems 3 and 4, denoted [(123)].

*Case 7:* **Segre symbol** [222]. Here we have two (projectively dual) subcases. In subcase 1, the equation of the complex is

$$2\lambda_1 p^{12}p^{34} + 2\lambda_2 p^{13}p^{42} + 2\lambda_3 p^{14}p^{23} + (p^{12})^2 + (p^{13})^2 + (p^{14})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1$, $dp^1 = 0$ we obtain the associated Monge equation,

$$(dp^2)^2 + (dp^3)^2 + (dp^4)^2 + 2\alpha p^4 dp^2 dp^3 + 2\beta p^3 dp^2 dp^4 + 2\gamma p^2 dp^3 dp^4 = 0,$$

where $\alpha = \lambda_2 - \lambda_1$, $\beta = \lambda_1 - \lambda_3$, $\gamma = \lambda_3 - \lambda_2$, so that the corresponding PDE is

$$u_{22} + u_{33} + u_{44} + 2\alpha u_4 u_{23} + 2\beta u_3 u_{24} + 2\gamma u_2 u_{34} = 0.$$

Setting $\alpha = \beta = \gamma = 0$, we obtain the linear equation. The corresponding Segre symbol is [(222)]. One can show that the above PDE is not integrable/conformally flat for nonzero values of constants. This is the linearizable subcase of Theorems 3 and 4.

In subcase 2, the equation of the complex is

$$2\lambda_1 p^{12}p^{34} + 2\lambda_2 p^{13}p^{42} + 2\lambda_3 p^{14}p^{23} + (p^{23})^2 + (p^{24})^2 + (p^{34})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1$, $dp^1 = 0$ we obtain the associated Monge equation,

$$
((p^3)^2 + (p^4)^2)(dp^2)^2 + ((p^2)^2 + (p^4)^2)(dp^3)^2 + ((p^2)^2 + (p^3)^2)(dp^4)^2
$$
$$
+ 2(\alpha p^4 - p^2 p^3)\, dp^2 dp^3 + 2(\beta p^3 - p^2 p^4)\, dp^2 dp^4 + 2(\gamma p^2 - p^3 p^4)\, dp^3 dp^4
$$
$$
= 0,
$$

so that the corresponding PDE is

$$
(u_3^2 + u_4^2)u_{22} + (u_2^2 + u_4^2)u_{33} + (u_2^2 + u_3^2)u_{44} + 2(\alpha u_4 - u_2 u_3)u_{23}
$$
$$
+ 2(\beta u_3 - u_2 u_4)u_{24} + 2(\gamma u_2 - u_3 u_4)u_{34} = 0.
$$

One can show that this subcase possesses no nondegenerate integrable/ conformally flat specializations (notice that for $\alpha = \beta = \gamma = 0$, this PDE becomes degenerate). Relabelling independent variables gives the seventh case of Theorem 2.

*Case 8:* **Segre symbol [15].** The equation of the complex is

$$
- \lambda_1(p^{12} - p^{34})^2 + \lambda_2(4p^{14}p^{23} + 4p^{13}p^{42} + (p^{12} + p^{34})^2) + 4p^{14}p^{42}
$$
$$
+ 2p^{13}(p^{12} + p^{34}) = 0.
$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1$, $dp^1 = 0$, we obtain the associated Monge equation

$$
\lambda(dp^2)^2 + [\lambda(p^4)^2 - 2p^4](dp^3)^2 + [\lambda(p^3)^2 - 4p^2](dp^4)^2
$$
$$
+ 2[\lambda p^4 + 1]dp^2 dp^3 + 2[2p^4 - \lambda p^3]dp^2 dp^4 + 2[p^3 - \lambda p^3 p^4]dp^3 dp^4 = 0,
$$

where $\lambda = \lambda_2 - \lambda_1$, so that the corresponding PDE is

$$
\lambda u_{22} + (\lambda u_4^2 - 2u_4)u_{33} + (\lambda u_3^2 - 4u_2)u_{44} + 2(\lambda u_4 + 1)u_{23} + 2(2u_4 - \lambda u_3)u_{24}
$$
$$
+ 2(u_3 - \lambda u_3 u_4)u_{34} = 0.
$$

One can show that this PDE possesses no integrable/conformally flat specializations. Relabelling independent variables gives the eighth case of Theorem 2.

*Case 9:* **Segre symbol** [24]**.** Here we have two (projectively dual) subcases. In subcase 1 the equation of the complex is

$$2\lambda_1 p^{12}p^{34} + (p^{12})^2 + 2\lambda_2(p^{14}p^{23} + p^{13}p^{42}) + 2p^{14}p^{42} + (p^{13})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1,\ dp^1 = 0$ we obtain the associated Monge equation

$$(dp^2)^2 + (dp^3)^2 - 2p^2(dp^4)^2 + 2\lambda p^4 dp^2 dp^3 + 2[p^4 - \lambda p^3]dp^2 dp^4 = 0,$$

where $\lambda = \lambda_2 - \lambda_1$, so that the corresponding PDE is

$$u_{22} + u_{33} - 2u_2 u_{44} + 2\lambda u_4 u_{23} + 2(u_4 - \lambda u_3)u_{24} = 0.$$

One can show that this subcase possesses no integrable/conformally flat specializations. In subcase 2 the equation of the complex is

$$2\lambda_1 p^{12}p^{34} + (p^{34})^2 + 2\lambda_2(p^{14}p^{23} + p^{13}p^{42}) + 2p^{13}p^{23} + (p^{42})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^3 = 1$ and $dp^3 = 0$, we obtain the associated Monge equation

$$(p^4)^2(dp^2)^2 + (1 + (p^2)^2)(dp^4)^2 + 2dp^1 dp^2 + 2\lambda p^2 dp^1 dp^4$$
$$- 2[\lambda p^1 + p^2 p^4]dp^2 dp^4 = 0,$$

where $\lambda = \lambda_2 - \lambda_1$, so that the corresponding PDE is

$$u_4^2 u_{22} + (1 + u_2^2)u_{44} + 2u_{12} + 2\lambda u_2 u_{14} - 2(\lambda u_1 + u_2 u_4)u_{24} = 0.$$

One can show that this PDE is not integrable, however, the corresponding conformal structure is flat for $\lambda = 0$. This Segre type is known as $[(24)]$, giving the corresponding subcase of Theorem 3. The associated Kummer surface consists of three planes, with a double plane at infinity. We point out that in subcase 1 the Kummer surface of the complex $[(24)]$ consists of a quadratic cone and a double plane at infinity. This gives an invariant characterization of subcase 2 of the complex $[(24)]$.

Relabelling independent variables gives the ninth case of Theorem 2.

*Case 10:* **Segre symbol [29].** The equation of the complex is

$$\lambda_1(4p^{31}p^{24} + (p^{12} + p^{34})^2) + 2p^{13}(p^{12} + p^{34}) + \lambda_2(4p^{23}p^{14} - (p^{12} - p^{34})^2) \\ + 2p^{14}(p^{12} - p^{34}) = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1$ and $dp^1 = 0$, we obtain the associated Monge equation

$$\lambda(dp^2)^2 + [\lambda(p^4)^2 - 2p^4](dp^3)^2 + [\lambda(p^3)^2 - 2p^3](dp^4)^2 + 2[\lambda p^4 + 1]dp^2 dp^3 \\ + 2[\lambda p^3 + 1]dp^2 dp^4 - 2[2\lambda p^2 + \lambda p^3 p^4 - p^3 - p^4]dp^3 dp^4 = 0,$$

where $\lambda = \lambda_1 - \lambda_2$, so that the corresponding PDE is

$$\lambda u_{22} + (\lambda u_4^2 - 2u_4)u_{33} + (\lambda u_3^2 - 2u_3)u_{44} + 2(\lambda u_4 + 1)u_{23} + 2(\lambda u_3 + 1)u_{24} \\ - 2(2\lambda u_2 + \lambda u_3 u_4 - u_3 - u_4)u_{34} = 0.$$

Relabelling independent variables gives the 10th case of Theorem 2. One can show that the conditions of integrability are equivalent to conformal flatness, leading to $\lambda = 0$,

$$u_4 u_{33} + u_3 u_{44} - u_{23} - u_{24} - (u_3 + u_4)u_{34} = 0.$$

The corresponding complex is denoted [(33)]. Introducing the new independent variables $x, y, t$ such that $\partial_3 = \partial_x + \partial_y$, $\partial_4 = \partial_x - \partial_y$, $\partial_2 = -2\partial_t$ one can reduce the above PDE to the canonical form

$$u_{xt} + u_x u_{yy} - u_y u_{xy} = 0.$$

This is the last case of Theorems 3 and 4.

*Case 11:* **Segre symbol [6].** Here we have two (projectively dual) subcases. In subcase 1, the equation of the complex is

$$2\lambda(p^{23}p^{14} + p^{31}p^{24} + p^{12}p^{34}) + 2p^{14}p^{34} + 2p^{12}p^{42} + (p^{13})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1$ and $dp^1 = 0$, we obtain the associated Monge equation,

$$2p_4(dp^2)^2 + (dp^3)^2 + 2p^3(dp^4)^2 - 2p^2 dp^2 dp^4 - 2p^4 dp^3 dp^4 = 0,$$

so that the corresponding PDE is

$$2u_4u_{22} + u_{33} + 2u_3u_{44} - 2u_2u_{24} - 2u_4u_{34} = 0.$$

In the second subcase, the equation of the complex is

$$2\lambda(p^{23}p^{14} + p^{31}p^{24} + p^{12}p^{34}) + 2p^{23}p^{12} + 2p^{34}p^{13} + (p^{42})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1$ and $dp^1 = 0$, we obtain the associated Monge equation,

$$((p^4)^2 - 2p_3)(dp^2)^2 - 2p^4(dp^3)^2 + (p^2)^2(dp^4)^2 + 2p^2 dp^2 dp^3 - 2p^2 p^4 dp^2 dp^4$$
$$+ 2p^3 dp^3 dp^4 = 0,$$

so that the corresponding PDE is

$$(u_4^2 - 2u_3)u_{22} - 2u_4u_{33} + u_2^2 u_{44} + 2u_2u_{23} - 2u_2u_4u_{24} + 2u_3u_{34} = 0.$$

One can show that both subcases are not integrable/conformally flat. Relabelling independent variables gives the last case of Theorem 2. This finished the proof of Theorems 2–4.

## 4. Remarks on the Cauchy problem for linearly degenerate PDEs

In $1+1$ dimensions, linearly degenerate systems are known to be quite exceptional from the point of view of solvability of the Cauchy problem: generic smooth initial data do not develop shocks in finite time [27, 41, 42, 44]. The conjecture of Majda [28], p. 89, suggests that the same statement should be true in higher dimensions, namely, for linearly degenerate systems the shock formation never happens for smooth initial data. To the best of our knowledge this conjecture is largely open, and has only been established for particular classes of multi-dimensional linearly degenerate PDEs, see [6, 8, 19–21] and references therein. We emphasize that the so-called "null condition" of Klainerman, which is instrumental for establishing global existence results for $3+1$ dimensional nonlinear wave equations with small initial data, is automatically satisfied for linearly degenerate PDEs. In the more subtle case of $2+1$ dimensions, the null condition implies long time existence and additional conditions (e.g., the second null condition, which also follows from linear degeneracy) are required to guarantee global existence [3]. The approach of [3, 8, 19–21] applies to second-order quasilinear

PDEs which can be viewed as nonlinear deformations of the wave equation,

$$(4.1) \qquad\qquad \Box u = g_{ij}(u_k)u_{ij}.$$

Here $\Box = \partial_1^2 - \partial_2^2 - ... - \partial_n^2$ is the wave operator, and the coefficients $g_{ij}$, which depend on the first-order derivatives of $u$, are required to vanish at the origin $u_k = 0$. Under the null conditions imposed on $g_{ij}$ (which are automatically satisfied for linearly degenerate PDEs of the form (4.1), in fact, these conditions follow from the requirement of linear degeneracy (1.3) in the vicinity of the origin), one has global existence of classical solutions with small initial data. Since some of the linearly degenerate examples from Theorem 2 can be put into the form (4.1), one can automatically guarantee global existence. For instance, the PDE for minimal hypersurfaces in $3 + 1$ dimensional Minkowski space is

$$
\begin{aligned}
u_{11} - u_{22} - u_{33} = &-(u_2^2 + u_3^2)u_{11} + (u_3^2 - u_1^2)u_{22} + (u_2^2 - u_1^2)u_{33} \\
&+ 2u_1 u_2 u_{12} + 2u_1 u_3 u_{13} - 2u_2 u_3 u_{23},
\end{aligned}
$$

$(4.2)$

take case [111111] of Theorem 2 and set $a_1 = -1$, $a_2 = a_3 = 1$, $\alpha = \beta = \gamma = 0$, $u \to iu$. It can be obtained as the Euler–Lagrange equation for the area functional, $\int \sqrt{1 + u_2^2 + u_3^2 - u_1^2}\, dx$. In this particular case, global existence was established in [26], in fact, this PDE fits into the general framework of [3]. Further examples of this type include the equation

$$(4.3) \qquad u_{11} - u_{22} - u_{33} = 2\alpha u_3 u_{12} + 2\beta u_2 u_{13} + 2\gamma u_1 u_{23}.$$

(Take case [222] of Theorem 2 and set $x_2 \to ix_2$, $x_3 \to ix_3$). For PDEs of this type, solutions with small initial data essentially behave like solutions of the linear wave equation. As an illustration we present Mathematica snapshots
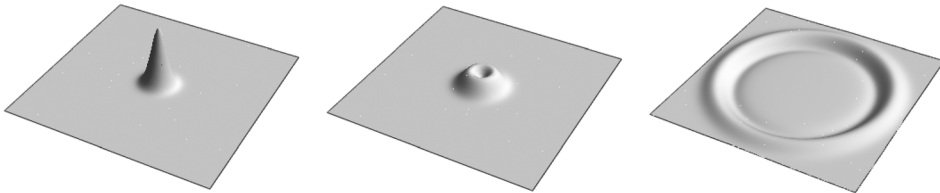


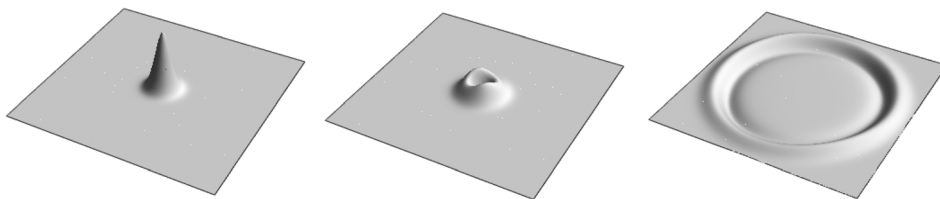Figure 1: Numerical solution of Equation (4.2) for $x_1 = 0, 1, 8$.

Figure 2: Numerical solution of Equation (4.3) for $x_1 = 0, 1, 8$.

of numerical solutions for Equations (4.2) and (4.3) ($\alpha = \beta = 1/2, \gamma = -1$) with hump-like initial data at $x_1 = 0$: $u = 0.8\mathrm{e}^{-x_2^2 - x_3^2}$, $u_{x_1} = 0$.

Although we observe some differences at the early stages of evolution, for large values of the "time" variable $x_1$ solutions become almost indistinguishable from analogous solutions of the linear wave equation. Note that both Equations (4.2) and (4.3) are not integrable.

We refer to [29, 30] for an alternative approach to the Cauchy problem for linearly degenerate *integrable* PDEs based on the novel version of the inverse scattering transform.

## Acknowledgments

## References

[1] V.E. Adler and A.B. Shabat, *Model equation of the theory of solitons*, Theor. Math. Phys. **153**(1) (2007), 1373–1387.

[2] M.A. Akivis and V.V. Goldberg, *Projective differential geometry of sub-manifolds*, Elsevier Science Publishers, Amsterdam, The Netherlands, 1993, 362pp.

[3] S. Alinhac, *The null condition for quasilinear wave equations in two space dimensions I*, Invent. Math. **145**(3) (2001), 597–618.

[4] D. Avritzer and H. Lange, *Moduli spaces of quadratic complexes and their singular surfaces*, Geom. Dedicata **127** (2007), 177–197.

[5] M.C. Beltrametti, E. Carletti, D. Gallarati, G. Monti Bragadin, *Lectures on curves, surfaces and projective varieties. A classical view of algebraic geometry*. Translated from the 2003 Italian original by Francis Sullivan. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2009, 491 pp.

[6] Y. Brenier, *Hydrodynamic structure of the augmented Born–Infeld equations*, Arch. Ration. Mech. Anal. **172** (1) (2004), 65–91.

[7] P.A. Burovskii, E.V. Ferapontov and S.P. Tsarev, *Second order quasilinear PDEs and conformal structures in projective space*, Int. J. Math. **21**(6) (2010), 799–841.

[8] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Commun. Pure Appl. Math. **39** (1986), 267–282.

[9] I.V. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge University Press, Cambridge, 2012, 639 pp.

[10] M. Dunajski, *A class of Einstein–Weyl spaces associated to an integrable system of hydrodynamic type*, J. Geom. Phys. **51**(1) (2004), 126–137.

[11] M. Dunajski and W. Krynski, *Einstein–Weyl geometry, dispersionless Hirota equation and Veronese webs*, arXiv:1301.0621.

[12] E.V. Ferapontov, *Integration of weakly nonlinear hydrodynamic systems in Riemann invariants*, Phys. Lett. A **158** (1991), 112–118.

[13] E.V. Ferapontov and K.R. Khusnutdinova, *On integrability of (2+1)-dimensional quasilinear systems*, Commun. Math. Phys. **248** (2004), 187–206.

[14] E. V. Ferapontov and B. Kruglikov, *Dispersionless integrable systems in 3D and Einstein–Weyl geometry*, J. Diff. Geom. **97** (2014), 215–254.

[15] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley and Sons, New York, 1978, 813 pp.

[16] D.K. Gvazava, *General integrals and initial-characteristic problems for second-order nonlinear equations with rectilinear characteristics*, Lithuanian Math. J. **40**(4) (2000), 352–363.

[17] R.W.H.T. Hudson, *Kummer's quartic surface*, Cambridge University Press, Cambridge, UK, 1905.

[18] C.M. Jessop, *A treatise on the line complex*, Chelsea Publishing Co., New York, 1969, 364 pp.

[19] F. John, *Existence for large times of strict solutions of nonlinear wave equations in three space dimensions for small initial data*, Commun. Pure Appl. Math. **40**(1), (1987), 79–109.

[20] F. John, *Nonlinear wave equations, formation of singularities*, Seventh Annual Pitcher Lectures delivered at Lehigh University, Bethlehem, Pennsylvania, April 1989. University Lecture Series, 2. American Mathematical Society, Providence, RI, 1990, 64 pp.

[21] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 293–326, Lect. Appl. Math. **23**, Amer. Math. Soc., Providence, RI, 1986.

[22] F. Klein, *Zur theorie der Liniencomplexe des ersten und zweiten Grades*, Math. Ann. **2** (1870), 198–226.

[23] F. Klein, *Über die Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien-Coordinaten auf eine canonische Form*, Math. Ann. **23**(4) (1884), 539–578.

[24] E. Kummer, *Über die Flächen vierten Grades mit sechzehn singulären Punkten*, Monatsberichte der Koniglichen Preusischen Akademie der Wissenschaften zu Berlin, 1864, 246–260; see also Collected papers, *Volume II: Function theory, geometry and miscellaneous*, Springer-Verlag, Berlin-New York, 1975, 877 pp.

[25] S. Lie, *Geometrie der Berührungstransformationen*, ed Georg Scheffers, 2nd Corrected edn., Chelsea Publishing Co., New York, 1977. xi+694 pp. ISBN: 0-8284-0291-4.

[26] H. Lindblad, *A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time*, Proc. Amer. Math. Soc. **132**(4) (2004), 1095–1102.

[27] T.P. Liu, *Development of singularities in the nonlinear waves for quasilinear hyperbolic PDEs*, J. Differ. Equ. **33** (1979), 92–111.

[28] A. Majda, *Compressible fluid flows and systems of conservation laws in several space variables*, Appl. Math. Sci., Springer-Verlag, NY, **53**, 1984, 159 pp.

[29] S.V. Manakov and P.M. Santini, *Inverse scattering problem for vector fields and the Cauchy problem for the heavenly equation*, Phys. Lett. A **359**(6) (2006), 613–619.

[30] S.V. Manakov and P.M. Santini, *On the solutions of the second heavenly and Pavlov equations*, J. Phys. A **42**(40) (2009), 404013, 11 pp.

[31] L. Martinez Alonso and A.B. Shabat, *Hydrodynamic reductions and solutions of a universal hierarchy*, Theor. Math. Phys. **140**(2) (2004), 1073–1085.

[32] O.F. Men'shikh, *Conservation laws and Bäcklund transformations associated with the Born–Infeld equation*, Math. Notes **77**(3–4) (2005), 510–522.

[33] O.I. Mokhov and Y. Nutku, *Bianchi transformation between the real hyperbolic Monge–Ampère equation and the Born–Infeld equation*, Lett. Math. Phys. **32**(2) (1994), 121–123.

[34] O. I. Morozov, *Recursion operators and nonlocal symmetries for integrable rmdKP and rdDym equations*, arXiv:1202.2308.

[35] F.Kh. Mukminov, *On the straightening of the characteristics of a second-order quasilinear equation*, Theor. Math. Phys. **75**(1) (1988), 340–345.

[36] P.E. Newstead, *Quadratic complexes.* II, Math. Proc. Cambridge Philos. Soc. **91**(2) (1982), 183–206.

[37] A. Odesskii and V. Sokolov, *Integrable (2+1)-dimensional systems of hydrodynamic type*, Theor. Math. Phys. **163**(2) (2010), 549–586.

[38] V. Ovsienko, *Bi-Hamiltonian nature of the equation $u_{tx} = u_{xy}u_y - u_{yy}u_x$*, Adv. Pure Appl. Math. **1**(1) (2010), 7–17.

[39] M.V. Pavlov, *Integrable hydrodynamic chains*, J. Math. Phys. **44** (2003), 4134–4156;
M.V. Pavlov, *Integrable dispersive chains and energy dependent Schrödinger operator*, arXiv:1402.3836.

[40] J. Plücker, *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement*, Bd. I, Teubner, Leipzig, 1868; ibid. Bd. II, 1869.

[41] B.L. Rozdestvenskii and A.D. Sidorenko, *On the impossibility of "gradient catastrophe" for weakly nonlinear systems*, Z. Vycisl. Math. I Math. Fiz. **7** (1967), 1176–1179.

[42] B.L. Rozdestvenskii and N.N. Yanenko, *Systems of quasilinear equations and their applications to gas dynamics*, translated from the 2nd Russian edn., ed. J.R. Schulenberger, Translations of Mathematical Monographs, **55** American Mathematical Society, Providence, RI, 1983, 676 pp.

[43] C. Segre, *Note sur les complexes quadratiques dont la surface singulière est une surface du 2e dégré double*, Math. Ann. **23**(2) (1884), 235–243.

[44] D. Serre, *Systems of conservation laws. 1. Hyperbolicity, entropies, shock waves*, Cambridge University Press, Cambridge, UK, 1999, 263 pp;
*Systems of conservation laws. 2. Geometric structures, oscillations, and initial-boundary value problems*, Cambridge University Press, Cambridge, UK, 2000, 269 pp.

[45] L.P. Safaryan, *Certain classes of manifolds of cones of order two in $P_n$*, (Russian) Akad. Nauk Armjan. SSR Dokl. **50** (1970), 83–90.

[46] J.G. Semple and L. Roth, *Introduction to algebraic geometry*, Clarendon Press, Oxford, 1949, 446 pp.

[47] A. Weiler, *Über die verschiedenen Gattungen der Complexe zweiten Grades*, Math. Ann. **7** (1873), 145–207.

[48] I. Zakharevich, *Nonlinear wave equation, nonlinear Riemann problem, and the twistor transform of Veronese webs*, arXiv:math-ph/0006001.

Department of Mathematical Sciences
Loughborough University
Loughborough
Leicestershire LE11 3TU
UK
*E-mail address*: E.V.Ferapontov@lboro.ac.uk; J.Moss@lboro.ac.uk