On the smoothness of the potential function in Riemannian optimal transport[*]

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On a closed Riemannian manifold, McCann proved the existence of a unique Borel map pushing a given smooth positive probability measure to another one while minimizing a related quadratic cost functional. The optimal map is obtained as the exponential of the gradient of a c-convex function u. The question of the smoothness of u has been intensively investigated. We present a self-contained partial differential equations approach to this problem. The smoothness question is reduced to a couple of a priori estimates, namely: a positive lower bound on the Jacobian of the exponential map (meant at each fixed tangent space) restricted to the graph of grad u; and an upper bound on the c-Hessian of u. By the Ma–Trudinger–Wang device, the former estimate implies the latter on manifolds satisfying the so-called A3 condition. On such manifolds, it only remains to get the Jacobian lower bound. We get it on simply connected positively curved manifolds which are, either locally symmetric, or two-dimensional with Gauss curvature C^2 close to 1.

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[*]This paper is dedicated to Neil Trudinger.

1. Introduction

Monge considered the problem of minimizing the total work necessary to push, in the Euclidean space, an assigned mass distribution to another one [52]. Brenier [3, 4] revived the problem by replacing the total work by the total kinetic energy. His theory was subsequently developed in the Riemannian setting by McCann [50] who obtained the following landmark result:

Theorem 1.1. *Let M be a closed Riemannian manifold and (μ_0, μ_1), a couple of probability measures on M, both absolutely continuous with respect to the volume measure dV. There exists a unique minimizer of the total cost functional $\mathcal{C}(\phi) = \int_M c(m, \phi(m))\, d\mu_0(m)$, among Borel maps $\phi : M \to M$ pushing μ_0 to μ_1. Moreover, if we compose that minimizer with the one obtained by switching μ_0 and μ_1, we get the identity at almost all points.*

Here, we have called closed (for short) a manifold which is compact connected without boundary, and we have taken for cost function c the two-point function $\frac{1}{2}d^2$, where d stands for the Riemannian distance function. The pushing condition means that $\mu_1(B) = \mu_0\big(\phi^{-1}(B)\big)$, for every Borel subset $B \subset M$, or else, that

$$(1.1) \qquad\qquad \int_M h\, d\mu_1 = \int_M (h \circ \phi)\, d\mu_0,$$

for every dV measurable function $h : M \to \mathbb{R}$. It is denoted by: $\phi_\# \mu_0 = \mu_1$. In the sequel, it will be convenient to call abusively Monge's problem the minimization problem described in the above statement and to denote by $\mathrm{Mc}(\mu_0, \mu_1)$ its solution.

1.1   The smoothness issue

The solution map $\mathrm{Mc}(\mu_0, \mu_1)$ has a special form, obtained by composing the gradient field of a c-convex potential function u with the exponential map of the manifold. To explain what c-convex means, one should first define the c-supremal convolution of a function $v : M \to \mathbb{R} \cup \{+\infty\}$ bounded at some point, as the function v^c of the same kind, given by: $v^c(m) = \sup_{p \in M}\big(-c(m, p) - v(p)\big)$. We say that u is c-convex if it is equal to v^c for a function v of the preceding type. If so, the function u satisfyies: $u = \big(u^c\big)^c$. In [50], McCann shows that c-convex functions must be Lipschitz, hence almost everywhere differentiable, and that their gradients are Borel maps. Furthermore, if the map $\mathrm{Mc}(\mu_0, \mu_1)$ is equal to $G_u := \exp(\mathrm{grad}\, u)$ with u c-convex, the inverse minimizer $\mathrm{Mc}(\mu_1, \mu_0)$ is given by G_{u^c}. The pair of solution potentials (u, u^c) itself is constructed by Kantorovich duality (see [50]).

   The potential function $u = u(\mu_0, \mu_1)$ of the map $\mathrm{Mc}(\mu_0, \mu_1) = G_u$ is called a Brenier solution of the Monge–Ampère equation:

$$(1.2) \qquad \rho_1\big(G_u(m)\big)\mathrm{Jac}(G_u)(m) = \rho_0(m),$$

where the ρ_i's stand for the Radon–Nikodym derivatives $\dfrac{d\mu_i}{dV}$, and Jac, formally for the Jacobian of the map $G_u : M \to M$ (see Section 2.4.3 below). The very meaning of $\mathrm{Jac}(G_u)$ in the present case is specified in [12]. This equation is nothing but the change of variables formula which one would expect from the pushing condition (1.1) for the invertible map $\phi = G_u$. If the given densities ρ_0, ρ_1 are L^∞ and positively pinched, possibly endowed with some degree of smoothness, can we infer that the Brenier solution $u(\mu_0, \mu_1)$ is correspondingly smooth? Looking for sharp conditions on the data M, μ_0, μ_1 under which the potential function $u(\mu_0, \mu_1)$ must be C^1 has been a topic of intense research in the recent years [24–26, 28, 40, 59]. Higher order smoothness of $u(\mu_0, \mu_1)$ was derived in the cases listed below, relying on the Ma–Trudinger–Wang estimate [49] or its subsequent improvements [45, 58], combined with a so-called stay-away estimate.

   Henceforth, we always work on a closed Riemannian manifold M, we fix a couple $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geqslant 2$, and we assume that the given

probability densities ρ_0 and ρ_1 are $C^{k,\alpha}$ nonvanishing on M. For short, we say that the given measures μ_0, μ_1 are $C^{k,\alpha}$ positive. We want to prove that the potential function $u(\mu_0, \mu_1)$ is $C^{k+2,\alpha}$. This is the question which motivates the present paper.

1.2 Known results

If the manifold M is flat, the potential function $u(\mu_0, \mu_1)$ must be $C^{k+2,\alpha}$. Using the Bieberbach theorem (see e.g. [42, Section V.4]) combined with the covering space result of [18, Appendix C], this result is readily seen to follow from Cordero–Erausquin's regularity theorem [11] and further elliptic regularity. The flat case lies on a borderline, though. Indeed, as soon as the sectional curvature function K is allowed to take a negative value at some tangent 2-plane, one can select smooth probability densities ρ_0, ρ_1 such that the map $\mathrm{Mc}(\mu_0, \mu_1)$ is not continuous [46]. If M has constant positive curvature, the potential function $u(\mu_0, \mu_1)$ must be smooth [18, 47]. But the assumption $K > 0$ by itself does not imply the same result; counterexamples can be constructed in any dimension, again with $\mathrm{Mc}(\mu_0, \mu_1)$ not continuous [38] (see also [48] in the light of [28]). Trudinger conjectured that the smoothness issue should be tractable anytime the curvature of M is positive and slowly varying [57] (see also [18, p. 67]). We thus expect that the class of positively curved Riemannian locally symmetric spaces be a privileged one. If M (or its universal cover) is a complex or quaternionic projective space, the potential function $u(\mu_0, \mu_1)$ must be $C^{k+2,\alpha}$ [24, 40, 45]. Sticking to Riemannian local symmetry, Figalli, Kim and McCann further took for M the Riemannian product of constant curvature spheres of various dimensions. This case is, again, on the borderline for optimal transport smoothness, since the sectional curvature of M is now just nonnegative. Their results in that case [24, 25], combined with an extension (yet unpublished) of the interior regularity theorem of [45], imply that the potential function $u(\mu_0, \mu_1)$ is $C^{k+2,\alpha}$. Finally, in Corollary 2 of [18], dropping local symmetry, we treated the case of nearly spherical manifolds with nontrivial topology. Recalling that $K : \mathrm{Gr}_2(M) \to \mathbb{R}$ stands for the sectional curvature function defined on the Grassmann bundle of tangent 2-planes, and denoting by R the Riemann tensor of M, by ∇, the Levi–Civita connection and by $|\nabla^k R|$, the Riemannian norm of the tensor $\nabla^k R$, we can state the main result of [18] as follows:

Theorem 1.2. *Let M be a closed positively curved Riemannian manifold with curvature normalized by $\min_{\mathrm{Gr}_2(M)} K = 1$. Assume that M has*

a nontrivial first fundamental group. We can find a universal constant $\epsilon > 0$ such that, if the functions $(K - 1), |\nabla R|$ and $|\nabla^2 R|$ are everywhere smaller than ϵ, given any couple $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geqslant 2$, the potential function of the optimal map $\mathrm{Mc}(\mu_0, \mu_1)$ must be $C^{k+2,\alpha}$ for every couple (μ_0, μ_1) of $C^{k,\alpha}$ positive probability measures.

In the simply connected case, that is, with M diffeomorphic to $\mathbb{S}^n$, under the stronger[1] condition that the metric of M be C^4 close to a constant curvature metric, the continuity of $\mathrm{Mc}(\mu_0, \mu_1)$ was established in [26–29]. The higher order smoothness of the potential function $u(\mu_0, \mu_1)$ remains to be proven in that case, due to an incomplete *stay-away* estimate. Specifically, Villani and his coworkers miss to control how far the image point $\mathrm{Mc}(\mu_0, \mu_1)(m)$ stays away from the first conjugate point p on the geodesic ray $t \in [0, +\infty) \to \exp_m(t \operatorname{grad} u(\mu_0, \mu_1)(m)) \in M$, in case there is no other minimizing geodesic from m to p. The condition $\delta(M) > 0$ of [48, Theorem 7.1] was designed to exclude that case. It is verified on constant curvature spheres but unrealistic when the curvature is perturbed. The stay-away property, in the form just stated, is essential for the smoothness issue (see the first requirement of Theorem 1.5). The nondegeneracy of the Monge–Ampère Equation (1.2), as an elliptic equation, is at stake. This can already be seen from the symbol of the first-order differential operator obtained by linearizing formally the map $u \mapsto G_u = \exp(\operatorname{grad} u)$. This symbol, calculated at a point $m \in M$, is equal to the linear map $\xi \in T_m^* M \to d(\exp_m)(\operatorname{grad} u(m))(\xi^\sharp) \in T_{G_u(m)} M$, where $\xi \in T_m^* M \to \xi^\sharp \in T_m M$ stands for the Riesz isomorphism associated to the Euclidean scalar product on $T_m M$. The symbol is an isomorphism if and only if the image point $G_u(m)$ does not reach the first conjugate point on the ray $t \in [0, +\infty) \to \exp_m(t \operatorname{grad} u(m)) \in M$.

1.3   A stability conjecture

In the spirit of Trudinger's conjecture, one should be able to improve Theorem 1.2 by allowing the triviality of $\pi_1(M)$ and by dropping the ϵ curvature pinching condition, that is, to treat the nearly Riemannian locally symmetric case. Besides, the smoothness result of [25] indicates that one should even be able to let the curvature K be merely nonnegative. In other words, we are led to state the following:

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[1]See the Appendix of the paper.

Conjecture. *Let M be a closed nonnegatively curved Riemannian manifold. If the C^1 norm of the tensor ∇R is sufficiently small, given any couple $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geqslant 2$, the potential function of the optimal map $\mathrm{Mc}(\mu_0, \mu_1)$ must be $C^{k+2,\alpha}$ for every couple (μ_0, μ_1) of $C^{k,\alpha}$ positive probability measures.*

In the present paper, we will provide a *method* (see Theorem 1.5 below) for attacking this program and we will *test* it successfully in two cases, namely: the locally symmetric case (Theorem 1.3), which is the starting point of the conjecture (letting $\epsilon = 0$), and the 2-sphere case (Theorem 1.4). With this method at hand, much work still lies ahead to prove the full conjecture (we hope to return to part of it soon).

1.4   Statement of results, outline of the paper

We are going to prove the following results:

Theorem 1.3. *If the manifold M is positively curved Riemannian locally symmetric, given any couple $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geqslant 2$, the potential function of the optimal map $\mathrm{Mc}(\mu_0, \mu_1)$ must be $C^{k+2,\alpha}$ for every couple (μ_0, μ_1) of $C^{k,\alpha}$ positive probability measures.*

Theorem 1.4. *We can find a universal constant $\epsilon > 0$ such that, on the manifold $M = \mathbb{S}^2$ endowed with a Riemannian metric of Gauss curvature K normalized by $\min_M K = 1$ and satisfying the bound $\left|K - 1\right|_{C^2(M)} \leqslant \epsilon$, given any couple $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geqslant 2$, the potential function of the optimal map $\mathrm{Mc}(\mu_0, \mu_1)$ must be $C^{k+2,\alpha}$ for every couple (μ_0, μ_1) of $C^{k,\alpha}$ positive probability measures.*

As mentioned above, Theorem 1.3 was known except for the case of the Cayley plane. Here, we will give a unified proof of it. Theorem 1.4 is new. In connection with the comment made after Theorem 1.2, let us mention that, for every Riemannian metric on the 2-sphere, the cut locus of each point *must* contain a conjugate point [53, 60]. The proofs of these two theorems share a common part (Sections 3, 4 and 6) valid for every A3 manifold as stated in Theorem 6.1 below. They depart only by their final step (Section 7) consisting of a suitable stay-away estimate (all these notions will be specified below).

We use a partial differential equations method which makes no use of McCann's potential function $u(\mu_0, \mu_1)$, except for its final identification with

a smooth potential function u obtained as a classical solution of the Monge–Ampère Equation (1.2). This method was partly sketched, with some gaps, in [17] and used in [18, 47]. It is presented here for the first time in a complete self-contained form. It involves only techniques of differential geometry combined with an intrinsic local analysis of critical and extremum differential conditions, in contrast with smoothness proofs given on closed manifolds, as yet, by other authors. It yields a smoothness scheme for the solution of (1.2) valid on every closed Riemannian manifold as stated in Theorem 1.5 below. The strategy of the proof may be described by the following steps.

Section 3. The classical solvability of the Monge–Ampère Equation (1.2) completely encodes the smoothness issue for Monge's problem: specifically, we show that the existence of a C^2 solution of (1.2) is equivalent to the optimal potential $u(\mu_0, \mu_1)$ being a C^2 function. Moreover, at such a C^2 solution, the Monge–Ampère equation must be nondegenerate and elliptic.

Section 4. We solve uniquely (1.2) by the continuity method, replacing the target probability density ρ_1 by the deformed one $\rho_t = t\rho_1 + (1-t)\rho_0$ and arguing by connectedness on the subset $\mathcal{T}$ of all values of $t \in [0,1]$ for which the resulting analog of (1.2), namely the continuity Equation (4.1), admits a classical solution $u_t \in C^{k+2,\alpha}(M, \mathbb{R})$. To ensure uniqueness, the solutions u_t are normalized by $\int_M u_t \, dV = 0$ (see Proposition 4.2). Showing that $\mathcal{T}$ coincides with $[0,1]$ boils down to proving that the supremum over $\mathcal{T}$ of the largest eigenvalue on the manifold M of the Hessian endomorphism field $m \mapsto \nabla\mathrm{grad}\, u_t(m)$ is finite. This finiteness requirement turns out equivalent to requirements 1 and 2 of the general smoothness result to which the continuity method leads, namely:

Theorem 1.5 (general smoothness result). *On a closed Riemannian manifold M, given $(k, \alpha) \in \mathbb{N} \times (0,1)$, with $k \geqslant 2$, the potential function $u(\mu_0, \mu_1)$ is $C^{k+2,\alpha}$ for every couple of $C^{k,\alpha}$ positive probability measures (μ_0, μ_1), if, for each such couple, the following two requirements are fulfilled:*

(1) *The infimum of $\min_{m \in M} \mathrm{Jac}(\exp_m)(\mathrm{grad}\, u_t(m))$, over the solution counterset $\mathcal{T}$ just defined, is positive;*

(2) *The supremum, over $\mathcal{T}$, of the largest eigenvalue on the manifold M of the symmetric endomorphism field $m \mapsto \mathcal{J}(m, \mathrm{grad}\, u_t(m)) + \nabla \mathrm{grad}\, u_t(m)$, is finite ($\mathcal{J}$ standing for the Jacobi endomorphism field, see Section 2.1.3).*

If, for a couple of measures (μ_0, μ_1) of the above type, either requirements fails, there exists a real $\tau \in [0,1]$ such that, setting μ_τ for the measure, still

of the same type, equal to $\tau\mu_1 + (1-\tau)\mu_0$, the potential function $u(\mu_0, \mu_\tau)$ is not C^2.

This statement is new. It will be established in the concluding part of Section 4. The proofs of Theorems 1.3 and 1.4 are now reduced to checking, on the corresponding manifolds, the two requirements stated in Theorem 1.5. For this purpose, we will have to derive from the Monge–Ampère equation satisfied by u_t, for $t \in \mathcal{T}$, a suitable sequence of uniform a priori estimates.

Remark 1.1 (specifications for the a priori estimates). The a priori estimates on $|\nabla^\ell u_t|(m)$ or on related quantities of order ℓ with respect to u_t, for $\ell = 0, 1, 2$, will be understood to be independent of $(t, m) \in \mathcal{T} \times M$ and controlled, unless otherwise specified, solely by appropriate geometric invariants of the Riemannian manifold M and by the largest C^ℓ norm of $\log \rho_t$ as $t \in [0, 1]$. Any such estimate will be said *under control*. Here, we should warn the reader that these estimates, derived by maximum principle techniques, will not be as sharp as the ones designed for nonsmooth densities. This was already clear in the two contrasted parts of [20]. It is the price to pay for the self-consistency of a partial differential equations method ignoring McCann's solution.

Checking the requirements 1 and 2 of Theorem 1.5 on each particular manifold will be achieved by carrying out the following two estimates:

Estimate 1 (first order). There exists a real $\delta > 0$ under control such that:

$$\mathrm{Jac}(\exp_m)(\mathrm{grad}\, u_t(m)) \geqslant \delta,$$

for every point $(t, m) \in \mathcal{T} \times M$.

Estimate 2 (second order). There exists a real $\Lambda > 0$ under control, possibly depending on the preceding constant δ, such that:

$$\mathcal{J}(m, \mathrm{grad}\, u_t(m)) + \nabla \mathrm{grad}\, u_t(m) \leqslant \Lambda\, \mathrm{Id}_m,$$

for every point $(t, m) \in \mathcal{T} \times M$ (where Id_m denotes the identity of $T_m M$).

The controls actually obtained on δ and Λ will be further specified for each estimate (see Lemma 6.1 for Λ, and Sections 7.1 and 7.2 for δ).

Section 6. Estimate 1 implies Estimate 2 anytime the manifold M fulfills the emblematic condition A3 discovered in [49], as do the manifolds of Theorems 1.3 and 1.4 [19, 21]. Indeed, this is the celebrated Ma–Trudinger–Wang estimate [49, Section 4]. For completeness, the latter is recast below in

intrinsic form, as a maximum principle for the trace of the tensor appearing in Estimate 2 (an alternative adaptation was proposed in [18, Appendix B]). So much for the common part of the proofs.

Section 7. We complete the proofs by checking Estimate 1 separately for each theorem. For Theorem 1.3, we do it by proving a maximum principle for $|\operatorname{grad} u_t|^2$ which extends the one proved for constant curvature spheres in [20, Section 3]. For Theorem 1.4, we must prove a genuine minimum principle for the full test-function $m \mapsto \operatorname{Jac}(\exp_m)(\operatorname{grad} u_t(m))$ instead, getting for it a positive lower bound under control. This estimate is the newest result of the paper.

Finally, let us say a few words about the sections which we did not mention yet. In *Section 5*, we revisit the known case of flat manifolds (cf. *supra*) since in that case, Estimate 1 trivially holds. Flat manifolds do not fulfill condition A3, nevertheless Estimate 2 holds as a particular case of an a priori estimate proven by the author [15, pp. 428–429].

Last, but not least, *Section 2* contains an appropriate introduction to the differential geometric tools which make natural the partial differential equations approach. Admittedly, it is quite a long preliminary, but we could not find the right Introduction in the literature. The heavy use of differential geometry for proving the smoothness of optimal potential functions can be conceived by noting that the operator $u \mapsto \operatorname{Jac}(G_u)$ factors through the exponential map. The background of Section 2 will serve to prove, for instance, the nondegeneracy and ellipticity of that operator, when restricted to the C^2 functions at which it does not vanish (see Proposition 3.1). Our efforts to construct a C^2 solution of the Monge–Ampère Equation (1.2) will mostly bear on getting a control on that nondegeneracy, by Estimate 1, and that ellipticity *via* Estimate 2. These a priori estimates will require deeper tools from geometry again, like Proposition 7.1 for tackling Estimate 1, or like the verification of condition A3 [19, 21] on which Estimate 2 relies *via* the Ma–Trudinger–Wang device. From the inescapable role played by differential geometry for solving (1.2) in the classical sense, we can realize that much work lies ahead in the research program mentioned above.

2. Preliminary

Let (M, g) be a closed unoriented n-dimensional Riemannian manifold. In the first part of this preliminary, we define partial differentials of functions or maps with source space the tangent bundle $\varpi_M : TM \to M$. It yields an intrinsic way of calculating the differential of a function on M, when given as the pullback by a vector field of a function defined on TM. Similar

calculations are made for the exponential map and for the Jacobi endomorphism field. The second part of the preliminary bears on solutions of initial value problems for the Jacobi equation, focusing on their representation in terms of the partial differentials of the exponential map just developed. Applications are drawn, including a derivation of the main properties of the Jacobi endomorphism field. In the last part, we define Jacobian determinants for various types of maps arising in the smoothness problem under study.

2.1. Partial differentials

2.1.1. Real functions. Throughout the paper, we set (m, V) for the generic point of TM, with the usual meaning that $m \in M$ is the base point and V, a tangent vector at m. Consistently with that notation, we would welcome a way of viewing uniquely every vector W, tangent to TM at a point (m, V), as a couple $W \simeq (\xi, \nu)$ of vectors in $T_m M \times T_m M$. Accordingly, we seek a global fashion of splitting the differential of a function $\psi : TM \to \mathbb{R}$ into the sum of two partial differentials, something like:

$$d\psi(m, V)(W) = \frac{\partial \psi}{\partial m}(m, V)(\xi) + \frac{\partial \psi}{\partial V}(m, V)(\nu).$$

Let us explain how it can be done with some care[2], using the Levi–Civita connection[3] of M. Once settled the case of real functions, we will be able to treat that of more general objects defined on TM. The present account is intended for nonexperts. Other readers may skip it and just record the final Equations (2.3) and (2.5).

In every natural chart $x = x(m), v = dx(m)(V)$ of TM, the differential of the function ψ is given by

$$d\psi(m, V) = \frac{\partial \psi}{\partial x^i}(x, v)\, dx^i + \frac{\partial \psi}{\partial v^i}(x, v)\, dv^i,$$

with Einstein's convention. In this local expression, only half of the terms are intrinsic. Indeed, the differential of the base point projection $\varpi_M : TM \to M$ reads $d\varpi_M(m, V) = \left(dx^1, \ldots, dx^n \right)$. Moreover, for each integer $i = 1, \ldots, n$, there is no difficulty to fix the point $x = x(m)$ and the tangent vector $v = dx(m)(V)$ but its ith component, and calculate the partial derivative

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[2]Compare the preceding formula with the displayed one that precedes (2.3) below.
[3]Here, one could deal similarly with any vector bundle of finite rank over M, using a linear connection.

$\frac{\partial \psi}{\partial v^i}(x,v)$. The $\frac{\partial \psi}{\partial v^i}$ are the components of the differential $d\big(\psi(m,.)\big)$ obtained by fixing the base point m and letting the tangent vector V vary in $T_m M$. This differential is an element of the *dual* of $T_m M$. Evaluated at the vector V, it thus reads $\frac{\partial \psi}{\partial v^i}(x,v)\,dx^i$. It is called the partial differential of ψ with respect to V for fixed m, denoted below by $\left(\frac{\partial \psi}{\partial V}\right)_m$. In contrast to dx^i and $\frac{\partial \psi}{\partial v^i}(x,v)$, the two other terms of the above formula, namely dv^i and $\frac{\partial \psi}{\partial x^i}(x,v)$, do not transform properly under a change of charts (unless the atlas of M is affine). They are not intrinsic. In order to deform the partial derivative $\frac{\partial \psi}{\partial x^i}(x,v)$ into an intrinsic expression, we need a tool to fix the tangent vector V while the ith coordinate of its base point m varies near the value $x^i(m)$, all other coordinates being frozen. The Levi–Civita connection ∇ is an appropriate tool. Using ∇, we will keep the vector V *parallel* to itself while transporting it along the path $t \mapsto \big(x^1(m),\ldots,x^{i-1}(m),x^i(m)+t,x^{i+1}(m),\ldots,x^n(m)\big)$, for $t \in \mathbb{R}$ small. Doing so yields the intrinsic notion of partial differential of ψ with respect to m keeping V parallel, denoted by $\left(\frac{\partial \psi}{\partial m}\right)_V^\nabla$, and defined at the point (m,V) by

$$(2.1) \quad \left(\frac{\partial \psi}{\partial m}\right)_V^\nabla (m,V) = \delta_i^\nabla \psi(x,v)\,dx^i, \quad \text{with: } \delta_i^\nabla = \frac{\partial}{\partial x^i} - \Gamma_{ij}^k(x)v^j \frac{\partial}{\partial v^k},$$

the Γ_{ij}^k standing for the Christoffel symbols of the connection ∇ in the chart x. Subtracting this expression from $d\psi$, we get

$$d\psi - \left(\frac{\partial \psi}{\partial m}\right)_V^\nabla (m,V) = \frac{\partial \psi}{\partial v^i}\,d^\nabla v^i, \quad \text{with: } d^\nabla v^i = dv^i + \Gamma_{kj}^i(x)v^j dx^k.$$

Altogether, we have recast the differential of the function ψ as follows:

$$(2.2) \qquad\qquad d\psi(m,V) = \delta_i^\nabla \psi(x,v)\,dx^i + \frac{\partial \psi}{\partial v^i}(x,v)\,d^\nabla v^i.$$

We need some geometry to interpret further this expression, since the partial differential $\left(\frac{\partial \psi}{\partial V}\right)_m (m,V)$ was defined above as an element of $T_m^* M$ and its components $\frac{\partial \psi}{\partial v^i}$ are now associated with the covariant differentials $d^\nabla v^i$. The kernel $\mathcal{V} = \bigcap_{i=1}^n \ker(dx^i)$ of the differential $d\varpi_M$ is a subbundle of $T(TM) \to TM$ called the vertical subbundle and locally spanned by the $\frac{\partial}{\partial v^i}$'s. The vector fields $\delta_i^\nabla, 1 \leqslant i \leqslant n$, span a subbundle $\mathcal{H}^\nabla$ of $T(TM) \to TM$, called the horizontal subbundle, locally given by $\mathcal{H}^\nabla = \bigcap_{i=1}^n \ker(d^\nabla v^i)$ and such that the following splitting holds:

$$T(TM) = \mathcal{H}^\nabla \oplus \mathcal{V}.$$

Moreover, the local fields of basis $\{\delta_1^\nabla, \ldots, \delta_n^\nabla, \frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^n}\}$ and cobasis $\{dx^1, \ldots, dx^n, d^\nabla v^1, \ldots, d^\nabla v^n\}$ are dual of each other. The horizontal projection $T(TM) \to \mathcal{H}^\nabla$ with kernel equal to $\mathcal{V}$ is given by $dm = dx^i \otimes \delta_i^\nabla$ and the vertical projection $T(TM) \to \mathcal{V}$ with kernel equal to $\mathcal{H}^\nabla$, by $d^\nabla V = d^\nabla v^i \otimes \frac{\partial}{\partial v^i}$. Every tangent vector $W = \xi^i \frac{\partial}{\partial x^i} + \eta^i \frac{\partial}{\partial v^i} \in T_{(m,V)}(TM)$ reads now uniquely as the sum of a horizontal vector $W_h = dm(W)$ and a vertical one $W_v = d^\nabla V(W)$, thus, respectively, equal to:

$$W_h = \xi^i \delta_i^\nabla \in \mathcal{H}_{(m,V)}^\nabla, \quad W_v = \left(\eta^i + \Gamma_{kj}^i(x)v^j\xi^k\right)\frac{\partial}{\partial v^i} \in \mathcal{V}_{(m,V)}.$$

Last but not least, we may identify the fiber $\mathcal{H}_{(m,V)}^\nabla$ with the tangent space $T_m M$ by means of the differential $d\varpi_M$ which induces the isomorphism from $\mathcal{H}_{(m,V)}^\nabla$ to $T_m M$ given by: $\delta_i^\nabla \to \frac{\partial}{\partial x^i}, i = 1, \ldots, n$. We further identify $\mathcal{V}_{(m,V)}$ with $T_m M$ by the canonical isomorphism from $\mathcal{V}_{(m,V)}$ to $T_m M$ given by the affine structure of $T_m M$, or else, by: $\frac{\partial}{\partial v^i} \to \frac{\partial}{\partial x^i}, i = 1, \ldots, n$. In this way, we may freely consider at (m, V) the projections dm and $d^\nabla V$, or else, the components W_h and W_v of $W \in T_{(m,V)}(TM)$, as valued in $T_m M$. Accordingly, we may view the partial differentials of ψ at (m, V) appearing in (2.2), that is $\left(\frac{\partial\psi}{\partial m}\right)_V^\nabla(m, V)$ and $\frac{\partial\psi}{\partial v^i}(x, v)\, d^\nabla v^i$, which are elements of the dual of $\mathcal{H}_{(m,V)}^\nabla$ and $\mathcal{V}_{(m,V)}$, respectively, as belonging to $T_m^* M$. Doing so, we may now write (2.2) as a global identity of the desired form, namely:

$$d\psi(m, V)(W) = \left(\frac{\partial\psi}{\partial m}\right)_V^\nabla(m, V)\left(W_h\right) + \left(\frac{\partial\psi}{\partial V}\right)_m(m, V)\left(W_v\right).$$

Pulling the function ψ back on M by a vector field $U \in \Gamma(TM)$, thus setting $U^*\psi(m) \equiv \psi\big(m, U(m)\big)$, we infer for every $m \in M, \xi \in T_m M$, the formula:

$$
\begin{aligned}
(2.3) \quad & d(U^*\psi)(m)(\xi) \\
&= \left(\frac{\partial\psi}{\partial m}\right)_V^\nabla(m, U(m))(\xi) + \left(\frac{\partial\psi}{\partial V}\right)_m(m, U(m))\big(\nabla_\xi U(m)\big),
\end{aligned}
$$

of constant use in the paper.

Remark 2.1. For later use, let us record here the existence of the so-called diagonal lift g^D of the metric g to the manifold TM, a Riemannian metric, often named after Sasaki (see [61] and references therein) and defined, using

the preceding identifications, by:

$$g^D_{(m,V)}(W, Z) = g_m(W_{\mathrm{h}}, Z_{\mathrm{h}}) + g_m(W_{\mathrm{v}}, Z_{\mathrm{v}}),$$

for every point $(m, V) \in TM$ and every couple of vectors W, Z in $T_{(m,V)}(TM)$. In particular, the factors $\mathcal{H}^\nabla$ and $\mathcal{V}$ of the splitting of $T(TM)$ are mutually orthogonal for the metric g^D.

2.1.2. Exponential map. We can deal similarly with the exponential map $\exp : TM \to M$, getting the identity:

$$(2.4) \quad d\exp(m, V)(W) = \left(\frac{\partial \exp}{\partial m}\right)^\nabla_V (m, V)(W_{\mathrm{h}}) + \left(\frac{\partial \exp}{\partial V}\right)_m (m, V)(W_{\mathrm{v}}).$$

Here, the partial differentials of the exponential map at the point (m, V) may be viewed as linear maps from $T_m M$ to $T_p M$, with $p = \exp_m(V)$, equal to the identity at $V = 0$. As the point (m, V) varies in TM, each of them gives rise to a section of the vector bundle $\mathrm{Mor}(\varpi^*_M TM, \exp^* TM) \to TM$ of morphisms covering the identity, $\varpi^*_M TM \to TM$ (resp. $\exp^* TM \to TM$) standing for the pullback of the tangent bundle $\varpi_M : TM \to M$ by the map ϖ_M (resp. by the exponential map). Equivalently, we may view the maps $\left(\frac{\partial \exp}{\partial m}\right)^\nabla_V$ and $\left(\frac{\partial \exp}{\partial V}\right)_m$ as sections of the bundle $\varpi^*_M T^* M \otimes \exp^* TM \to TM$.

Pulling the exponential map back to the base manifold M by means of a vector field $U \in \Gamma(TM)$, we infer from (2.4) the further identity:

$$(2.5) \quad d(U^* \exp)(m)(\xi)$$
$$= \left(\frac{\partial \exp}{\partial m}\right)^\nabla_V (m, U(m))(\xi) + \left(\frac{\partial \exp}{\partial V}\right)_m (m, U(m))(\nabla_\xi U(m)),$$

where $U^* \exp(m) \equiv \exp_m(U(m))$.

2.1.3. Jacobi endomorphism. Let $\mathcal{N}_{\mathrm{conj}}$ denote the open neighborhood of the zero section of TM consisting of the points $(m, V) \in TM$ such that the linear map:

$$\left(\frac{\partial \exp}{\partial V}\right)_m (m, V) : T_m M \to T_p M, \ \text{with} \ p = \exp_m(V),$$

remains an isomorphism. Given $(m, V) \in \mathcal{N}_{\mathrm{conj}}$ and $\xi \in T_m M$, the equation:

$$\left(\frac{\partial \exp}{\partial m}\right)^\nabla_V (m, V)(\xi) = \left(\frac{\partial \exp}{\partial V}\right)_m (m, V)(\nu),$$

can be solved uniquely for $\nu \in T_m M$. It thus defines an endomorphism $\xi \in T_m M \to \mathcal{J}(m, V)(\xi) = \nu \in T_m M$, often called the Jacobi endomorphism of $T_m M$ at (m, V). This endomorphism arises naturally in the description of the *fibers* of the exponential map, when the latter is restricted to $\mathcal{N}_{\text{conj}}$. Specifically, the tangent space to $\exp^{-1}(p)$ at $(m, V) \in \mathcal{N}_{\text{conj}}$ is the subspace of $T_{(m,V)}(TM)$ given by the graph of the linear map from $\mathcal{H}^{\nabla}_{(m,V)}$ to $\mathcal{V}_{(m,V)}$ defined, with the usual identifications, by $-\mathcal{J}(m, V)$. The main properties of this endomorphism will be derived in Section 2.3.2 below, using the Jacobi equation. Here, we just note that $\mathcal{J}(m, 0)$ must be equal to the identity.

The map $(m, V) \in TM \to \mathcal{J}(m, V) \in \text{End}(T_m M)$ is a section, over $\mathcal{N}_{\text{conj}}$, of the pullback bundle $\varpi_M^* \text{End}(TM) \to TM$. Let us pause and explain a natural way of differentiating on TM maps like the ones considered so far, namely $\left(\frac{\partial \exp}{\partial m} \right)^{\nabla}_V$, $\left(\frac{\partial \exp}{\partial V} \right)_m$ and $\mathcal{J}$.

2.1.4. Linear connections on vector bundles.   The material presented here for convenience is fairly well known [54]. A linear connection ∇ on a vector bundle $p: E \to Q$ over a closed manifold Q is a map which associates, to each couple $((q, W), S)$ formed by a tangent vector $(q, W) \in TQ$ at a point $q \in Q$ and a section S of the bundle $E \to Q$, a vector $\nabla_W S$ of the fiber $E_q = p^{-1}(q)$, in such a way that:

(i) given any integer k, for every C^k vector field W on the base manifold Q and every C^{k+1} section S of the bundle $E \to Q$:
   (i.1) the section $\nabla_W S$ of that bundle is of class C^k and the map $(W, S) \mapsto \nabla_W S$ so defined is smooth;
   (i.2) the map $W \mapsto \nabla_W S$, defined by fixing the section S and letting the vector field W vary, is linear over the algebra $C^k(Q)$ of real functions of class C^k on Q;

(ii) for each fixed tangent vector (q, W), the map $S \mapsto \nabla_W S$ obtained by letting the section S vary, is $\mathbb{R}$-linear and satisfies the chain rule, that is, the equation:

$$\nabla_W(fS) = df(W)\, S + f\, \nabla_W S,$$

holds identically at the point q, for every real function $f \in C^1(Q)$.

Given a linear connection ∇ on a vector bundle $E \to Q$, we can define new linear connections out of it by duality, tensor product and pullback. The dual connection ∇^* is the linear connection defined on the *dual* bundle $E^* \to Q$

by the equation:

$$\nabla^*_W \ell(S) = d(\ell(S))(W) - \ell(\nabla_W S),$$

valid for every vector field W on Q and sections S of E, ℓ of E^*. The product connection $\nabla^\otimes$ on the tensor product $E^* \otimes E \to Q$, for instance, is given by:

$$\nabla^\otimes_W (\ell \otimes S) = (\nabla^* \ell) \otimes S + \ell \otimes (\nabla_W S).$$

A similar definition can be given for more general tensor products, of course. In either case, we are guided by the chain rule for the defining equation and the axioms $(i)(ii)$ of a linear connection can readily be verified.

   The pullback device is a bit trickier; it goes as follows. Given a map $\varphi : P \to Q$ between two closed manifolds and a vector bundle $E \to Q$ of rank r endowed with a linear connection ∇, we can define on the pullback bundle $\varphi^* E \to P$ a linear connection ∇^φ as follows. Fix a point $p_0 \in P$ and a local frame field $q \in Q \to \big(S_1(q), \ldots, S_r(q)\big)$ spanning the fiber of E over the point q, for q near the image point $q_0 = \varphi(p_0)$. Such a frame field yields, for every section ξ of the pullback bundle $\varphi^* E \to P$, local real functions $\xi^1, \ldots, \xi^r$ defined near the point p_0 in the source manifold P and such that the section ξ reads: $\xi(p) = \xi^a(p) \, S_a(q)$, with $q = \varphi(p)$. Now, for each vector field V on P, a natural candidate for the covariant derivative $\nabla^\varphi_V \xi$ is given near the point p_0 by:

(2.6) $$\nabla^\varphi_V \xi(p) = d\xi^a(p)(V(p)) \, S_a(q) + \xi^a(p)(\nabla_W S_a)(q),$$

with $q = \varphi(p)$ and $W = d\varphi(p)(V(p)) \in T_q Q$. The right-hand side of this equation turns out to be independent of the choice of the local frame S_i, hence of that of the point p_0 as well. So we have actually defined a global section $\nabla^\varphi_V \xi$ of the pullback bundle $\varphi^* E \to P$. Furthermore, one can readily check that ∇^φ fulfills the axioms $(i)(ii)$ of a linear connection on the vector bundle $\varphi^* E \to P$.

   In case the source manifold P is an interval (resp. the product of two intervals) of the real line, with generic point t (resp. (τ, t)), it is common to write $\frac{\nabla}{dt}$ (resp. $(\frac{\nabla}{\partial \tau}, \frac{\nabla}{\partial t})$) instead of $\nabla^\varphi_{\frac{d}{dt}}$ (resp. $\big(\nabla^\varphi_{\frac{\partial}{\partial \tau}}, \nabla^\varphi_{\frac{\partial}{\partial t}}\big)$). If the map $\varphi : P \to Q$ is equal to, either $\varpi_M : TM \to M$ or $\exp : TM \to M$, and we take for the vector bundle $E \to Q$ the tangent bundle $TM \to M$ equipped with the Levi–Civita connection, we will omit the superscript φ of ∇^φ and write ∇ anytime no confusion may occur. Let us specify what are the coefficients of these connections in local charts. Sticking to the notations of Section 2.1.1,

we infer from (2.6) that they are given by:

$$\nabla_{\delta_i^\nabla}^{\varpi_M}\left(\frac{\partial}{\partial x^j}\right) = \Gamma_{ij}^k(x)\frac{\partial}{\partial x^k}, \quad \nabla_{\frac{\partial}{\partial v^i}}^{\varpi_M}\left(\frac{\partial}{\partial x^j}\right) = 0, \text{ and:}$$

$$\nabla_{\delta_i^\nabla}^{\exp}\left(\frac{\partial}{\partial \bar{x}^j}\right) = (\delta_i^\nabla E^k)(x,v)\,\overline{\Gamma}_{jk}^\ell(E(x,v))\frac{\partial}{\partial \bar{x}^\ell},$$

$$\nabla_{\frac{\partial}{\partial v^i}}^{\exp}\left(\frac{\partial}{\partial \bar{x}^j}\right) = \frac{\partial E^k}{\partial v^i}(x,v)\,\overline{\Gamma}_{jk}^\ell(E(x,v))\frac{\partial}{\partial \bar{x}^\ell}.$$

Here, we have used a chart x (resp. $\bar{x}$) at the source point m (resp. target point $\overline{m} = \exp_m(V)$), set $E(x,v) = \bar{x}(\exp_m(V))$, where $x = x(m), v = dx(m)(V)$, and denoted by Γ_{ij}^k (resp. $\overline{\Gamma}_{ij}^k$) the Christoffel symbols of the Levi–Civita connection of M in the chart x (resp. $\bar{x}$). Combining the knowledge of these coefficients with the duality and tensor product devices described above, we are now in position to differentiate sections like $\left(\frac{\partial \exp}{\partial m}\right)_V^\nabla$, $\left(\frac{\partial \exp}{\partial V}\right)_m$ and $\mathcal{J}$. In the latter case, for instance, we find:

$$(2.7) \quad \nabla_{\delta_i^\nabla}\mathcal{J}_j^k = \delta_i^\nabla \mathcal{J}_j^k + \Gamma_{i\ell}^k(x)\mathcal{J}_j^\ell - \Gamma_{ij}^\ell(x)\mathcal{J}_\ell^k = \nabla_i\mathcal{J}_j^k - \Gamma_{ih}^\ell(x)v^h\frac{\partial \mathcal{J}_j^k}{\partial v^\ell},$$

$$\text{and } \nabla_{\frac{\partial}{\partial v^i}}\mathcal{J}_j^k = \frac{\partial \mathcal{J}_j^k}{\partial v^i},$$

with $\nabla_i\mathcal{J}_j^k = \frac{\partial \mathcal{J}_j^k}{\partial x^i} + \Gamma_{i\ell}^k(x)\mathcal{J}_j^\ell - \Gamma_{ij}^\ell(x)\mathcal{J}_\ell^k$. Let us emphasize that the local functions $\nabla_i\mathcal{J}_j^k$ are not the components of a tensor, unlike the functions $\nabla_{\delta_i^\nabla}\mathcal{J}_j^k$. Nevertheless, it will be convenient to use them in some calculations.

Remark 2.2 (general erratum). In his previous works, anytime the author had to differentiate twice a section of a vector bundle over TM (obtained by pulling back a tensor bundle over M) for the purpose of a *second*-order estimate performed in a normal chart of M, he made a meaningless use of the original Levi–Civita connection ∇, instead of using the appropriate pullback connection. However, his estimates are still valid. Indeed, every rectification of this mistake yields only *first*-order terms with curvature coefficients, which are inessential for the estimate. Section 6.2 of the present paper illustrates the correct way to proceed.

2.2. C^ℓ norms

We need to define C^ℓ norms of real functions, or of sections of pullback tensor bundles, defined on (the closure of) some smooth bounded domain Ω of TM.

For real functions, we do it using the Sasaki metric g^D (see Remark 2.1) and its Levi–Civita connection. For sections of $\varpi_M^* TM$, we first note that this vector bundle inherits an obvious avatar of the metric g. Moreover, the connection ∇^{ϖ_M} is compatible with the latter. Combining the preceding tools, we can define the Banach space $C^\ell(\overline{\Omega}, \varpi_M^* TM)$. A similar definition can be given for the space $C^\ell(\overline{\Omega}, \varpi_M^* \mathrm{End}(TM))$.

2.3. Jacobi fields and applications

2.3.1. Jacobi equation. Given $(m_0, V_0) \in TM$, we set $\mathcal{X}_{\gamma_0}$ for the vector space of all vector fields $t \mapsto Y(t) \in T_{\gamma_0(t)}M$ along the geodesic path $t \mapsto \gamma_0(t) = \exp_{m_0}(tV_0) \in M$, and let $\mathfrak{J}_{\gamma_0} \in \mathrm{End}(\mathcal{X}_{\gamma_0})$ denote the Jacobi operator along γ_0. Recall that the latter reads:

$$Y \mapsto \mathfrak{J}_{\gamma_0}(Y)(t) = \frac{\nabla^2 Y}{dt^2} + R_{\gamma_0(t)}(Y, \dot{\gamma}_0(t))\dot{\gamma}_0(t),$$

where R stands for the curvature (or Riemann) tensor[4], of the Levi–Civita connection ∇ and the dot, as usual, for the time derivative. If we change the initial time from 0 to τ, setting $\gamma_\tau(t) = \exp_{\gamma_0(\tau)}(t\dot{\gamma}_0(\tau))$, we have $\gamma_\tau(t - \tau) \equiv \gamma_0(t)$, hence $\dot{\gamma}_\tau(t - \tau) \equiv \dot{\gamma}_0(t)$ and:

$$R_{\gamma_0(t)}(., \dot{\gamma}_0(t))\dot{\gamma}_0(t) \equiv R_{\gamma_\tau(\theta)}(., \dot{\gamma}_\tau(\theta))\dot{\gamma}_\tau(\theta) \text{ with } \theta = t - \tau.$$

We infer that, for every $Y_0 \in \mathcal{X}_{\gamma_0}$, the vector field $Y_\tau \in \mathcal{X}_{\gamma_\tau}$ defined by $Y_\tau(t - \tau) = Y_0(t)$ satisfies identically $\mathfrak{J}_{\gamma_\tau}(Y_\tau)(t - \tau) \equiv \mathfrak{J}_{\gamma_0}(Y_0)(t)$.

If a vector field $Y_0 \in \mathcal{X}_{\gamma_0}$ satisfies the Jacobi equation $\mathfrak{J}_{\gamma_0} Y_0 = 0$, it is called a Jacobi field. Note that: $Y_0 \in \ker \mathfrak{J}_{\gamma_0} \iff Y_\tau \in \ker \mathfrak{J}_{\gamma_\tau}$. Regarding the special way in which Jacobi fields arise, we refer the reader to Riemannian geometry textbooks [10, pp. 14–15] [6, Chapter 5]. Here, we would like to point out that $Y_0 \in \mathcal{X}_{\gamma_0}$ is a Jacobi field if and only it satisfies the following initial value identity:

$$(2.8) \quad Y_0(t) \equiv \left(\frac{\partial \exp}{\partial m}\right)_V^\nabla (m_0, tV_0)(Y_0(0)) + t\left(\frac{\partial \exp}{\partial V}\right)_m (m_0, tV_0)\left(\frac{\nabla Y_0}{dt}(0)\right).$$

Indeed, this identity expresses exactly what geometers prove about Jacobi fields, for instance in the books just quoted.

---

[4]With the sign convention given by: $R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]}W$, for every triple of vector fields U, V, W.

Remark 2.3. As is well known and easy to check, the velocity field $\dot{\gamma}_0$ is a Jacobi field along the geodesic path γ_0, and so is the field $t \mapsto t\dot{\gamma}_0(t)$. From (2.8), we see that the former is equal to $\left(\frac{\partial \exp}{\partial m}\right)^{\nabla}_V(m_0, tV_0)(V_0)$ and the latter, to $t\left(\frac{\partial \exp}{\partial V}\right)_m(m_0, tV_0)(V_0)$. Besides, for every Jacobi field Y_0, one readily verifies that the scalar product $g_{\gamma_0(t)}\big(Y_0(t), \dot{\gamma}_0(t)\big)$ must be an affine function of t. If $V_0 \neq 0$, it follows that the linear maps $\left(\frac{\partial \exp}{\partial m}\right)^{\nabla}_V(m_0, tV_0)$ and $\left(\frac{\partial \exp}{\partial V}\right)_m(m_0, tV_0)$ send $V_0^\perp$ to $\dot{\gamma}_0(t)^\perp$.

Let us describe some applications of the identity (2.8) to be used in the paper.

2.3.2. Applications.

Explicit calculations. For each point $(m, V) \in TM$, the endomorphism of $T_m M$ given by: $Y \mapsto R_m(Y, V)V$, often called the Jacobi curvature endomorphism along the vector V, is symmetric. If U is a nontrivial eigenvector orthogonal to V, the corresponding eigenvalue is equal to $|V|^2 K(p)$, p standing for the tangent 2-plane at m defined by the vectors U and V, and K, for the sectional curvature of the manifold M in the direction of that 2-plane.

If $V_0 \neq 0$ and the manifold M is Riemannian locally symmetric, that is, if the Riemann tensor is parallel, it is possible to take along the geodesic γ_0 a parallel orthonormal frame field $(e_1(t), \ldots, e_n(t))$, with $\dot{\gamma}_0(t) = |V_0|e_n(t)$, in which the matrix of the Jacobi curvature endomorphism along $\dot{\gamma}_0(t)$ remains diagonal and constant as t varies. Setting $Y_0(t) = \sum_{i=1}^n Y_0^i(t)e_i(t)$ and $R_{m_0}\big(e_\alpha(0), V_0\big)V_0 = |V_0|^2 \kappa_\alpha \, e_\alpha(0), \alpha = 1, \ldots, n-1$, the Jacobi equation $\mathfrak{J}_{\gamma_0}(Y_0) = 0$ becomes: $\frac{d^2 Y_0^\alpha}{dt^2} + |V_0|^2 \kappa_\alpha Y_0^\alpha = 0, \alpha < n$, and $\frac{d^2 Y_0^n}{dt^2} = 0$. So the Cauchy problem can be solved explicitly. In particular, we can calculate the matrices of the maps $\left(\frac{\partial \exp}{\partial m}\right)^{\nabla}_V(m_0, V_0)$ and $\left(\frac{\partial \exp}{\partial V}\right)_m(m_0, V_0)$, as diagonal matrices with the αth entries depending only on $|V_0|$ and the sectional curvature κ_α, and with the nth entries both equal to 1.

Inhomogeneous Jacobi equations. Dropping the local symmetry assumption on M, let us apply the identity (2.8) to solve in $\mathcal{X}_{\gamma_0}$ the inhomogeneous Cauchy problem:

$$(2.9) \qquad \mathfrak{J}_{\gamma_0}(Y) = U, \quad Y(0) = \xi, \quad \frac{\nabla Y}{dt}(0) = \nu,$$

where ξ, ν in $T_m M$ and $U \in \mathcal{X}_{\gamma_0}$ are given. Specifically, writing the equation $\mathfrak{J}_{\gamma_0}(Y) = U$ as a first-order system in $\binom{Y}{Z}$ with $Z = \frac{\nabla Y}{dt}$, we note that the

resolvent $\mathcal{R}(t,\tau)$ of the homogeneous system is given by:

$$
\begin{pmatrix}
\left(\frac{\partial \exp}{\partial m}\right)^{\nabla}_V (\gamma_0(\tau),(t-\tau)\dot{\gamma}_0(\tau)) & (t-\tau)\left(\frac{\partial \exp}{\partial V}\right)_m (\gamma_0(\tau),(t-\tau)\dot{\gamma}_0(\tau)) \\
\frac{\nabla}{\partial t}\left[\left(\frac{\partial \exp}{\partial m}\right)^{\nabla}_V (\gamma_0(\tau),(t-\tau)\dot{\gamma}_0(\tau))\right] & \frac{\nabla}{\partial t}\left[(t-\tau)\left(\frac{\partial \exp}{\partial V}\right)_m (\gamma_0(\tau),(t-\tau)\dot{\gamma}_0(\tau))\right]
\end{pmatrix}.
$$

For instance, from (2.8), we have $W(t) = \mathcal{R}(t,0)W(0)$, with $W = \begin{pmatrix} Y_0 \\ \frac{\nabla Y_0}{dt} \end{pmatrix}$.
The map $t \mapsto \mathcal{R}(t,\tau)$ satisfies the initial condition $\mathcal{R}(\tau,\tau) = \mathrm{Id}$ and the composition identity: $\mathcal{R}(t,\theta)\mathcal{R}(\theta,\tau) \equiv \mathcal{R}(t,\tau)$. Now, the method of variation of constants [8, 33] yields for the solution of (2.9) the representation formula:

$$
(2.10) \quad Y(t) = Y_0(t) + \int_0^t (t-\tau)\left(\frac{\partial \exp}{\partial V}\right)_m (\gamma_0(\tau),(t-\tau)\dot{\gamma}_0(\tau))(U(\tau))\,d\tau\,,
$$

where Y_0 is the Jacobi field given by the first component of $W(t) = \mathcal{R}(t,0)\begin{pmatrix}\xi \\ \nu\end{pmatrix}$.

Estimates. From the Jacobi equation combined with Schwarz inequality, we can find a constant $C > 0$ depending only on n and $\max_{\mathrm{Gr}_2(M)} |K|$, still setting $\mathrm{Gr}_2(M)$ for the Grassmann bundle of tangent 2-planes on M, such that the energy of every Jacobi field $Y_0 \in \mathcal{X}_{\gamma_0}$ satisfies the differential inequality: $\frac{d}{dt}\left(|Y_0(t)|^2 + \left|\frac{\nabla Y_0}{dt}(t)\right|^2\right) \leqslant C(1+|V_0|^2)\left(|Y_0(t)|^2 + \left|\frac{\nabla Y_0}{dt}(t)\right|^2\right)$, hence the estimate:

$$
(2.11) \qquad |Y_0(t)|^2 + \left|\frac{\nabla Y_0}{dt}(t)\right|^2 \leqslant e^{C(1+|V_0|^2)t}\left(|Y_0(0)|^2 + \left|\frac{\nabla Y_0}{dt}(0)\right|^2\right).
$$

In particular, with the convention that the norm of the identity is equal to $\sqrt{n}$, we infer from (2.8) that the norms of the linear maps $\left(\frac{\partial \exp}{\partial m}\right)^{\nabla}_V (m_0, V_0)$ and $\left(\frac{\partial \exp}{\partial V}\right)_m (m_0, V_0)$ are bounded above by $e^{C(1+|V_0|^2)/2}\sqrt{n}$.

If the eigenvalues of the curvature endomorphism along the geodesic segment $t \in [0,1] \to \gamma_0(t) \in M$ are nonnegative and if that segment contains no conjugate point, the Rauch comparison theorem [6, p. 215] [10, p. 29] yields a sharper bound, namely: $t\left|\left(\frac{\partial \exp}{\partial V}\right)_m (m_0, tV_0)\right| \leqslant t\sqrt{n}$. If so, we infer from (2.10) the estimate:

$$
(2.12) \qquad \max_{[0,t]} |Y(\tau) - Y_0(\tau)| \leqslant \frac{t^2}{2}\sqrt{n}\max_{[0,t]} |U(\tau)|,
$$

for the solution of the initial value problem (2.9).

Main properties of the Jacobi endomorphism. We infer at once from Remark 2.3 (read at $t = 1$) the:

Proposition 2.1. *The identity $\mathcal{J}(m_0, V_0)V_0 = V_0$ holds at every tangent point $(m_0, V_0) \in \mathcal{N}_{\text{conj}}$. Furthermore, if $V_0 \neq 0$, the Jacobi endomorphism $\mathcal{J}(m_0, V_0)$ maps the subspace $V_0^\perp$ to itself.*

Let us fix a point $(m_0, V_0) \in \mathcal{N}_{\text{conj}}$, set $p_0 = \exp_{m_0}(V_0)$ and consider the equation $\exp_m(V) = p_0$ for $(m, V) \in TM$ close to (m_0, V_0). Since $\left(\frac{\partial \exp}{\partial V}\right)_m$ $(m_0, V_0) : T_{m_0} M \to T_{p_0} M$ is an isomorphism, we may apply the implicit function theorem at (m_0, V_0) to that equation. It yields a local vector field $\widehat{V}$, equal to V_0 at the point m_0, and such that the identity $\exp_m(\widehat{V}(m)) \equiv p_0$ holds for m close enough to m_0. We take m so close to m_0 that the graph of the vector field $\widehat{V}$ remains inside $\mathcal{N}_{\text{conj}}$. By differentiating the preceding identity, the reader will find (exercise) that the derivative of the local vector field $\widehat{V}$ is given by:

$$(2.13) \qquad \nabla_\xi \widehat{V}(m) = -\mathcal{J}\big(m, \widehat{V}(m)\big)(\xi),$$

for every vector $\xi \in T_m M$. We further study the vector field $\widehat{V}$ itself, by introducing the local function $\hat{c}(m) = \frac{1}{2}\big|\widehat{V}(m)\big|^2$. The latter represents half the squared length of the geodesic path joining the point m to the fixed endpoint p_0, with initial velocity $\widehat{V}(m)$ close to V_0. From (2.13), the derivative of that local squared length function is given by: $d\hat{c}(m)(\xi) = g_m\big(-\widehat{V}(m),$ $\mathcal{J}\big(m, \widehat{V}(m)\big)(\xi)\big)$. From Proposition 2.1, the right-hand side of this equation is equal to $g_m\big(-\widehat{V}(m), \xi\big)$. Since the vector $\xi \in T_m M$ is arbitrary, we have proved the local identity:

$$\widehat{V}(m) = -\operatorname{grad} \hat{c}(m).$$

Plugging the latter in (2.13), we further get:

$$(2.14) \qquad \mathcal{J}\big(m, \widehat{V}(m)\big) \equiv \nabla \operatorname{grad} \hat{c}(m).$$

In particular, letting $m = m_0$, we infer the

Proposition 2.2. *The Jacobi endomorphism $\mathcal{J}(m_0, V_0) : T_{m_0} M \to T_{m_0} M$ is symmetric, for every $(m_0, V_0) \in \mathcal{N}_{\text{conj}}$.*

Let us now focus on the behavior of the eigenvalues of the symmetric endomorphism $\mathcal{J}^\perp(m_0, V_0)$ induced on $V_0^\perp$ by $\mathcal{J}(m_0, V_0)$. The following proposition extends Lemma 3.12 and Proposition 2.5 of [12], from $\mathcal{N}_{\text{cut}}$ (defined below) to $\mathcal{N}_{\text{conj}}$, with a simpler proof based on the Index Lemma.

Proposition 2.3. *For every* $(m_0, V_0) \in \mathcal{N}_{\mathrm{conj}}, V_0 \neq 0$, *setting* $\mathcal{J}^\perp(m_0, V_0)$ *for the endomorphism of* $V_0^\perp$ *induced by* $\mathcal{J}(m_0, V_0)$, *the following upper bounds hold in the sense of symmetric endomorphisms:*

   (i) $\mathcal{J}^\perp(m_0, V_0) \leqslant \mathrm{Id}$, *if the Jacobi curvature endomorphism along* $\dot\gamma_0(t)$ *is nonnegative for all* $t \in [0,1]$;

   (ii) $\mathcal{J}^\perp(m_0, V_0) \leqslant \big(|V_0|\sqrt{\kappa}\coth|V_0|\sqrt{\kappa}\big)\,\mathrm{Id}$, *if* $-\kappa < 0$ *stands for a lower bound on the eigenvalues of the Jacobi curvature endomorphism along* $\dot\gamma_0(t)$, *valid for all* $t \in [0,1]$.

Furthermore, for every converging sequence $(V_k)_{k\in\mathbb{N}}$ *of* $T_{m_0}M$ *such that* (m_0, V_k) *lies in* $\mathcal{N}_{\mathrm{conj}}$ *for each* $k \in \mathbb{N}$ *and the limit point* (m_0, V_∞) *lies on the boundary of* $\mathcal{N}_{\mathrm{conj}}$, *the sequence of symmetric endomorphisms* $\Big(\mathcal{J}(m_0, V_k)\Big)_{k\in\mathbb{N}}$ *is unbounded below.*

Proof. We denote henceforth by S_m the sectional curvature tensor (or covariant Riemann tensor) of the manifold at the point m. Following [12], we use the index form of the geodesic $t \in [0,1] \to \gamma_0(t) = \exp_{m_0}(tv_0) \in M$, namely the bilinear form I_{γ_0} given by:

$$I_{\gamma_0}(Y, Z) = \int_0^1 \left(g_{\gamma_0(t)}\Big(\frac{\nabla Y}{dt}, \frac{\nabla Z}{dt}\Big) - S_{\gamma_0(t)}\big(Y(t), \dot\gamma_0(t), Z(t), \dot\gamma_0(t)\big) \right) dt,$$

which acts on the subspace $\mathcal{X}_{\gamma_0}^\perp$ of $\mathcal{X}_{\gamma_0}$ consisting of vector fields everywhere orthogonal to γ_0. We fix a unit vector $\xi \in V_0^\perp$ and consider the Jacobi field J_ξ equal to:

$$J_\xi(t) = \Big(\frac{\partial\exp}{\partial m}\Big)_V^\nabla(m_0, tV_0)(\xi) - t\Big(\frac{\partial\exp}{\partial V}\Big)_m(m_0, tV_0)\big(\mathcal{J}(m_0, V_0)(\xi)\big).$$

By reading (2.13) at $m = m_0$, we know that this field vanishes at $t = 1$. The combination of Remark 2.3 with Proposition 2.1 further shows that it belongs to $\mathcal{X}_{\gamma_0}^\perp$. Moreover, integration by parts yields for the field J_ξ the identity:

$$(2.15) \qquad\qquad I_{\gamma_0}\big(J_\xi, J_\xi\big) = g_{m_0}\big(\mathcal{J}(m_0, V_0)(\xi), \xi\big).$$

For every vector field $Z \in \mathcal{X}_{\gamma_0}^\perp$ with the same endpoint values as J_ξ, namely $Z(0) = \xi, Z(1) = 0$, the Index Lemma [1, p. 19] [6, pp. 212–215] [10, pp.

24–25], applied after a time reversion, provides the fundamental inequality:

$$(2.16) \qquad I_{\gamma_0}\big(J_\xi, J_\xi\big) \leqslant I_{\gamma_0}(Z, Z).$$

Here, time reversion means that γ_0 is replaced by the geodesic $t \in [0, 1] \to \widetilde{\gamma}_0(t) = \gamma_0(1 - t)$ and each vector field $Y \in \mathcal{X}_{\gamma_0}^\perp$, by the vector field $\widetilde{Y} \in \mathcal{X}_{\widetilde{\gamma}_0}^\perp$ given by $\widetilde{Y}(t) = Y(1 - t)$. It is straightforward to verify that Y is a Jacobi field along γ_0 if and only if $\widetilde{Y}$ is so along $\widetilde{\gamma}_0$.

Under assumption (i), we apply (2.16) with $Z(t) = (1 - t)\xi_{\gamma_0}(t)$, where $\xi_{\gamma_0}(t)$ denotes the parallel transport of ξ from m_0 to $\gamma_0(t)$ along $\tau \in [0, t] \to \gamma_0(\tau)$. The desired conclusion comes from the obvious bound $I_{\gamma_0}(Z, Z) \leqslant 1$. If we assume (ii) instead, the following upper bound holds:

$$I_{\gamma_0}(Z, Z) \leqslant \int_0^1 \left(\left| \frac{\nabla Z}{dt} \right|^2_{\gamma_0(t)} + \kappa |V_0|^2 |Z|^2_{\gamma_0(t)} \right) dt,$$

for every $Z \in \mathcal{X}_{\gamma_0}^\perp$. The right-hand side corresponds formally to the index form that would occur if the Jacobi curvature endomorphism field along the geodesic γ_0 had $n - 1$ eigenvalues constant equal to $-\kappa$. If it were so, on the one hand, the Jacobi field J_ξ would formally coincide with the field:

$$Z(t) = \Big(\cosh(t|V_0|\sqrt{\kappa}) - \coth(|V_0|\sqrt{\kappa}) \sinh(t|V_0|\sqrt{\kappa}) \Big) \xi_{\gamma_0}(t);$$

on the other hand, the eigenvalues of the formal analogue of $\mathcal{J}^\perp(m_0, V_0)$ would be constant equal to $\big(|V_0|\sqrt{\kappa} \coth |V_0|\sqrt{\kappa}\big)$. Plugging this expression of Z in the above right-hand integral, we thus know from (2.15) that we must get the desired conclusion (exercise: compute the integral). The first part of Proposition 2.3 is established.

In order to prove its second part, following [12], we fix a small real $\alpha > 0$ and argue by contradiction, assuming the existence of a sequence $(V_k)_{k \in \mathbb{N}}$, as stated, and of a constant $C > 0$ such that the lower bound: $g_{m_0}\big(\mathcal{J}(m_0, V_k)(\xi), \xi\big) \geqslant -C$ holds, for every integer k and unit vector $\xi \in T_{m_0}M$. Setting $\gamma_k(t) = \exp_{m_0}(tV_k)$, we infer from (2.15) and (2.16) that the bound: $I_{\gamma_k}(Z_k, Z_k) \geqslant -C\alpha^2$, holds uniformly with respect to $k \in \mathbb{N}$ and to the vector fields $Z_k \in \mathcal{X}_{\gamma_k}^\perp$ satisfying: $Z_k(0) = \alpha\xi, Z_k(1) = 0$. Letting k tend to ∞, we get:

$$(2.17) \qquad I_{\gamma_\infty}(Z, Z) \geqslant -C\alpha^2,$$

where $\gamma_\infty(t) = \exp_{m_0}(tV_\infty)$, a lower bound valid for every vector field $Z \in \mathcal{X}_{\gamma_\infty}^\perp$ satisfying: $Z(0) = \alpha\xi, Z(1) = 0$. We now mimick [12] for the rest of the

proof. From our assumption on the limit point (m_0, V_∞), we may take ξ in the kernel of the map $\left(\frac{\partial \exp}{\partial V}\right)_m (m_0, V_\infty)$. Fixing ξ so, we choose the vector field Z equal to: $Z(t) = Y_\xi(t) + \alpha(1-t)\xi_{\gamma_\infty}(t)$, with $Y_\xi(t) = t\left(\frac{\partial \exp}{\partial V}\right)_m (m_0, tV_\infty)(\xi)$. The latter is such that the integral $I_{\gamma_\infty}(Y_\xi, Y_\xi)$ must vanish [10, p. 21]. Besides, we readily find $I_{\gamma_\infty}(Y_\xi, (1-t)\xi_{\gamma_\infty}) = -1$. It follows that (2.17), read with our choice of Z, implies the inequality:

$$-2\alpha + \left(I_{\gamma_\infty}\big((1-t)\xi_{\gamma_\infty}, (1-t)\xi_{\gamma_\infty}\big) + C \right)\alpha^2 \geqslant 0,$$

which is absurd, provided the real $\alpha > 0$ is taken small enough. □

2.4. Jacobians

2.4.1. Calibrated determinants. Let X be a nonoriented real vector space of dimension n and $\pm\omega$, a nonzero n-covector on X defined up to sign. Let $(X', \pm\omega')$ and $(X'', \pm\omega'')$ be similarly defined with the same integer n. The calibrated determinant of a linear map $L : X \to X'$ is given by:

$$(2.18) \qquad \det{}_{|\omega|,|\omega'|}(L) = \left| \frac{\omega'(Le_1, \ldots, Le_n)}{\omega(e_1, \ldots, e_n)} \right|,$$

where $\{e_1, \ldots, e_n\}$ stands for a basis of X. This definition does not depend on the choice of that basis, nor on the sign $\pm$ chosen before ω and ω'. If $X = X'$ and $|\omega| = |\omega'|$, the definition becomes independent of the choice of $\pm\omega$ itself and reduces, up to the absolute value, to the well known notion of determinant of an endomorphism. Like the latter, the calibrated determinant has a nice multiplication property, namely:

$$(2.19) \qquad \det{}_{|\omega|,|\omega''|}(L' \circ L) \equiv \det{}_{|\omega|,|\omega'|}(L)\det{}_{|\omega'|,|\omega''|}(L'),$$

for every couple of linear maps $L : X \to X'$ and $L' : X' \to X''$. Indeed, both sides vanish if L is not an isomorphism and, if it is, we just have to write:

$$\left| \frac{\omega''\big((L' \circ L)e_1, \ldots, (L' \circ L)e_n\big)}{\omega(e_1, \ldots, e_n)} \right| \equiv \left| \frac{\omega''(L'v_1, \ldots, L'v_n)}{\omega'(v_1, \ldots, v_n)} \right|$$

$$\times \left| \frac{\omega'(Le_1, \ldots, Le_n)}{\omega(e_1, \ldots, e_n)} \right|,$$

with $v_i = Le_i$, to make (2.19) obvious.

Objects like $(X, \pm \omega)$ arise naturally in Riemannian geometry. Indeed, at each point m of our closed nonoriented Riemannian manifold (M, g), the tangent space $T_m M$ is endowed with a nonzero n-covector defined up to sign, namely the restriction to $T_m M$ of the volume form ω_g. The latter is defined by $\omega_g(m) = \pm \theta^1 \wedge \cdots \wedge \theta^n$, where $\{\theta^1, \ldots, \theta^n\}$ stands for an orthonormal basis of $T_m^* M$.

2.4.2. Exponential map at a point. Fixing the point $m \in M$ and letting the vector V vary in $T_m M$, consider the family of linear maps $\left(\frac{\partial \exp}{\partial V}\right)_m$ $(m, V) : T_m M \to T_p M$, with $p = \exp_m(V)$. The *Jacobian* of the map $\exp_m : T_m M \to \mathbb{R}$ at $V \in T_m M$ is defined by:

$$\mathrm{Jac}(\exp_m)(V) = \det_{|\omega_g(m)|, |\omega_g(p)|}\left(\left(\frac{\partial \exp}{\partial V}\right)_m (m, V)\right).$$

Letting now $m \in M$ vary, we get a smooth nonnegative function on TM, equal to 1 along the zero section and nonvanishing on $\mathcal{N}_{\mathrm{conj}}$. Let us pause and illustrate briefly the use of that function.

Henceforward, we denote by dV_g (or just dV when not confusing) the volume measure of the metric g and by $\mathcal{N}_{\mathrm{cut}}$, the open neighborhood of the zero section of TM consisting of the points $(m, V) \in TM$ such that the geodesic segment $t \in [0, 1] \to \exp_m(tV) \in M$ contains no cut point. As is well known, the inclusion $\mathcal{N}_{\mathrm{cut}} \subseteq \mathcal{N}_{\mathrm{conj}}$ holds; moreover, the map $\exp_m$ sends the tangential domain $\mathcal{N}_{\mathrm{cut}}(m) = \mathcal{N}_{\mathrm{cut}} \cap T_m M$ diffeomorphically onto the open subset of full measure $M \setminus \mathrm{Cut}(m)$, where $\mathrm{Cut}(m)$ denotes the cut locus of the point m [6, 10].

The function $(m, V) \mapsto \mathrm{Jac}(\exp_m)(V)$ occurs anytime one computes an integral like $\int_M h(p) dV_g(p)$, $h \in L^1(M, dV_g)$. Using the change of variables $\overline{m} = \exp_m(V), V \in \mathcal{N}_{\mathrm{cut}}$, the integral becomes:

$$\int_M h(\overline{m}) dV_g(\overline{m}) = \int_{\mathcal{N}_{\mathrm{cut}}(m)} (h \circ \exp_m) \mathrm{Jac}(\exp_m)(V) dv_{g_m}(V),$$

where dv_{g_m} stands for the Lebesgue measure on $T_m M$ canonically associated to the n-covector $\omega_g(m)$. If the manifold M is Riemannian locally symmetric, as pointed out in the first paragraph of Section 2.3.2, we can calculate explicitly the expression of the Jacobi fields along the geodesic $t \mapsto \exp_m(tV)$. The reader can do the exercise or find the solution in [2, p. 82] in the positively curved case. In the notations of Section 7.1, we get in that case:

$$(2.20) \qquad \mathrm{Jac}(\exp_m)(V) = \left(\frac{\sin |V|}{|V|}\right)^{\nu} \left(\frac{\sin \frac{|V|}{2}}{\frac{|V|}{2}}\right)^{n - \nu - 1}.$$

This expression factors as a function of the sole radial variable $r = |V|$, decreasing from 1 to 0 on $[0, \pi)$. The computation of the above integral is now tractable. Taking for instance $h = 1$ on a geodesic ball of radius $r < \pi$, and zero elsewhere, we can compute in this way the volume of that ball.

Dropping local symmetry and letting the point (m, V) vary in TM, the expression of the function $(m, V) \mapsto \mathrm{Jac}(\exp_m)(V)$ itself is given in any couple of charts by:

$$\mathrm{Jac}(\exp_m)(V) = \sqrt{\frac{\det\left(g_{\bar{i}\bar{j}}(E(x,v))\right)}{\det\left(g_{ij}(x)\right)}}\left|\det\left(\frac{\partial E^{\bar{i}}}{\partial v^j}(x,v)\right)\right|.$$

Here, we have used a chart x (resp. $\bar{x}$) at the source point m (resp. target point $\overline{m} = \exp_m(V)$), set $E(x, v) = \bar{x}(\exp_m(V))$, where $x = x(m), v = dx(m)(V)$, and denoted by $g_{ij}(x)$ (resp. $g_{\bar{i}\bar{j}}(\bar{x})$) the local components of the metric tensor.

2.4.3. Maps between Riemannian manifolds. Let $\phi : M \to M'$ be a map ranging in another unoriented Riemannian n-manifold (with metric g'). For every point $m \in M$, setting for short $\omega_m = \omega_g(m), \omega'_m = \omega_{g'}(\phi(m))$, the *Jacobian* of the map ϕ is the nonnegative function $\mathrm{Jac}(\phi)$ defined on the manifold M by:

$$(2.21) \qquad\qquad \mathrm{Jac}(\phi)(m) = \det_{|\omega_m|, |\omega'_m|}\left(d\phi(m)\right).$$

If $\psi : M' \to M''$ is a further map between unoriented Riemannian manifolds, the following multiplication property:

$$(2.22) \qquad\qquad \mathrm{Jac}(\psi \circ \phi) \equiv \mathrm{Jac}(\phi).\left(\mathrm{Jac}(\psi) \circ \phi\right),$$

can be infered from (2.19) in a straightforward way. In case $\phi : M \to M'$ is a diffeomorphism, the pullback tensor $\phi^* g'$ is a Riemannian metric on M and the identity $dV_{\phi^* g'} = \mathrm{Jac}(\phi)dV_g$ holds between the volume measures of the metrics g and $\phi^* g'$, due to (2.21). So the change of variable formula $\int_{M'} h'\, dV_{g'} = \int_M (h' \circ \phi)dV_{\phi^* g'}$, valid for every function $h' \in L^1(M', dV_{g'})$, becomes:

$$(2.23) \qquad\qquad \int_{M'} h'\, dV_{g'} = \int_M (h' \circ \phi)\mathrm{Jac}(\phi)dV_g .$$

Besides, the volume measure of $\phi^* g'$ coincides with the pushforward of the measure $dV_{g'}$ by the inverse mapping $\phi^{-1} : M' \to M$, that is:

$$(2.24) \qquad dV_{\phi^* g'} = (\phi^{-1})_\# dV_{g'}, \text{ or else, } (\phi^{-1})_\# dV_{g'} = \mathrm{Jac}(\phi)dV_g.$$

Indeed, on the one hand, setting $h' = h \circ \phi^{-1}$, with $h \in L^1(M, dV_g)$, the above change of variable formula yields the identity $\int_{M'} (h \circ \phi^{-1})\, dV_{g'} = \int_M h\, dV_{\phi^* g'}$. On the other hand, we have $\int_{M'} (h \circ \phi^{-1})\, dV_{g'} = \int_M h\big((\phi^{-1})_\# dV_{g'}\big)$, by the very definition of the pushforward measure.

Finally, the expression of the function $m \mapsto \mathrm{Jac}(\phi)(m)$ in any couple of charts is given by:

$$\mathrm{Jac}(\phi)(m) = \sqrt{\frac{\det\big(g_{\bar{i}\bar{j}}(\phi(x))\big)}{\det\big(g_{ij}(x)\big)}}\left|\det\left(\frac{\partial \phi^{\bar{i}}}{\partial x^j}(x)\right)\right|.$$

Here, we have used a chart x (resp. $\bar{x}$) at the source point m (resp. target point $\bar{m} = \phi(m)$), set abusively $\phi(x) = \bar{x}(\phi(m))$, where $x = x(m)$, and denoted by $g_{ij}(x)$ (resp. $g_{\bar{i}\bar{j}}(\bar{x})$) the local components of the metric tensor.

2.4.4. Maps factorizing through nonconjugating vector fields. In the sequel, it will be convenient to call nonconjugating, a vector field U on M which ranges in $\mathcal{N}_{\mathrm{conj}}$ (see Section 2.1.3). Abusively, one may call so the map $(U^* \exp) : M \to M$, as well. Similarly, we call the vector field U (and the map $U^* \exp$) noncutting, provided U ranges in $\mathcal{N}_{\mathrm{cut}} \subseteq \mathcal{N}_{\mathrm{conj}}$ (see Section 2.4.2). Finally, we call U (and $U^* \exp$, as well) strictly minimizing, if for every point $m \in M$, there exists a unique minimizing geodesic joining m to its image point $\exp_m(U(m))$.

Let U be a nonconjugating vector field on M. Since U is so, we may recast the expression (2.5) of the differential of the map $(U^* \exp)$ as follows:

$$d(U^* \exp)(m)(\xi) = \left(\frac{\partial \exp}{\partial V}\right)_m (m, U(m)) \circ \Big(\mathcal{J}(m, U(m))(\xi) + \nabla_\xi U(m)\Big),$$

for every $m \in M, \xi \in T_m M$. From the multiplication property (2.19), we infer for the Jacobian of that map the expression:

$$(2.25) \qquad \mathrm{Jac}(U^* \exp)(m) = \mathrm{Jac}(\exp_m)\big(U(m)\big)$$
$$\times \left|\det\big(\mathcal{J}(m, U(m))(.) + \nabla_. U(m)\big)\right|.$$

A similar expression was derived differently in [5, pp. 632–635], under the more restrictive assumption that U is noncutting.

3. Foundations of the partial differential equations approach

Back to the Monge problem described in the Introduction, and sticking to the notations adopted there, let us investigate the smoothness issue raised

by McCann's theorem. If, for every couple of $C^{k,\alpha}$ positive probability measures (μ_0, μ_1), the potential function $u(\mu_0, \mu_1)$ is $C^{k+2,\alpha}$, given such a couple, the optimal maps $\mathrm{Mc}(\mu_0, \mu_1)$ and $\mathrm{Mc}(\mu_1, \mu_0)$ are $C^{k+1,\alpha}$. From the last part of Theorem 1.1, these maps are now inverse of each other at every point of the manifold M, hence diffeomorphisms of M. It follows that the potential function $u(\mu_0, \mu_1)$ must be a classical solution of the Monge–Ampère Equation (1.2). This is what we infer from McCann's theorem; but we do not want to rely on that theorem.

In this section, forgetting the map $\mathrm{Mc}(\mu_1, \mu_0)$ and allowing provisionally the given densities ρ_0, ρ_1 to be merely continuous on the manifold, we will establish the following result:

Theorem 3.1. *Let M be a closed Riemannian manifold and (μ_0, μ_1), a couple of continuous positive probability measures on M. A C^2 function $u : M \to \mathbb{R}$ is such that the gradient-mapping $G_u = \exp(\mathrm{grad}\, u) : M \to M$ pushes μ_0 to μ_1 if and only if u is a classical solution of the Monge–Ampère Equation (1.2). Moreover, if so, the map G_u must be a noncutting diffeomorphism and the function u must be c-convex.*

This statement yields the founding ansatz of the partial differential equations approach, namely:

Corollary 3.1. *Let M be a closed Riemannian manifold and (μ_0, μ_1), a couple of $C^{k,\alpha}$ positive probability measures on McCann's potential function $u(\mu_0, \mu_1)$ is $C^{k+2,\alpha}$ if and only if the Monge–Ampère Equation (1.2) admits a $C^{k+2,\alpha}$ solution.*

Proof of the Corollary. By the preceding theorem, if $u(\mu_0, \mu_1)$ is $C^{k+2,\alpha}$, it must solve (1.2). Conversely, if u is a $C^{k+2,\alpha}$ solution of (1.2), it must be c-convex and the map G_u, be a diffeomorphism, hence push μ_0 to μ_1. In the latter case, McCann's uniqueness theorem [50, Theorem 8] implies that the maps G_u and $\mathrm{Mc}(\mu_0, \mu_1)$ coincide. So do the gradients of the functions u and $u(\mu_0, \mu_1)$, because the map G_u is noncutting. Therefore McCann's potential function $u(\mu_0, \mu_1)$ must be $C^{k+2,\alpha}$. $\qquad\square$

The rest of the section is devoted to the proof of Theorem 3.1. We will proceed stepwise, starting with a series of auxiliary results of independent interest, then combining them all to derive the theorem itself. The first lemma was the object of [23].

Lemma 3.1 (Figalli). *Let $\phi : M \to M$ be a C^1 map pushing μ_0 to μ_1. The Jacobian of ϕ must be bounded below by $\dfrac{\min_M \rho_0}{\max_M \rho_1} > 0$.*

Proof. Claim 1: the open subset $\{\mathrm{Jac}(\phi) > 0\}$ is nonempty.

Indeed, if it were empty, the image of ϕ would have zero volume measure, by Sard's theorem [51]. For every Borel subset B of $M \setminus \phi(M)$, we would get: $\mu_1(B) = \mu_0\big(\phi^{-1}(B)\big) = 0$, contradicting the positivity assumption made on the target measure μ_1.

Claim 2: the subsets $\{\mathrm{Jac}(\phi) > 0\}$ and $\left\{\mathrm{Jac}(\phi) \geqslant \frac{\min_M \rho_0}{\max_M \rho_1}\right\}$ coincide.

Fix a point $m_0 \in \{\mathrm{Jac}(\phi) > 0\}$. The inverse function theorem, applied to the map ϕ at m_0, yields a neighborhood $\widetilde{\mathcal{U}}_0$ of m_0 such that ϕ induces a diffeomorphism $\widetilde{\phi} : \widetilde{\mathcal{U}}_0 \to \phi(\widetilde{\mathcal{U}}_0)$. Setting $\widetilde{\mu}_0$ for the restriction of the measure μ_0 to the neighborhood $\widetilde{\mathcal{U}}_0$, we consider the measure $\widetilde{\mu}_1 = \widetilde{\phi}_{\#}\widetilde{\mu}_0$ on $\phi(\widetilde{\mathcal{U}}_0)$. Since $\widetilde{\phi}$ is a diffeomorphism, the measure $\widetilde{\mu}_1$ must be smooth positive; we set $\widetilde{\rho}_1 = \dfrac{d\widetilde{\mu}_1}{dV}$. We observe that $\widetilde{\rho}_1 \leqslant \rho_1$, due to the assumption $\phi_{\#}\mu_0 = \mu_1$ and the obvious inclusion $\widetilde{\phi}^{-1}(B) \subset \phi^{-1}(B)$ holding for every Borel subset B of $\phi(\widetilde{\mathcal{U}}_0)$. Furthermore, we may write for $\widetilde{\phi}$ the change of variables formula:

$$\widetilde{\rho}_1\big(\widetilde{\phi}(m)\big)\mathrm{Jac}\big(\widetilde{\phi}\big)(m) = \rho_0(m),$$

valid at every point $m \in \widetilde{\mathcal{U}}_0$. From $\widetilde{\rho}_1 \leqslant \rho_1$, we get the lower bound: $\mathrm{Jac}\big(\widetilde{\phi}\big)(m) \geqslant \dfrac{\rho_0(m)}{\rho_1\big(\widetilde{\phi}(m)\big)}$. Since Jac is a local operator, we may drop the tilda over ϕ in that inequality. We infer that $\widetilde{\mathcal{U}}_0$ is contained in $\Big\{\mathrm{Jac}(\phi) \geqslant \dfrac{\min_M \rho_0}{\max_M \rho_1}\Big\}$. In particular, Claim 2 is proved.

Conclusion: From the preceding claims, we know that the open subset $\{\mathrm{Jac}(\phi) > 0\}$ is nonempty and closed. It must coincide with the manifold M, since the latter is connected. So the lemma holds on the whole of M, due to Claim 2. □

Lemma 3.2. *If a C^2 function $u : M \to \mathbb{R}$ is such that the Jacobian of its gradient-mapping G_u nowhere vanishes on M, the map $G_u : M \to M$ must be a diffeomorphism.*

Proof. By the inverse function theorem [44], the map G_u must be a local diffeomorphism. Its image is thus open. By compactness of the manifold M and continuity of the map G_u, the image $G_u(M)$ must also be closed. It follows that $G_u(M) = M$, since the manifold is connected. So $G_u : M \to M$ is a covering map. If the manifold is simply connected, we are done: the covering map G_u must be 1-sheeted, hence a diffeomorphism. If not, let $p : \widetilde{M} \to M$ denote the universal covering space of M. We endow the manifold $\widetilde{M}$ with the pullback Riemannian metric $\widetilde{g}$. The projection p is now an isometry

and so is every element $\gamma : \widetilde{M} \to \widetilde{M}$ of the covering transformation group $\widetilde{\Gamma}$ (isomorphic to $\pi_1(M)$). Let us consider the function $\widetilde{u} : \widetilde{M} \to \mathbb{R}$ given by: $\widetilde{u} = u \circ p$. Set $\widetilde{G} : \widetilde{M} \to \widetilde{M}$ for the corresponding gradient-mapping. From the invariance of the function $\widetilde{u}$ under the action of the group $\widetilde{\Gamma}$, it is not difficult to figure out that $\gamma \circ \widetilde{G} = \widetilde{G} \circ \gamma$, for every deck transformation $\gamma \in \widetilde{\Gamma}$. Furthermore, the naturality of gradient-mappings yields the identity: $p \circ \widetilde{G} = G_u \circ p$. In particular, the Jacobian of the map $\widetilde{G}$ nowhere vanishes on $\widetilde{M}$ therefore, by our previous argument, this map must be a diffeomorphism. We will use this result to prove that the covering map G_u must be 1-sheeted. Arguing by contradiction, let us suppose the existence of a point $\overline{m} \in M$ and two distinct points $m \neq m'$ in M, such that (dropping the subscript u): $G(m) = G(m') = \overline{m}$. Pick a point $\widetilde{m}$ in the fiber $p^{-1}(m)$ and another one $\widetilde{m}'$ in $p^{-1}(m')$. Their images $\widetilde{G}(\widetilde{m})$ and $\widetilde{G}(\widetilde{m}')$ both lie in the fiber $p^{-1}(\overline{m})$, so there exists a covering transformation $\gamma : \widetilde{M} \to \widetilde{M}$ such that $\widetilde{G}(\widetilde{m}) = \gamma \circ \widetilde{G}(\widetilde{m}')$. But $\gamma \circ \widetilde{G}(\widetilde{m}') = \widetilde{G}(\gamma(\widetilde{m}'))$ and the map $\widetilde{G} : \widetilde{M} \to \widetilde{M}$ is one-to-one, therefore $\widetilde{m} = \gamma(\widetilde{m}')$ hence $m = m'$, which is absurd. $\square$

Lemma 3.3. *If a C^2 function $u : M \to \mathbb{R}$ is such that $\mathrm{Jac}(G_u)$ nowhere vanishes, it must be c-convex.*

Proof. For every C^1 function $u : M \to \mathbb{R}$ and every point $\overline{m} \in M$, the function $p \in M \to -c(p, \overline{m}) - u(p) \in \mathbb{R}$ is continuous. It thus assumes, at some point $p = m \in M$, a global maximum which coincides with the value $u^c(\overline{m})$ taken at the point $\overline{m}$ by the c-supremal convolution[5] of u. For $p \in M$ close to the point m, we have: $c(p, \overline{m}) - c(m, \overline{m}) \geqslant u(m) - u(p)$, hence the function $p \mapsto c(p, \overline{m})$ is subdifferentiable at the point m. Being also superdifferentiable [50], it must be differentiable at m, from what we may conclude that $\partial c(m, \overline{m}) = du(m)$ hence also, by the proof given on top p. 599 of [50], that $\overline{m} = G_u(m)$. Here, in the spirit of a convenient notation used in [39, 40], we have set $dc = \partial c + \overline{\partial} c$ for the splitting of the differential of the two-point cost function c into $\partial c(m, \overline{m})$ and $\overline{\partial} c(m, \overline{m})$, respectively, covectors at the points m and $\overline{m}$.

Now, if u is C^2 and $\mathrm{Jac}(G_u)$ does not vanish, the map G_u must be a diffeomorphism by the preceding lemma. We thus get: $m = G_u^{-1}(\overline{m})$, with m a differentiability point of the function $c(., \overline{m})$. Letting the point $\overline{m}$ vary in the manifold, the function $\overline{m} \mapsto u^c(\overline{m})$ is now given by: $u^c(\overline{m}) = -c\big(G_u^{-1}(\overline{m}), \overline{m}\big) - u\big(G_u^{-1}(\overline{m})\big)$. It is thus differentiable.

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[5]Defined in the Introduction.

Playing with u^c the game just played with u, we fix the point $m \in M$ and consider the function $q \in M \to -c(m,q) - u^c(q) \in \mathbb{R}$. It assumes at some point $q = \overline{m}$ a global maximum equal to $(u^c)^c(m)$. Recalling the general inequality $u^c(\overline{m}) \geqslant -c(m,\overline{m}) - u(m)$, we see that $(u^c)^c(m) \leqslant u(m)$. If this inequality were strict, the definition of $(u^c)^c$ would imply that $u(m) > -c(m,q) - u^c(q)$, for every point $q \in M$. But this is false at $q = G_u(m)$. Therefore $(u^c)^c(m) = u(m)$ and since the point $m \in M$ is arbitrary, we conclude that u is, indeed, c-convex. $\qquad\square$

Lemma 3.4. *If a C^2 function $u : M \to \mathbb{R}$ is such that $\mathrm{Jac}(G_u)$ nowhere vanishes, the map $G_u : M \to M$ must be strictly minimizing.*

Proof. If $\mathrm{Jac}(G_u)$ does not vanish, we know from the preceding proof that the function $q \in M \to -c(m,q) - u^c(q) \in \mathbb{R}$ must assume a global maximum at the point $q = G_u(m)$. In the neighborhood of that point, we thus have: $c(m,q) - c(m, G_u(m)) \geqslant u^c(G_u(m)) - u^c(q)$, which shows that the function $q \mapsto c(m,q)$ is subdifferentiable, hence differentiable [50], at the very point $q = G_u(m)$. This result implies (see top p. 232 of [12]) the nonexistence of two distinct minimizing geodesics going from m to $G_u(m)$, as claimed. $\quad\square$

Proposition 3.1. *If a C^2 function $u : M \to \mathbb{R}$ is such that $\mathrm{Jac}(G_u)$ does not vanish, the map $\exp_m : T_m M \to M$ must have full rank at the tangent vector $\mathrm{grad}\, u(m)$ and the symmetric endomorphism $\mathcal{H}u(m)$ defined by:*

$$\mathcal{H}u(m) = \mathcal{J}\big(m, \mathrm{grad}\, u(m)\big) + \nabla \mathrm{grad}\, u(m)$$

must be strictly positive, for every point $m \in M$.

Consistently with the term used in [18, p. 69], we call the endomorphism $\mathcal{H}u(m) : T_m M \to T_m M$ the c-Hessian endomorphism of the function u at the point m.

Proof. Fix a point $m_0 \in M$ where the function u assumes a local minimum. The conclusions of the lemma are satisfied at the point m_0. Let $S_u(m_0)$ be the connected component of the open subset $S_u \subset M$, given by:

$$S_u = \big\{ m \in M,\ \mathrm{Jac}(\exp_m)(\mathrm{grad}\, u(m)) > 0 \text{ and } \mathcal{H}u(m) > 0 \big\},$$

containing the point m_0. It suffices to prove that $S_u(m_0)$ is closed. Indeed, if it is so, the lemma holds because the manifold M itself is connected. Accordingly, let us consider a sequence $\big(m_i\big)_{i \in \mathbb{N}}$ of $S_u(m_0)$ with limit point $m_\infty = \lim_{i \to \infty} m_i \in M$. For fixed $i \in \mathbb{N}$, Lemma 3.4 and the definition of the

subset S_u imply that the point $(m_i, \operatorname{grad} u(m_i))$ lies in $\mathcal{N}_{\mathrm{cut}}$. Besides, from the positivity of $\mathcal{H}u(m_i)$, we have: $\mathcal{J}\big(m_i, \operatorname{grad} u(m_i)\big) \geqslant -C_2 \operatorname{Id}_{m_i}$, where the real $C_2 \geqslant 0$ stands for the largest eigenvalue on M of the Hessian endomorphism field $\nabla \operatorname{grad} u$. Letting i tend to infinity, we get at the limit point m_∞ three inequalitites, namely: $\operatorname{Jac}(\exp_{m_\infty})(\operatorname{grad} u(m_\infty)) \geqslant 0$, $\mathcal{H}u(m_\infty) \geqslant 0$, and $\mathcal{J}\big(m_\infty, \operatorname{grad} u(m_\infty)\big) \geqslant -C_2 \operatorname{Id}_{m_\infty}$. From the latter combined with the last part of Proposition 2.3, we infer that the point $(m_\infty, \operatorname{grad} u(m_\infty))$ lies in $\mathcal{N}_{\mathrm{conj}}$. It follows that the first inequality must be strict. But the function $\operatorname{Jac}(G_u)$ does not vanish on M. Therefore, reading Equation (2.25) at $m = m_\infty$ with the vector field $U = \operatorname{grad} u$, we conclude that the middle inequality, that is $\mathcal{H}u(m_\infty) \geqslant 0$, must also be strict. $\qquad\square$

Lemma 3.5. *If a C^2 function $u : M \to \mathbb{R}$ is such that $\operatorname{Jac}(G_u)$ nowhere vanishes, the map $G_u : M \to M$ must be noncutting.*

Proof. For every point $m \in M$, we know by Lemma 3.4 that the image point $G_u(m)$ can be joined to m by a unique minimizing geodesic, namely $t \in [0,1] \to G_{tu}(m) \in M$. By the first part of Proposition 3.1, we further know that this geodesic contains no conjugate point. Therefore, indeed, it contains no cut point. $\qquad\square$

Proof of Theorem 3.1. With the preceding lemmas at hand, the proof of the theorem is not difficult. Still, it should be provided since it contains a major step of our method, namely the reduction of the Monge problem to the classical solvability of the Monge–Ampère Equation (1.2).

Proof of the first part of Theorem 3.1. Let the function u lie in $C^2(M, \mathbb{R})$ and be such that the gradient-mapping G_u pushes μ_0 to μ_1. The combination of Lemmas 3.1 and 3.2 implies that the map $G_u : M \to M$ is a diffeomorphism. Applying, on the one hand the change of variable formula (2.23) read with $M' = M, \phi = G_u, h' = h\rho_1, h \in L^1(M, dV)$; on the other hand, the pushing formula (1.1) read with $\phi = G_u$, we see that the equation:

$$(3.1) \qquad \int_M (h \circ G_u)(\rho_1 \circ G_u) \operatorname{Jac}(G_u)\, dV = \int_M (h \circ G_u) \rho_0 \, dV,$$

holds for every function $h \in L^1(M, dV)$. We infer from this identity that the C^2 function u must satisfy (1.2) pointwise. Conversely, if the latter conclusion is now our assumption, on the one hand the identity (3.1) holds, on the other hand the map G_u must be a diffeomorphism due to Lemma 3.2. But if G_u is so, the left-hand side of (3.1) can be equated to $\int_M h\, d\mu_1$ by

means of the change of variable formula (2.23) read as we did above. It follows that the map G_u pushes the measure μ_0 to the measure μ_1. The first part of the theorem is proved.

The *proof of the second part* of Theorem 3.1 is now straightforward. Indeed, we just proved in either case that the gradient-mapping G_u must be a diffeomorphism. So we may apply to it Lemmas 3.3 and 3.5 and get the desired conclusion. □

Remark 3.1. The uniqueness, up to addition of a constant, of the C^2 solution of the Monge–Ampère Equation (1.2), is another aspect of the partial differential equations approach, consistent with the uniqueness of the optimal map $\mathrm{Mc}(\mu_0, \mu_1)$ proven in [50]. It will be established in Section 4.3 (Proposition 4.2).

4. Continuity method

Recall that we have fixed in the Introduction a smoothness degree, namely a couple $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geqslant 2$. Accordingly, we focus on the construction of a $C^{k+2,\alpha}$ solution $u : M \to \mathbb{R}$ of the Monge–Ampère Equation (1.2) posed on a closed Riemannian manifold M, with volume measure dV, endowed with a couple of $C^{k,\alpha}$ positive probability densities (ρ_0, ρ_1). By Corollary 3.1, this construction is equivalent to showing that McCann's optimal potential function $u(\rho_0\, dV, \rho_1\, dV)$ is $C^{k+2,\alpha}$. We will perform it by means of the continuity method.

Specifically, we will argue by connectedness of the subset $\mathcal{T} \subset [0, 1]$ consisting of the all values of the parameter $t \in [0, 1]$ for which one can find a $C^{k+2,\alpha}$ solution u_t of the deformed equation:

$$(4.1) \qquad \rho_t\big(G_u(m)\big)\mathrm{Jac}(G_u)(m) = \rho_0(m),$$

where ρ_t stands for the $C^{k,\alpha}$ positive probability density given by $\rho_t = t\rho_1 + (1 - t)\rho_0$. For $t = 1$, the latter coincides with the original Equation (1.2) to be solved. For $t = 0$, it has the obvious solution $u_0 = 0$, so the subset $\mathcal{T}$ is nonempty and we will prove successively that it is relatively open in $[0, 1]$ and closed.

4.1. Openess part of the method

The idea of that part is to use an implicit function theorem argument. Let us describe the related functional analytic setting. We say that a C^2

function $u : M \to \mathbb{R}$ is *admissible*, if $\mathrm{Jac}(G_u)$ nowhere vanishes. For each integer ℓ and real number a, we set $C_a^{\ell,\alpha}$ for the affine Banach space of $C^{\ell,\alpha}$ real functions with average on M equal to a and denote by $\mathcal{A}^{k+2,\alpha}$ the open subset of all admissible functions in $C_0^{k+2,\alpha}$. By Lemma 3.2 and the observation that, if u solves (4.1), so does $u+$constant, we may take the solutions of (4.1) in $\mathcal{A}^{k+2,\alpha}$ with no loss of generality. Besides, we consider the open subset $P_1^{k,\alpha}$ of all positive functions in $C_1^{k,\alpha}$. We note that the tangent bundles to $\mathcal{A}^{k+2,\alpha}$ and to $P_1^{k,\alpha}$ are both trivial, with constant fiber consisting of the Banach space $C_0^{\ell,\alpha}$, respectively, with $\ell = k + 2$ and $\ell = k$. Finally, we let $\Phi : [0,1] \times \mathcal{A}^{k+2,\alpha} \to C^{k,\alpha}(M,\mathbb{R})$ be the operator given by: $\Phi(t,u)(m) = \rho_t\big(G_u(m)\big)\mathrm{Jac}(G_u)(m)$, for every point $m \in M$.

Lemma 4.1. *The operator Φ ranges in $P_1^{k,\alpha}$. Moreover, for fixed $t \in [0,1]$, the differential $d\Phi(t,.)$ is an isomorphism from $C_0^{k+2,\alpha}$ to $C_0^{k,\alpha}$, at every function $u \in \mathcal{A}^{k+2,\alpha}$.*

Proof. Fix a couple $(t,u) \in [0,1] \times \mathcal{A}^{k+2,\alpha}$. The $C^{k,\alpha}$ function $\Phi(t,u)$ must be positive on the manifold because so are the density ρ_t and the Jacobian of the map G_u. Indeed, the latter cannot vanish because G_u is a diffeomorphism. For the same reason, the density $\rho_u = \frac{d\mu_u}{dV}$ of the probability measure μ_u defined by $\mu_u = (G_u)_{\#}\mu_0$ must belong to $P_1^{k,\alpha}$. The identity $\Phi(t,u) \equiv \Big(\big(\frac{\rho_t}{\rho_u}\big) \circ G_u\Big)\rho_0$ is thus satisfied on the manifold. Integrating it over M and applying the pushing condition $(G_u)_{\#}\mu_0 = \mu_u$, we get: $\int_M \Phi(t,u)\,dV = \int_M \frac{\rho_t}{\rho_u}\,d\mu_u = \int_m \rho_t\,dV = 1$. Therefore the map Φ ranges in $P_1^{k,\alpha}$, as claimed. Let us pause for a remark of constant use in the sequel.

Remark 4.1 (actual expression of the Monge–Ampère operator). From the combination of Proposition 3.1 with the expression of $\mathrm{Jac}(G_u)$ given by (2.25) read at the vector field $U = \mathrm{grad}\,u$, we infer that the operator Φ is actually given by:

$$\Phi(t,u)(m) = \rho_t\big(G_u(m)\big)\mathrm{Jac}(\exp_m)\big(\mathrm{grad}\,u(m)\big)\det\big(\mathcal{H}u(m)\big),$$

at every admissible function $u \in \mathcal{A}^{k+2,\alpha}$ and point $(t,m) \in [0,1] \times M$. Here, the c-Hessian endomorphism field $\mathcal{H}u$, which was defined in Proposition 3.1 by $\mathcal{H}u(m) = \mathcal{J}(m, \mathrm{grad}\,u(m)) + \nabla\mathrm{grad}\,u(m)$, is symmetric and must stay positive everywhere on the manifold.

   Back to the interrupted proof, we set for short L for the differential of the map $\Phi(t,.)$ at u. The map L is thus a continuous linear map from

$C_0^{k+2,\alpha}$ to $C_0^{k,\alpha}$. We note that it can be viewed as a second-order scalar differential operator $\mathcal{L} : C^{\ell+2,\alpha}(M,\mathbb{R}) \to C^{\ell,\alpha}(M,\mathbb{R})$, for every integer $\ell \leqslant k$, that is, an operator with $C^{k,\alpha}$ coefficients. Moreover, from Remark 4.1, the symbol of the operator $\mathcal{L}$ must be positive definite at every point of the manifold. So $\mathcal{L}$ is *elliptic*. Besides, on the one hand, since $\Phi(t,u) \equiv \Phi(t, u + \text{constant})$, the operator $\mathcal{L}$ must vanish on constant functions. On the other hand, since the integral $\int_m \mathcal{L}v\, dV$ vanishes for every function $v \in C^{k+2,\alpha}(M,\mathbb{R})$, the formal $L^2(M, dV)$ adjoint $\mathcal{L}^*$ of the operator $\mathcal{L}$ must also vanish on constant functions. Hopf's maximum principle [30, 34, 55] implies that the kernels of the operators $\mathcal{L}$ and $\mathcal{L}^*$ are both exactly made of the constant functions on M. So the map $L : C_0^{k+2,\alpha} \to C_0^{k,\alpha}$ is not only one-to-one but also, by the Fredholm alternative combined with standard elliptic regularity, onto. The lemma now follows from the open mapping theorem. $\square$

We are ready to prove that the subset $\mathcal{T}$ is relatively open in $[0,1]$. If $t_0 \in \mathcal{T}$, letting $u_{t_0} \in \mathcal{A}^{k+2,\alpha}$ be a solution of (4.1) read at $t = t_0$, we can apply the implicit function theorem [44] to the equation $\Phi(t,u) = \rho_0$ at the point (t_0, u_{t_0}). Indeed, this point solves that equation and Lemma 4.1 tells us that the partial differential $\frac{\partial \Phi}{\partial u}(t_0, u_{t_0})$ is an isomorphism from the tangent space to $\mathcal{A}^{k+2,\alpha}$ at u_{t_0} to the tangent space to $P_1^{k,\alpha}$ at ρ_0. We infer the existence of a real $\epsilon > 0$ such that $(t_0 - \epsilon, t_0 + \epsilon) \cap [0,1]$ is contained in $\mathcal{T}$, as desired.

4.2. Closedness part of the method

Our present aim is to reduce the proof that the subset $\mathcal{T}$ is closed to a uniform upper bound on the Hessian of the classical solutions u_t of (4.1) for $t \in \mathcal{T}$. Specifically, we will establish the following statement:

Proposition 4.1. *The subset $\mathcal{T}$ is closed provided there exists a constant $C_2 > 0$ such that $\nabla \operatorname{grad} u_t \leqslant C_2 \operatorname{Id}$, for every $t \in \mathcal{T}$ and every solution $u_t \in \mathcal{A}^{k+2,\alpha}$ of the continuity Equation (4.1).*

Before proving the proposition, we will record straightforward, geometric or algebraic, a priori estimates bearing on admissible functions. The proof of the proposition itself is deferred till the end of the section.

Lemma 4.2. *For every changing sign admissible function u, the following estimates hold:*

$$\max_M |u| \leqslant (D_M)^2, \ \max_M \big|\operatorname{grad} u\big| \leqslant D_M,$$

where D_M stands for the diameter of the manifold M.

Proof. Let the function u be admissible. By Lemma 3.5, the map G_u is noncutting, so the length of the gradient of u cannot exceed D_M. This estimate implies that the oscillation of u is bounded above by D_M^2. So must be $\max_M |u|$ if u changes sign. □

Lemma 4.3. *There exists a geometric constant of the manifold $C_M > 0$ such that $\nabla \operatorname{grad} u \geqslant -C_M \operatorname{Id}$, for every admissible function u.*

Proof. If u is admissible, we know from Proposition 3.1 that, for every point $m_0 \in M$, the point $(m_0, \operatorname{grad} u(m_0))$ must lie in $\mathcal{N}_{\text{conj}}$ and the Hessian endomorphism $\nabla \operatorname{grad} u(m_0)$ must be bounded below by $-\mathcal{J}\big(m_0, \operatorname{grad} u(m_0)\big)$. So the lemma follows from the first part of Proposition 2.3, read at $V_0 = \operatorname{grad} u(m_0)$, combined with the gradient bound of Lemma 4.2. Specifically, we take $C_M = 1$ if the manifold M is nonnegatively curved and, if not, we take $C_M = D_M \sqrt{\kappa_M} \coth(D_M \sqrt{\kappa_M})$ where we have set $-\kappa_M < 0$ for the minimum of the sectional curvature function and used Lemma 4.2. □

For the next statement, given a C^2 real function u on M, it is convenient to set $c_2(u) \geqslant 0$ for the maximum over the manifold of the largest eigenvalue of the Hessian endomorphism $\nabla \operatorname{grad} u$. We will stick to the notation ρ_u used above for the density of the probability measure $\mu_u = \big(G_u\big)_{\#}\mu_0$.

Lemma 4.4. *For every admissible function u, the following lower bound holds:*

$$\min_M \operatorname{Jac}(\exp_m)\big(\operatorname{grad} u(m)\big) \geqslant \frac{\min_M \rho_0}{\max_M \rho_u}\big(C_M + c_2(u)\big)^{-n},$$

where $n = \dim(M)$. Moreover, for every point $m \in M$, the following pinching holds in the sense of symmetric endomorphisms of the tangent space at m:

$$\frac{\min_M \rho_0}{B_M \max_M \rho_u}\big(C_M + c_2(u)\big)^{1-n} \operatorname{Id}_m \leqslant \mathcal{H}u(m) \leqslant \big(C_M + c_2(u)\big) \operatorname{Id}_m,$$

where $B_M > 0$ denotes a geometric constant of the manifold.

Proof. Fixing an admissible function u, we note that the upper bound asserted on the Hessian endomorphism of u in the lemma is trivial. From the definition of the probability density ρ_u, at every point m, we may write:

$$\mathrm{Jac}(\exp_m)\big(\mathrm{grad}\,u(m)\big) = \frac{\rho_0(m)}{\rho_u\big(G_u(m)\big)}\big(\det(\mathcal{H}u(m))\big)^{-1}.$$

This identity combined with the trivial upper bound on $\mathcal{H}u(m)$ yields the first lower bound of the lemma. Writing the identity the other way around, namely:

$$\det(\mathcal{H}u(m)) = \frac{\rho_0(m)}{\rho_u\big(G_u(m)\big)}\Big(\mathrm{Jac}(\exp_m)\big(\mathrm{grad}\,u(m)\big)\Big)^{-1},$$

and recalling the positivity of the eigenvalues of the symmetric endomorphism $\mathcal{H}u(m)$, we infer that the smallest eigenvalue of $\mathcal{H}u(m)$ is bounded below by: $\frac{\rho_0(m)}{\rho_u\big(G_u(m)\big)}\Big(\mathrm{Jac}(\exp_m)\big(\mathrm{grad}\,u(m)\big)\Big)^{-1}\big(C_M + c_2(u)\big)^{1-n}$. Looking here for an upper bound on $\mathrm{Jac}(\exp_m)\big(\mathrm{grad}\,u(m)\big)$, we note that, by Lemma 3.5, the tangent point $(m, \mathrm{grad}\,u(m))$ lies in the neighborhood $\mathcal{N}_{\mathrm{cut}}$ of the zero section of TM. Bishop's comparison theorem [32, p. 47] thus implies that $\mathrm{Jac}(\exp_m)\big(\mathrm{grad}\,u(m)\big)$ is bounded above by the constant B_M equal to 1 if the manifold has nonnegative Ricci curvature and, if not, equal to $\left(\frac{\sinh(D_M\sqrt{r_M})}{D_M\sqrt{r_M}}\right)^{n-1}$, where we have set $-(n-1)r_M < 0$ for the smallest eigenvalue of the Ricci curvature on the manifold and used Lemma 4.2. The desired lower bound on $\mathcal{H}u(m)$ follows. $\qquad\square$

Remark 4.2. The proof of Lemma 4.4 seems to indicate that the estimates 1 and 2 stated in the Introduction are redundant, since one could take the constant δ in Estimate 1 equal to $\frac{\min_M \rho_0}{\max_M \rho_u}\Lambda^{-n}$. Let us explain why one cannot do so. Keeping in mind Remark 1.1, we allow $\max_{t\in[0,1]}\big|\log\rho_t\big|_{C^2}$ to enter in the control the constant Λ, whereas only $\max_{t\in[0,1]}\big|\log\rho_t\big|_{C^1}$ should enter in the control of δ. Worse, the constant Λ in Estimate 2 is allowed to depend on δ. In practice, it *will* depend on δ, because Estimate 2 will be derived from a maximum principle argument exploiting Equation (4.2), in particular, the uniform boundedness of its first-order term $\psi_t(m, \mathrm{grad}\,u(m))$. This boundedness will follow from Estimate 1. So one will really have to establish both estimates, one after the other, to apply successfully the continuity method.

Proof of Proposition 4.1. Recalling Remark 4.1, it is convenient to recast the continuity Equation (4.1) as:

$$(4.2) \qquad F[u](m) - \psi_t\big(m, \operatorname{grad} u(m)\big) = 0, \ \text{with} \ u \in \mathcal{A}^{k+2,\alpha},$$

where $F[u](m) = \log \det\big(\mathcal{H}u(m)\big)$ and $\psi_t : \mathcal{N}_{\mathrm{conj}} \subset TM \to \mathbb{R}$ denotes the function given by:

$$\psi_t(m, V) = \log\left\{ \frac{\rho_0(m)}{\rho_t(\exp_m V)\operatorname{Jac}(\exp_m)(V)} \right\}, \ \text{with} \ t \in [0, 1].$$

Letting $(t_i)_{i \in \mathbb{N}}$ be a sequence of $\mathcal{T}$ and setting $t_\infty = \lim_{i \to \infty} t_i \in [0, 1]$, we must prove that t_∞ belongs to the subset $\mathcal{T}$. To each $i \in \mathbb{N}$, we associate a function $u_i \in \mathcal{A}^{k+2,\alpha}$ solving (4.2) at $t = t_i$. Under our present assumption, the three preceding lemmas hold for the function u_i, the last one with $c_2(u_i)$ replaced by the uniform constant C_2 of our statement. The sequence $(u_i)_{i \in \mathbb{N}}$ is thus *bounded* in $C^2(M, \mathbb{R})$. Since (4.2) is a second-order equation, we would like to extract a subsequence of $(u_i)_{i \in \mathbb{N}}$ converging at least in $C^2(M, \mathbb{R})$. So we need to bound the original sequence in a stronger norm than the C^2 one. This will be achieved by means of the interior Hölder regularity theory for second derivatives of Evans–Trudinger [22, 30, 56]. We thus focus on verifying stepwise the conditions which this theory requires (see [30, Section 17.4]) in the case of (4.2).

First, let us deal with the ψ_t term. By Lemma 4.4, for every couple $(i, m) \in \mathbb{N} \times M$, the tangent point $(m, \operatorname{grad} u_i(m))$ lies in a compact subset $\mathcal{K}_1$ (depending on the constant C_2) of the open neighborhood $\mathcal{N}_{\mathrm{conj}} \subset TM$. The restriction of the function ψ_t to the subset $\mathcal{K}_1$ is *smooth*.

Let us now record three properties of the fully nonlinear differential operator F acting on admissible functions. The differential of the operator F at an admissible function u is given by:

$$(4.3) \qquad dF[u](v) = \operatorname{trace}\big(\mathcal{F}_{[u]} \circ \nabla_{\!\boldsymbol{.}}\operatorname{grad} v\big),$$

where $\mathcal{F}_{[u]}$ stands for the endomorphism field on M such that, at each point $m \in M$, $\mathcal{F}_{[u]}(m)$ is the symmetric endomorphism of the tangent space $T_m M$ inverse of the Hessian endomorphism $\mathcal{H}u(m)$. It follows from Lemma 4.4 that the symbol $\sigma\big(dF[u]\big) : T^*M \to \mathbb{R}$ of that operator at a cotangent point

(m, ξ) satisfies the pinching:

$$(4.4) \qquad \left(C_M + c_2(u)\right)^{-1}|\xi|^2 \leqslant \sigma\big(dF[u]\big)(m, \xi)$$
$$\leqslant \frac{B_M \max_M \rho_u}{\min_M \rho_0}\left(C_M + c_2(u)\right)^{n-1}|\xi|^2.$$

The operator F itself factors through the function $f = \log \det$, acting on the open subset $\mathrm{End}^+_{\mathrm{sym}}(TM)$ of the vector bundle $\mathrm{End}_{\mathrm{sym}}(TM) \to M$ of symmetric endomorphisms, consisting of the ones which are positive. The restriction of the function f to each fiber of the bundle is *concave* (for the canonical flat connection of the fiber). Back to the sequence $(u_i)_{i \in \mathbb{N}}$, we may infer from the pinching (4.4) read at $u = u_i$ a *uniform* pinching on the symbol $\sigma\big(dF[u_i]\big)(m, \xi)$, namely the one obtained by replacing the constant $c_2(u_i)$ by the constant C_2 of Proposition 4.1. As already noted, we can do similarly for the pinching of Lemma 4.4. Therefore the section $\mathcal{H}u_i$ of the vector bundle $\mathrm{End}_{\mathrm{sym}}(TM)$ ranges in a compact subset $\mathcal{K}_2$ of $\mathrm{End}^+_{\mathrm{sym}}(TM)$, depending on the constant C_2 but independent of $i \in \mathbb{N}$. The restriction of the function f to the subset $\mathcal{K}_2$ is *smooth*.

We are now in position to apply the interior regularity theorem of Evans–Trudinger [30, Theorem 17.14] to Equation (4.2) read at $u = u_i$. It readily yields a real $\beta \in (0, 1)$, independent of $i \in \mathbb{N}$, such that the whole sequence $(u_i)_{i \in \mathbb{N}}$ is bounded in $C^{2,\beta}(M, \mathbb{R})$. By the Arzelà–Ascoli theorem, fixing $\gamma \in (0, \beta)$, we can extract a subsequence of $(u_i)_{i \in \mathbb{N}}$ which converges in $C^{2,\gamma}(M, \mathbb{R})$ toward a function u_∞. This limit function has, like each function u_i, zero average. It must satisfy the continuity Equation (4.1) read at $t = t_\infty$. In particular, by Lemma 3.2, it must be admissible. We infer that the differential operator $dF[u_\infty]$ is elliptic on the manifold. Finally, since the given probability densities ρ_0 and ρ_1 lie in $C^{k,\alpha}(M, \mathbb{R})$ and the Riemannian manifold M itself is smooth, the Giraud–Hopf regularity theorem for nonlinear second-order elliptic equations [31, 35] implies that u_∞ belongs to $C^{k+2,\alpha}(M, \mathbb{R})$, hence to $\mathcal{A}^{k+2,\alpha}$. In other words, the limit number t_∞ belongs to the subset $\mathcal{T}$. The proof of Proposition 4.1 is thus complete. $\qquad\square$

4.3. Conclusion: proof of the general smoothness theorem

We wish to prove the general smoothness result stated in the Introduction, namely Theorem 1.5. So far, using the continuity method, given a couple of $C^{k,\alpha}$ positive probability densities (ρ_0, ρ_1), we have reduced the existence of a classical solution $u \in \mathcal{A}^{k+2,\alpha}$ of the Monge–Ampère Equation (1.2), to the assumption of Proposition 4.1, that is, to the existence of a constant

$C_2 > 0$ such that $\nabla \operatorname{grad} u_t \leqslant C_2 \operatorname{Id}$, for every $t \in \mathcal{T}$ and $u_t \in \mathcal{A}^{k+2,\alpha}$ solving the continuity Equation (4.1).

Proof of the "if" part of Theorem 1.5. Let us show that our present existence result is equivalent to the 'if' part of Theorem 1.5, in other words, that the existence of a uniform upper bound on $\nabla \operatorname{grad} u_t$ is equivalent to the existence of a couple of positive constants (δ, Λ) such that the following bounds hold:

$$(4.5a) \qquad\qquad \operatorname{Jac}(\exp_m)(\operatorname{grad} u_t(m)) \geqslant \delta \,,$$

$$(4.5b) \qquad\quad \mathcal{J}(m, \operatorname{grad} u_t(m)) + \nabla \operatorname{grad} u_t(m) \leqslant \Lambda \operatorname{Id}_m \,,$$

uniformly with respect to the point $(t, m) \in \mathcal{T} \times M$.

Indeed, if the preceding bound holds, we infer from Lemma 4.4, read with $u = u_t$, $\rho_u = \rho_t$ and $c_2(u)$ replaced by the constant C_2, that the bounds (4.5) hold as well, respectively, with $\delta = \dfrac{\min_M \rho_0}{\max_M \rho_t}(C_M + C_2)^{-n}$ and $\Lambda = C_M + C_2$. Conversely, if the bounds (4.5) hold, we infer from (4.5a) combined with Lemma 4.2 and the estimates on $\left(\frac{\partial \exp}{\partial m}\right)_V^\nabla (m, \operatorname{grad} u_t(m))$ and $\left(\frac{\partial \exp}{\partial V}\right)_m (m, \operatorname{grad} u_t(m))$ which follow from (2.11), that the norm of the Jacobi endomorphism at $(m, \operatorname{grad} u_t(m))$ is bounded above by:

$$\frac{1}{\delta}\, e^{\frac{nC}{2}(1+D_M^2)}\, n^{n/2},$$

where C stands for the geometric constant of the manifold appearing in (2.11). In particular, a uniform lower bound on $\mathcal{J}(m, \operatorname{grad} u_t(m))$ follows. The latter combined with (4.5b) provides an upper bound $\nabla \operatorname{grad} u_t \leqslant C_2 \operatorname{Id}$ on the Hessian endomorphism, with:

$$C_2 = \Lambda + \frac{1}{\delta}\, e^{\frac{nC}{2}(1+D_M^2)}\, n^{n/2}.$$

The "if" part of Theorem 1.5 is thus established.

Before giving the proof of the "only if" part of Theorem 1.5, we require the following result:

Proposition 4.2 (uniqueness). *Any two C^2 solutions of the Monge–Ampère Equation (1.2) on a closed manifold M must differ by a constant.*

This result is a particular case of a general one presented in [17, Theorem A.1]. For completeness, let us provide a proof of it.

Proof. Let w_0 and w_1 be C^2 solutions of (1.2). By Lemma 3.2, each of them must be admissible. By Proposition 3.1, the c-Hessian of each must be positive definite everywhere on the manifold. Set $w_\theta = w_0 + \theta w$ with $\theta \in [0,1]$ and $w = w_1 - w_0$, and consider the nonempty closed subset $S = \{m \in M, w(m) = \min_M w\}$. Let us argue by connectedness in the manifold M and show that the subset S is open. Fixing a point $m_0 \in S$, we note that $\operatorname{grad} w(m_0)$ vanishes while the symmetric endomorphism $\nabla \operatorname{grad} w(m_0)$ is nonnegative. We infer that the latter vanishes. Indeed, writing (1.2) at the point m_0 for the admissible functions w_0 and w_1, and noting that: $G_{w_0}(m_0) = G_{w_1}(m_0)$, $\operatorname{Jac}(\exp_{m_0})(\operatorname{grad} w_0(m_0)) = \operatorname{Jac}(\exp_{m_0})(\operatorname{grad} w_1(m_0))$, $\mathcal{J}(m_0, \operatorname{grad} w_0(m_0)) = \mathcal{J}(m_0, \operatorname{grad} w_1(m_0))$, we infer that $\det(\mathcal{H}w_0(m_0)) = \det(\mathcal{H}w_1(m_0))$, from which the vanishing of $\nabla \operatorname{grad} w(m_0)$ follows. Since the functions w_0 and w_1 are C^2, we can find a geodesic ball B_0 centered at the point m_0 on which the differential of the auxiliary operator $u \in C^2(M, \mathbb{R}) \to \widetilde{\Phi}(u) = (\rho_1 \circ G_u)\operatorname{Jac}(G_u) \in C^0(M, \mathbb{R})$, calculated at the function $u = w_\theta$, stays elliptic with positive symbol, for every $\theta \in [0,1]$. So the linear operator $\widetilde{L}$, given by $\widetilde{L} = \int_0^1 d\widetilde{\Phi}(w_\theta) \, d\theta$, shares the same property. Moreover, the operators $d\widetilde{\Phi}(w_\theta)$ and $\widetilde{L}$ are linear second-order differential operators with continuous coefficients on the ball B_0 and they vanish on constant functions. The identity $\widetilde{\Phi}(w_1) - \widetilde{\Phi}(w_0) \equiv \widetilde{L}(w)$, combined with the vanishing of its left-hand side and Hopf's maximum principle [30, 34, 55], thus implies that the function w is *constant* on B_0, equal to $w(m_0) = \min_M w$. In other words, the ball B_0 must be contained in the subset S. It follows that S is open hence it must coincide with the whole manifold. $\qquad\square$

Proof of the "only if" part of Theorem 1.5. By the "only if" part of the theorem, we mean its final part, of course. So let us suppose the existence of a couple (ρ_0, ρ_1) of $C^{k,\alpha}$ positive probability densities on M such that either bounds (4.5) fails. By Proposition 4.2, we may consider the solution map $t \in \mathcal{T} \to u_t \in \mathcal{A}^{k+2,\alpha}$ defined by the continuity Equation (4.1). The proof given above for the "if" part shows that the corresponding family of functions $\{u_t\}_{t \in \mathcal{T}}$ must be unbounded in $C^2(M, \mathbb{R})$. Accordingly, for each nonzero integer i, we can find a real $t_i \in \mathcal{T}$ such that the C^2 norm of $u_i := u_{t_i}$ is larger than i. Setting $\tau = \lim_{i \to \infty} t_i \in [0,1]$, we claim that McCann's potential function $u(\rho_0 \, dV, \rho_\tau \, dV)$ is not C^2. We prove it by contradiction. If this function were C^2, the combination of Lemmas 3.1 and 3.2 would imply that it is admissible, thus satisfying the continuity Equation (4.1) read at $t = \tau$. Moreover, the nonlinear elliptic regularity argument used in the last part of the proof of Proposition 4.1 would apply to

the function $u(\rho_0\, dV, \rho_\tau\, dV)$ and show that it belongs to $C^{k+2,\alpha}(M,\mathbb{R})$. We would thus infer that τ belongs to the solution counterset $\mathcal{T}$ and, recalling Proposition 4.2, that the difference between $u(\rho_0\, dV, \rho_\tau\, dV)$ and the solution $u_\tau \in \mathcal{A}^{k+2,\alpha}$ is constant. Finally, by Lemma 4.1 read at $t = \tau$, the sequence $\left(u_i\right)_{i\in\mathbb{N}^*}$ would converge toward u_τ in $C^{k+2,\alpha}(M,\mathbb{R})$. But this convergence would contradict the blow up of the corresponding sequence of C^2 norms. $\quad\square$

5. Continuity method applied to flat manifolds

Let us digress briefly from our main topic and record a straightforward application of Theorem 1.5, namely:

Theorem 5.1. *If the closed Riemannian manifold M is* flat*, given a couple $(k,\alpha) \in \mathbb{N} \times (0,1)$, with $k \geqslant 2$, the potential function $u(\mu_0,\mu_1)$ must be $C^{k+2,\alpha}$ for every couple (μ_0,μ_1) of $C^{k,\alpha}$ positive probability measures.*

As mentioned in the Introduction, this theorem is known. It follows from Cordero–Erausquin's regularity result [11] combined with further elliptic regularity, once applied to the manifold M the Bieberbach theorem (see e.g. [42, Section V.4]) combined with the covering space result of [18, Appendix C]. Here, we present an alternative, direct proof of it.

Proof. Let us fix a couple (μ_0,μ_1) of $C^{k,\alpha}$ positive probability measures on M and apply Theorem 1.5. If M is flat, the following identities hold:

$$\mathrm{Jac}(\exp_m)(V) \equiv 1, \quad \mathcal{J}(m,V) \equiv \mathrm{Id}_m,$$

at every tangent point (m,V). In particular, the open neighborhood $\mathcal{N}_{\mathrm{conj}}$ consists of the whole tangent bundle and the first requirement of Theorem 1.5 is trivially fulfilled. Moreover, the continuity equation reads:

$$\log\det\big(\mathrm{Id}_m + \nabla\mathrm{grad}\, u_t(m)\big) = \psi_t(m, \mathrm{grad}\, u_t(m)),$$

with $\psi_t(m,V) = \log\rho_0(m) - \log\rho_t(\exp_m V)$. Here, let us recall that the function u_t satisfies $\int_M u_t\, dV = 0$, hence also, by Lemma 4.2, the bounds: $\left|\mathrm{grad}\, u_t\right| \leqslant D_M$, $|u_t| \leqslant D_M^2$. Therefore the function u_t solves on the manifold M a Monge–Ampère equation of the type studied in Corollaries 2 and 3 of [14]. The second requirement of Theorem 1.5 thus follows from the a priori estimate carried out in [15, pp. 428–429] to fix the proof of Lemma 6 of [14, p.351]. So, by Theorem 1.5, we are done. $\quad\square$

6. Continuity method applied to A3 manifolds

6.1. Main statements

Definition 6.1. A closed Riemannian manifold (M, g) will be called an *A3 manifold* if there exists a constant $\theta > 0$ such that the following inequality holds:

$$-\frac{d^2}{dt^2}\Big(g\big(\mathcal{J}(m, V + tW)(U), U\big)\Big)_{t=0} \geqslant \theta |U|^2 |W|^2,$$

for every tangent point (m, V) lying in $\mathcal{N}_{\text{cut}}$ and every couple of tangent vectors $(U, W) \in T_m M \times T_m M$ orthogonal to each other.

The inequality itself originates in the condition A3 of [49, p. 154], as will be clear from the proof of Lemma 6.1 given below. Read along the zero section of TM, it implies that the sectional curvature of M is bounded below by $\frac{3}{2}\theta$ [18, p. 101]. Since $\theta > 0$, we infer by Rauch comparison [6, 10, 36, 41] that the closure of the neighborhood $\mathcal{N}_{\text{conj}}$ in TM is *compact*. In particular, for every real $\delta > 0$ small, so is the closure of the smooth strict subdomain $\mathcal{N}_{\text{conj}}^{\delta}$ of $\mathcal{N}_{\text{conj}}$ given by:

$$\mathcal{N}_{\text{conj}}^{\delta} = \big\{(m, V) \in \mathcal{N}_{\text{conj}}, \, \text{Jac}(\exp_m)(V) > \delta\big\}.$$

Fixing such a real δ, one may thus define on the closure of $\mathcal{N}_{\text{conj}}^{\delta}$ Banach spaces of C^2 functions or sections of the pullback bundle $\varpi_M^* TM$, as explained in Section 2.2. We will denote below the corresponding norms simply by $|.|_2^{(\delta)}$. Our present aim is to derive from Theorem 1.5 the following result:

Theorem 6.1. *Let M be a closed A3 manifold. Given $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geqslant 2$, the potential function $u(\mu_0, \mu_1)$ is $C^{k+2,\alpha}$ for every couple of $C^{k,\alpha}$ positive probability measures (μ_0, μ_1), if, for each such couple, the infimum of $\min_{m \in M} \text{Jac}(\exp_m)(\text{grad}\, u_t(m))$, over the solution counterset $\mathcal{T}$, is positive.*

Proof. To prove Theorem 6.1, we fix a couple $(\mu_0, \mu_1) = (\rho_0\, dV, \rho_1\, dV)$ of $C^{k,\alpha}$ positive probability measures on M and set $\delta > 0$ for the infimum, over the solution counterset $\mathcal{T}$, of $\min_{m \in M} \text{Jac}(\exp_m)(\text{grad}\, u_t(m))$ where u_t denotes the solution in $\mathcal{A}^{k+2,\alpha}$ of the continuity Equation (4.1) corresponding to the data (ρ_0, ρ_1) and the value $t \in \mathcal{T}$ of the deformation parameter. All we have to do for the proof is to combine Theorem 1.5 with the following key lemma, formulated according to Remark 1.1:

Lemma 6.1 (Ma–Trudinger–Wang estimate). *There exists a constant Θ under control, depending on δ, such that the trace $\mathfrak{T}_t(m)$ of the c-Hessian endomorphism $\mathcal{H}u_t(m)$ is bounded above by Θ, for every point $(t, m) \in \mathcal{T} \times M$.*

Indeed, by Proposition 3.1, this lemma implies Estimate 2 of the Introduction. Granted the lemma, the second requirement of Theorem 1.5 is thus fulfilled and the proof of Theorem 6.1 is complete. As for the proof of Lemma 6.1 itself, recalling Lemma 3.5, it may be reduced essentially to the estimate carried out in [49, Section 4]. This reduction was performed in [18, Appendix B]; so we are done.   □

We would like to point out that Theorem 1.4 of [48] follows at once from Theorem 6.1. In other words, granted the Ma–Trudinger–Wang estimate, the partial differential equations approach makes that result obvious. Here, the reader should not be confused by the proof outlined in [48, pp. 436–437]. Indeed, a key fact was unobserved in [48] although clear from [18, Appendix B], namely: on every A3 manifold such that $\mathcal{N}_{\mathrm{cut}} = \mathcal{N}_{\mathrm{cut}} \cap \mathcal{N}_{\mathrm{conj}}^{\delta}$ for some $\delta > 0$, the two-point function $c = \frac{1}{2}d^2$ fulfills the assumptions made on the cost function in [49]. There is thus no estimate to pursue any further.

6.2. Maximum principle à la Ma–Trudinger–Wang

Instead of relying as above on the reduction to [49, Section 4] operated in [18, Appendix B], it is instructive to give a proof of Lemma 6.1 featuring an intrinsic control on the constant Θ of the lemma. For completeness, we present such a proof in this section.

Outline of an intrinsic proof of Lemma 6.1 Fix $t \in \mathcal{T}$, pick a point $m_0 \in M$ such that $\mathfrak{T}_t(m_0) = \max_{m \in M} \mathfrak{T}_t(m)$ and write successively the critical equality and the maximum inequality, namely (dropping the subscript t):

(6.1a) $$\operatorname{grad} \mathfrak{T}(m_0) = 0,$$
(6.1b) $$dF[u](\mathfrak{T})(m_0) \leqslant 0.$$

To derive from (6.1) a controlled upper bound on $\mathfrak{T}(m_0)$, we must first cope with the third and fourth-order terms appearing in (6.1b). This will be achieved by using, besides (6.1a), the continuity Equation (4.2) differentiated twice. We will be left with an inequality involving terms of order at most two. To reach the desired bound on $\mathfrak{T}(m_0)$, we will sort these terms according to a criterion inspired from [49, bottom p. 162]. Let us define it, assuming

with no loss of generality that $\mathfrak{T}(m_0) \geqslant 1$, sticking to the notations used in the proof of Proposition 4.1 (see Section 4.2) and using the norms $|.|_2^{(\delta)}$ introduced above (after Definition 6.1):

Definition 6.2. Any of the terms of order 0, 1 or 2 under study at the point m_0 will be called *MTW-bounded* if it can be bounded above by:

$$C \left(1 + \mathfrak{T}^2(m_0) + \mathfrak{T}(m_0) \operatorname{trace}\big(\mathcal{F}_{[u]}(m_0)\big) \right),$$

for some positive constant C controlled by the dimension n and the diameter D_M of the manifold M, the C^1 norm of its Riemann tensor, the norm $|\mathcal{J}|_2^{(\delta)}$ of the Jacobi endomorphism $\mathcal{J}$ and $\max_{t \in [0,1]} |\psi_t|_2^{(\delta)}$, where the real function ψ_t is the one appearing in the continuity Equation (4.2).

With this outline in mind, let us get to a detailed proof of Lemma 6.1. **Some local notations.** Given a chart $(x^1, \ldots, x^n)$ of the manifold M and the natural chart $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ associated to it on TM, we will set for short:

$$\frac{\partial}{\partial x^i} = \partial_i, \quad \frac{\partial^2}{\partial x^i \partial x^j} = \partial_{ij}, \quad \frac{\partial}{\partial v^i} = D_i,$$
$$\frac{\partial^2}{\partial v^i \partial v^j} = D_{ij}, \quad \frac{\partial^2}{\partial x^i \partial v^j} = \partial_i D_j, \ldots.$$

We will take the chart $(x^1, \ldots, x^n)$ *normal* at the point m_0 and such that the matrix of the endomorphism $\mathcal{H}u_t(m_0)$ is diagonal, with diagonal entries $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ (for simplicity, we drop the subscript t until the end of the proof). Components of relevant tensors will be denoted by:

$$\operatorname{grad} u = \nabla^j u(x)\, \partial_j, \; \nabla \operatorname{grad} u = \nabla_i^j u(x)\, dx^i \otimes \partial_j, \; \mathcal{J} = \mathcal{J}_i^j(x,v)\, dx^i \otimes \partial_j,$$
$$\text{and } \mathcal{H}u = \mathcal{H}_i^j\, dx^i \otimes \partial_j, \; \mathcal{F}_{[u]} = \mathcal{F}_i^j\, dx^i \otimes \partial_j, \text{ thus with } \mathcal{H}_i^k \mathcal{F}_k^j = \delta_i^j.$$

Calculation of the conditions (6.1). Using (2.7), we find for the components of $\operatorname{grad} \mathfrak{T}$ the expression:

$$\nabla^j \mathfrak{T} = \nabla^j \mathcal{J}_k^k - g^{jp} \Gamma_{pq}^\ell \nabla^q u\, D_\ell \mathcal{J}_k^k + D_\ell \mathcal{J}_k^k \nabla^{j\ell} u + \nabla^j \nabla_k^k u.$$

After commuting third derivatives of u, the condition (6.1a) thus reads:

$$(6.2) \qquad -\nabla_k^k \nabla^j u = \partial_j \mathcal{J}_k^k + \sum_{\ell=1}^n D_\ell \mathcal{J}_k^k\, \partial_{j\ell} u - R_{j\ell}\, \partial_\ell u, \text{ at } x = 0,$$

where the first derivatives of the local function $\mathcal{J}_k^k(x, v)$ are computed at $x = 0, v^i = \partial_i u(0)$, and $R_{j\ell}\, dx^j \otimes dx^\ell$ stands for the Ricci tensor. Differentiating the function $\mathfrak{T}$ once more yields for (6.1b) the expression:

$$(6.3) \qquad \mathcal{F}_j^i\big(\nabla_i^j \nabla_k^k u + \nabla_i^j \nabla^\ell u\, D_\ell \mathcal{J}_k^k\big) + \text{l.o.t.} \ \leqslant 0, \ \text{at } x = 0,$$

with the lower order terms (l.o.t) given at $x = 0$ by:

$$(6.4) \qquad \text{l.o.t.} = \mathcal{F}_j^i \Big(\partial_{ij} \mathcal{J}_k^k - \frac{1}{3}\big(R_{qij}^\ell + R_{jiq}^\ell\big) D_\ell \mathcal{J}_k^k\, \partial_q u$$

$$+\, \partial_i D_\ell \mathcal{J}_k^k\, \partial_{j\ell} u + \partial_j D_\ell \mathcal{J}_k^k\, \partial_{i\ell} u + D_{h\ell}\mathcal{J}_k^k\, \partial_{ih} u\, \partial_{j\ell} u \Big),$$

where one should sum over all indices and, again, compute the derivatives of $\mathcal{J}_k^k(x, v)$ at $x = 0, v^i = \partial_i u(0)$. Here, we have used the Riemann–Cartan formula $\partial_i \Gamma_{jq}^\ell = \frac{1}{3}\big(R_{qij}^\ell + R_{jiq}^\ell\big)$, valid at the center of any normal chart [7, p. 244]. Let us observe that, at $x = 0$, the following identities hold:

$$(6.5\text{a}) \qquad \mathcal{F}_j^i \partial_{i\ell} u = \delta_j^\ell - \mathcal{F}_j^i \mathcal{J}_i^\ell, \ \ \mathcal{F}_j^i \partial_{j\ell} u = \delta_\ell^i - \mathcal{F}_j^i \mathcal{J}_\ell^j,$$

$$(6.5\text{b}) \qquad \mathcal{F}_j^i \partial_{ih} u\, \partial_{j\ell} u = \partial_{h\ell} u - \mathcal{J}_\ell^h + \mathcal{F}_j^i \mathcal{J}_i^h \mathcal{J}_\ell^j.$$

They imply that the above scalar l.o.t. is MTW-bounded. We thus focus on the other terms of the inequality (6.3), namely the third and fourth-order terms. We will calculate them by differentiating twice (4.2).

Differentiating twice the continuity equation. Applying the gradient operator to (4.2) yields in our chart x, for each $\ell = 1, \ldots, n$, the equation:

$$\mathcal{F}_j^i \nabla^\ell \mathcal{H}_i^j - \nabla^\ell\big(\psi_t(x, \operatorname{grad} u(x))\big) = 0,$$

or else, with the subscript of ψ_t dropped:

$$\mathcal{F}_j^i \nabla^\ell \nabla_i^j u = -\mathcal{F}_j^i \nabla^\ell\big(\mathcal{J}_i^j(x, \operatorname{grad} u(x))\big) + \nabla^\ell\big(\psi(x, \operatorname{grad} u(x))\big).$$

Commuting the third derivatives of u and using (2.2), (2.7) to calculate the first derivatives of $\mathcal{J}_i^j$ and ψ, we get:

$$\mathcal{F}_j^i \nabla_i^j \nabla^\ell u = \mathcal{F}_j^i\big(g^{\ell h} R_{qih}^j \nabla^q u - \nabla^\ell \mathcal{J}_i^j + g^{\ell h}\Gamma_{qh}^p \nabla^q u D_p \mathcal{J}_i^j - D_h \mathcal{J}_i^j \nabla^{\ell h} u\big)$$

$$+\, \nabla^\ell \psi - g^{\ell i}\Gamma_{qi}^j \nabla^q u D_j \psi + D_j \psi \nabla^{\ell j} u.$$

At $x = 0$, we thus obtain the n equations:

$$\mathcal{F}_j^i \nabla_i^j \nabla^\ell u = \mathcal{F}_j^i\big(R_{qi\ell}^j \partial_q u - \partial_\ell \mathcal{J}_i^j - D_h \mathcal{J}_i^j \partial_{h\ell} u\big) + \partial_\ell \psi + D_j \psi \partial_{j\ell} u,$$

where one should sum over all repeated indices and compute the first derivatives of $\mathcal{J}_i^j(x, v)$ and $\psi(x, v)$ at the point $x = 0, v^i = \partial_i u(0)$. We infer from these equations combined with (6.5a) that the term $\mathcal{F}_j^i \nabla_i^j \nabla^\ell u \, D_\ell \mathcal{J}_k^k$ appearing in (6.3) is MTW-bounded.

We are left with treating the fourth-order term of (6.3). To do so, we apply the Laplace operator to (4.2) and get:

$$(6.6) \qquad \mathcal{F}_j^i \nabla_k^k \nabla_i^j u = \mathcal{F}_p^i \mathcal{F}_j^q \nabla_k \mathcal{H}_q^p \nabla^k \mathcal{H}_i^j$$
$$- \mathcal{F}_j^i \nabla_k^k \big(\mathcal{J}_i^j(x, \operatorname{grad} u(x)) \big) - \Delta \big(\psi(x, \operatorname{grad} u(x)) \big).$$

After commutation of the fourth derivatives of u, the left-hand side becomes equal to $\mathcal{F}_j^i \nabla_i^j \nabla_k^k u + \text{l.o.t.}$, with

$$(6.7) \quad \text{l.o.t.} = \mathcal{F}_j^i \big((\nabla^k R_{\ell k i}^j + \nabla_i R_\ell^j) \nabla^\ell u + 2 R_{\ell k i}^j \nabla^{k\ell} u + R_{\ell i} \nabla^{\ell j} u + R_\ell^j \nabla_i^\ell u \big),$$

an expression which, evaluated at $x = 0$, is MTW-bounded. Let us now calculate the second derivatives of $\mathcal{J}_i^j$ and ψ appearing in the right-hand side of (6.6). Using (2.7) and the identity $\sum_{k=1}^n \partial_k \Gamma_{ik}^\ell = -\frac{1}{3} R_i^\ell$, valid at $x = 0$ due to the Riemann–Cartan formula [7, p. 244], we find that the term $-\mathcal{F}_j^i \nabla_k^k \big(\mathcal{J}_i^j(x, \operatorname{grad} u(x)) \big)$, evaluated at $x = 0$, is equal to:

$$-\mathcal{F}_j^i (D_\ell \mathcal{J}_i^j \nabla_k^k \nabla^\ell u + D_{h\ell} \mathcal{J}_i^j \nabla_k^\ell u \nabla^{kh} u) + \text{ further terms.}$$

The further terms in this equation are given by the following expression:

$$-\mathcal{F}_j^i \left(\partial_{kk} \mathcal{J}_i^j + \frac{1}{3} (R_i^\ell \mathcal{J}_\ell^j - R_\ell^j \mathcal{J}_i^\ell + R_k^\ell D_\ell \mathcal{J}_i^j \, \partial_k u) + \partial_k D_\ell \mathcal{J}_i^j \, \partial_{k\ell} u \right),$$

where one should sum over all indices and compute the derivatives of $\mathcal{J}_i^j(x, v)$ at $x = 0, v^i = \partial_i u(0)$. So, we see that the further terms into question are MTW-bounded. Besides, using (2.2), we get:

$$-\Delta \big(\psi(x, \operatorname{grad} u(x)) \big)_{x=0} = D_\ell \psi \, \nabla_k^k \nabla^\ell u + \text{l.o.t.},$$

with l.o.t. $= \partial_{kk} \psi + \frac{1}{3} R_k^\ell D_\ell \psi \, \partial_k u + 2 \partial_k D_\ell \psi \, \partial_{k\ell} u + D_{h\ell} \psi \, \partial_{k\ell} u \, \partial_{kh} u$, where one should sum over all indices and compute the derivatives of $\psi(x, v)$ at $x = 0, v^i = \partial_i u(0)$. It is clear that the terms abbreviated here by l.o.t. are MTW-bounded. In order to cope with the third-order terms $-\mathcal{F}_j^i D_\ell \mathcal{J}_i^j \nabla_k^k \nabla^\ell u$ and $D_\ell \psi \nabla_k^k \nabla^\ell u$ at $x = 0$, we use the critical condition (6.2). The latter implies that these terms are MTW-bounded. Last, we note that the fourth-order

term $-\mathcal{F}_j^i D_{h\ell}\mathcal{J}_i^j\nabla_k^\ell u\nabla^{kh}u$ is equal to $-\mathcal{F}_j^i D_{h\ell}\mathcal{J}_i^j\mathcal{H}_\ell^k\mathcal{H}_k^h$ up to additional MTW-bounded terms. Altogether, we infer from (6.6) read at $x = 0$ the inequality:

$$(6.8) \qquad \mathcal{F}_j^i\nabla_i^j\nabla_k^k u \geqslant -\sum_{j\neq k}\mathcal{F}_j^j D_{kk}\mathcal{J}_j^j\big(\mathcal{H}_k^k\big)^2 + \text{ MTW-bounded terms,}$$

in which one should sum over all indices and compute the derivatives $D_{kk}\mathcal{J}_j^j$ at the point $x = 0, v^i = \partial_i u(0)$. Here, we have used the fact that the matrices $\mathcal{F}_j^i$ and $\mathcal{H}_\ell^k$ are *diagonal* at $x = 0$. Since these matrices are inverses of each other, we have dropped the part of the sum $-\sum_{k=1}^n\mathcal{F}_k^k D_{kk}\mathcal{J}_k^k\big(\mathcal{H}_k^k\big)^2$ obtained for $j = k$, because it is MTW-bounded.

Back to the maximum condition. Since we have assumed that the manifold M is A3, Definition 6.1 implies the existence of a real $\theta > 0$ such that the sum $-\sum_{j\neq k}\mathcal{F}_j^j D_{kk}\mathcal{J}_j^j\big(\mathcal{H}_k^k\big)^2$ is bounded below by $\theta\sum_{j\neq k}\mathcal{F}_j^j\big(\mathcal{H}_k^k\big)^2$. Combining this bound with (6.8) and (6.3) and recalling that $\lambda_1\leqslant\cdots\leqslant\lambda_n$ stand for the eigenvalues of the symmetric (and positive) endomorphism $\mathcal{H}u(m_0)$, each repeated with its multiplicity, we conclude that there exists a constant C, controlled as in Definition 6.2, such that the following inequality holds:

$$\theta\sum_{j\neq k}\frac{\lambda_k^2}{\lambda_j} \leqslant C\left(1 + \left(\sum_{k=1}^n\lambda_k\right)^2 + \sum_{k=1}^n\lambda_k\sum_{j=1}^n\frac{1}{\lambda_j}\right).$$

The left-hand side is larger than $\theta\dfrac{\lambda_n^2}{\lambda_1}$, while the right-hand side is bounded above by $C\big(1 + \lambda_n^2 + \dfrac{\lambda_n}{\lambda_1}\big)$, with another suitably controlled constant C (say, equal to n^2 times the preceding one). The resulting inequality reads:

$$(6.9) \qquad\qquad \lambda_n^2\Big(\frac{\theta}{2\lambda_1} - C\Big) + \frac{\lambda_n}{\lambda_1}\Big(\frac{\theta}{2}\lambda_n - C\Big) \leqslant C.$$

Completion of the proof of Lemma 6.1. Let us distinguish two cases:

  Case 1: either $\big(\frac{\theta}{2\lambda_1} - C\big)$ or $\big(\frac{\theta}{2}\lambda_n - C\big)$ is less than 1.
If $\big(\frac{\theta}{2}\lambda_n - C\big) \leqslant 1$, we get $\mathfrak{T}(m_0) \leqslant \frac{2}{\theta}n(C+1)$. If $\big(\frac{\theta}{2\lambda_1} - C\big) \leqslant 1$, we combine this inequality with the continuity equation written as: $\prod_{i=1}^n\lambda_i = e^\psi$, where the function $\psi(x,v)$ is computed at $x = 0, v^i = \partial_i u(0)$. It yields $\lambda_n \leqslant e^\psi\big(\frac{2}{\theta}(C+1)\big)^{n-1}$, hence $\mathfrak{T}(m_0) \leqslant \frac{C}{\theta^{n-1}}$, with another suitable constant C. In either subcase, we are done.

Case 2: $\left(\frac{\theta}{2\lambda_1} - C\right) > 1$ _and_ $\left(\frac{\theta}{2}\lambda_n - C\right) > 1.$
Here, we infer from (6.9) that $\lambda_n^2 + \frac{\lambda_n}{\lambda_1} \leqslant C$, which yields $\mathfrak{T}(m_0) \leqslant C$ with another suitably controlled constant C and, again, we are done. $\square$

7. Proofs completion: first-order (stay-away) estimates

So far, we know by Theorem 6.1 that, on a closed A3 manifold M, the potential function of the optimal map $\mathrm{Mc}(\mu_0, \mu_1)$ is $C^{k+2,\alpha}$, for every couple of $C^{k,\alpha}, k \geqslant 2$, positive probability measures (μ_0, μ_1) given on M, provided Estimate 1 of the Introduction can be established for each such couple (μ_0, μ_1), when we use the continuity method with the path $\mu_t = t\mu_1 + (1-t)\mu_0$. In this section, we will prove Theorems 1.3 and 1.4 by establishing Estimate 1 on the manifolds successively considered in these two theorems, all shown to be A3 manifolds [19, 21, 40].

7.1. Positively curved Riemannian locally symmetric spaces

In this subsection, we let the closed n-dimensional manifold M be Riemannian locally symmetric, positively curved. We take it simply connected with no loss of generality [18, Appendix C]. Whenever a manifold is so, it must be a symmetric space (according to Cartan's theorem, see [10, 42, 43]) and each cut point must be a conjugate point [13, 37]. Furthermore, all the geodesics must be simply closed of constant length [9, 37]. We take this length equal to 2π. With this normalization, we know [9, 37] that, for every point $(m, V) \in TM$ with $V \neq 0$, the Jacobi curvature endomorphism $Y \in T_m M \to R_m(Y, V)V \in T_m M$ (see the beginning of Section 2.3.2) admits ν eigenvalues equal to $|V|^2$ and $(n-1-\nu)$ eigenvalues equal to $\frac{1}{4}|V|^2$. Furthermore, according to Cartan's classification (see [37]), the integer ν must be equal to $n-1$ if M is the sphere $\mathbb{S}^n$, or equal to 1 if M is the complex projective space $\mathbb{C}P^k$, or equal to 3 if M is the quaternionic projective space $\mathbb{H}P^k$, or equal to 7 if M is the Cayley projective plane $\mathbb{O}P^2$, each with the appropriate sectional curvature (usual, holomorphic, quaternionic, or octonionic) equal to 1.

Proof of Estimate 1. Sticking to the notations used in Sections 4 and 6, we fix $t \in \mathcal{T}$, that is, we fix a value of $t \in [0, 1]$ for which there exists a solution $u_t \in \mathcal{A}^{k+2,\alpha}$ of the continuity Equation (4.1). Let $m_0 \in M$ be a point of the manifold where the function $m \mapsto \mathrm{Jac}(\exp_m)(\mathrm{grad}\, u_t(m))$ assumes its global minimum. We seek a positive lower bound δ for that minimum. From (2.20),

we know that the simpler function w_t given by $w_t(m) = \frac{1}{2}|\operatorname{grad} u_t(m)|^2$ must assume a global maximum at the point m_0. Accordingly, recalling the ellipticity at $u = u_t$ of the differential operator $dF[u]$ given by (4.2), (4.3), we may write the following critical equality and maximum inequality (dropping freely the subscript t):

(7.1a) $\operatorname{grad} w(m_0) = 0,$

(7.1b) $dF[u](w)(m_0) \leqslant 0.$

If $w(m_0)$ vanishes, we get $\operatorname{Jac}(\exp_m)(\operatorname{grad} u(m_0)) = 1$, so we are done. If $w(m_0) > 0$, let us choose a convenient chart to express the conditions (7.1). We take along the geodesic path $t \in \mathbb{R} \to \gamma_0(t) = \exp_{m_0}(t \operatorname{grad} u(m_0))$ a parallel orthonormal frame field $t \mapsto (e_1(t), \ldots, e_n(t))$ of the type described in the first paragraph of Section 2.3.2. Specifically, setting $r_0 = |\operatorname{grad} u(m_0)|$, we choose it such that $e_n(t) = \dfrac{1}{r_0}\dot{\gamma}_0(t)$ and the vectors $(e_1(t), \ldots, e_\nu(t))$ (resp. $(e_{\nu+1}(t), \ldots, e_{n-1}(t))$) span the eigenspace of the curvature endomorphism $R_{\gamma_0(t)}(., \dot{\gamma}_0(t))\dot{\gamma}_0(t)$ associated to the eigenvalue r_0^2 (resp. $\frac{1}{4}r_0^2$). As the time parameter t varies in a small enough open interval I containing the segment $[0,1]$, this construction yields a Fermi chart defined in a thin enough tubular neighborhood $\mathfrak{N}$ of the image $\gamma_0(I)$ in the manifold M. The interval I and the neighborhood $\mathfrak{N}$ are chosen such that, for every point $m \in \mathfrak{N}$, we can find a unique value $x^n = x^n(m)$ of the arc length parameter $x^n = r_0 t$ and a unique $(n-1)$-tuple of reals $(x^1, \ldots, x^{n-1})$ for which the following defining equality holds:

$$m = \exp_{\gamma_0(t)}(x^1 e_1(t) + \cdots + x^{n-1} e_{n-1}(t)).$$

In the Fermi chart $(x^1, \ldots, x^n)$ so defined, the coordinates x^i may be viewed as cylindrical coordinates about the axis of the chart, namely the geodesic γ_0. Along the latter, the Riemannian metric of the manifold is osculating to the local Euclidean metric $\sum_{i=1}^n (dx^i)^2$. In particular, the Christoffel symbols vanish: $\Gamma_{ij}^k(0, \ldots, 0, r_0 t) \equiv 0$. We keep writing $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ for the natural chart of TM associated to the chart x, and ∂_i (resp. D_i) for the local vector fields $\frac{\partial}{\partial x^i}$ (resp. $\frac{\partial}{\partial v^i}$). We refer the reader to [1, 32] or [18, section 2] for an account on Fermi charts. In the present Fermi chart, the critical condition (7.1a) reads:

(7.2) $\partial_{in} u(0) = 0 \, ,$

for every integer $i = 1, \ldots, n$. Besides, we get:

$$\nabla_i^j w(0) = r_0 \partial_{nij} u(0) + r_0^2 R_{injn}(0) + \sum_{\gamma=1}^{n-1} \partial_{i\gamma} u(0) \partial_{j\gamma} u(0),$$

after commuting third covariant derivatives of the function u, where $R_{ijk\ell}(x)$ stand for the chart components of the covariant Riemann tensor. Recalling (4.3), the local expression of $dF[u](w)(m_0)$ is obtained by contracting the right-hand side of the preceding equation with the symmetric matrix $\mathcal{F}_j^i(0)$. Here comes the reason why we have chosen the elliptic operator $dF[u]$ to express the maximum condition in (7.1b). Indeed, we can take advantage of the continuity Equation (4.2) itself, differentiated with respect to the arc length coordinate x^n at $x = 0$, to calculate the third-order term $r_0 \mathcal{F}_j^i(0) \partial_{nij} u(0)$ and express it by l.o.t. Specifically, we infer for $\mathcal{F}_j^i(0) \partial_{nij} u(0)$ the expression:

$$\mathcal{F}_j^i(0) \partial_{nij} u(0) = -\mathcal{F}_j^i(0) \partial_n \big(\mathcal{J}_i^j(x, \operatorname{grad} u(x))\big)\big|_{x=0} + \partial_n \big(\psi_t(x, \operatorname{grad} u(x))\big)\big|_{x=0}.$$

Combining (2.7) with (7.2), we find that each term $\partial_n\big(\mathcal{J}_i^j(x, \operatorname{grad} u(x))\big)$ reduces at $x = 0$ to $(\partial_n \mathcal{J}_i^j)(0, \operatorname{grad} u(0))$. Moreover, the local symmetry of the manifold M forces the latter derivative to vanish. Indeed, the components of the Jacobi endomorphism $\mathcal{J}(\gamma_0(t), \dot{\gamma}_0(t))$ in the parallel frame $(e_1(t), \ldots, e_n(t))$ must be *constant*. Specifically, by the special choice of the frame, one can readily verify (as in [21]) that the vector $e_i(t)$ is an eigenvector of $\mathcal{J}(\gamma_0(t), \dot{\gamma}_0(t))$ associated to the eigenvalue λ_i with: $\lambda_i = r_0 \cot r_0$ if $i \leqslant \nu$, $\lambda_i = \frac{r_0}{2} \cot \frac{r_0}{2}$ if $i \in \{\nu + 1, \ldots, n - 1\}$ and $\lambda_n = 1$ as always (by Proposition 2.1). So, the section $t \mapsto \mathcal{J}(\gamma_0(t), \dot{\gamma}_0(t))$ of the pullback bundle $\gamma_0^* \operatorname{End}(TM) \to \mathbb{R}$ is parallel. The equation $\frac{\nabla}{dt} \mathcal{J}(\gamma_0(t), \dot{\gamma}_0(t)) = 0$, expressed using (2.7), read at $t = 0$ in the Fermi chart yields $(\partial_n \mathcal{J}_i^j)(0, \operatorname{grad} u(0)) \equiv 0$, as claimed. Besides, the term $\partial_n\big(\psi_t(x, \operatorname{grad} u(x))\big)$ calculated at $x = 0$ is merely equal to:

$$\partial_n \big(\psi_t(x, \operatorname{grad} u(x))\big)\big|_{x=0} = \frac{\partial_n \rho_0}{\rho_0}(0) - \frac{\partial_n \rho_t}{\rho_t}(0, \ldots, 0, r_0),$$

because $x = 0$ is a critical point of the chart expression of the function $m \mapsto \operatorname{Jac}(\exp_m)(\operatorname{grad} u(m))$. We conclude that the maximum condition (7.1b)

reads:

$$(7.3) \qquad 0 \geqslant \mathcal{F}_j^i(0)\Big(r_0^2 R_{injn}(0) + \sum_{\gamma=1}^{n-1} \partial_{i\gamma}u(0)\partial_{j\gamma}u(0)\Big)$$

$$+ r_0\Big(\frac{\partial_n\rho_0}{\rho_0}(0) - \frac{\partial_n\rho_t}{\rho_t}(0,\dots,0,r_0)\Big).$$

By (6.5b) and the special choice of our chart, we get:

$$\sum_{\gamma=1}^{n-1}\mathcal{F}_j^i(0)\partial_{i\gamma}u(0)\partial_{j\gamma}u(0) = \sum_{\gamma=1}^{n-1}\partial_{\gamma\gamma}u(0) - \mathcal{J}_\gamma^\gamma + \mathcal{F}_\gamma^\gamma(0)\big(\mathcal{J}_\gamma^\gamma\big)^2,$$

where the components $\mathcal{J}_\gamma^\gamma$ should be evaluated at $(0, \operatorname{grad} u(0))$. But $\partial_{\gamma\gamma}u(0)$ is equal to $\mathcal{H}_\gamma^\gamma(0) - \mathcal{J}_\gamma^\gamma(0, \operatorname{grad} u(0))$. By Proposition 3.1, it is thus bounded below by $-\mathcal{J}_\gamma^\gamma(0, \operatorname{grad} u(0))$. As for the term $\sum_{\gamma=1}^{n-1}\mathcal{F}_\gamma^\gamma(0)\big(\mathcal{J}_\gamma^\gamma\big)^2$, we know by (4.4) that it is nonnegative. So is the term $r_0^2\mathcal{F}_j^i(0)R_{injn}(0)$, because the manifold is positively curved. Altogether, we infer from (7.3) the inequality:

$$0 \geqslant -2\sum_{\gamma=1}^{n-1}\mathcal{J}_\gamma^\gamma(0, \operatorname{grad} u(0)) + r_0\Big(\frac{\partial_n\rho_0}{\rho_0}(0) - \frac{\partial_n\rho_t}{\rho_t}(0,\dots,0,r_0)\Big).$$

Recalling Lemma 4.2, it yields the lower bound:

$$\sum_{\gamma=1}^{n-1}\mathcal{J}_\gamma^\gamma(0, \operatorname{grad} u(0)) \geqslant -\frac{1}{2}D_M\big(\max_M|d\log\rho_0| + \max_M|d\log\rho_t|\big).$$

Here, the diameter D_M of the manifold is equal to π and the expression of the left-hand side is known [21], given by:

$$\sum_{\gamma=1}^{n-1}\mathcal{J}_\gamma^\gamma(0, \operatorname{grad} u(0)) = \nu\, r_0\cot r_0 + (n-\nu-1)\frac{r_0}{2}\cot\frac{r_0}{2}.$$

We thus obtain the uniform lower bound:

$$\nu\, r_0\cot r_0 + (n-\nu-1)\frac{r_0}{2}\cot\frac{r_0}{2} \geqslant -\pi \max_{t\in[0,1]}\max_M|d\log\rho_t|.$$

It yields the existence of a small real $\alpha > 0$, independent of the point $m_0 \in M$ and of the value t of the deformation parameter in $\mathcal{T} \subset [0,1]$, such that: $r_0 = |\operatorname{grad} u_t(m_0)| \leqslant \pi - \alpha$. Recalling (2.20), we infer Estimate

1 with $\delta = \dfrac{1}{(\pi - \alpha)^{n-1}} (\sin \alpha)^{\nu} \left(2 \cos \dfrac{\alpha}{2}\right)^{n-1-\nu}$. The proof of Theorem 1.3 is complete. $\qquad\square$

7.2. Nearly spherical surfaces

In this section, the manifold M is the 2-sphere endowed with a Riemannian metric g of positive Gauss curvature K normalized by $\min_M K = 1$. We set henceforth $\varepsilon_K = |K - 1|_{C^2(M)}$ and assume that ε_K is small. By Myers theorem [10], the diameter of M satisfies: $D \leqslant \pi$. In this setting, before attacking Estimate 1, we require estimates on further geometric quantities.

7.2.1. Two-dimensional perturbation tools. For each point $(m, V) \in TM$ and $i = 0, 1$, let $t \in [0, +\infty) \to f_i(m, V, t) \in \mathbb{R}$ denote the solution of the Cauchy problem:

$$(7.4) \qquad \ddot{f}_i + |V|^2 K\big(\exp_m(tV)\big) f_i = 0, \ \ f_i(0) = \delta_{i0}, \ \dot{f}_i(0) = \delta_{i1},$$

where the dot stands for $\frac{\partial}{\partial t}$. The differential equation which appears in (7.4) is the scalar Jacobi equation. The functions f_0, f_1 so defined on $TM \times \mathbb{R}$ yield the solution of the Cauchy problem for the, possibly inhomogeneous, scalar Jacobi equation. Let us describe how. Fix a point $(m, V) \in TM$ with $V \neq 0$, set $\gamma(t) = \exp_m(tV)$ and let $t \mapsto (e_1(t), e_2(t))$ be the direct orthonormal *parallel* frame field along the path $t \mapsto \gamma(t)$, such that $\dot{\gamma}(t) = |V| e_2(t)$. If $Y(t) = Y^1(t) e_1(t) + Y^2(t) e_2(t)$ is a Jacobi field along the geodesic γ, its first component satisfies the above scalar Jacobi equation while the component Y^2 must be an affine function of the time parameter t. We infer from (2.8) that the matrices of the linear maps $\left(\frac{\partial \exp}{\partial m}\right)_V^{\nabla}(m, tV)$, and $t\left(\frac{\partial \exp}{\partial V}\right)_m (m, tV)$, in the source frame $(e_1(0), e_2(0))$ and the target frame $(e_1(t), e_2(t))$, are, respectively, equal to:

$$(7.5) \qquad \begin{pmatrix} f_0(m, V, t) & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} f_1(m, V, t) & 0 \\ 0 & t \end{pmatrix}.$$

For each prescribed couple of reals (a, b) and continuous function $t \mapsto h(t)$, it follows now from (2.10), read with $U(t) = h(t) e_1(t), \xi = a e_1(0), \nu = b e_1(0)$, that the solution of the Cauchy problem:

$$\ddot{f} + |V|^2 K\big(\exp_m(tV)\big) f = h, \ \ f(0) = a, \ \dot{f}(0) = b,$$

must be given by:

$$(7.6) \quad f(t) = af_0(m, V, t) + bf_1(m, V, t) + \int_0^t f_1\big(\gamma(\tau), \dot{\gamma}(\tau), t - \tau\big)h(\tau)d\tau.$$

Another consequence of (7.5) is the identity:

$$(7.7) \qquad\qquad \mathrm{Jac}(\exp_m)(V) = f_1(m, V, 1),$$

from which we anticipate the need, for Estimate 1, to control the function $f_1(m, V, 1)$ together with its first and second derivatives. Before focussing on that control problem, we observe that the neighborhhood $\mathcal{N}_{\mathrm{conj}}$ admits in dimension two the following characterization:

$$\mathcal{N}_{\mathrm{conj}} = \big\{(m, V) \in TM, \forall t \in [0, 1], f_1(m, V, t) > 0\big\}.$$

Finally, if the point (m, V) lies in $\mathcal{N}_{\mathrm{conj}}$, we get at once from (7.5) the expression of the Jacobi endomorphism at (m, V) in the tangent basis $(e_1(0), e_2(0))$, namely:

$$(7.8) \qquad\qquad \mathcal{J}(m, V) = \begin{pmatrix} \frac{f_0(m,V,1)}{f_1(m,V,1)} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us set $\bar{f}_0$ and $\bar{f}_1$ for the functions defined analogously to f_0, f_1 when the function K is replaced by 1 in (7.4), that is, we set: $\bar{f}_0 = \cos(|V|t)$ and $\bar{f}_1(m, V, t) = \dfrac{\sin(|V|t)}{|V|}$. By Sturm comparison [6, pp. 238–239], the curvature lower bound $K \geqslant 1$ implies that the upper bound:

$$(7.9) \qquad\qquad f_1(m, V, t) \leqslant \bar{f}_1(m, V, t),$$

holds, provided the point (m, tV) lies in $\mathcal{N}_{\mathrm{conj}}$. If so, we infer at once the further bound:

$$f_1(m, V, t) \leqslant t.$$

Plugging this bound under the integral sign in (7.6), we get the estimate:

$$(7.10) \qquad\qquad \|f - af_0(m, V, .) - bf_1(m, V, .)\| \leqslant \frac{1}{2}\|h\|,$$

bearing on the solution f of the inhomogeneous Cauchy problem, where the norm $\|.\|$ is defined by: $\|h\| = \max_{t \in [0,1]} |h(t)|$. Let us apply this estimate to the function $f(t) = f_1(m, V, t) - \bar{f}_1(m, V, t)$, thus with $a = b = 0$ and $h(t) =$

$|V|^2 (1 - K(\exp_m(tV))) \bar{f}_1(m, V, t)$. With this choice of f, the combination of (7.10) with the curvature pinching yields the lower bound:

$$(7.11) \qquad f_1(m, V, t) \geqslant \bar{f}_1(m, V, t) - \pi^2 \varepsilon_K,$$

valid for every $(m, V) \in \mathcal{N}_{\text{conj}}$ and $t \in [0, 1]$. If $|V| \geqslant \frac{3\pi}{4}$, this inequality read at $t = 1$ implies:

$$\pi - |V| \leqslant \arcsin(\pi^3 \varepsilon_K + \pi f_1(m, V, 1)),$$

hence the existence of a universal constant $C > 0$ (equal to $\pi^3 \sqrt{2}$) such that:

$$(7.12) \qquad \pi - |V| \leqslant C \left(\varepsilon_K + f_1(m, V, 1) \right).$$

So, the smaller ε_K and $f_1(m, V, 1)$ are, the closer the length of the vector V must get to π.

   If the map $t \in [0, 1] \to \gamma(t) = \exp_m(tV) \in M$ is one to one, we may associate to the orthonormal moving frame $(e_1(t), e_2(t))$ defined above a Fermi chart $x = (x^1, x^2)$ along the corresponding segment. In that chart, the Riemannian metric g reads:

$$g = dx^1 \otimes dx^1 + \mathfrak{G}(x^1, x^2) \, dx^2 \otimes dx^2,$$

with $\mathfrak{G}(0, x^2) = 1$, $\partial_1 \mathfrak{G}(0, x^2) = 0$. In the domain of the chart, the Christoffel symbols vanish except the following ones:

$$\Gamma_{22}^1 = -\frac{1}{2} \partial_1 \mathfrak{G}, \quad \Gamma_{12}^2 = \frac{\partial_1 \mathfrak{G}}{2\mathfrak{G}}, \quad \Gamma_{22}^2 = \frac{\partial_2 \mathfrak{G}}{2\mathfrak{G}}.$$

Moreover, the Gauss curvature is given by: $K = -\dfrac{\partial_{11} \sqrt{\mathfrak{G}}}{\sqrt{\mathfrak{G}}}$. We infer that, along the x^2-axis, the first and second partial derivatives with respect to x^1 of the Christoffel symbols can be expressed in terms of K as follows:

$$(7.13) \qquad \begin{aligned} &\partial_1 \Gamma_{22}^1 = -\partial_1 \Gamma_{12}^2 = K, \ \partial_1 \Gamma_{22}^2 = 0, \text{ and} \\ &\partial_{11} \Gamma_{22}^1 = -\partial_{11} \Gamma_{12}^2 = \partial_1 K, \ \partial_{11} \Gamma_{22}^2 = -\partial_2 K. \end{aligned}$$

We set $(x, v) = (x^1, x^2, v^1, v^2)$ for the natural chart of TM associated to the chart x and, $X = X(x, v, t) = \left(X^1(x, v, t), X^2(x, v, t) \right) = x\left(\exp_m(tV) \right)$, with $x = x(m), v = dx(m)(V)$, for the local expression of the geodesic motion.

Besides, we still set $D_i = \dfrac{\partial}{\partial v^i}$ (while ∂_i stands as above for $\frac{\partial}{\partial x^i}$). From (7.5), we get at once:

$$(7.14) \quad \begin{pmatrix} \partial_1 X^1 & \partial_2 X^1 \\ \partial_1 X^2 & \partial_2 X^2 \end{pmatrix} = \begin{pmatrix} f_0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} D_1 X^1 & D_2 X^1 \\ D_1 X^2 & D_2 X^2 \end{pmatrix} = \begin{pmatrix} f_1 & 0 \\ 0 & t \end{pmatrix},$$

where each entry of the matrices should be calculated at the point $(x, v, t) = (0, 0, 0, |V|, t)$, and the functions $f_i(x, v, t), i = 0, 1$, stand abusively for the local expressions of the global functions f_i's precedingly defined.

When the point (m, V) lies in $\mathcal{N}_{\mathrm{conj}}$, the identity (7.7) prompts us to evaluate the first and second derivatives with respect to the Fermi variables x^j and v^k, at the point $(x, v, t) = (0, 0, 0, |V|, t)$, of the function f_1. If we first do it for the function $\bar{f}_1$ instead, we find that these derivatives, evaluated at $(0, 0, 0, |V|, t)$, all vanish except the following ones:

$$D_2 \bar{f}_1, \partial_{11} \bar{f}_1, D_{11} \bar{f}_1, D_{22} \bar{f}_1.$$

As the length of the vector V grows to π, the limits of the latter at $t = 1$ are, respectively, equal to: $-\dfrac{1}{\pi}, 1, -\dfrac{1}{\pi^2}, \dfrac{2}{\pi^2}$. Let us now describe correspondingly the behavior of the function f_1.

Proposition 7.1. *Let (m, V) be a point of $\mathcal{N}_{\mathrm{conj}}$ such that the geodesic path $t \in [0, 1] \to \exp_m(tV) \in M$ is one to one. Let x be the Fermi chart associated to it as done above. There exists a universal constant $C > 0$ such that, for each $t \in [0, 1]$, all the first and second partial derivatives of the function f_1 with respect to the Fermi variables (x^1, x^2, v^1, v^2), evaluated at the point $(x, v, t) = (0, 0, 0, |V|, t)$, have their absolute value bounded above by $C\varepsilon_K$, but the following ones:*

$$D_2 f_1, \partial_{11} f_1, D_{11} f_1, D_{22} f_1.$$

Regarding the latter, evaluated at $(0, 0, 0, |V|, 1)$ with $|V| \geqslant \frac{3\pi}{4}$, the following estimates hold:

$$(7.15\mathrm{a}) \qquad \left| D_2 f_1 + \frac{1}{\pi} \right| \leqslant C\big(\varepsilon_K + f_1(0, 0, 0, |V|, 1)\big),$$

$$(7.15\mathrm{b}) \qquad \left| \partial_{11} f_1 - 1 \right| \leqslant C\big(\varepsilon_K + f_1(0, 0, 0, |V|, 1)\big),$$

$$(7.15\mathrm{c}) \qquad \left| D_{11} f_1 + \frac{1}{\pi^2} \right| \leqslant C\big(\varepsilon_K + f_1(0, 0, 0, |V|, 1)\big),$$

$$(7.15\mathrm{d}) \qquad \left| D_{22} f_1 - \frac{2}{\pi^2} \right| \leqslant C\big(\varepsilon_K + f_1(0, 0, 0, |V|, 1)\big).$$

This may be the right place to mention that the curvature of the piece of curve in $T_m M$ given by the local equation $f_1(0, 0, .., ., 1) = f_1(0, 0, 0, |V|, 1)$ is equal, at the tangent vector $v = (0, |V|)$, to $k = -\dfrac{D_{11} f_1}{D_2 f_1}(0, 0, 0, |V|, 1)$. We thus see from (7.15a) to (7.15c) that, the smaller ε_K and $f_1(0, 0, 0, |V|, 1)$ are, the closer to $-\dfrac{1}{\pi}$ is the curvature of that piece of curve, that is, the more that piece of curve looks in the tangent plane $T_m M$ like a piece of the circle of radius π centered at the origin. For details on this matter, see [19, Corollary 2]. This geometrical fact will appear essential for Estimate 1.

Proof for Proposition 7.1. The proof of the proposition is a lengthy step by step procedure similar to the one used for proving the first part of Lemma 1 in [19]. For completeness, we indicate it to the reader as a sequence of ten guided exercises.

Exercise 1. Find the expression $X(x, v, t)$ of the geodesic motion, when $x = 0$ and $v = (0, |V|)$. Using the latter and setting $\mathcal{K}(x, v, t) = K(X(x, v, t))$, calculate $\dot{\mathcal{K}}$ and $\ddot{\mathcal{K}}$ (the dot standing for $\frac{\partial}{\partial t}$) when $x = 0$ and $v = (0, |V|)$. Using (7.14), calculate the first partial derivatives of $\mathcal{K}$ with respect to the x and v variables at $x = 0, v = (0, |V|)$.

Exercise 2. Viewing the differential equation satisfied by the $\bar{f_i}$'s as a perturbation of the actual scalar Jacobi equation and using (7.6), (7.10), show the existence of a universal constant C (in the sequel of the proof, we will freely denote so any constant of the sort) such that, for $i = 0, 1$, the norms $\|f_i - \bar{f_i}\|$ and $\|\dot{f_i} - \dot{\bar{f_i}}\|$ are bounded above by $C\varepsilon_K$ (this was already done above for $\|f_1 - \bar{f_1}\|$ to derive (7.11)).

Exercise 3. The first partial derivatives of the function $f_1(x, v, t)$ with respect to the x and v variables, evaluated at $x = 0, v = (0, |V|)$, satisfy inhomogeneous scalar Jacobi equations and have zero Cauchy data at $t = 0$. Using (7.6), (7.10), show that, if we ignore the case of $D_2 f_1$, the $\|.\|$ norm of each of the remaining partial derivatives of f_1 is bounded above by $C\varepsilon_K$.

Exercise 4. Show that $\|D_2 f_1 - D_2 \bar{f_1}\|$, evaluated at $x = 0, v = (0, |V|)$, is bounded above by $C\varepsilon_K$ and, using (7.12), infer the estimate (7.15a).

Exercise 5. Using (7.14), show that, if $i = j = 1$, the second partial derivatives: $\partial_{2j} X^i, \partial_2 D_j X^i, D_2 \partial_j X^i, D_{2j} X^i$, evaluated at $(0, 0, 0, |V|, t)$, are, respectively, equal to: $\partial_2 f_0, \partial_2 f_1, D_2 f_0, D_2 f_1$, evaluated at the same point while, if $(i, j) \neq (1, 1)$, they vanish. For each of the nonzero ones, using (7.13), write down the, possibly inhomogeneous, scalar Jacobi equation that it solves. Combining Exercises 1 and 2 with (7.6), (7.10), prove that each of the preceding nonzero second partial derivative of X^i has a $\|.\|$ norm at the point $x = 0, v = (0, |V|)$ bounded above by $C(1 + \varepsilon_K)$.

Exercise 6. Regarding the six remaining second partial derivatives of X^i, evaluated at $(0,0,0,|V|,t)$, the reader will admit that they satisfy the following inhomogeneous scalar Jacobi equations (obtained by differentiating twice the geodesic equation, see Equation (49) of [18] read with $n=2$):

$$\frac{\partial^2}{\partial t^2}\big(\partial_{11}X^1\big) + |V|^2\mathcal{K}\,\partial_{11}X^1 = -|V|^2\big(\partial_1 K \circ X\big)f_0^2\,,$$

$$\frac{\partial^2}{\partial t^2}\big(\partial_{11}X^2\big) = |V|^2\big(\partial_2 K \circ X\big)f_0^2 + 4|V|\mathcal{K}f_0\dot{f}_0\,,$$

$$\frac{\partial^2}{\partial t^2}\big(\partial_1 D_1 X^1\big) + |V|^2\mathcal{K}\,\partial_1 D_1 X^1 = -|V|\big(\partial_1 K \circ X\big)f_0 f_1\,,$$

$$\frac{\partial^2}{\partial t^2}\big(\partial_1 D_1 X^2\big) = |V|^2\big(\partial_2 K \circ X\big)f_0 f_1 + 2|V|\mathcal{K}\big(f_0\dot{f}_1 + \dot{f}_0 f_1\big)\,,$$

$$\frac{\partial^2}{\partial t^2}\big(D_{11}X^1\big) + |V|^2\mathcal{K}\,D_{11}X^1 = -|V|^2\big(\partial_1 K \circ X\big)f_1^2\,,$$

$$\frac{\partial^2}{\partial t^2}\big(D_{11}X^2\big) = |V|^2\big(\partial_2 K \circ X\big)f_1^2 + 4|V|\mathcal{K}f_1\dot{f}_1\,.$$

In the same way, we did at the end of the preceding exercise, show that each of these six second partial derivative of X has a $\|.\|$ norm at the point $x=0, v=(0,|V|)$ bounded above by $C(1+\varepsilon_K)$.

Exercise 7. As a sequel to Exercise 1, calculate the second partial derivatives of the function $\mathcal{K}$ with respect to the x and v variables at $(0,0,0,|V|,t)$. Infer from Exercises 5 and 6 that each of these second partial derivative of $\mathcal{K}$ has a $\|.\|$ norm at the point $x=0, v=(0,|V|)$ bounded above by $C\varepsilon_K$.

Exercise 8. The second partial derivatives of the function $f_1(x,v,t)$ with respect to the variables x,v, at $x=0, v=(0,|V|)$, satisfy inhomogeneous scalar Jacobi equations and have zero Cauchy data at $t=0$. Write down these Jacobi equations. Combining Exercise 7 with (7.6), (7.10) and ignoring $\partial_{11}f_1, D_{11}f_1, D_{22}f_1$, show that the $\|.\|$ norm of each of the remaining second partial derivatives is bounded above above by $C\varepsilon_K$.

Exercise 9. Write down the inhomogeneous ODE's of Jacobi type satisfied by $\partial_{11}\bar{f}_1, D_{11}\bar{f}_1, D_{22}\bar{f}_1$, at $x=0, v=(0,|V|)$. Calculate the expressions of these second derivatives.

Exercise 10. Treating each of the ODE's found in the preceding exercise as a perturbation of the scalar Jacobi equation satisfied by the corresponding second partial derivative of f_1, prove with the help of (7.6), (7.10) that the $\|.\|$ norms of $\partial_{11}(f_1 - \bar{f}_1), D_{11}(f_1 - \bar{f}_1), D_{22}(f_1 - \bar{f}_1)$, at $x=0, v=(0,|V|)$,

are bounded above by $C\varepsilon_K$. Combining (7.12) with the last part of Exercise 9, infer the estimates (7.15b), (7.15c) and (7.15d).

We will also require results of the same kind bearing on the function f_0. We record them here without proof:

Proposition 7.2. *Under the assumption made on the point (m, V) at the beginning of Proposition 7.1, there exists a universal constant $C > 0$ such that, for each $t \in [0, 1]$, the first partial derivatives of the function f_0 with respect to the Fermi variables x^1, x^2, v^1 (excepting v^2), evaluated at the point $(x, v, t) = (0, 0, 0, |V|, t)$, have their absolute value bounded above by $C\varepsilon_K$. Moreover, if $|V| \geqslant \frac{3\pi}{4}$, the following estimates hold:*

$$(7.16a) \qquad \left| f_0(0, 0, 0, |V|, 1) + 1 \right| \leqslant C\big(\varepsilon_K + f_1(0, 0, 0, |V|, 1)\big),$$

$$(7.16b) \qquad \left| D_2 f_0(0, 0, 0, |V|, 1) \right| \leqslant C\big(\varepsilon_K + f_1(0, 0, 0, |V|, 1)\big).$$

7.2.2. Analytic setup at the minimum point. The start of Estimate 1 is standard. We fix $t \in \mathcal{T}$ and let $m_0 \in M$ be a point where the function $m \mapsto \mathrm{Jac}(\exp_m)(\mathrm{grad}\, u_t(m))$ assumes its minimum. We seek a positive lower bound under control on that minimum. We will derive it by writing at the point m_0 the critical equality and the minimum inequality satisfied by the preceding function and by combining them with the continuity Equation (4.2) satisfied by the potential function u_t. In this process, we will have the freedom to shrink the curvature pinching parameter ε_K as long as its size stays independent of the given density functions ρ_0, ρ_t.

By (7.7), we know that the test function which we are minimizing on the manifold is nothing but the function $m \mapsto f_1(m, \mathrm{grad}\, u_t(m), 1)$. Henceforth, we drop freely the subscript t and set for short:

$$\phi_1(m) = f_1(m, \mathrm{grad}\, u(m), 1), \delta_0 = \phi_1(m_0), V_0 = \mathrm{grad}\, u(m_0), r_0 = |V_0|.$$

Here, we will be able to shrink the parameter δ_0 as much as needed, provided it remains under control (in the sense of Remark 1.1, still). We assume that the length r_0 of the vector V_0 is greater than $\frac{3\pi}{4}$. We may do so because (7.11), read at $t = 1$ with $|V| \leqslant \frac{3\pi}{4}$, yields the lower bound $f_1(m, V, 1) \geqslant \frac{2\sqrt{2}}{3\pi} - \pi^2 \varepsilon_K$ which can be made positive by taking ε_K small enough. Besides, it is clear from Lemma 3.5 that the geodesic path $t \in [0, 1] \to \gamma_0(t) = \exp_{m_0}(tV_0) \in M$ contains no cut point. In particular, this path is one to one and the point (m_0, tV_0) remains in the neighborhood $\mathcal{N}_{\mathrm{conj}}$ of the zero section of TM. Along the geodesic γ_0, we may thus take a Fermi chart x as the

one constructed above and use the estimates of Propositions 7.1 and 7.2, including (7.15) and (7.16).

We first calculate the components of the gradient of the test function ϕ_1 in the Fermi chart x and get:

$$\operatorname{grad}\phi_1 = \left(\partial_j f_1 + D_k f_1\left(\partial_j g^{k\ell}\partial_\ell u + g^{k\ell}\partial_{j\ell}u\right)\right)g^{ij}\partial_i,$$

where the $g^{k\ell}(x)$ stand for the components of the matrix inverse of that of the Riemannian metric g, that is, where $g^{k\ell}g_{\ell i} = \delta_i^k$. Letting $x = 0$, we find that the critical condition $\operatorname{grad}\phi_1(m_0) = 0$ reads:

$$(7.17) \qquad \partial_j f_1 + \sum_{k=1}^{2} D_k f_1\,\partial_{jk}u(0) = 0, \ \ j = 1, 2,$$

with the derivatives of f_1 evaluated at the point $(x, v, t) = (0, 0, 0, r_0, 1)$. The estimate (7.15a) prompts us to recast these equations as follows:

$$(7.18a) \qquad \partial_{12}u(0) = a_{12}\,\partial_{11}u(0) + b_{12},$$
$$(7.18b) \qquad \partial_{22}u(0) = a_{22}\,\partial_{11}u(0) + b_{22},$$

with:

$$a_{12} = -\frac{D_1 f_1}{D_2 f_1}, \quad b_{12} = -\frac{\partial_1 f_1}{D_2 f_1}, \quad a_{22} = \left(\frac{D_1 f_1}{D_2 f_1}\right)^2,$$
$$b_{22} = -\frac{\partial_2 f_1}{D_2 f_1} + \frac{(D_1 f_1)(\partial_1 f_1)}{(D_2 f_1)^2}.$$

By Proposition 7.1, we infer from (7.18a) to (7.18b) the existence of a universal constant C such that:

$$(7.18c) \quad \left|\partial_{12}u(0)\right| \leqslant C\varepsilon_K\left(1 + \partial_{11}u(0)\right), \quad \left|\partial_{22}u(0)\right| \leqslant C\varepsilon_K^2\left(1 + \partial_{11}u(0)\right).$$

Differentiating once more the test function ϕ_1 in the Fermi chart x, we find that the minimum condition $0 \leqslant dF[u](\phi_1)(m_0)$ reads as follows:

$$(7.19) \qquad 0 \leqslant I + II + III + IV + V,$$

with:

$$I = \sum_{i,j=1}^{2} \mathcal{F}_j^i(0)\partial_{ij} f_1,$$

$$II = r_0 \sum_{i,j,k=1}^{2} \mathcal{F}_j^i(0)(D_k f_1)\partial_{ij} g^{k2}(0),$$

$$III = 2 \sum_{i,j,k=1}^{2} \mathcal{F}_j^i(0)(D_k \partial_j f_1)\,\partial_{ik} u(0),$$

$$IV = \sum_{i,j,k,\ell=1}^{2} \mathcal{F}_j^i(0)(D_{k\ell} f_1)\partial_{ik} u(0)\partial_{j\ell} u(0),$$

$$V = \sum_{i,j,k=1}^{2} \mathcal{F}_j^i(0)(D_k f_1)\partial_{ijk} u(0),$$

where the derivatives of f_1 are still evaluated at $(x, v, t) = (0, 0, 0, r_0, 1)$.

Combining (4.1) with Remark 4.1 and (7.8), we know that the potential function u satisfies at $x = 0$ the equation:

$$(7.20a) \qquad\qquad \delta_0 \det\big(\mathcal{H}_i^j(0)\big) = \frac{\rho_0(0,0)}{\rho_t(0, r_0)},$$

with the matrix $\mathcal{H}_i^j(0)$ symmetric positive definite, given by

$$(7.20b) \qquad \mathcal{H}_i^j(0) = \begin{pmatrix} \frac{1}{\delta_0} f_0(0,0,0,r_0,1) + \partial_{11} u(0) & \partial_{12} u(0) \\ \partial_{12} u(0) & 1 + \partial_{22} u(0) \end{pmatrix}.$$

The positive definiteness of that matrix combined with (7.16a) implies that the minimum δ_0 under study satisfies the inequality:

$$(7.21) \qquad\qquad \delta_0 \geqslant \frac{1 - C\varepsilon_K}{C + \partial_{11} u(0)}.$$

The latter shows that we may suppose $\partial_{11} u(0)$ larger than any suitable constant under control: for the moment, we take it larger than 1. Accordingly, we will freely replace $1 + \partial_{11} u(0)$ merely by $\partial_{11} u(0)$ in the estimates (7.18c). Here, we should warn the reader that plugging the expressions (7.18) of $\partial_{12} u(0)$ and $\partial_{22} u(0)$ into (7.20a) yields for $\partial_{11} u(0)$ an *affine* equation which, by itself, provides no controlled upper bound on $\partial_{11} u(0)$.

We will require below the expressions at $x = 0$ of two further matrices, namely:

$$(7.22) \qquad \mathcal{J}_j^i(0,0,0,r_0) = \begin{pmatrix} \frac{1}{\delta_0} f_0(0,0,0,r_0,1) & 0 \\ 0 & 1 \end{pmatrix},$$

which is known from (7.8), and:

$$(7.23) \quad \mathcal{F}_j^i(0) = \frac{\rho_t(0,r_0)}{\rho_0(0,0)} \begin{pmatrix} \delta_0\big(1 + \partial_{22}u(0)\big) & -\delta_0\,\partial_{12}u(0) \\ -\delta_0\,\partial_{12}u(0) & f_0(0,0,0,r_0,1) + \delta_0\,\partial_{11}u(0) \end{pmatrix},$$

obtained from (7.20) by inversion.

7.2.3. Second-order terms calculations.

In this section, we will calculate successively each of the terms I to IV which appear in the minimum condition (7.19), making use of Equations (7.17), (7.20a), the estimates (7.18c) and the expressions (7.20b), (7.22), and (7.23). For short, we will not specify repeatedly that the derivatives of the functions f_1, f_0 are evaluated at the point $(x, v, t) = (0, 0, 0, r_0, 1)$; it will be implicitly understood throughout the calculations.

The term I We calculate the term I using (7.23) and find:

$$I = \frac{\rho_t(0,r_0)}{\rho_0(0,0)} \Big(\delta_0\big(1 + \partial_{22}u(0)\big)\,\partial_{11}f_1$$
$$- 2\delta_0\,\partial_{12}u(0)\,\partial_{12}f_1 - \big(f_0 + \delta_0\,\partial_{11}u(0)\big)\,\partial_{22}f_1 \Big).$$

Recalling Proposition 7.1 and the estimates (7.16a), (7.18c), we infer the existence of a universal constant C such that the following upper bound holds:

$$(7.24) \qquad I \leqslant C\,\frac{\rho_t(0,r_0)}{\rho_0(0,0)} \Big(\varepsilon_K + \delta_0\,\partial_{11}u(0) + \delta_0\big(1 + \partial_{22}u(0)\big) \Big).$$

The term II Direct calculation yields for the term II the expression: $II = -r_0 D_2 f_1\,\mathcal{F}_1^1(0)\partial_{11}\mathfrak{G}(0)$. Noting that $\partial_{11}\mathfrak{G}(0) = -2K(0)$, we infer that $\partial_{11}\mathfrak{G}(0)$ is negative. Furthermore, by (7.15a), taking $\varepsilon_K + \delta_0$ smaller than a universal positive constant, we may consider that $D_2 f_1(0,0,0,r_0,1)$ is negative. The whole term II is thus negative and, as such, *negligible* in the minimum condition (7.19).

The term III Using (6.5a), the term III becomes:

$$III = 2 \sum_{k=1}^{2} D_k \partial_k f_1 - 2 \sum_{i,j,k=1}^{2} \mathcal{F}_j^i(0) \mathcal{J}_i^k(0,0,0,r_0) D_k \partial_j f_1,$$

hence the combination of (7.23) with (7.22) yields:

$$III = 2 \sum_{k=1}^{2} D_k \partial_k f_1 - 2 \frac{\rho_t(0,r_0)}{\rho_0(0,0)} \Big\{ f_0 \big(1 + \partial_{22} u(0)\big) D_1 \partial_1 f_1$$
$$- f_0\, \partial_{12} u(0) D_1 \partial_2 f_1 - \delta_0 \partial_{12} u(0) D_2 \partial_1 f_1 + \big(f_0 + \delta_0 \partial_{11} u(0)\big) D_2 \partial_2 f_1 \Big\}.$$

From Proposition 7.1 and the estimates (7.16a), (7.18c), we infer the existence of a universal constant C such that the following upper bound holds:

$$(7.25) \quad III \leqslant C + C \frac{\rho_t(0,r_0)}{\rho_0(0,0)} \Big(\delta_0\, \partial_{11} u(0) + 1 + \partial_{22} u(0) + \varepsilon_K \big| \partial_{12} u(0) \big| \Big).$$

The term IV Using (6.5b), the term IV may be recast as follows:

$$IV = \sum_{k,\ell=1}^{2} \Big(\mathcal{H}_\ell^k(0) - 2\mathcal{J}_\ell^k + \sum_{i,j=1}^{2} \mathcal{F}_j^i(0) \mathcal{J}_i^k \mathcal{J}_\ell^j \Big) D_{k\ell} f_1,$$

where the components of the Jacobi endomorphism field $\mathcal{J}$ are evaluated at the point $(x,v) = (0,0,0,r_0)$. Combining (7.20b) with (7.22) and (7.23), we find this term equal to:

$$\Big(\partial_{11} u(0) + \frac{\rho_t(0,r_0)}{\delta_0 \rho_0(0,0)} f_0^2 \big(1 + \partial_{22} u(0)\big) - \frac{f_0}{\delta_0} \Big) D_{11} f_1$$
$$+ 2\partial_{12} u(0) \Big(1 - \frac{\rho_t(0,r_0)}{\rho_0(0,0)} f_0 \Big) D_{12} f_1$$
$$+ \Big(\delta_0 \frac{\rho_t(0,r_0)}{\rho_0(0,0)} \partial_{11} u(0) + \partial_{22} u(0) + \frac{\rho_t(0,r_0)}{\rho_0(0,0)} f_0 - 2 \Big) D_{22} f_1.$$

From (7.15c), (7.15d), and (7.16a), we may take for granted at the point $(x,v,t) = (0,0,0,r_0,1)$ the inequalities:

$$D_{11} f_1 \leqslant -\frac{1}{2\pi^2}, \quad -\frac{3}{4} \leqslant f_0 \leqslant -\frac{1}{2}, \quad \frac{1}{\pi^2} \leqslant D_{22} f_1 \leqslant \frac{3}{\pi^2},$$

provided $\varepsilon_K + \delta_0$ is assumed smaller than a universal positive constant. The combination of Proposition 7.1 with (7.18) thus implies the existence of a

universal constant C such that the following upper bound holds:

$$(7.26) \quad IV \leqslant -\frac{1}{2\pi^2}\left(\partial_{11}u(0) + \frac{\rho_t(0,r_0)}{4\delta_0\,\rho_0(0,0)}\big(1 + \partial_{22}u(0)\big) + \frac{1}{2\delta_0}\right)$$
$$+ C\varepsilon_K^2\,\partial_{11}u(0) + \frac{\rho_t(0,r_0)}{\rho_0(0,0)}\left(C\varepsilon_K|\partial_{12}u(0)| + \frac{3}{\pi^2}\delta_0\,\partial_{11}u(0)\right).$$

The terms appearing in the first part of the right-hand side, all large negative for $\delta_0 > 0$ small enough, are the ones that will make our proof of Estimate 1 work.

7.2.4. Third-order terms calculations.   In this section, we will calculate the term V of the minimum condition (7.19). To do so, we cannot use directly Equations (7.17), (7.20a), the estimates (7.18c) and the expressions (7.20b), (7.22), and (7.23), as done for the preceding terms, because the term V is a linear combination of *third* derivatives of the potential function u. We will overcome this difficulty by using the continuity Equation (4.1), differentiated once, to derive an expression of the term V involving only *lower order* derivatives of u. We will then complete as above the calculation of that expression.

Differentiating the continuity equation at the minimum point. We first describe a convenient way of writing the continuity equation near the minimum point m_0. Let $x \mapsto (\zeta_1(x), \zeta_2(x))$ be the local direct orthonormal frame field given in the Fermi chart by:

$$\zeta_2 = \frac{\operatorname{grad} u}{|\operatorname{grad} u|} = \frac{1}{|\operatorname{grad} u|}\left(\partial_1 u\,\partial_1 + \frac{1}{\mathfrak{G}}\partial_2 u\,\partial_2\right),$$
$$\text{and thus: } \zeta_1 = \frac{1}{\sqrt{\mathfrak{G}}|\operatorname{grad} u|}\big(\partial_2 u\,\partial_1 - \partial_1 u\,\partial_2\big).$$

Setting $\mathcal{H}(\zeta_i) = H_i^j \zeta_j$, we claim that the continuity Equation (4.1), explicited as in Remark 4.1, reads in the Fermi chart as follows:

$$(7.27) \qquad\qquad \phi_1(x)\,\det\big(H_i^j(x)\big) = \frac{\rho_0(x)}{\rho_t(G_u(x))},$$

with:

$$H_i^j(0) = \begin{pmatrix} \frac{f_0}{f_1} + \nabla du(\zeta_1,\zeta_1) & \nabla du(\zeta_1,\zeta_2) \\ \nabla du(\zeta_2,\zeta_1) & \mathfrak{G}(x) + \nabla du(\zeta_2,\zeta_2) \end{pmatrix},$$

where the functions f_0 and f_1 are evaluated at the point:

$$(x, v, t) = (x, \operatorname{grad} u(x), 1) = \left(x^1, x^2, \partial_1 u(x), \frac{1}{\mathfrak{G}(x)} \partial_2 u(x), 1 \right),$$

and $G_u(x)$ stands for simplicity for the local expression of the gradient-mapping G_u, equal to $X\left(x^1, x^2, \partial_1 u(x), \frac{1}{\mathfrak{G}(x)} \partial_2 u(x), 1 \right)$. Indeed, the claim is straightforward at $x = 0$ and it can be checked elsewhere by using the Fermi chart constructed there similarly to ours at $x = 0$.

Taking the logarithm of both sides of (7.27) and differentiating the resulting equation at $x = 0$ (where the test function ϕ_1 assumes its minimum δ_0), we obtain the 1-forms equality:

$$(7.28) \qquad \mathcal{F}_j^i(0) dH_i^j(0) = \frac{d\rho_0}{\rho_0}(0,0) - \frac{d\rho_t}{\rho_t}(0, r_0) \circ dG_u(0).$$

Let us proceed to expressing this equation in the Fermi chart. Here, and in the sequel of this subsection, it is implicitly understood that the functions f_0, f_1 and their derivatives are evaluated at the point $(x, v, t) = (0, 0, 0, r_0, 1)$. In particular, the function f_1 itself is equal to δ_0. From (7.14), we already know that the matrix of the endomorphism $dG_u(0)$ in the Fermi chart is equal to $\begin{pmatrix} f_0 + \delta_0 \partial_{11} u(0) & \delta_0 \partial_{12} u(0) \\ \partial_{21} u(0) & 1 + \partial_{22} u(0) \end{pmatrix}$. Let us calculate the components of the $dH_i^j(0)$ using the vanishing of $d\mathfrak{G}(0)$ and $d\phi_1(0)$. We get successively:

$$\partial_1 H_1^1(0) = \frac{\partial_1 f_0}{\delta_0} - \frac{2}{r_0} \partial_{11} u(0) \partial_{12} u(0) + \partial_{111} u(0),$$

$$\partial_1 H_1^2(0) = \partial_1 H_2^1(0) = r_0 K(0) + \frac{1}{r_0} \partial_{11} u(0) \big(\partial_{11} u(0) - \partial_{22} u(0) \big) + \partial_{112} u(0),$$

$$\partial_1 H_2^2(0) = \frac{2}{r_0} \partial_{11} u(0) \partial_{12} u(0) + \partial_{122} u(0), \text{ and:}$$

$$\partial_2 H_1^1(0) = \frac{\partial_2 f_0}{\delta_0} - \frac{2}{r_0} \big(\partial_{12} u(0) \big)^2 + \partial_{112} u(0),$$

$$\partial_2 H_1^2(0) = \partial_2 H_2^1(0) = \frac{1}{r_0} \partial_{12} u(0) \big(\partial_{11} u(0) - \partial_{22} u(0) \big) + \partial_{122} u(0),$$

$$\partial_2 H_2^2(0) = \frac{2}{r_0} \big(\partial_{12} u(0) \big)^2 + \partial_{222} u(0).$$

Let us plug in (7.28) these expressions of $dG_u(0)$ and the $dH_i^j(0)$, together with those of the $\mathcal{F}_j^i(0)$ given by (7.23). Solving each resulting equation for

the third-order terms, we find:

(7.29a)
$$\sum_{i,j=1}^{2} \mathcal{F}_j^i(0)\partial_{1ij}u(0) = \frac{\partial_1\rho_0}{\rho_0} - \frac{1}{\rho_t}\Big(\partial_1\rho_t(f_0 + \delta_0\partial_{11}u) + \partial_2\rho_t\,\partial_{12}u\Big)$$

$$+ \frac{\rho_t}{\rho_0}\Big(-\partial_1 f_0 - D_1 f_0\,\partial_{11}u + (2\delta_0 r_0 K(0) - D_2 f_0)\partial_{12}u - \partial_1 f_0\,\partial_{22}u$$

$$+ \frac{2}{r_0}(\delta_0 - f_0)\partial_{11}u\,\partial_{12}u - D_1 f_0\,\partial_{11}u\,\partial_{22}u - D_2 f_0\,\partial_{12}u\,\partial_{22}u\Big),$$

(7.29b)
$$\sum_{i,j=1}^{2} \mathcal{F}_j^i(0)\partial_{2ij}u(0) = \frac{\partial_2\rho_0}{\rho_0} - \frac{1}{\rho_t}\Big(\delta_0\partial_1\rho_t\,\partial_{12}u + \partial_2\rho_t(1 + \partial_{22}u)\Big)$$

$$+ \frac{\rho_t}{\rho_0}\Big(-\partial_2 f_0 - D_1 f_0\,\partial_{12}u - (\partial_2 f_0 + D_2 f_0)\partial_{22}u + \frac{2}{r_0}(\delta_0 - f_0)\big(\partial_{12}u\big)^2$$

$$- D_1 f_0\,\partial_{12}u\,\partial_{22}u - D_2 f_0\big(\partial_{22}u\big)^2\Big),$$

where, of course, the prescribed density functions ρ_0, ρ_t and their first partial derivatives are evaluated, respectively, at $x = 0$ and $x = (0, r_0)$, while the second partial derivatives of the potential function u are evaluated at $x = 0$. *Remarkably, in the process of this calculation, the* cubic *terms in $\partial^2 u$ which the reader would expect to appear in* (7.29) *have cancelled.*
The term V. With (7.29) at hand, we are ready to calculate the term $V = \sum_{k=1}^{2} D_k f_1 \left(\sum_{i,j=1}^{2} \mathcal{F}_j^i(0)\partial_{ijk}u(0)\right)$. *Here, three simplifications occur based on the combination of the critical conditions* (7.17) *with* (7.29). Indeed, on the one hand, we gather the terms in $\partial^2 u$ involving the gradient of the prescribed density function ρ_t, namely:

$$-\frac{1}{\rho_t}\Big(\delta_0\partial_1\rho_t(D_1 f_1\partial_{11}u + D_2 f_1\partial_{12}u) + \partial_2\rho_t(D_1 f_1\partial_{12}u + D_2 f_1\partial_{22}u)\Big)$$

and, using (7.17), we find their sum equal to:

$$\frac{1}{\rho_t}(\delta_0\partial_1\rho_t\partial_1 f_1 + \partial_2\rho_t\partial_2 f_1).$$

On the other hand, we gather the quadratic terms in $\partial^2 u$, namely:

$$\frac{\rho_t}{\rho_0}\left(\frac{2}{r_0}(\delta_0 - f_0)(D_1 f_1 \partial_{11} u + D_2 f_1 \partial_{12} u)\partial_{12} u\right.$$
$$\left. - \big(D_1 f_0(D_1 f_1 \partial_{11} u + D_2 f_1 \partial_{12} u) + D_2 f_0(D_1 f_1 \partial_{12} u + D_2 f_1 \partial_{22} u)\big)\partial_{22} u\right),$$

and find, by (7.17) again, that their sum becomes *linear* in $\partial^2 u$, equal to:

$$\frac{\rho_t}{\rho_0}\left(-\frac{2}{r_0}(\delta_0 - f_0)\partial_1 f_1 \partial_{12} u + \big(D_1 f_0 \partial_1 f_1 + D_2 f_0 \partial_2 f_1\big)\partial_{22} u\right).$$

Finally, we gather the terms in $D_j f_0 D_k f_1 \partial_{jk} u$ and get similarly:

$$-\frac{\rho_t}{\rho_0}\sum_{j,k=1}^{2} D_j f_0 D_k f_1 \partial_{jk} u = \frac{\rho_t}{\rho_0}\sum_{j=1}^{2} D_j f_0 \partial_j f_1.$$

Altogether, we obtain for the term V the following simplified expression:

$$V = D_1 f_1\left(\frac{\partial_1 \rho_0}{\rho_0} - \frac{\partial_1 \rho_t}{\rho_t} f_0\right) + D_2 f_1\left(\frac{\partial_2 \rho_0}{\rho_0} - \frac{\partial_2 \rho_t}{\rho_t}\right)$$
$$+ \frac{\rho_t}{\rho_0}\big(D_1 f_0 \partial_1 f_1 - D_1 f_1 \partial_1 f_0 + D_2 f_0 \partial_2 f_1 - D_2 f_1 \partial_2 f_0\big)(1 + \partial_{22} u)$$
$$+ \frac{\rho_t}{\rho_0}\left(2\delta_0 r_0 K(0) D_1 f_1 - \frac{2}{r_0}(\delta_0 - f_0)\partial_1 f_1\right)\partial_{12} u.$$

Combining Propositions 7.1 and 7.2, we infer the existence of a universal constant C such that the following upper bound holds:

$$(7.30) \qquad V \leqslant C\left(\max_M |d\log \rho_0| + \max_M |d\log \rho_t|\right)$$
$$+ C\frac{\rho_t(0, r_0)}{\rho_0}\left(\varepsilon_K\big(1 + \partial_{22} u(0)\big) + \varepsilon_K |\partial_{12} u(0)|\right).$$

7.2.5. Minimum condition: final discussion. Let us sum up the main reductions operated so far at the minimum point m_0 of the function ϕ_1. We know the existence of two universal positive constants, C (large) and ι (small), such that, if the geometric parameter $\varepsilon_K = |K - 1|_{C^2(M)}$ is smaller than ι, either the positive minimum under study $\delta_0 = \min_M \phi_1$ is bounded below by ι, in which case we are done, or if not, the length r_0 of the vector $V_0 = \operatorname{grad} u(m_0)$ must be larger than $\frac{3\pi}{4}$ (due to (7.11) read at $t = 1$). Moreover, in the Fermi chart constructed along the geodesic segment $\gamma_0(t) =$

$\exp_{m_0}(tV_0), t \in [0, 1]$, the second partial derivative $\partial_{11}u(0)$ must be larger than 1 (due to (7.21)) and the following inequality[6] holds:

$$(7.31) \qquad 0 \leqslant -\frac{1}{2\pi^2}\left(\partial_{11}u(0) + \frac{\rho_t(0, r_0)}{4\delta_0\,\rho_0(0,0)}\big(1 + \partial_{22}u(0)\big) + \frac{1}{2\delta_0}\right)$$
$$+ C\varepsilon_K^2\,\partial_{11}u(0) + C\big(\max_M |d\log\rho_0| + \max_M |d\log\rho_t|\big)$$
$$+ C\,\frac{\rho_t(0, r_0)}{\rho_0(0,0)}\Big(\delta_0\,\partial_{11}u(0) + 1 + \partial_{22}u(0) + \varepsilon_K\big|\partial_{12}u(0)\big|\Big).$$

This inequality is obtained from the minimum condition (7.19) after plugging in the upper bounds derived in Sections 7.2.3 and 7.2.4, namely: $II \leqslant 0$ and (7.24), (7.25), (7.26), and (7.30). To discuss (7.31) complying with the statement of Theorem 1.4, we should beware that the size of the parameter ε_K must stay independent of the given densities ρ_0, ρ_1. We thus fix the maximum value of ε_K in (7.31) once for all equal to $\min\left(\iota, \frac{1}{\sqrt{6\pi^2 C}}\right)$. This choice, combined with (7.31), yields the inequality:

$$(7.32)$$
$$0 \leqslant \partial_{11}u(0)\left(\delta_0 C\frac{\max_M \rho_t}{\min_M \rho_0} - \frac{1}{6\pi^2}\right) + \frac{\rho_t(0, r_0)}{\rho_0(0,0)}\big(1 + \partial_{22}u(0)\big)\left(C - \frac{1}{16\pi^2\delta_0}\right)$$
$$+ \left(C\left(\max_M |d\log\rho_0| + \max_M |d\log\rho_t|\right) - \frac{1}{4\pi^2\delta_0}\right)$$
$$+ \left(\frac{\rho_t(0, r_0)}{\rho_0(0,0)}\left(C|\partial_{12}u(0)| - \frac{1}{16\pi^2\delta_0}\big(1 + \partial_{22}u(0)\big)\right) - \frac{1}{6\pi^2}\partial_{11}u(0)\right),$$

where we have intentionally split the right-hand side into four pieces. Let us focus provisionaly on the *last* one. We require the pinching:

$$(7.33) \qquad\qquad \frac{\pi}{2} \leqslant \frac{1}{|D_2 f_1|} \leqslant 2\pi,$$

which is valid due to (7.15a) and the smallness of the constant ι. Here and below, the derivatives of the function f_1 are still understood evaluated at the point $(x, v, t) = (0, 0, 0, r_0, 1)$. On the one hand, from (7.18a) combined with Proposition 7.1 and (7.33), we may write: $\big|\partial_{12}u(0)\big| \leqslant 2\pi\big|D_1 f_1\big|\,\partial_{11}u(0) + C\varepsilon_K$. On the other hand, from (7.18b) combined with Proposition 7.1 and

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[6]In which we know, from (7.20b), that the factor $(1 + \partial_{22}u(0))$ is positive.

(7.33), the lower bound: $1 + \partial_{22}u(0) \geqslant \frac{\pi^2}{4}|D_1f_1|^2\partial_{11}u(0)$ holds. The last piece under study is thus bounded above by:

$$\partial_{11}u(0)\left[\frac{\max_M \rho_t}{\min_M \rho_0}|D_1f_1|\left(2\pi C - \frac{1}{64\delta_0}|D_1f_1|\right) - \frac{1}{6\pi^2}\right] + C\frac{\max_M \rho_t}{\min_M \rho_0},$$

hence also by:

$$\partial_{11}u(0)\left[\delta_0 64\pi^2 C^2\frac{\max_M \rho_t}{\min_M \rho_0} - \frac{1}{6\pi^2}\right] + C\frac{\max_M \rho_t}{\min_M \rho_0}.$$

Replacing by the latter expression the last piece of the right-hand side of (7.32), we obtain the inequality:

$$\begin{aligned} 0 \leqslant \partial_{11}u(0)&\left[\delta_0\left(C + 64\pi^2 C^2\right)\frac{\max_M \rho_t}{\min_M \rho_0} - \frac{1}{3\pi^2}\right] \\ &+ \frac{\rho_t(0,r_0)}{\rho_0(0,0)}\left(1 + \partial_{22}u(0)\right)\left[C - \frac{1}{16\pi^2\delta_0}\right] \\ &+ \left[C\left(\max_M |d\log\rho_0| + \max_M |d\log\rho_t| + \frac{\max_M \rho_t}{\min_M \rho_0}\right) - \frac{1}{4\pi^2\delta_0}\right], \end{aligned}$$

from which we conclude that δ_0, or else $\min_M \mathrm{Jac}(\exp_m)\left(\mathrm{grad}\, u_t(m)\right)$, must be bounded below by:

$$\min_{t\in[0,1]}\left(\iota, \frac{1}{16\pi^2 C}, \frac{\min_M \rho_0}{3\pi^2 C(1 + 64\pi^2 C)\max_M \rho_t},\right.$$

$$\left.\frac{1}{4\pi^2 C\left(\max_M |d\log\rho_0| + \max_M |d\log\rho_t| + \frac{\max_M \rho_t}{\min_M \rho_0}\right)}\right).$$

This minimum is therefore bounded below by the constant $\delta > 0$ defined by:

$$\delta = \min\left(\iota, \frac{1}{16\pi^2 C}, \frac{e^{-2\mathfrak{M}_1}}{3\pi^2 C(1 + 64\pi^2 C)}, \frac{1}{4\pi^2 C(2\mathfrak{M}_1 + e^{2\mathfrak{M}_1})}\right),$$

with $\mathfrak{M}_1 = \max_{t\in[0,1]}\left|\log\rho_t\right|_{C^1(M)}$. This constant is, indeed, under control in the sense of Remark 1.1. The proof of Estimate 1, in the case of Theorem 1.4, is thus complete. $\square$

Appendix A. Assumptions comparison: curvature closeness versus metric closeness

So far, two different assumptions bearing on the Riemannian metric g of $\mathbb{S}^2$ have been used to study the regularity of the optimal transport potential function: a metric closeness assumption [26, Theorem 1.2] [27, Theorem 5.1] and the curvature closeness assumption used in Theorem 1.4 and in [19, Theorems 1 and 2]. They go, respectively, as follows:

(i) the metric g is C^4 close to a Riemannian metric g_0 of constant Gauss curvature, say, equal to 1;

(ii) the Gauss curvature of g is C^2 close to 1.

The distinction between these two assumptions is tacitly ignored[7] in [26, bottom p. 1695, top p. 1696] [28, p. 872, section 7]. To clarify this topic, we will prove that conditions (i) and (ii) are not equivalent. The former implies the latter, but the converse is not true. Here is an example:

Lemma A.1. *Let g_0 be a metric on $\mathbb{S}^2$ with constant Gauss curvature $K_0 = 1$. There exists a sequence $(g_i)_{i \in \mathbb{N}^*}$ of metrics on $\mathbb{S}^2$, pointwise conformal to g_0, converging to g_0 in $C^{3,\alpha}$ norm, for every $\alpha \in (0,1)$, but not in C^4 norm, and such that the corresponding sequence of Gauss curvature functions $(K_i)_{i \in \mathbb{N}^*}$ converges to 1 in C^2 norm.*

Proof. Working with the metric g_0 and setting Δ_0 for its positive Laplacian, take a function $f : \mathbb{S}^2 \to \mathbb{R}$ of zero average, twice continuously differentiable and not better, and such that the solution $\varphi : \mathbb{S}^2 \to \mathbb{R}$ of the Poisson equation $\Delta_0 \varphi = f$ is not C^4. Set p_1 for the orthogonal projection of L^2 on the eigenspace Λ_1 associated to the first positive eignevalue of Δ_0, and $u \mapsto \mathcal{K}[u] = e^{-2u}(\Delta_0 u + 1)$, for the conformal Gaussian curvature operator. For every $\alpha \in (0,1)$, the map:

$$u \in C^{3,\alpha}\big(\mathbb{S}^2\big) \to F[u] = \mathcal{K}[u] + p_1(u) \in C^{1,\alpha}\big(\mathbb{S}^2\big)$$

is locally invertible near $u = 0$ [16, p. 35] hence, for each small enough real t, there exists a unique function u_t close to 0 in $C^{3,\alpha}\big(\mathbb{S}^2\big)$, solving the equation $F[u_t] = 1 + tf$. The solution u_t vanishes for $t = 0$ and is differentiable with respect to t. We infer that its derivative $\dot{u}_0 = \frac{du_t}{dt}\big|_{t=0}$ satisfies

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[7]Worse, the assumption quoted in [26, pp. 1695–1696] is not the condition (ii) but a mere positive *curvature pinching*, namely the one required in [18, Theorem 1].

the equation: $\Delta_0 \dot{u}_0 - 2\dot{u}_0 + p_1(\dot{u}_0) = f$, or else: $\Delta_0(\varphi - \dot{u}_0) = p_1(\dot{u}_0) - 2\dot{u}_0$, where the right-hand side lies in $C^{3,\alpha}(\mathbb{S}^2)$. The Schauder estimates yield $(\varphi - \dot{u}_0) \in C^{5,\alpha}(\mathbb{S}^2)$.

Claim: We can find a sequence $(t_i)_{i\in\mathbb{N}^*}$, with $t_i \in \left(0, \frac{1}{i}\right)$, such that, for each $i \in \mathbb{N}^*$, the function u_{t_i} does not belong to $C^4(\mathbb{S}^2)$.

Indeed, by contradiction, if not, there exists $i \in \mathbb{N}^*$ such that all the functions u_t belong to $C^4(\mathbb{S}^2)$ provided t lies in $\left(0, \frac{1}{i}\right)$. If so, the limit of the ratio $\dfrac{u_t}{t}$ as $t \to 0$, equal to $\dot{u}_0$, must belong to $C^4(\mathbb{S}^2)$ as well. Recalling the above Schauder regularity result, we conclude that $\varphi \in C^4(\mathbb{S}^2)$, getting a contradiction. So the claim is verified.

By construction, the sequence of $C^{3,\alpha}(\mathbb{S}^2)$ conformal metrics defined by $g_i = e^{2u_{t_i}} g_0$, for $i \in \mathbb{N}^*$, satisfies $\left|K_i - 1\right|_{C^2} \equiv \left|t_i f - p_1(u_{t_i})\right|_{C^2}$ hence $\lim_{i\to\infty}\left|K_i - 1\right|_{C^2} = 0$, while $\left|g_i - g_0\right|_{C^4}$ is infinite, which proves the lemma. $\square$

Remark A.1. In the condition (ii) above, we did not mention the normalization $\min_{\mathbb{S}^2} K = 1$ used in Theorem 1.4 because we did not care quantifying the closeness condition in a scale invariant fashion. In any case, given a metric g with Gauss curvature K sufficiently close to 1 in C^2 norm, one may consider the rescaled metric $g' = g \min_{\mathbb{S}^2} K$. The Gauss curvature K' of g' satisfies the normalization $\min_{\mathbb{S}^2} K' = 1$ and one can readily verify that it is still C^2 close to 1. The reader may wish to apply this remark, for i large, to the metric g_i just constructed. The resulting sequence of metrics will satisfy the conclusion of Lemma A.1 as well.

On the 2-sphere endowed with a Riemannian metric like[8] g_i (for i large), how can we tackle the smoothness issue in optimal transport ? The continuity results [26, Theorem 1.2] and [28, Theorem 1.3] combined with [27, Theorem 5.1] cannot be applied because the metric is not C^4. We cannot apply Theorem 1.4, as stated, either. This difficulty motivates the following *ad hoc* extension of the results of the present paper:

Proposition A.1. *If we allow the positive probability measures (μ_0, μ_1) to be only C^2, the conclusions of Theorems 1.3 and 5.1 remain valid with $k = 1$ and $\alpha \in (0, 1)$ arbitrary. Moreover, if we let the Riemannian metric g be only C^3, but require that its curvature tensor be C^2, keeping unchanged the other assumptions possibly made on the manifold (like having Gauss curvature C^2 close to 1, or like being an A3 manifold), the conclusions of Theorems 1.4, 1.5 and 6.1 remain valid with $k = 1$ and $\alpha \in (0, 1)$ arbitrary.*

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[8]Rescaled as just explained.

Proof. Let M be a closed manifold endowed with a C^3 Riemannian metric g, the curvature tensor of which is C^2. By a standard result in ordinary differential equations theory about regularity with respect to initial conditions [8, 33], applied to the geodesic equation, the exponential map must be C^2. Actually, it is even C^3 because we have assumed that the curvature is C^2. Indeed, a standard result in ordinary differential equations theory about regularity with respect to a parameter [8, 33], applied to the Jacobi equation (see Section 2.3.1), implies that the differential of the exponential map is C^2. In dimension 2, it further implies that the functions f_0 and f_1 defined by (7.4) are C^2. Therefore one can still carry through the proof of [19, Theorem 2]. Moreover, in any dimension, the positive function $(m, V) \mapsto \mathrm{Jac}(\exp_m)(V) \in \mathbb{R}$ and the Jacobi endomorphism field (see Section 2.1.3) are C^2 on the domain $\mathcal{N}_{\mathrm{conj}} \subset TM$. In particular, it makes sense to say that the manifold M is A3 (see Definition 6.1). Finally, given a couple (ρ_0, ρ_1) of C^2 positive probability densities on the Riemannian manifold (M, g), the real function $(m, V) \mapsto \psi_t(m, V)$ introduced in the log form of the continuity Equation (4.2) is also C^2 on the domain $\mathcal{N}_{\mathrm{conj}}$. For every fixed Hölder exponent $\alpha \in (0, 1)$, we may thus consider the operator Φ defined in Section 4.1 as a mapping from $[0, 1] \times \mathcal{A}^{3,\alpha}$ to $P_1^{1,\alpha}$. In this setting, all the arguments used in Section 4 remain valid. Besides, keeping in mind the preceding smoothness observations, one can verify that the proofs given in Section 7 remain fully applicable. As regards Section 6, although, the test function $\mathfrak{T}_t$ is not C^2 any more, but only $C^{1,\alpha}$. So we must justify the maximum inequality (6.1b) on which the proof of the Ma–Trudinger–Wang estimate given in Section 6.2 is based. A similar difficulty arises at the start of the estimate of [15, pp. 428–429] quoted in the proof of Theorem 5.1. Here, we will explain how it can be overcome in the case of Section 6.2. One can proceed along the same lines to fix the proof of [15, pp. 428–429], hence that of Theorem 5.1. The argument goes as follows.

For convenience, we set $\tilde{\psi}_t$ for the pullback on M by the vector field $\mathrm{grad}\, u_t$ of the function ψ_t. The function $\tilde{\psi}_t$ is C^2. Dropping the subscript t, we claim that the function $\mathfrak{T}$ satisfies *weakly* on the manifold the equation $dF[u](\mathfrak{T}) = Q[u] - \Delta\tilde{\psi}$, where the auxiliary differential operator $Q[u]$ is defined by:

$$Q[u] := \mathcal{F}_j^i D_\ell \mathcal{J}_k^k \nabla_i^j \nabla^\ell u + \mathcal{F}_p^i \mathcal{F}_j^q \nabla_k \mathcal{H}_q^p \nabla^k \mathcal{H}_i^j - \mathcal{F}_j^i D_\ell \mathcal{J}_i^j \nabla_k^k \nabla^\ell u + \text{l.o.t.},$$

l.o.t. standing here for the lower order terms given by (6.4) minus those given by (6.7). Let us take the claim for granted and defer its proof. Importantly for us, the function $Q[u] - \Delta\tilde{\psi}$ is *continuous*. We want to prove that it is

nonpositive at the point m_0 where the test function $\mathfrak{T}$ assumes its maximum. Let us argue by contradiction and suppose that $(Q[u] - \Delta\tilde{\psi})(m_0) > 0$. If so, by continuity, the inequality $Q[u] - \Delta\tilde{\psi} \geqslant 0$ holds in a neighborhood of m_0. By the strong maximum principle for weak subsolutions [30, Chap. 8], the test function $\mathfrak{T}$ must be *constant* on that neighborhood. It follows that $Q[u] - \Delta\tilde{\psi}(m_0) = 0$, hence a contradiction. Therefore the inequality $(Q[u] - \Delta\tilde{\psi})(m_0) \leqslant 0$ holds, indeed. Relying on this inequality, one may now carry through the proof given in Section 6.2.

We are left with proving the above claim. Let $(u_N)_{N \in \mathbb{N}}$ be a sequence of C^4 functions converging in C^3 toward the potential function u. For N large, the function u_N is admissible and it is convenient to set $\mathcal{H}_N = \mathcal{H}u_N$, $\mathcal{T}_N = \text{trace}(\mathcal{H}_N)$, $\mathcal{F}_N = \mathcal{F}_{[u_N]}$, $G_N = G_{u_N}$, $\rho_N \, dV_g = (G_N)_{\#}\mu_0$, $Q_N = Q[u_N]$ and $\tilde{\psi}_N = \log \rho_0 - \log\big((\rho_N \circ G_N) \, \text{Jac}(\exp)(\text{grad } u_N)\big)$. For later use, we note that, as N goes to ∞, the function Q_N tends to $Q[u]$ in C^0 while the tensor $\mathcal{F}_N$ and the trace $\mathcal{T}_N$ tend, respectively, to the tensor $\mathcal{F}_{[u]}$ and to the test function $\mathcal{T} = \text{trace}(\mathcal{H}u)$ in C^1. Furthermore, the probability density ρ_N and the function $\tilde{\psi}_N$ tend, respectively, to ρ_t and to $\tilde{\psi} = \tilde{\psi}_t$ in C^1. Fixing an arbitrary C^1 function v, let us consider the integral $I_v[u_N]$ defined by:

$$I_v[u_N] := - \int_M \left(\left(\mathcal{F}_N\right)^i_j \nabla_i v + v \nabla_i \left(\mathcal{F}_N\right)^i_j\right) \nabla^j \mathcal{T}_N \, dV_g.$$

This integral is, by construction, equal to $\int_M v \, dF[u_N](\mathcal{T}_N) \, dV_g$ since the function u_N is C^4. The calculations of Section 6.2, performed with u_N instead of u, imply that $dF[u_N](\mathcal{T}_N) = Q_N - \Delta\tilde{\psi}_N$. We have singled out the function $\tilde{\psi}$ throughout this argument because there is no C^0 convergence available for the sequence $(\Delta\tilde{\psi}_N)_{N \in \mathbb{N}}$ whereas there is one for the sequence $(Q_N)_{N \in \mathbb{N}}$. We are thus led to write:

$$I_v[u_N] = \int_M \left(v Q_N - \nabla_k v \, \nabla^k \tilde{\psi}_N\right) dV_g.$$

Letting the integer N go to ∞ and recalling the various C^0 and C^1 limits recorded above, we infer that:

$$I_v[u] = \int_M \left(v Q[u] - \nabla_k v \, \nabla^k \tilde{\psi}\right) dV_g,$$

or else, since the function $\tilde{\psi}$ is C^2 :

$$I_v[u] = \int_M v(Q[u] - \Delta\tilde{\psi}) dV_g.$$

Now, this equality holds for every function $v \in C^1$, so the claim is proved. $\square$

Proposition A.1 tells us that, with a C^3 Riemannian metric having a C^2 curvature, and with C^2 positive probability measures, all the results of the paper hold and yield $C^{3,\alpha}$ potential functions $u(\mu_0, \mu_1)$, for every $\alpha \in (0,1)$. In particular, regarding our preceding discussion, the following extension of Theorem 1.4 is now known to hold:

Theorem A.1. *Let g be a C^3 Riemannian metric on $\mathbb{S}^2$ with Gauss curvature K of class C^2 satisfying the normalization $\min_{\mathbb{S}^2} K = 1$. There exists a universal real $\epsilon > 0$ such that, if $\big| K - 1 \big|_{C^2} \leqslant \epsilon$, for every $\alpha \in (0,1)$ and every couple (μ_0, μ_1) of C^2 positive probability measures on $\mathbb{S}^2$, the potential function of the optimal transportation map $\mathrm{Mc}(\mu_0, \mu_1)$ is of class $C^{3,\alpha}$.*

Acknowledgments

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