

Bridge spectra of iterated torus knots

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We define the bridge spectrum $\mathbf{b}(K)$ of a knot K in S^3 to be $\mathbf{b}(K) = (b_0(K), b_1(K), \dots)$, where $b_g(K)$ is the bridge number of K with respect to a genus g Heegaard surface for S^3 . A well-known construction shows that when $b_{g-1}(K)$ is positive, $b_g(K) \leq b_{g-1}(K) - 1$; hence, we say that $\mathbf{b}(K)$ has a gap at index g if this inequality is strict. An open question of Yo'av Rieck asks whether there are knots in S^3 whose bridge spectra have more than one gap. We determine the bridge spectra of many iterated torus knots, answering Rieck's question by showing that for every n , there are infinitely many iterated torus knots K , such that $\mathbf{b}(K)$ has precisely n gaps. In addition, we prove a structural lemma about the decomposition of a strongly irreducible bridge surface induced by cutting along a collection of essential surfaces.

1. Introduction

Given a knot K in a compact, orientable 3-manifold M , a bridge splitting of (M, K) often characterizes topological and geometric aspects of K in M . Defined by Doll [3] and Morimoto and Sakuma [14], bridge surfaces can be seen as analogs to Heegaard surfaces in 3-manifold theory: a bridge surface cuts the pair (M, K) into two simple topological pieces, reducing the essence of (M, K) to a gluing map.

Every knot has infinitely many bridge surfaces, but we may narrow our search for structure by looking for *irreducible* bridge surfaces, which are not the result of a generic modification to another surface. Although there is a significant body of research concerning the set of irreducible Heegaard splittings of a 3-manifold, there are few examples of knots $K \subset S^3$ with many irreducible bridge surfaces. Several classes of knots are known to have unique irreducible bridge spheres, up to isotopy as unoriented surfaces. Knots whose nonminimal bridge spheres are reducible are called *destabilizable*, a definition due to Ozawa and Takao [17], and classes of knots known to have this

property include

- the unknot [15],
- 2-bridge knots [15],
- torus knots [16],
- iterated torus knots and iterated cables of 2-bridge knots [28], and
- more generally, cables of an mp -small destabilizable knot [28].

In contrast, Ozawa and Takao have recently produced the first example of a knot $K \subset S^3$, such that K has two irreducible bridge spheres with different bridge numbers [17], and Jang has exhibited 3-bridge links with infinitely many distinct 3-bridge spheres [8]. In terms of higher genus bridge surfaces, we are aware of only one result: Scharlemann and Tomova have shown that 2-bridge knots have unique irreducible bridge surfaces up to unoriented isotopy [21].

Following Doll [3], we define the genus g bridge number $b_g(K)$ of a knot $K \subset S^3$

$$b_g(K) = \min\{b : K \text{ admits a } (g, b)\text{-splitting}\},$$

and we introduce the *bridge spectrum* $\mathbf{b}(K)$,

$$\mathbf{b}(K) = (b_0(K), b_1(K), \dots).$$

A construction called *meridional stabilization* (see Section 2) transforms a (g, b) -surface into a $(g + 1, b - 1)$ -surface; hence, the bridge spectrum is bounded above by the sequence $(b_0(K), b_0(K) - 1, \dots, 1, 0)$, with respect to the lexicographical order. In view of this property, we say that $\mathbf{b}(K)$ has a *gap* at index g if $b_g(K) < b_{g-1}(K) - 1$, and it is a simple verification that if $\mathbf{b}(K)$ has such a gap, the corresponding (g, b) -surface is irreducible.

It is well known that the bridge spectra of torus knots have a single gap. In the present work, we set out to answer the following question, first proposed to us by Yo'av Rieck.

Question 1. *Are there examples of knots in S^3 whose bridge spectra have more than one gap?*

A theorem of Tomova completely characterizes the bridge spectra of high distance knots.

Theorem 1.1 ([26]). *Suppose K is a knot in S^3 with a $(0, b)$ -bridge sphere Σ of sufficiently high distance (with respect to b). Then any (g', b') -bridge*

surface Σ' satisfying $b' = b_{g'}(K)$ is the result of meridional stabilizations performed on Σ . Thus,

$$\mathbf{b}(K) = (b, b - 1, b - 2, \dots, 0).$$

It follows that the bridge spectrum of a “generic” knot K is rather uninteresting. We will show that, in contrast to high-distance knots, the bridge spectra of iterated torus knots exhibit different behavior. The main theorem is as follows.

Theorem 1.2. *Suppose that $K_n = ((p_0, q_0), \dots, (p_n, q_n))$ is an iterated torus knot, whose cabling parameters satisfy $|p_i - p_{i-1}q_{i-1}q_i| > 1$ for any i . Then,*

$$b_g(K_n) = \begin{cases} q_n \cdot b_g(K_{n-1}) & \text{if } g < n, \\ \min\{|p_n - p_{n-1}q_{n-1}q_n|, q_n\} & \text{if } g = n, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$\mathbf{b}(K_n) = q_n \cdot \mathbf{b}(K_{n-1}) + \min\{|p_n - p_{n-1}q_{n-1}q_n|, q_n\} \cdot \mathbf{e}_n.$$

It follows that the bridge spectrum of K_n has a gap at every index from 1 to $n + 1$, providing a positive answer to Question 1 above and yielding for any n the first examples of a knot K_n in S^3 having more than n irreducible bridge surfaces. In the course of proving the main theorem, we establish Theorem 4.2, previously unknown for $n > 1$: the tunnel number of such K_n is $n + 1$. This proof uses a result of Schultens (Theorem 4.1) concerning Heegaard splittings of graph manifolds [24].

The main theorem is related to a classical result of Schubert [22], with a modern proof given by Schultens [23].

Theorem 1.3 ([22, 23]). *Let K be a satellite knot with companion J and pattern of index n . Then*

$$b_0(K) \geq n \cdot b_0(J).$$

A (g, b) -bridge surface for a knot K is *minimal* if $b = b_g(K)$. Schultens’ proof of the theorem reveals that after isotopy, a minimal bridge sphere Σ for K can be made to intersect the companion torus in meridian disks, so that each bridge of J contributes at least n bridges to K . The proof of

Theorem 1.2 yields that whenever $g < n$, a minimal genus g bridge surface for K_n can be made to intersect the companion torus corresponding to K_{n-1} in meridian disks. However, there are knots K_n , such that a minimal genus n bridge surface meets the companion torus in an annulus; hence, we cannot, in general, hope for an extension of Theorem 1.3 to surfaces of higher genus.

The proof of Theorem 1.2 requires a lemma regarding strongly irreducible bridge surfaces, which may be of independent interest. Roughly, this lemma says that a strongly irreducible bridge surface cut along properly embedded essential surfaces will decompose into at most one exceptional component which is strongly irreducible, along with some number of incompressible components. In the notation appearing below, $\eta(\cdot)$ is an open regular neighborhood, $N(\cdot)$ is a closed regular neighborhood, $\Sigma_J = \Sigma \setminus \eta(J)$, and $M(J) = M \setminus \eta(J)$. The terms *Q-essential*, *Q-strongly irreducible*, and *almost transverse* are defined in Section 5.

Lemma 1.1. *Let M be a compact, orientable, irreducible 3-manifold, $J \subset M$ a properly embedded 1-manifold, and $Q = \partial N(J) \subset \partial M(J)$. Suppose Σ is a strongly irreducible bridge splitting surface for (M, J) , and let $S \subset M(J)$ be a collection of properly embedded essential surfaces, such that for each component c of ∂S , either $c \subset Q$ or $c \subset \partial M$. Then one of the following must hold:*

1. *After isotopy, Σ_J is transverse to S and each component of $\Sigma_J \setminus \eta(S)$ is Q -essential in $M(J) \setminus \eta(S)$.*
2. *After isotopy, Σ_J is transverse to S , one component of $\Sigma_J \setminus \eta(S)$ is Q -strongly irreducible and all other components are Q -essential in $M(J) \setminus \eta(S)$.*
3. *After isotopy, Σ_J is almost transverse to S , and each component of $\Sigma_J \setminus \eta(S)$ is Q -essential in $M(J) \setminus \eta(S)$.*

Combined with results on strongly irreducible bridge surfaces by Hayashi and Shimokawa [7], the above lemma can easily be seen to provide alternate proofs of the theorems in [16, 28].

Corollary 1.1. *If K is a torus knot or a cable of a nontrivial destabilizable knot, then K is destabilizable.*

The paper is organized as follows: in Section 2, we introduce relevant definitions and background material. Section 3 discusses topological properties

of iterated torus knots, while Section 4 contains a complete analysis of the tunnel number of such knots. Section 5 presents the above lemma about strongly irreducible bridge surfaces, setting up the proof of the main theorem in Section 6. Section 7 includes a worked example with figures, and finally Section 8 poses some open problems that may be of interest.

2. Preliminaries

Throughout, all 3-manifolds and surfaces will be orientable. We will let $\eta(\cdot)$ and $N(\cdot)$ denote open and closed regular neighborhoods, respectively, in an ambient manifold that should be clear from context. Let S be a properly embedded surface in a 3-manifold M . A *compressing disk* D for S is an embedded disk such that $D \cap S = \partial D$ but ∂D does not bound a disk $D' \subset S$. A *∂ -compressing disk* Δ for S is an embedded disk, such that $\partial\Delta$ is the endpoint union of arcs γ_1 and γ_2 such that $\Delta \cap S = \gamma_1$, γ_1 is essential in S , and $\Delta \cap \partial M = \gamma_2$. The surface S is said to be *incompressible* if there does not exist a compressing disk D for S and *∂ -incompressible* if there does not exist a ∂ -compressing disk Δ for S . Further, S is *essential* if S is incompressible, ∂ -incompressible, and not parallel into ∂M .

Suppose now that M a 3-manifold containing a properly embedded 1-manifold J , and denote the *exterior* of J in M by $M(J) = M \setminus \eta(J)$ (if $M = S^3$, we write $E(J) = M(J)$). Let Σ be a properly embedded surface transverse to J , with Σ_J denoting $\Sigma \setminus \eta(J)$. A *compressing disk* D for Σ_J in (M, J) is an embedded disk D such that $D \cap \Sigma_J = \partial D$, $D \cap J = \emptyset$, and ∂D does not bound a disk in Σ_J . A *bridge disk* Δ for Σ_J in (M, J) is an embedded disk, such that $\partial\Delta$ is the endpoint union of two arcs γ_1 and γ_2 , where $\Delta \cap \Sigma = \gamma_1$ and $\Delta \cap J = \gamma_2$. We also think of $\Delta \cap M(J)$ as a ∂ -compressing disk for Σ_J in $M(J)$ with $\Delta \cap \partial M(J) \subset \partial N(J)$. Finally, a *cut disk* E for Σ_J in (M, J) is an embedded disk E , such that $E \cap \Sigma_J = \partial E$, $E \cap J$ is a single point, and ∂E does not bound a disk in Σ_J .

Let V be a compression body and $\alpha \subset V$ a collection of properly embedded arcs. We say that α is *trivial* if every arc is either vertical or isotopic into $\partial_+ V$. A *bridge splitting* of (M, J) with *bridge surface* Σ is the decomposition of (M, J) as $(V, \alpha) \cup_\Sigma (W, \beta)$, where each V and W are compression bodies containing collections α and β of trivial arcs. In the special case that α and β contain only boundary parallel arcs, we say that Σ is a (g, b) -*bridge splitting surface*, where $g = g(\Sigma)$ and $b = |\alpha| = |\beta|$.

Given a bridge splitting $(M, J) = (V, \alpha) \cup_\Sigma (W, \beta)$, there are several generic methods to construct new bridge surfaces for (M, J) . To increase the

genus of the splitting, let $\gamma \subset V$ be a ∂_+ -parallel arc, such that $\gamma \cap \alpha = \emptyset$. Defining $V' = V \setminus \eta(\gamma)$, $W' = W \cup N(\gamma)$, and $\Sigma' = \partial_+ V' = \partial_+ W'$, we have $(M, J) = (V', \alpha) \cup_{\Sigma'} (W', \beta)$ is another splitting of (M, J) . This process is called *elementary stabilization*. Note that the surface Σ' has compressing disks in V' and W' that intersect in a single point. If, in the reverse direction, there are compressing disks D in (V, α) and E in (W, β) for Σ_J , such that $|D \cap E| = 1$, then $\partial N(D \cup E)$ is a 2-sphere which bounds a ball and which intersects Σ in a single curve. Compressing Σ along $\Sigma \cap \partial(N(D \cup E))$ yields a bridge surface Σ'' of lower genus, and we say that Σ is *stabilized*.

We may also increase the number of trivial arcs in the splitting by adding an extra pair of canceling trivial arcs to α and β near some point of $J \cap \Sigma$. The resulting surface Σ' is called an *elementary perturbation* of Σ . Conversely, if there are bridge disks Δ and Δ' in (V, α) and (W, β) such that $\Delta \cap \Delta'$ is a single point contained in J , we may construct an isotopy which cancels two arcs of α and β , creating a new surface Σ'' , and we say that Σ is *perturbed*. If Σ is a (g, b) -splitting surface and Σ_1 is an elementary stabilization of Σ , then Σ_1 is a $(g + 1, b)$ -surface. If Σ_2 is an elementary perturbation of Σ , then Σ_2 is a $(g, b + 1)$ -surface. Given any two bridge surfaces Σ_1 and Σ_2 for (M, J) , there is a third bridge surface Σ^* which can be obtained from either Σ_i by elementary perturbations and stabilizations [6, 27].

If there are bridge disks Δ and Δ' in (V, α) and (W, β) such that $\Delta \cap \Delta'$ is two points contained in J , then a component of J is isotopic into Σ and we say Σ is *cancelable*. Here we set the convention that a $(g, 0)$ -bridge surface for a knot K in M is a Heegaard surface Σ for M such that $K \subset \Sigma$. In some settings, it is required in addition that K be primitive with respect to one of the handlebodies, but we do not make that restriction here. Note that a $(g, 0)$ -surface may be perturbed to a $(g, 1)$ -surface.

A slightly more complicated way to construct a new surface from Σ is the following: fix a ∂_+ -parallel component α_1 of α . Letting $V' = V \setminus \eta(\alpha_1)$ and $W' = W \cup N(\alpha_1)$, we have that $\Sigma' = \partial V' = \partial W'$ is a bridge surface for (M, J) . We call this process *meridional stabilization*. If Σ is a (g, b) -surface, the resulting surface Σ' will be a $(g + 1, b - 1)$ -surface. On the other hand, if there exists a compressing disk D for Σ_J in (V, α) and a cut disk C for Σ_J in (W, β) (or vice versa), such that $|C \cap D| = 1$, we say that Σ is *meridionally stabilized* and we may reverse the above process to construct a new splitting Σ'' of (M, J) .

Recall that the *genus g bridge number* $b_g(K)$ of a knot $K \subset S^3$ is defined as

$$b_g(K) = \min\{b : K \text{ admits a } (g, b)\text{-splitting}\}.$$

Observe that $b_0(K)$ is the classical bridge number of K . Further, using meridional stabilization, we have $b_{g+1}(K) \leq b_g(K) - 1$. It follows that $b_g(K) = 0$ for $g \geq b_0(K)$.

In order to consider all genus g bridge numbers of K simultaneously, we define the *bridge spectrum* $\mathbf{b}(K)$ of K :

$$\mathbf{b}(K) = (b_0(K), b_1(K), b_2(K), \dots).$$

Although $\mathbf{b}(K) \in \mathbb{Z}^\infty$, the above argument implies that $\mathbf{b}(K)$ must have finitely many nonzero entries, and $\mathbf{b}(K)$ is bounded above by the sequence $(b_0(K), b_0(K) - 1, b_0(K) - 2, \dots)$ with respect to the lexicographical order.

A (g, b) -surface for a knot K is said to be *irreducible* if it is not stabilized, perturbed, meridionally stabilized, or cancelable. From the point of view of the bridge spectrum, if $b_g(K) < b_{g-1}(K) - 1$, a genus (g, b) -surface Σ satisfying $b = b_g(K)$ must be irreducible. For this reason, suppose $b_g(K) < b_{g-1}(K) - 1$ and let $n = b_{g-1}(K) - b_g(K)$. In this case, we say that $\mathbf{b}(K)$ has a *gap of order n at index g* .

Several results over (g, b) -splittings can be adapted to statements about bridge spectra. In [11], Minsky *et al.* prove a theorem which we can state as follows.

Theorem 2.1. *For every n and m such that $n \geq 2$ and $m \geq 1$, there exist knots K such that $\mathbf{b}(K)$ has a gap of order at least n at index m .*

The knots produced in Theorem 2.1 have the property that $b_m(K) = 0$. More generally, in [26] Tomova shows that

Theorem 2.2. *For every n, m , and l , where $n \geq 2$ and $m, l \geq 1$, there exist knots K , such that $\mathbf{b}(K)$ has a gap of order at least n at index m and $b_m(K) = l$. Further, there exist knots K whose bridge spectra have no gaps.*

The knots K from Theorem 2.2, whose spectra have gaps at index m , have the additional property that $b_g(K) = l + m - g$ for $g \geq m$, and corresponding minimal surfaces are meridional stabilizations of an (m, l) -surface. We note that the above examples are produced by exhibiting sufficiently complicated bridge and Heegaard surfaces, where complexity is measured by examining distance between disks sets in the curve complex of the surface. These methods, however, are not suited to answer Question 1 regarding knots whose spectra have more than one gap.

In order to classify bridge spectra, we employ generalizations of Heegaard splittings developed by Scharlemann and Thompson [20] and adapted to

bridge splittings by Hayashi and Shimokawa [7]. Let M be a 3-manifold containing a properly embedded 1-manifold J . We will describe the theories of generalized Heegaard and bridge splittings simultaneously by using the bridge splitting terminology and noting all results hold in the case that $J = \emptyset$, in which case bridge surfaces become Heegaard surfaces.

Let Σ be a bridge surface for (M, J) which yields the splitting $(M, J) = (V, \alpha) \cup_{\Sigma} (W, \beta)$. We say that Σ is *weakly reducible* if there exist disjoint compressing or bridge disks $D \subset (V, \alpha)$ and $D' \subset (W, \beta)$ for Σ_J . If Σ is not weakly reducible, perturbed, or cancelable, we say Σ is *strongly irreducible*. By considering bridge disks as embedded in $M(J)$, we can see that perturbed and cancelable surfaces will be weakly reducible; hence, in $M(J)$, Σ is strongly irreducible if and only if it is not weakly reducible.

Now, suppose (M, J) contains a collection $\mathcal{S} = \{\Sigma_0, S_1, \Sigma_1, \dots, S_d, \Sigma_d\}$ of disjoint surfaces transverse to J and such that (M, J) cut along \mathcal{S} is a collection of compression bodies containing trivial arcs $\{(C_0, \tau_0), (C'_0, \tau'_0), \dots, (C_d, \tau_d), (C'_d, \tau'_d)\}$, where

- $(C_i, \tau_i) \cup_{\Sigma_i} (C'_i, \tau'_i)$ is a bridge splitting of some submanifold (M_i, J_i) for each i ,
- $\partial_- C_i = \partial_- C'_{i-1} = S_i$ for $1 \leq i \leq d$,
- $\partial M = \partial_- C_0 \cup \partial_- C'_d$, and
- $J = \bigcup (\tau_i \cup \tau'_i)$.

We call this decomposition a *multiple bridge splitting* and the surfaces Σ_i are called *thick*, whereas the surfaces S_j are called *thin*. The thick surface Σ_i is strongly irreducible if it is strongly irreducible in the manifold $(C_i, \tau_i) \cup_{\Sigma_i} (C'_i, \tau'_i)$. A multiple splitting is called *strongly irreducible* if each thick surface is strongly irreducible and no compression body is trivial (homeomorphic to $\Sigma_i \times I$ with τ_i only vertical arcs). The following crucial theorem comes from Hayashi and Shimokawa [7] (and in the special case that $J = \emptyset$, it is proved by Scharlemann and Thompson [20]). Other proofs of this fact are given in the contexts of α -sloped Heegaard surfaces [2] and in the more general setting of embedded graphs [25].

Theorem 2.3. *Let M be a 3-manifold containing a properly embedded 1-manifold J . If (M, J) has a strongly irreducible multiple bridge splitting, then ∂M_J and every thin surface is incompressible. On the other hand, if ∂M_J is incompressible in $M(J)$ and Σ is a weakly reducible bridge splitting for (M, J) , then (M, J) has a strongly irreducible multiple bridge splitting*

satisfying

$$(1) \quad g(\Sigma) = \sum g(\Sigma_i) - \sum g(S_i).$$

Although we will not go into details here, the multiple splitting given by the above theorem is obtained from the weakly reducible surface Σ via a process known as *untelescoping*. Conversely, if \mathcal{S} yields a multiple bridge splitting, we may construct a bridge surface Σ satisfying (1) via a process called *amalgamation*.

3. Iterated torus knots

We will focus on a class of knots known as iterated torus knots, which make up a subset of a collection of knots called cable knots. Let V be a unknotted torus standardly embedded in S^3 , with $T = \partial V$. For a curve $c \subset T$, $[c]$ may be parameterized as $(p, q) = p[\mu] + q[\lambda]$ in $H_1(T)$, where μ bounds a meridian disk of V and λ is the preferred longitude of V bounding a disk outside of V . Choosing $q \geq 2$, let $\hat{K}_{p,q}$ be a copy of a (p, q) -curve on T pushed into $\text{int}(V)$. Now, suppose that K is a knot in S^3 and let $\varphi(V) \rightarrow S^3$ be a knotted embedding of V , such that if C is a core of V , then $\varphi(C)$ is isotopic to K . In this case, $K_{p,q} = \varphi(\hat{K}_{p,q})$ is called a (p, q) -cable of K . In addition, we stipulate that the cable has the preferred framing, that is, we have $[\varphi(\lambda)] = 0$ in $H_1(E(K))$.

Using the same symbols as above, we follow [4] to define a (p, q) -cable space $C_{p,q}$ by

$$C_{p,q} = V \setminus \eta(\hat{K}_{p,q}).$$

Observe that $E(K_{p,q})$ decomposes as $E(K) \cup_T C_{p,q}$, where we consider the torus T as both ∂V and $\partial E(K)$. The space $C_{p,q}$ is Seifert fibered with base space an annulus and a single exceptional fiber C (for further discussion, see [18]). We distinguish between the outer boundary $\partial_+ C_{p,q} = \partial V$ and the inner boundary $\partial_- C_{p,q} = \partial N(\hat{K}_{p,q})$. Note that $H_1(\partial_- C_{p,q})$ inherits a natural basis by virtue of $\hat{K}_{p,q}$ being a torus knot. See figure 1.

The class of *iterated torus knots* is defined inductively: define K_0 to be a (p_0, q_0) -torus knot. Then for any $n \geq 1$, define K_n to be a (p_n, q_n) -cable of K_{n-1} . We call the knot K_n an *iterated torus knot*, denoted by $K_n = ((p_0, q_0), \dots, (p_n, q_n))$. Here, $E(K_n)$ decomposes as

$$E(K_n) = E(K_0) \cup_{T_1} C_{p_1, q_1} \cup_{T_2} \cdots \cup_{T_n} C_{p_n, q_n},$$

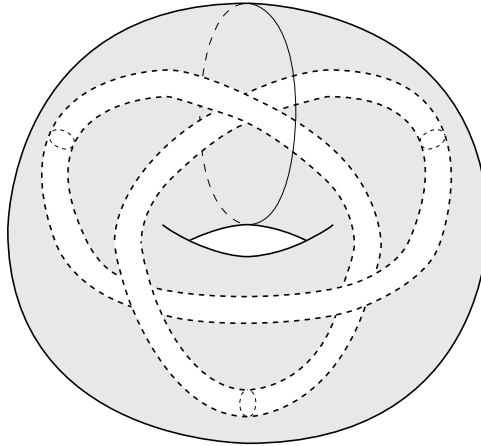


Figure 1: The cable space $C_{3,2}$.

where $T_i = \partial_+ C_i = \partial_- C_{i-1}$ for $i \geq 2$ and $T_1 = \partial_+ C_1 = \partial E(K_0)$. The longitude-meridian bases of $H_1(\partial_+ C_i)$ and $H_1(\partial_- C_{i-1})$ coincide; hence, we specify a preferred basis (μ_i, λ_i) for $H_1(T_i)$ without ambiguity. For a curve $c \subset T_i$ with $[c] = a[\mu_i] + b[\lambda_i]$, we say that c has slope $\frac{a}{b}$.

By [5], if M is a Seifert fibered space and S is incompressible in M , then S must either be a union of fibers, called a *vertical surface*, or transverse to all fibers, called a *horizontal surface*. There is a natural projection map $\pi : M \rightarrow B$, where B is the base orbifold of M , and if S is a horizontal surface, $\pi|_S$ is an orbifold covering map.

The following lemma, due to Gordon and Litherland [4], provides a complete classification of incompressible surfaces in cable spaces, viewed as Seifert fibered spaces. It will be critical for our understanding of such surfaces in $E(K_n)$. For two curves with slopes $\frac{a}{b}$ and $\frac{c}{d}$, we will denote their intersection number by $\Delta(\frac{a}{b}, \frac{c}{d}) = |ad - bc|$.

Lemma 3.1. *Suppose that $S \subset C_{p,q}$ is incompressible and not ∂ -parallel.*

- (1) *If S is vertical, then S is an annulus, and any component of $S \cap \partial_+ C_{p,q}$ has slope $\frac{p}{q}$, while any component of $S \cap \partial_- C_{p,q}$ has slope pq .*
- (2) *If S is horizontal, then there are coprime integers m and n (with $n \geq 0$) such that the total boundary slope of $S \cap \partial_+ C_{p,q}$ is $\frac{n+mp}{qm}$ and the total boundary slope of $S \cap \partial_- C_{p,q}$ is $\frac{q(n+mp)}{m}$. In this case, $\chi(S) = n(1 - q)$.*
- (3) *If S is horizontal and planar, then either*

- (a) *there are q components of $S \cap \partial_+ C_{p,q}$, each of slope $\frac{l}{m}$, and one component of $S \cap \partial_- C_{p,q}$, of slope $\frac{lq^2}{m}$, where l and m satisfy $\Delta(\frac{l}{m}, \frac{p}{q}) = 1$; or*
- (b) *there is one component of $S \cap \partial_+ C_{p,q}$, of slope $\frac{1+kpq}{kq^2}$, and q components of $S \cap \partial_- C_{p,q}$, each of slope $\frac{1+kpq}{k}$, where k is an integer.*

We may employ Lemma 3.1 in the proof of the statements below. Note that for any cable knot $K_{p,q}$, the companion torus T is essential in $E(K_{p,q})$, hence in the iterated torus knot K_n , each torus T_i is an essential surface.

Lemma 3.2. *Suppose $S \subset C_{p,q}$ is incompressible. If each component of $S \cap \partial_+ C_{p,q}$ has integral slope, then each component of $S \cap \partial_- C_{p,q}$ also has integral slope.*

Proof. Suppose without loss of generality that S is connected. Since $S \cap \partial_+ C_{p,q}$ has integral slope, S is not vertical, and so by Lemma 3.1, the total slope of $S \cap \partial_+ C_{p,q}$ is $\frac{n+mp}{qm}$ for some coprime integers m and n , that is, there are $r = \gcd(n + mp, qm)$ components of $S \cap \partial_+ C_{p,q}$, each of slope $\frac{a}{b}$, where $a = \frac{n+mp}{r}$ and $b = \frac{qm}{r}$. By assumption, $b = \pm 1$; thus $r = \pm qm$ and m divides $n + mp$. It follows that m divides n , hence $m = \pm 1$. The total slope of $S \cap \partial_- C_{p,q}$ is $\frac{q(m+np)}{m}$ and as such is also integral. □

A boundary component of a surface is said to be *meridional* if its boundary slope is $\frac{1}{0}$.

Lemma 3.3. *Suppose $S \subset C_{p,q}$ is incompressible. Then $S \cap \partial_+ C_{p,q}$ is meridional if and only if $S \cap \partial_- C_{p,q}$ is meridional.*

Proof. It is clear that if either set of boundary components is meridional, then S is horizontal. In this case, by Lemma 3.1, the total outer boundary slope is $\frac{n+mp}{qm}$ while the total inner boundary slope is $\frac{q(n+mp)}{m}$. If $m = 0$, then both outer and inner boundary components of S are meridional; otherwise neither is meridional. □

In light of Lemma 3.3, we say that a horizontal surface S in $C_{p,q}$ is *meridional* if any component of ∂S is meridional.

Lemma 3.4. *Suppose $S \subset C_{p,q}$ is incompressible. If $S \cap \partial_+ C_{p,q}$ is a single component of slope $\frac{1}{n}$ and S is planar, then S is meridional.*

Proof. By Lemma 3.1, we have $\frac{1}{n} = \frac{1+kpq}{kq^2}$, hence $k = 0$. □

Let \mathcal{T} denote the union of the tori T_i and let T_{n+1} denote $\partial E(K_n)$.

Lemma 3.5. *Suppose $S \subset E(K_n)$ is an essential surface. If S is not isotopic to some T_i , then $S \cap \partial E(K_n)$ is nonempty and has integral slope.*

Proof. After isotopy, we may suppose that $|S \cap \mathcal{T}|$ is minimal. First, we note that each component of $S \cap C_i$ or $S \cap E(K_0)$ is incompressible in C_i or $E(K_0)$: if D is a compressing disk for $S \cap C_i$ in C_i , then ∂D bounds a disk $D' \subset S$ by the incompressibility of S , where $D' \cap \mathcal{T} \neq \emptyset$. By the irreducibility of $E(K_n)$, there is an isotopy of S pushing D' onto D which reduces $|S \cap \mathcal{T}|$, a contradiction.

If $S \cap \mathcal{T} = \emptyset$, then $S \subset C_i$ for some i or $S \subset E(K_0)$. However, in this case S must be vertical and closed, and the only such surfaces in C_i or $E(K_0)$ are boundary parallel. It follows that S is isotopic to T_i for some i .

Thus, suppose that $S \cap \mathcal{T} \neq \emptyset$. If $S \cap T_1 \neq \emptyset$, then $S \cap E(K_0)$ is either a Seifert surface with boundary slope 0 or an essential annulus with slope p_0q_0 . If $S \cap T_1 = \emptyset$, let i be the smallest i such that $S \cap T_i \neq \emptyset$. Lemma 3.1 dictates that $S \cap C_{i-1}$ must be a vertical annulus in C_{i-1} which intersects T_i in a curve with slope p_iq_i (otherwise $S \cap T_{i-1} \neq \emptyset$). In either case, we have that $S \cap T_i$ has integral slope. By a repeated application of Lemma 3.2, $S \cap T_j$ has integral slope for all $j \geq i$, completing the proof. \square

4. The tunnel number of iterated torus knots

The *tunnel number* $t(K)$ of a knot $K \subset S^3$ is defined as $t(K) = g(E(K)) - 1$, where $g(E(K))$ is the Heegaard genus of $E(K)$. Our goal in this section is to determine the minimal genus of a Heegaard splitting of $E(K_n)$ for an iterated torus knot K_n , and our main tool is the classification of a Heegaard splittings of graph manifolds by Schultens [24].

A *graph manifold* is a 3-manifold M containing a nonempty collection of essential tori T , such that $M \setminus \eta(T)$ is a disjoint union of Seifert fibered spaces. A connected surface S properly embedded in a Seifert fibered space M' is *pseudohorizontal* if after isotopy there exists a fiber f of M' such that $S \cap (M' \setminus \eta(f))$ is a horizontal surface in the space $M' \setminus \eta(f)$, and $S \cap N(f)$ is an annulus which is a bicollar of f . Note that each boundary component of a horizontal surface has the same orientation; thus, if a pseudohorizontal surface S is to be orientable, then the horizontal piece $S \cap (M' \setminus \eta(f))$ must have two components.

Suppose that S and S' are connected surfaces properly embedded in M and γ is an arc with one endpoint in each of S and S' and interior

disjoint from $S \cup S'$. Then $N(S \cup S' \cup \gamma)$ has three boundary components, one isotopic to S , another isotopic to S' , and third we define to be the result of *tubing S to S' along γ* .

We may now state one of Schultens' main results over the classification of graph manifolds [24].

Theorem 4.1. *Suppose M is a graph manifold containing a strongly irreducible Heegaard surface Σ , and let T be a collection of essential tori splitting M into Seifert fibered spaces. After an isotopy of Σ , there is a submanifold M' of T such either M' is a component of $M \setminus \eta(T)$ or $M = T' \times I$ for a torus $T' \in T$, called the active component of M , with the following properties:*

1. *Each component of $\Sigma \setminus (\eta(T) \cup \eta(M'))$ is incompressible in $M \setminus (\eta(T) \cup \eta(M'))$.*
2. *If $M' \neq T' \times I$ for some $T' \in T$, then $\Sigma \cap M'$ is a pseudohorizontal surface in M' .*
3. *Otherwise, $M' = T' \times I$, and $\Sigma \cap M'$ is either*
 - (a) *the result of tubing an annulus parallel into $T' \times \{0\}$ with boundary slope s_0 to an annulus parallel into $T' \times \{1\}$ with boundary slope s_1 along a vertical arc in $T' \times I$, where $\Delta(s_0, s_1) = 1$, or*
 - (b) *the result of tubing two vertical annuli in $T' \times I$ along an arc contained in $T \times \{\frac{1}{2}\}$.*

From this point forward, for an iterated torus knot $K_n = ((p_0, q_0), (p_1, q_1), \dots, (p_n, q_n))$, we make the restriction that $|p_i - p_{i-1}q_{i-1}q_i| > 1$ for all i . The motivation for this is as follows: a regular fiber f_{i-1} of C_{i-1} has slope $s_{i-1} = p_{i-1}q_{i-1}$ in T_i , whereas a regular fiber f_i of C_i has slope $s_i = \frac{p_i}{q_i}$ in T_i . In order for the arguments below to hold, we need that $|f_i \cap f_{i-1}| > 1$. In other words, we require

$$|p_i - p_{i-1}q_{i-1}q_i| = \Delta(s_i, s_{i-1}) > 1.$$

The goal of this section is to show that for such iterated torus knots K_n , we have $t(K_n) = n + 1$. We will use Theorem 2.3 in conjunction with Theorem 4.1 to bound the genus of any Heegaard surface for $E(K_n)$ from below. For this, we must understand pseudohorizontal surfaces in $E(K_0)$ and the cable spaces C_i .

Lemma 4.1. *Suppose that S is a pseudohorizontal surface in $E(K_0)$. Then either $\chi(S) \leq -4$ or ∂S has meridional slope.*

Proof. Without loss of generality, let $p_0 \geq 2$ and let f be a fiber of $E(K_0)$, such that $S' = S \cap (E(K_0) \setminus \eta(f))$ is horizontal. First, suppose that f is a regular fiber, and let B' be the base space of the Seifert fibered space $E(K_0) \setminus \eta(f)$, so that B' is a topological annulus with two cone points. By the observation above, S' has two components S_1 and S_2 , each an orbifold cover of B' of degree $d \geq p_0q_0$. Following the discussion in [18], for instance, we let B^* denote B' with a neighborhood of the cone points removed, thus

$$\begin{aligned} \chi(S_i) &= d\chi(B^*) + \frac{d}{p_0} + \frac{d}{q_0} \\ &\leq -2p_0q_0 + p_0 + q_0 \\ &= p_0(1 - q_0) + q_0(1 - p_0) \leq -7. \end{aligned}$$

It follows that $\chi(S) = \chi(S') \leq -14$.

In the second case, suppose that f is a critical fiber. Without loss of generality, we may suppose that $E(K_0) \setminus \eta(f) = C_{p_0,q_0}$. After isotopy, S intersects $N(f)$ in an annulus which is a bicollar of f , thus $S \cap N(f)$ has integral boundary slope on the solid torus $N(f)$. In the coordinates of $\partial_+C_{p_0,q_0}$, each component of $S \cap \partial_+C_{p_0,q_0}$ has slope $\frac{1}{k}$ for some integer k . Further, S' has two components S_1 and S_2 , each with a single boundary component in $\partial_+C_{p_0,q_0}$. By Lemma 3.4, if each S_i is planar, then S has meridional boundary. Otherwise, $\chi(S_i) \leq -2$, and $\chi(S) = \chi(S') \leq -4$, as desired. \square

We carry out a similar analysis of pseudohorizontal surfaces in a cable space $C_{p,q}$.

Lemma 4.2. *Suppose that S is a pseudohorizontal surface in the cable space $C_{p,q}$. Then $\chi(S) \leq -4$. If, in addition, each component of $S \cap \partial_+C_{p,q}$ has boundary slope $\frac{1}{l}$, where $\Delta(l, \frac{p}{q}) > 1$, then either $\chi(S) \leq -6$, the slope of $S \cap \partial_-C_{p,q}$ is meridional, or the slope of $S \cap \partial_-C_{p,q}$ is integral.*

Proof. Let f be a fiber of $C_{p,q}$, such that $S' = S \cap (C_{p,q} \setminus \eta(f))$ is horizontal. As above, we first suppose that f is a regular fiber, and let B' be the base space of $C_{p,q} \setminus \eta(f)$, where B' is a thrice-punctured sphere with one exceptional fiber. Again, S' has two components S_1 and S_2 , each of which is a degree $d \geq q$ orbifold cover of B' , so that

$$\chi(S_i) = -2d + \frac{d}{q} = d \left(-2 + \frac{1}{q} \right) \leq -2q + 1 \leq -3,$$

so $\chi(S) = \chi(S') \leq -6$.

Otherwise, suppose f is the unique exceptional fiber of $C_{p,q}$, and note that $H_1(\partial N(f))$ inherits the same natural basis as $H_1(\partial_+ C_{p,q})$. Letting $S' = S_1 \cup S_2$, we have that $S_i \cap \partial N(f)$ is a single curve with slope $\frac{u}{1}$ for some u . Hence, S_i extends to a horizontal surface S_i^* in the manifold M^* which results from Dehn filling $C_{p,q} \setminus \eta(f)$ along the slope $\frac{u}{1}$. Then, $\chi(S_i^*) = \chi(S_i) + 1$ and M^* is another cable space $C_{r,s}$. Immediately, we have that $\chi(S_i^*) \leq -1$, thus $\chi(S) = \chi(S') = \chi(S_1) + \chi(S_2) \leq -4$.

Suppose further that the slope of each component of $S \cap \partial_+ C_{p,q}$ is $\frac{l}{1}$. If S_i^* is not planar, then $\chi(S_i^*) \leq -2$ and $\chi(S) \leq -6$ are desired, so suppose S_i^* is planar. Since the curves of $S \cap \partial_+ C_{p,q}$ intersect regular fibers of $C_{p,q}$ (which are also regular fibers of $C_{r,s}$) more than once, S_i^* is not a horizontal planar surface of type 3(a) given by Lemma 3.1. Note that curves of slope $\frac{u}{1}$ and $\frac{1}{0}$ in $\partial_+ C_{p,q}$ will have respective slopes of $\frac{1}{0}$ and $\frac{v}{-1}$ for some integer v in the Dehn filled manifold $C_{r,s}$. The 2×2 matrix representing this change of basis is

$$\begin{bmatrix} v & 1 - uv \\ -1 & u \end{bmatrix}.$$

It follows that the slope of each component of $S_i^* \cap \partial_+ C_{r,s}$ is $\frac{1-v(u-l)}{u-l}$, and by Lemma 3.1, we have

$$\frac{1 - v(u - l)}{u - l} = \frac{1 + krs}{kr^2}.$$

Thus $-vkr^2 = -v(u - l) = krs$. As r and s are relatively prime, we must have either $krs = 0$ or $krs = \pm 1$. If $krs = 0$, then the slope of $S_i^* \cap \partial_- C_{r,s} = S \cap \partial_- C_{p,q}$ is meridional by Lemma 3.4 (since $C_{p,q}$ and $C_{r,s}$ have the same regular fibers). On the other hand, if $krs = \pm 1$, then $kr^2 = \pm 1$, so $S_i^* \cap \partial_+ C_{r,s}$ has integral slope, implying $S_i^* \cap \partial_- C_{r,s}$ has integral slope by Lemma 3.2. This in turn shows that $S \cap \partial_- C_{p,q}$ has integral slope, completing the proof. □

For ease of notation but at the risk of confusion, we will let C_0 denote $E(K_0)$ despite the fact that $E(K_0)$ is not a cable space, and for $0 \leq l \leq m \leq n$, define

$$C_l^m = C_l \cup C_{l+1} \cup \dots \cup C_m.$$

We prove one final lemma before the main theorem of this section.

Lemma 4.3. *Suppose that Σ is a Heegaard surface for C_l^m , such that $\Sigma \cap C_l$ and $\Sigma \cap C_{l+1}$ are incompressible. Then $\chi(\Sigma \cap (C_l \cup C_{l+1})) \leq -4$.*

Proof. First, suppose that $S \cap C_l$ is a horizontal surface (this can only occur in the case $l = 0$), so each component of $\Sigma \cap C_l$ has negative Euler characteristic. As Σ is separating, there are at least two components, hence $\chi(\Sigma \cap C_l) \leq -2$. In addition, $\Sigma \cap T_{l+1}$ has integral slope, so $\Sigma \cap C_{l+1}$ is horizontal and has at least two components, implying $\chi(\Sigma \cap C_{l+1}) \leq -2$ as well.

On the other hand, suppose that $\Sigma \cap C_l$ consists of vertical annuli. By Lemma 3.1, $\Sigma \cap C_{l+1}$ cannot be of type 3(a) since the regular fibers of C_l and C_{l+1} intersect more than once, and $\Sigma \cap C_{l+1}$ cannot be of type 3(b) since the slope of $\Sigma \cap T_{l+1}$ is integral. It follows that each component Σ_i of $\Sigma \cap C_{l+1}$ satisfies $\chi(\Sigma_i) \leq -2$; thus $\chi(\Sigma \cap (C_l \cup C_{l+1})) = \chi(\Sigma \cap C_{l+1}) \leq -4$. \square

Theorem 4.2. *For the iterated torus knot K_n , with cabling parameters satisfying $\Delta(\frac{p_i}{q_i}, p_{i-1}q_{i-1}) > 1$, we have $t(K_n) = n + 1$.*

Proof. First, we note that each C_i has a minimal genus 2 Heegaard surface Σ_i , and together with the essential tori T_i , the collection $\{\Sigma_i\} \cup \{T_i\}$ yields a generalized Heegaard splitting of $E(K_n)$. Amalgamating this splitting gives a Heegaard surface with genus $\sum g(\Sigma_i) - \sum g(T_i) = 2(n + 1) - n = n + 2$. Thus, $t(K_n) \leq n + 1$.

Fix j such that $1 \leq j \leq n$ and suppose by way of induction that $g(C_l^{m'}) = m' - l' + 2$, whenever $m' - l' < j$. The above argument covers the base case $m' - l' = 0$. Let Σ be a Heegaard surface for some C_l^m , such that $m - l = j$ and suppose first that Σ is weakly reducible. By Theorem 2.3, untelescoping Σ yields a generalized Heegaard splitting with at least one essential thin surface S , and by Lemma 3.5, we have we have that $S = T_i$ for some i , where $l + 1 \leq i \leq m$. It follows that

$$g(\Sigma) \geq g(C_l^{i-1}) + g(C_i^m) - g(T_i) = (i - 1 - l + 2) + (m - i + 2) - 1 = m - l + 2,$$

completing the proof.

Hence, suppose that Σ is strongly irreducible. By Theorem 4.1, there is a submanifold $M' = C_i$ or $T_i \times I$ of C_l^m , such that

1. Each component of $\Sigma \setminus (\eta(\mathcal{T}) \cup \eta(M'))$ is incompressible in $M \setminus (\eta(\mathcal{T}) \cup \eta(M'))$.
2. If $M' = C_i$ for some i , then $\Sigma \cap M'$ is a pseudohorizontal surface in M' .
3. Otherwise, $M' = T_i \times I$, and $\Sigma \cap M'$ is either

- (a) the result of tubing an annulus parallel into $T_i \times \{0\}$ with boundary slope s_0 to an annulus parallel into $T_i \times \{1\}$ with boundary slope s_1 along a vertical arc in $T_i \times I$, where $\Delta(s_0, s_1) = 1$, or
- (b) the result of tubing two vertical annuli in $T_i \times I$ along an arc contained in $T_i \times \{\frac{1}{2}\}$.

Case 1: $M' = C_i$ for some i .

By the above, $\Sigma \cap C_i$ is pseudohorizontal and $\Sigma \cap C_j$ is incompressible for $j \neq i$. Observe that $i \neq m$, and if $i = l$, then $l = 0$, as a pseudohorizontal surface intersects all boundary components of C_i . Hence $\Sigma \cap C_m$ is a vertical annulus and $\chi(\Sigma \cap C_m) = 0$.

We will show that $\chi(\Sigma) \leq -2(m - l) - 2$. If $\Sigma \cap C_j$ is a vertical annulus where $i < j < m$, then $\Sigma \cap \partial_+ C_{j+1}$ has integral slope. By Lemma 3.2, $\Sigma \cap \partial_- C_m \neq \emptyset$, contradicting that Σ is a Heegaard surface for C_l^m . If $i \neq l$, then $\Sigma \cap C_l$ is either a vertical annulus or a Seifert surface for K_0 (if $l = 0$), so by Lemma 3.2, $\Sigma \cap T_j$ has integral slope for $1 \leq j \leq i$; thus $\Sigma \cap C_j$ is horizontal. Thus, we may suppose that $\Sigma \cap C_j$ is horizontal whenever $j \neq i, l, m$. As Σ is separating, $\Sigma \cap C_j$ has at least two components, so $\chi(\Sigma \cap C_j) \leq -2$.

Suppose first that $i = 0$. By Lemma 4.1, we have

$$\begin{aligned} \chi(\Sigma) &= \chi(\Sigma \cap C_0) + \chi(\Sigma \cap C_1) + \cdots + \chi(\Sigma \cap C_{m-1}) + \chi(\Sigma \cap C_m) \\ &\leq -4 - 2 - \cdots - 2 - 0 \\ &\leq -2(m - l) - 2. \end{aligned}$$

Next, suppose $i = l + 1$. By the remark above, $\Sigma \cap C_l$ is incompressible. If each component of $\Sigma \cap C_l$ is horizontal, then $\chi(\Sigma \cap C_l) \leq -2$ and by Lemma 4.2, $\chi(\Sigma \cap C_{l+1}) \leq -4$. If $\Sigma \cap C_l$ is vertical annuli, then Lemma 4.2 provides that $\chi(\Sigma \cap C_{l+1}) \leq -6$ (or else $\Sigma \cap \partial_- C_m \neq \emptyset$ by Lemmas 3.2 and 3.3, a contradiction). In either case, we have

$$\begin{aligned} \chi(\Sigma) &= \chi(\Sigma \cap (C_l \cup C_{l+1})) + \sum_{j=l+2}^{m-1} \chi(\Sigma \cap C_j) \leq -6 - 2(m - l - 2) \\ &= -2(m - l) - 2. \end{aligned}$$

Finally, suppose that $i > l + 1$. Then $\Sigma \cap C_l$ and $\Sigma \cap C_{l+1}$ are incompressible, so by Lemma 4.3, $\chi(\Sigma \cap (C_l \cup C_{l+1})) \leq -4$. In addition, by Lemma 4.2, $\chi(\Sigma \cap C_i) \leq -4$. Otherwise, for $j \neq l, l + 1, i, m$, components

of $\Sigma \cap C_j$ are horizontal and $\Sigma \cap C_j$ contributes at most -2 to $\chi(\Sigma)$. Thus

$$\begin{aligned} \chi(\Sigma) &\leq \chi(\Sigma \cap (C_l \cup C_{l+1})) + \sum_{i=l+2}^{m-1} \chi(\Sigma \cap C_j) \\ &\leq -4 - 2(m - l - 2) - 2 \\ &= -2(m - l) - 2. \end{aligned}$$

Case 2: $M' = T_i \times I$ for some i .

Suppose that $T_i \times I \subset C_i$, so that $T_i = T_i \times \{0\}$ and let $T'_i = T_i \times \{1\}$. In addition, set $C'_i = C_i \setminus \eta(T_i \times I)$ and for $j \neq i$, set $C'_j = C_j$, so that

$$C_l^m = C'_l \cup_{T_{l+1}} \cdots \cup_{T_{i-1}} C'_{i-1} \cup_{T_i} (T_i \times I) \cup_{T'_i} C'_i \cup_{T_{i+1}} \cdots \cup_{T_m} C'_m.$$

By Theorem 4.1, we have that $\chi(\Sigma \cap (T_i \times I)) = -2$, $\Delta((\Sigma \cap T_i), (\Sigma \cap T'_i))$ is either 0 or 1, and $\Sigma \cap C'_j$ is incompressible for all j . By the argument above, each $\Sigma \cap C'_j$ must be a horizontal surface for $l \leq j < m$, with the possible exception of $\Sigma \cap C'_l$.

Suppose first that $i = l + 1$. If $\Sigma \cap C'_l$ is horizontal, then we have $\chi(\Sigma \cap C'_j) \leq -2$ for all j , hence

$$\chi(\Sigma) \leq \chi(\Sigma \cap (T_{l+1} \times I)) + \sum_{j=l}^{m-1} \chi(\Sigma \cap C'_j) \leq -2 - 2(m - l).$$

On the other hand, suppose that $\Sigma \cap C'_l$ is vertical, so that the slope of $\Sigma \cap T_{l+1}$ is $p_l q_l$. If the slope of $\Sigma \cap T'_{l+1}$ is also $p_l q_l$, then by Lemma 2, $\Sigma \cap \partial_- C_m \neq \emptyset$, a contradiction. It follows that the slope of $\Sigma \cap T'_{l+1}$ is $\frac{r}{s}$, where $r - p_l q_l s = \pm 1$.

Observe that $\Sigma \cap C'_{l+1}$ has at least two components, and if these are not planar, $\chi(\Sigma \cap C'_{l+1}) \leq -4$, and again we have $\chi(\Sigma) \leq -2 - 2(m - l)$. If $\Sigma \cap C'_{l+1}$ is planar, it must consist of two components each having one boundary component on T'_{l+1} . By Lemma 3.1, it follows that $\frac{r}{s} = \frac{1+kp_{l+1}q_{l+1}}{kq_{l+1}^2}$. Thus, $1 + kp_{l+1}q_{l+1} - kp_l q_l q_{l+1}^2 = \pm 1$ and rearranging yields

$$kq_{l+1}(p_{l+1} - p_l q_l q_{l+1}) = \pm 1 - 1.$$

By assumption $q_{l+1} \geq 2$ and $|p_{l+1} - p_l q_l q_{l+1}| > 1$, and so we must have $k = 0$. Therefore $\Sigma \cap C'_i$ is meridional, and by Lemma 3.3, $\Sigma \cap \partial_- C^m \neq \emptyset$, a contradiction.

Finally, suppose $i > l + 1$. Then $\Sigma \cap (C_l \cup C_{l+1}) = \Sigma \cap (C'_l \cup C'_{l+1})$ and by Lemma 4.3, $\chi(\Sigma \cap (C'_l \cup C'_{l+1})) \leq -4$. Hence

$$\begin{aligned} \chi(\Sigma) &\leq \chi(\Sigma \cap (C'_l \cup C'_{l+1})) + \chi(\Sigma \cap (T_i \times I)) + \sum_{j=l+2}^{m-1} \chi(\Sigma \cap C'_j) \\ &\leq -4 - 2 - 2(m - l - 2) \\ &= -2(m - l) - 2, \end{aligned}$$

as desired. □

5. Strongly irreducible bridge surfaces

One valuable feature of strongly irreducible Heegaard surfaces for 3-manifolds is a “no-nesting” property, demonstrated by Scharlemann in [19]. We adapt the proof of this important lemma to show a version of no-nesting for bridge surfaces below.

Lemma 5.1. *Suppose Σ is a strongly irreducible bridge surface in (M, J) , where $(M, J) = (V, \alpha) \cup_{\Sigma} (W, \beta)$.*

1. *If c is an essential curve in Σ_J such that c bounds a disk $D \subset M(J)$, where a collar of c in D is disjoint from Σ_J , then c bounds a compressing disk D' in (V, α) or (W, β) .*
2. *If γ is an essential arc in Σ_J such that γ cobounds a disk $\Delta \subset M(J)$ with an essential arc $\mu \subset \partial N(J)$, where a collar of γ in Δ is disjoint from Σ_J , then γ cobounds a bridge disk in (V, α) or (W, β) .*

Proof. 1. Choose a disk D with $\partial D = c$ and such that $|D \cap \Sigma_J|$ is minimal. If $\text{int}(D) \cap \Sigma_J = \emptyset$, we are done. If not, D intersects Σ_J in some number of simple closed curves. Any curves which are inessential in Σ_J may be removed by isotopy; hence we may suppose that each curve of $D \cap \Sigma_J$ which is innermost curve in D bounds a compressing disk for Σ_J .

Pick an innermost nested pair of curves δ and ϵ , so that ϵ is innermost in D and δ cobounds a component P of $D \setminus \eta(\Sigma_J)$ with ϵ and possibly some other curves $\epsilon_1, \dots, \epsilon_n$, each of which is innermost in D and thus bounds a compressing disk for Σ_J . Suppose without loss of generality all of these disks lie in (W, β) . Compressing Σ along this collection of disks yields a surface Σ' and a bridge splitting $(M', J') =$

$(V, \alpha) \cup_{\Sigma} (C, \beta')$, where $\Sigma' = \partial_- C$, and C is chosen so that $D \cap C$ contains a vertical annulus A cobounded by δ and a curve $\delta' \subset D \cap \Sigma'$ (or in the case that $\delta = c$, the disk D can be extended to Σ' via such a vertical annulus A).

Clearly, δ' bounds a disk in $M'(J')$. If δ' is essential in $\Sigma'_{J'}$, then by Theorem 2.3, Σ cannot be strongly irreducible, a contradiction. Thus, δ' bounds a disk $D' \subset \Sigma'_{J'}$, and in (M, J) we may perform surgery on D by gluing D' along δ' . However, this reduces the number of intersections of $\text{int}(D)$ with Σ_J , another contradiction. We conclude that $\text{int}(D) \cap \Sigma_J = \emptyset$, completing the first part of the proof.

2. The second statement follows easily by viewing Δ as a disk in M with a boundary arc contained in J and by observing that $N(\Delta) \subset M$ is a 3-ball, where $c = \partial(N(\Delta) \cap \Sigma)$ bounds a compressing disk $D \subset \partial N(\Delta)$ satisfying (1). Thus, c bounds a disk D' in (V, α) or (W, β) . It must then be true that the arc $\mu \subset \partial N(J)$ intersects Σ_J only in its endpoints, hence D' is the frontier in V or W of a regular neighborhood of a bridge disk Δ' , where $\gamma \subset \partial \Delta'$. \square

Next, we adapt a lemma from the theory of Heegaard surfaces in the context of bridge surfaces [1, 10]. The lemma asserts that strongly irreducible surfaces behave much like incompressible surfaces; namely, cutting a strongly irreducible surface A along a collection of incompressible surfaces splits A into a number of pieces, all of which are incompressible with the exception of at most one strongly irreducible component. We make this statement rigorous in the next lemma, whose proof is modeled on the proof of Lemma 3.7 in [10]. We need several definitions before proceeding.

First, we weaken the definition of an essential surface. Let M be a 3-manifold with boundary, $P \subset \partial M$ a subsurface, and $A \subset M$ a properly embedded surface. A P - ∂ -compressing disk for A is a ∂ -compressing disk Δ for A such that $\Delta \cap \partial M \subset P$. We say that A is P -essential if A is incompressible and there does not exist a P - ∂ -compressing for A in M . On the other hand, if A is separating and admits compressing or P - ∂ -compressing disks on either side but admits no pair of disjoint disks on opposite sides, we say that A is P -strongly irreducible. As an example, a strongly irreducible bridge surface for a knot K in M is $\partial N(K)$ -strongly irreducible in $M(K)$. Finally, we say that two surfaces A and B are *almost transverse* if A is transverse to B except for a single saddle tangency.

The lemma below is stated in the greater generality than is needed here; however, we include it in its entirety for anticipated use in future work.

Lemma 5.2. *Let M be a compact 3-manifold and J a properly embedded 1-manifold, with $Q = \partial N(J)$ in $M(J)$. Suppose Σ is a strongly irreducible bridge splitting surface for (M, J) , and let $S \subset M(J)$ be a collection of properly embedded essential surfaces, such that for each component c of ∂S , either $c \subset Q$ or $c \subset \partial M$. Then one of the following must hold:*

- (1) *After isotopy, Σ_J is transverse to S and each component of $\Sigma_J \setminus \eta(S)$ is Q -essential in $M(J) \setminus \eta(S)$.*
- (2) *After isotopy, Σ_J is transverse to S , one component of $\Sigma_J \setminus \eta(S)$ is Q -strongly irreducible and all other components are Q -essential in $M(J) \setminus \eta(S)$,*
- (3) *After isotopy, Σ_J is almost transverse to S and each component of $\Sigma_J \setminus \eta(S)$ is Q -essential in $M(J) \setminus \eta(S)$.*

Proof. Let $(M, J) = (V, \alpha) \cup_{\Sigma} (W, \beta)$ be a bridge splitting with strongly irreducible bridge surface Σ , and let G_V and G_W denote cores of V and W , respectively. If $V = \Sigma \times I$, define G_V to be a small arc with one endpoint on $\partial_- V$ and one endpoint in $\text{int}(V)$ (similarly with W and G_W). After isotopy, we may assume that G_V and G_W miss each arc in α and β . Let b_{α} be the number of ∂ -parallel arcs in (V, α) , and define Γ_V to be the union of G_V , $(\partial_- V)_J$, b_{α} meridional curves contained in $\partial N(\alpha) \subset V(\alpha)$, and b_{α} unknotted arcs connecting the meridional curves to G_V , so that $V(\alpha) \setminus \Gamma_V \cong \Sigma_J \times (0, 1]$. We may define $\Gamma_W \subset W(\beta)$ similarly.

Then $M(J) \setminus (\Gamma_V \cup \Gamma_W) \cong \Sigma_J \times (0, 1)$. This induces a sweepout $f : M(J) \rightarrow I$, such that $f^{-1}(0) = \Gamma_V$, $f^{-1}(1) = \Gamma_W$, and for every $t \in (0, 1)$, $\Sigma_t = f^{-1}(t)$ is isotopic to Σ_J . We let $(M, J) = (V_t, \alpha_t) \cup_{\Sigma_t} (W_t, \beta_t)$ denote the bridge splitting induced by Σ_t . Assume that S is transverse to $\Gamma_V \cup \Gamma_W$. Choosing ϵ small enough, we then have that both $S \cap ((V_{\epsilon}, \alpha_{\epsilon}) \cup (W_{1-\epsilon}, \beta_{1-\epsilon}))$ is a collection of compressing and bridge disks.

After a small perturbation, the function $h = f|_S$ is Morse on $(-1, 1)$, where all critical points of h occur at different levels, each meridional component of ∂S is contained in a unique level away from critical values of h (these levels will also be considered critical values, viewed as truncated extrema), and for any other component γ of ∂S , $h|_{\gamma}$ is monotone on $(-1, 1)$, so that for any two components $c_1 \subset \partial \Sigma_t$ and $c_2 \subset \partial S$, we have $|c_1 \cap c_2|$ is minimal up to isotopy.

We assign to each $t \in (-1, 1)$ some subset of the labels $\{\nu, \omega\}$ by the following method: if $\Sigma_t \setminus \eta(S)$ contains a curve which bounds a compressing disk in (V_t, α_t) or an arc which cobounds a bridge disk in (V_t, α_t) , we assign t the label ν . The label ω is defined analogously with (W_t, β_t) . Note that

t may have either, both, or no labels. In addition, we observe that ϵ has the label ν and $1 - \epsilon$ has the label ω , which follows from the fact that S is transverse to Γ_V and Γ_W , and both cores are nonempty.

Next, we claim that if some regular value t of h has no label, then (1) holds. Suppose t has no label. We assert that each curve and arc of $S \cap \Sigma_t$ is essential in both S and Σ_t or inessential in S and Σ_t . By the incompressibility and ∂ -incompressibility of S , no curve or arc is essential in S but inessential in Σ_t . Suppose that $S \cap \Sigma_t$ contains a curve or arc c which is essential in Σ_t but inessential in S . By Lemma 5.1 and the strong irreducibility of Σ_t , this implies that c bounds a compressing disk or cobounds a bridge disk in either (V_t, α_t) or (W_t, β_t) , contradicting the assumption that Σ_t has no label. Thus, curves and arcs in $S \cap \Sigma_t$ are either essential or inessential in both.

If $S \cap \Sigma_t$ contains an arc γ which is inessential in both Σ_t and S , then both endpoints of γ are contained in single component c_1 of $\partial\Sigma_t$ and c_2 of ∂S . However, this implies that $\partial\gamma \subset c_1 \cap c_2$ contains two points of opposite algebraic intersection number, contradicting that $|c_1 \cap c_2|$ is minimal. Thus, arcs of $S \cap \Sigma_t$ are essential in both surfaces.

Let c be a curve in $S \cap \Sigma_t$ which is innermost among inessential curves in S . Then c bounds a disk $\Delta \subset S$ which misses Σ_t , and performing surgery on Σ_t along c gives a surface isotopic to Σ_t with fewer intersections with S . Finitely many iterations of this operation yields a bridge surface Σ'_t isotopic to Σ_t and such that each curve and arc in $\Sigma'_t \cap S$ is essential in both surfaces. Moreover, $\Sigma'_t \cap S \subset \Sigma_t \cap S$.

Suppose now that some component of $\Sigma'_t \setminus \eta(S)$ is Q -compressible. Then there is a curve or arc $c \subset \Sigma'_t \setminus \eta(S)$ such that c bounds a disk or cobounds a bridge disk in (V_t, α_t) or (W_t, β_t) . Since each curve of $\Sigma'_t \cap S$ is essential in Σ'_t , it follows that c is essential in Σ'_t . We may isotope c so that $c \subset \Sigma_t$, but this means that t is labeled ν or ω , a contradiction. We conclude that each component of $\Sigma'_t \setminus \eta(S)$ is Q -essential in its respective submanifold.

Our second major claim is that if a regular value t is labeled ν and ω , then (2) holds. As above, we assert that $S \cap \Sigma_t$ contains curves and arcs that are either essential in both S and Σ_t or inessential in both S and Σ_t . The incompressibility and ∂ -incompressibility of S rules out curves and arcs essential in S but inessential in Σ_t . Suppose c is a curve or arc in $S \cap \Sigma_t$ which is essential in Σ_t but inessential in S . Then c bounds or cobounds a disk in $M(J)$, and by Lemma 5.1, c bounds or cobounds a disk in (V_t, α_t) or (W_t, β_t) . However, since t is labeled ν and ω , there exist curves or arcs c_V and c_W disjoint from S (and thus from c) which bound or cobound disks in (V_t, α_t) and (W_t, β_t) , respectively. This contradicts the strong irreducibility

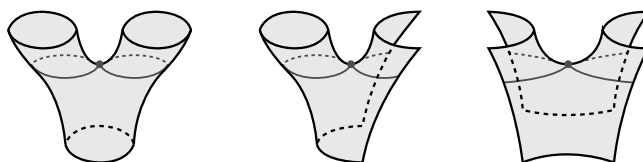


Figure 2: Possible components of S corresponding to a saddle.

of Σ_t . Hence, curves and arcs in $S \cap \Sigma_t$ are either essential or inessential in both surfaces.

As above, $S \cap \Sigma_t$ cannot contain arcs inessential in both surfaces, and we may construct a surface Σ'_t such that $S \cap \Sigma'_t$ contains only curves and arcs which are essential in both S and Σ'_t . Let Σ' be the component of $\Sigma'_t \setminus \eta(S)$ which contains c_V . By the strong irreducibility of Σ'_t , $c_V \cap c_W \neq \emptyset$ and thus $c_W \subset \Sigma'$ and, more generally, Σ' must be Q -strongly irreducible. Let Σ'' be any other component of $\Sigma'_t \setminus \eta(S)$. If an essential curve or arc c bounds or cobounds a disk for (V_t, α_t) or (W_t, β_t) , this gives rise to a compressing or bridge disk for Σ_t disjoint from c_V and c_W , another contradiction. We conclude that Σ'' is Q -essential in $M(J) \setminus \eta(S)$.

In the final remaining case, suppose that h has a critical value $t \in (\epsilon, 1 - \epsilon)$ such that $t - \delta$ is labeled ν and $t + \delta$ is labeled ω , with curves or arcs c_V and c_W contained in $\Sigma_{t \pm \delta} \setminus \eta(S)$ bounding or cobounding disks in $(V_{t-\delta}, \alpha_{t-\delta})$ and $(W_{t+\delta}, \beta_{t+\delta})$, respectively. If c corresponds to minimum, maximum, or a level component of ∂S , then there exists an isotopy pushing c_V to $c'_V \in \Sigma_{t+\delta}$ or c_W to $c'_W \in \Sigma_{t-\delta}$ bounding disks in $(V_{t+\delta}, \alpha_{t+\delta})$ or $(W_{t-\delta}, \beta_{t-\delta})$, respectively, contradicting either the assumption that $t + \delta$ is labeled ω or the assumption that $t - \delta$ is labeled ν . We conclude that t corresponds to a saddle.

We may regard a small closed regular neighborhood $N(S)$ of in $M(J)$ as $S \times I$. Then $\Sigma_t \cap (S \times I)$ contains components of the form $\gamma \times I$, where γ is curve or arc in $\Sigma_t \cap S$, in addition to one exceptional component E containing the saddle point. The surface E must be a pair of pants, an annulus or a disk, depending on whether $S \cap \Sigma_t$ is a figure eight curve, a curve wedged with an arc, or the wedge of two arcs. figure 2 depicts the possible configurations corresponding to the saddle point along with potential singular sets $S \cap \Sigma_t$. Note while the figure depicts subsets of S on which h is a Morse function, each of these pieces is isotopic to the corresponding exceptional component $E \subset \Sigma_t$.

We assert that if a curve or arc $\gamma \subset \Sigma_t \setminus \eta(S)$ is essential in Σ_t , then γ does not bound or cobound a disk in M . Otherwise, by the no-nesting

Lemma γ bounds or cobounds a disk in (V_t, α_t) or (W_t, β_t) , implying that $\Sigma_{t-\delta}$ and $\Sigma_{t+\delta}$ have the a common label, since for small δ , $\Sigma_{t\pm\delta} \setminus \eta(S)$ is parallel to $\Sigma_t \setminus \eta(S)$.

Next, we claim that curves and arcs in $\Sigma_t \cap \partial N(S)$ are either essential or inessential in both $\partial N(S)$ and Σ_t . Since $\partial N(S)$ is essential, no curve or arc $\gamma \in \partial N(S) \cap \Sigma_t$ is essential in $\partial N(S)$ and inessential in Σ_t . Additionally, by the assertion above, no such γ is essential in Σ_t and inessential in $\partial N(S)$. As above, there cannot be arcs in $\partial N(S) \cap \Sigma_t$ which are inessential in both surfaces. If there exists a curve of intersection which is inessential in both surfaces, cutting and pasting along a curve which is innermost in $\partial N(S)$ yields a surface isotopic to Σ_t with fewer intersections with $\partial N(S)$, and finitely many repetitions produces a surface Σ'_t isotopic to Σ_t and such that $\partial N(S) \cap \Sigma'_t$ contains only curves and arcs which are essential in both surfaces.

Here we show that every component of $\Sigma'_t \setminus \eta(S)$ is Q -essential. If γ is a curve or arc which is essential in $\Sigma'_t \setminus \eta(S)$, then γ must be essential in Σ'_t as all components of $\Sigma'_t \cap \partial N(S)$ are essential curves or arcs. Thus, γ cannot bound or cobound a disk in M . Otherwise, after isotopy we may assume $\gamma \subset \Sigma_t$ and by the no nesting Lemma, γ bounds or cobounds a disk in (V_t, α_t) or (W_t, β_t) , contradicting our assertion above.

To finish the proof, we must show that Σ'_t is either tangent or almost tangent to S . In process of capping off disks to get Σ'_t from Σ_t , we may have altered the exceptional component E if some curve in ∂E was inessential in $\partial N(S)$. Let $E' = E \cap \Sigma'_t$ (that is, E' is the essential subsurface of Σ'_t resulting from capping off inessential curves and arcs of E). There are several cases to consider: if $E' = \emptyset$, then (1) holds. If $E' = E$ then (3) holds.

If $E' \neq \emptyset$ and $E' \neq E$, then the above process must have capped off exactly one curve component of ∂E , and $\partial E'$ has two components, c_0 and c_1 . If c_0 and c_1 are contained in different components of $\partial N(S)$, then E' can be expressed as $c_0 \times I$ in $N(S)$, and (1) holds. Otherwise, c_0 and c_1 are in the same component, call it S_0 , of $\partial N(S)$. We may replace S with S_0 (since the two surfaces are isotopic), noting that Σ'_t is transverse to S_0 and each component of $\Sigma'_t \setminus \eta(S_0)$ is Q -essential in its respective submanifold; that is, (1) holds, as desired. □

We note that if the exceptional component E of Σ_t in the final part of the above proof is contained in an essential torus T or annulus A in S , then (3) cannot occur, as at least one of the curves in ∂E must be inessential in S . Thus, if S is a collection of tori and annuli, we have that either statement (1) or statement (2) holds.



Figure 3: Replacing $K \subset \Sigma$ with its cable.

6. Analyzing the bridge spectra of K_n

In Section 4, we utilized Theorem 4.1 to understand minimal genus Heegaard surfaces for $E(K_n)$. In a similar vein, in this section we will employ Lemma 5.2 to characterize minimal bridge surfaces for K_n , although the analysis here is significantly less complicated than that of Section 4. We begin with a lemma concerning embeddings of cables on Heegaard surfaces.

Lemma 6.1. *Let K be a knot in a 3-manifold M and $K_{p,q}$ a (p, q) -cable of K . If $\Sigma \subset M$ is a Heegaard surface, such that $K \subset \Sigma$ and D is a compressing disk for Σ such that $|D \cap K| = 1$, then there exists an embedding of $K_{p,q}$ in M such that $K_{p,q} \subset \Sigma$.*

Proof. Suppose $M = V \cup_{\Sigma} W$, with $D \subset V$, and let $D \times I$ be a collar neighborhood of D in V , noting that $K \cap (\partial D \times I)$ is a single essential arc. In addition, let $X = N(K) \cap V$. Then both X and $X' = X \cup (D \times I)$ are solid tori whose cores are isotopic to K . Now $X' \cap S = \partial X' \cap S$ is a once-punctured torus. Since every torus knot can be embedded on a once punctured torus, every (p, q) -cable of K can be embedded on $X' \cap S$. See figure 3. □

As a consequence of Lemma 6.1, we may demonstrate that $b_g(K_n) = 0$ whenever $g > n$: By Theorem 4.2, $t(K_{n-1}) = n$, which implies that $E(K_{n-1})$ has a genus $n + 1$ Heegaard splitting. In this case, the ∞ -sloped Dehn filling of K_{n-1} yields a genus $n + 1$ Heegaard surface Σ of S^3 in which K_{n-1} is contained in a core of one of the handlebodies cut out by Σ . It follows that K_{n-1} is isotopic into Σ , and there is a compressing disk D such that $|D \cap K_{n-1}| = 1$. By Lemma 6.1, there is an embedding of K_n such that $K_n \subset \Sigma$. Hence $b_{n+1}(K_n) = 0$, and using stabilization, $b_g(K_n) = 0$ for all $g > n$.

Conversely, if K_n is isotopic into a genus g Heegaard surface Σ' , then a perturbation followed by a meridional stabilization of Σ' yields a genus $g + 1$ surface Σ'' such that there is a compressing disk D' for Σ'' with $|D' \cap K_n| = 1$. In this case, K_n is isotopic into the core of a handlebody cut out by Σ'' , implying that $g(E(K_n)) \leq g + 1$. This implies that $t(K_n) \geq g$; thus by

Theorem 4, $n + 1$ must be the smallest g for which $b_g(K_n) = 0$, and Σ is not cancelable for $g \leq n$.

For any knot $K \subset S^3$, the smallest g for which $b_g(K) = 0$ is called the h -genus $h(K)$ of K , following Morimoto [12]. In this work, Morimoto partitions the set of all knots into sets $A_n, B_n,$ and C_n related to tunnel number, h -genus, and another invariant known as 1-bridge genus. He conjectures that each of these sets is nonempty, and the above argument verifies that for all K_n , we have $h(K_n) = t(K_n) = n + 1$ and $K_n \subset A_{n+1}$.

Before we arrive at the proof of the main theorem, recall that $E(K_n)$ decomposes as

$$E(K_n) = C_0 \cup_{T_1} \cup \cdots \cup_{T_n} C_n,$$

and that \mathcal{T} denotes the collection $\{T_1, \dots, T_n\}$. Note also that C_n contains an essential vertical annulus A such that $\partial A \subset \partial E(K_n)$, where $C_n \setminus \eta(A)$ is the union of $T_n \times I$ and a solid torus X . Thus, if we set $\mathcal{T}' = \mathcal{T} \cup \{A\}$, then $E(K_n)$ cut along \mathcal{T}' is the union of C_0 , the cable spaces C_1, \dots, C_{n-1} , the product region $T_n \times I$, and the solid torus X .

Define $V_0 = S^3$, and for every l with $1 \leq l \leq n$, let V_l denotes the solid torus in S^3 bounded by T_l , so that $V_l \setminus \eta(K_n) = C_l^n$ (using the terminology of Section 4).

Lemma 6.2. *Suppose that Σ is a strongly irreducible (g, b) -bridge surface for (V_l, K_n) with $0 \leq l \leq n - 1$ and $b \geq 1$. Then one of the following holds:*

- (1) Σ is a (g, b') -bridge surface for (V_l, K_{n-1}) , where $b \geq q_n \cdot b'$ and $b' \geq 1$,
- (2) $g > n - l$, or
- (3) $l = n - 1, g = 1$, and $b \geq |p_n - p_{n-1}q_{n-1}q_n|$.

Proof. Let $Q = \partial N(K_n)$. By Lemma 5.2, there is an isotopy of Σ after at most one of $\Sigma \cap C_l, \dots, \Sigma \cap C_{n-1}, \Sigma \cap (T_n \times I), \Sigma \cap V$ is Q -strongly irreducible, while the remaining surfaces are Q -essential in their respective submanifolds. Suppose first that $\Sigma \cap X$ is Q -essential. Since $\Sigma \cap K_n \neq \emptyset$, we have that $\Sigma \cap X$ cannot be a ∂ -parallel annulus or disk, as these surfaces are either disjoint from K_n or are Q - ∂ -compressible. Thus each component of $\Sigma \cap X$ is a meridian disk, which intersects K_n at least q_n times. This implies that Σ is a (g, b') -bridge surface for a core K_{n-1} of X intersecting Σ transversely, implying $b \geq q_n \cdot b'$ and $b' \geq 1$.

On the other hand, suppose that $\Sigma \cap X$ is Q -strongly irreducible. Then $\Sigma \cap (T_n \times I)$ is incompressible, and thus must be a collection of vertical

annuli. This implies that $\Sigma \cap T_n$ is essential simple closed curves and parameterizing ∂X as T_n , slopes of $\Sigma \cap \partial X$ and $\Sigma \cap T_n$ are equal. In addition, each $\Sigma \cap C_j$ is essential. It follows that $\Sigma \cap C_l$ must be a collection of vertical annuli or Seifert surfaces for K_0 (if $l = 0$); thus by Lemma 3.2, $\Sigma \cap T_j$ has integral slope for all j . Thus, $\Sigma \cap X$ has no disk components and $\chi(\Sigma \cap X) \leq 0$. If $l \leq n - 2$, then by Lemma 4.3 $\chi(\Sigma \cap (C_l \cup C_{l+1})) \leq -4$ and $\chi(\Sigma \cap C_j) \leq -2$ whenever $l + 1 < j < n$. Therefore

$$\chi(\Sigma) = \chi(\Sigma \cap (C_l \cup C_{l+1})) + \sum_{j=l+2}^{n-1} \chi(\Sigma \cap C_j) \leq -4 - 2(n - l - 2),$$

$g(\Sigma) \geq n - l + 1$, and (2) holds.

Thus, if $l \leq n - 2$, then (1) or (2) holds. Now suppose $l = n - 1$, so if (2) does not hold, then Σ is a $(1, b)$ -surface. As above, if $\Sigma \cap X$ is meridian disks, (1) holds. Otherwise, $\Sigma \cap C_{n-1}$ is a vertical annulus, as is $\Sigma \cap (T_n \times I)$. Viewing K_n as a regular fiber of C_n , we see that each component of $\Sigma \cap T_n$ must contribute $|p_n - p_{n-1}q_{n-1}q_n|$ intersections with K_n , and since there are at least two such components, we have $b \geq |p_n - p_{n-1}q_{n-1}q_n|$. \square

Note that by [22, 23], the bridge spectrum of the (p_0, q_0) -torus knot K_0 is $\mathbf{b}(K_0) = (\min\{p_0, q_0\}, 0)$. We describe the spectrum of K_n inductively.

Theorem 6.1. *Suppose that K_n is an iterated torus knot, whose cabling parameters satisfy $|p_i - p_{i-1}q_{i-1}q_i| > 1$. Then*

$$b_g(K_n) = \begin{cases} q_n \cdot b_g(K_{n-1}) & \text{if } g < n, \\ \min\{|p_n - p_{n-1}q_{n-1}q_n|, q_n\} & \text{if } g = n, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$\mathbf{b}(K_n) = q_n \cdot \mathbf{b}(K_{n-1}) + \min\{|p_n - p_{n-1}q_{n-1}q_n|, q_n\} \cdot \mathbf{e}_n.$$

Proof. By Theorem 1.3, $b_0(K_n) = q_n \cdot b_0(K_{n-1})$, and by the above arguments $b_g(K_n) = 0$ if and only if $g > n$. First, we exhibit bridge surfaces satisfying the above equalities. Any (g, b) -bridge surface Σ for K_{n-1} can also be seen as a $(g, q_n \cdot b)$ -surface for K_n by replacing each trivial arc of K_{n-1} with q_n trivial arcs of its cable K_n . It follows that $b_g(K_n) \leq q_n \cdot b_g(K_{n-1})$ for $g < n$, and since $b_n(K_{n-1}) = 0$, perturbing this surface yields an $(n, 1)$ -surface for K_{n-1} , implying $b_n(K_n) \leq q_n$.

Let $p_n^* = |p_n - p_{n-1}q_{n-1}q_n|$. Following the proof of Lemma 6.2, we can see that (V_{n-1}, K_n) has a $(1, p_n^*)$ -surface Σ' constructed by taking the union of a vertical annulus in C_{n-1} , a vertical annulus in $T_n \times I$, and a ∂ -parallel annulus in X . If $n = 1$, this is a $(1, p_n^*)$ -surface for (S^3, K_1) ; otherwise, by Theorem 4.2, $E(K_{n-2})$ has a genus n Heegaard surface Σ'' , and thus $\{\Sigma', T_{n-1}, \Sigma''\}$ is a generalized bridge splitting for (S^3, K_n) whose amalgamation Σ^* is an (n, p_n^*) -bridge surface. It follows that $b_n(K_n) \leq p_n^*$.

Now, let $(S^3, K_n) = (V, \alpha) \cup_\Sigma (W, \beta)$ be a (g, b) -bridge splitting with $g \leq n$, so that $b \geq 1$ by above arguments. If Σ is strongly irreducible, then by Lemma 6.2, either $b \geq q_n \cdot b_g(K_{n-1})$, or $g = n = 1$ and $b \geq \min\{p_n^*, q_n\}$. Otherwise, Σ is weakly reducible and may be untelescoped to yield a generalized bridge splitting $\{\Sigma_0, S_1, \dots, S_d, \Sigma_d\}$, where each Σ_i is strongly irreducible and each S_i is essential. By Theorem 6.6 of [28], $E(K_n)$ contains no essential meridional surfaces, so $S_i \cap K_n = \emptyset$ for all i and thus by Lemma 3.5, each S_i must be isotopic to some T_j .

As K_n lies on the same side of T_j for all j , either $\Sigma_0 \cap K_n \neq \emptyset$ or $\Sigma_d \cap K_n \neq \emptyset$. Assume $\Sigma_d \cap K_n \neq \emptyset$ and consider let $T_l = S_d$. Then Σ_d is a strongly irreducible bridge surface for some V_l , Σ_d has the same bridge number as Σ , and $\{\Sigma_0, S_1, \dots, S_{d-1}, \Sigma_{d-1}\}$ is a generalized Heegaard splitting for $E(K_{l-2})$. By Theorem 4.2, we have

$$\sum_{i=0}^{d-1} g(\Sigma_i) - \sum_{i=1}^{d-1} g(S_i) \geq g(E(K_{l-2})) = l + 1.$$

Consider the three possibilities afforded to Σ_d by Lemma 6.2: in the first case, $b \geq \max\{q_n \cdot b_g(K_{n-1}), q_n\}$. If $g(\Sigma_d) > n - l$, then we have

$$g(\Sigma) = g(\Sigma_d) - g(T_l) + \sum_{i=0}^{d-1} g(\Sigma_i) - \sum_{i=1}^{d-1} g(S_i) > (n - l) - 1 + (l + 1) = n,$$

a contradiction.

In the final case, $l = n - 1$, $g(\Sigma_d) = 1$, and $b \geq p_n^*$. This implies

$$g(\Sigma) = g(\Sigma_d) - g(T_{n-1}) + \sum_{i=0}^{d-1} g(\Sigma_i) - \sum_{i=1}^{d-1} g(S_i) \geq 1 - 1 + n = n,$$

hence $g(\Sigma) = n$ and $b_n(K_n) \geq p_n^*$, completing the proof of the theorem. \square

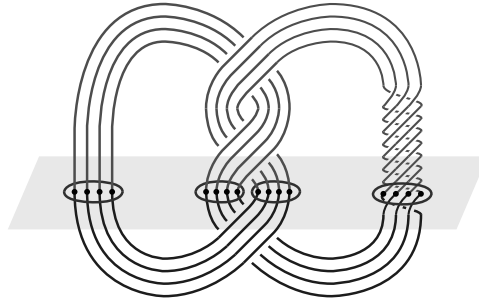


Figure 4: A $(0, 8)$ -surface Σ_0 for K_1 , with curves in Σ_0 depicting $T_n \cap \Sigma_0$.

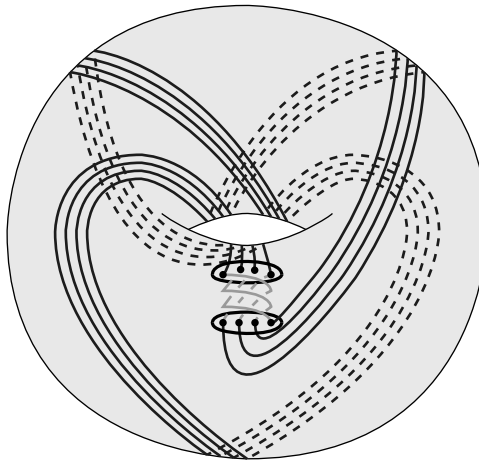


Figure 5: A $(1, 4)$ -surface Σ_1 for K_1 , with curves in Σ_1 depicting $T_n \cap \Sigma_1$.

7. An example

Here we apply Theorem 6.1 to produce the bridge spectrum of $K_1 = ((3, 2), (21, 4))$, with illustrations. Note that K_1 is a $(21, 4)$ -cable of the trefoil $K_0 = ((3, 2))$, so by Theorem 1.3, we have $b_0(K_1) = 8$. An illustration of a minimal $(0, 8)$ -surface Σ_0 appears in figure 4, where $X \cap \Sigma_0$ is a collection of meridian disks, each intersecting K_1 four times.

Turning to the genus one case, taking an obvious cabling of $(1, 1)$ -surface for K_0 yields a $(1, 4)$ -surface Σ_1 for K_1 , where $X \cap \Sigma_1$ is a collection of meridian disks, each of which hits K_1 four times. See figure 5.

However, the surface is not minimal; by Theorem 6.1,

$$b_1(K_1) = \min\{|21 - 3 \cdot 2 \cdot 4|, 4\} = 3.$$

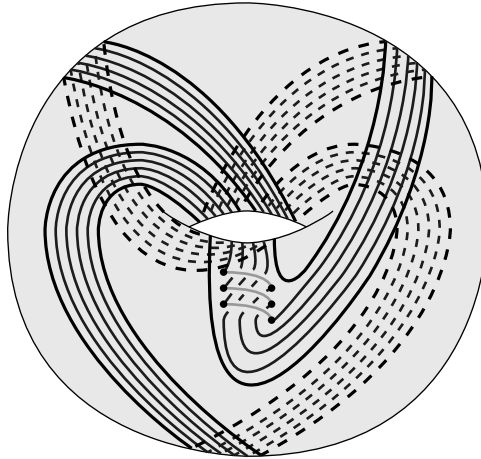


Figure 6: A $(1,3)$ -surface Σ'_1 for K_1 , with curves in Σ'_1 depicting $T_n \cap \Sigma'_1$.

Thus, there is a $(1,3)$ -surface Σ'_1 for K_1 , shown in figure 6, where $X \cap \Sigma'_1$ is a boundary parallel annulus. It is not difficult to observe that Σ_1 is a perturbation of Σ'_1 .

Finally, K_1 is isotopic into a genus g Heegaard surface for S^3 whenever $g \geq 2$, and we have

$$\mathbf{b}(K_1) = (8, 3, 0);$$

hence, the bridge spectrum of K_1 has two gaps.

8. Questions

We conclude with several open questions that may be of interest.

Question 2. *What other spectra can be realized by knots in S^3 ? Specifically, for any decreasing sequence \mathbf{v} of positive integers, is there a knot K such that $\mathbf{b}(K) = \mathbf{v}$?*

Question 3. *What other families of knots have interesting bridge spectra? For instance, what is the bridge spectrum of a twisted torus knot?*

While it is relatively straightforward to exhibit candidate bridge surfaces for twisted torus knots, showing these positions to be minimal is a more complicated task. In general, Lemma 5.2 does not apply to hyperbolic twisted torus knots, and so a new strategy would likely be required.

The next question is posited in [3] by Doll.

Question 4. For knots K_1 and K_2 in S^3 , is there a relationship between $\mathbf{b}(K_1)$, $\mathbf{b}(K_2)$, and $\mathbf{b}(K_1\#K_2)$?

A simple construction shows that

$$b_g(K_1\#K_2) \leq \min\{b_{g_1}(K_1) + b_{g_2}(K_2) - 1 : g_1 + g_2 = g\}.$$

However, for any n , there are knots K_1 and K_2 such that $t(K_1\#K_2) < t(K_1) + t(K_2) - n$ (see [9, 13]); hence, these knots have the property that for some g the inequality above is strict. It may be possible that the inequalities above become equalities when we restrict to the class of meridionally small knots, and Lemma 5.2 may be of use here.

We can also examine the overall bridge structure of iterated torus knots.

Question 5. Is there an iterated torus knot K_n with an irreducible (g, b) -bridge surface Σ such that $b > b_g(K_n)$?

In [21], it is shown that every bridge surface for a 2-bridge knot is the result of stabilization, perturbation, and meridional stabilization performed on a $(0, 2)$ -surface. Is it possible that all bridge surfaces for iterated torus knots are derived in this way from the bridge surfaces exhibited here, or are there any unexpected surfaces?

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