Dressing transformations of constrained Willmore surfaces

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We use the dressing method to construct transformations of constrained Willmore surfaces in arbitrary codimension. An adaptation of the Terng–Uhlenbeck theory of dressing by simple factors to this context leads us to define Bäcklund transforms of these surfaces for which we prove Bianchi permutability. Specializing to codimension 2, we generalize the Darboux transforms of Willmore surfaces via Riccati equations, due to Burstall–Ferus–Leschke–Pedit–Pinkall, to the constrained Willmore case and show that they amount to our Bäcklund transforms with real spectral parameter.

1. Introduction

The Willmore functional is a conformally invariant functional, defined on immersed surfaces in the conformal n-sphere, with a long, interesting history [1, 7, 21, 39, 41]. Its extrema are the *Willmore surfaces* and these have attracted a lot of attention in recent years, in large part due to interest in the celebrated Willmore conjecture, now affirmed by Marques and Neves [28].

One approach to Willmore surfaces is via a certain Gauss map: to any surface in S^n, one may associate [1, 7] its *central sphere congruence.* Geometrically, this is a family of 2-spheres tangent to the surface and sharing mean curvature vectors with it. Alternatively, it may be viewed as a map into the space of 2-spheres which is a certain Grassmannian of 4-planes [36]. A key result is that a surface is Willmore if and only if its central sphere congruence is a harmonic map [1, 7, 18, 35] and then the well-developed theory of harmonic maps applies. In particular, the machinery of integrable systems theory becomes available.

The starting point of the integrable systems approach to harmonic maps is the observation [30, 40, 42, 43] that the harmonic map equations amount to the flatness of a family of flat connections depending on a *spectral parameter.* This structure gives rise to symmetries, of which we shall say much more

below, and constructions via completely integrable Hamiltonian systems and algebraic curves [11, 23].

This zero curvature representation of the harmonic map equations allows one to deduce two kinds of symmetry. Firstly, harmonic maps admit a *spectral deformation* [40] by exploiting a scaling freedom in the spectral parameter. Secondly, harmonic maps admit *dressing transformations* [38, 40] which arise by applying carefully chosen gauge transformations to the family of flat connections.

A larger class of surfaces arises when one imposes the weaker requirement that a surface extremize the Willmore functional only with respect to variations which preserve the conformal structure: these are the *constrained Willmore surfaces*. Now the Euler–Lagrange equations include a Lagrange multiplier in the form of a holomorphic quadratic differential [4] and, while the central sphere congruence is no longer harmonic, a loop of flat connections is still available [2, 14].

It is the purpose of the present paper to apply the theory of dressing transformations to constrained Willmore surfaces in S^n. Two considerations prevent this from being a routine exercise in the existing theory for harmonic maps: first, that theory applies to the central sphere congruence qua map into a Grassmannian and it requires some care to see that the transformed map is the central sphere congruence of a surface at all, let alone a constrained Willmore one. Moreover, the theory must be extended somewhat to cover the possibly non-harmonic central sphere congruences of constrained Willmore surfaces.

To address the second problem, we introduce and study the notion of a *k-perturbed harmonic map* which includes (with $k = 2$) the central sphere congruences of constrained Willmore surfaces. The idea is that the flat connections associated to harmonic maps are characterized by having simple poles (with respect to the spectral parameter) at 0 and ∞. Relaxing this requirement to allow poles of order k leads us to k-perturbed harmonic maps. These maps are a geometric incarnation of the kth elliptic integrable system associated with a symmetric space in the sense of Terng [37] (see also Khemar [24]) and also fit into the framework of Brander and Dorfmeister [5]. As such, k-perturbed harmonic maps make sense with any symmetric space as target but we focus on the space of 2-spheres that is of immediate interest to us. We define spectral deformations and dressing transformations of such maps, the latter mediated by gauge transformations we call *dressing gauges*.

We define an energy density for k-perturbed harmonic maps which coincides with the harmonic map energy density when $k = 1$ and prove that this changes by an explicit exact term under our dressing transformations.

This result, of possible independent interest, appears to be new, even for harmonic maps.

We then show that our dressing transformations preserve the class of central sphere congruences of constrained Willmore surfaces. In this setting, we see that the Willmore density changes by an exact term only and, moreover, the holomorphic quadratic differential that acts as Lagrange multiplier is fixed by the dressing transformations.

We complete our general theory by adapting the Terng–Uhlenbeck [38] theory of dressing by simple factors to our context. Thus we construct dressing gauges algebraically from parallel subbundles of one of the given flat connections. We call the resulting dressing transformations *Bäcklund transformations* and prove a Bianchi permutability theorem for them.

Finally, we restrict attention to the case of surfaces in S^4. Here, there is a well-developed quaternionic formalism [2, 3, 10, 19] that can be brought to bear. In particular, arguing by analogy with Darboux transforms of constant mean curvature surfaces, a Darboux transform of Willmore surfaces in S^4 is presented in [10] which proceeds by solving a Riccati equation. For us, the key point about this codimension 2 setting is that the space of (oriented) 2-spheres is a (pseudo-)Hermitian symmetric space. Consequently, one may apply a canonical gauge transformation, arising from the ambient complex structure, to replace the family of flat connections with a simpler *untwisted* family with poles of about half the order of the original family. Working with the untwisted family, we find that the Bäcklund transformations specialize precisely to simple factor dressing in the sense of Terng–Uhlenbeck and, moreover, can be derived from solutions of a Riccati equation. In particular, we simultaneously extend the Darboux transforms of [10] to the constrained Willmore case and identify them with the Bäcklund transforms with real spectral parameter. These results generalize a similar analysis of harmonic maps into the 2-sphere and the constant mean curvature surfaces of which they are Gauss maps that is carried out in [9].

The results of the paper are based, in part, on those in the second author's PhD thesis [31, 32] and some of them were announced in [33].

2. Constrained Willmore surfaces and perturbed harmonicity

2.1. Conformal submanifold geometry

2.1.1. Conformal geometry of the sphere   We are going to study submanifolds of the conformal n-sphere S^n and find a convenient conformally

invariant viewpoint on the latter in Darboux's light-cone model [16, Chapitre VI]. For this, contemplate the Lorentzian space-time $\mathbb{R}^{n+1,1}$ with inner product (,) and light-cone $\mathcal{L}$. The corresponding quadric $\mathbb{P}(\mathcal{L})$ is a conformal manifold: each section σ of $\mathcal{L}^{\times} \to \mathbb{P}(\mathcal{L})$ provides a metric g_{σ} via $g_{\sigma}(X,Y) = (d_X\sigma, d_Y\sigma)$ and $g_{\mathrm{e}^u\sigma} = \mathrm{e}^{2u}g_{\sigma}$. Fixing unit-timelike $t_0 \in \mathbb{R}^{n+1,1}$ yields a section σ with $(\sigma, t_0) \equiv -1$ and so an isometry $\sigma : (\mathbb{P}(\mathcal{L}), g_{\sigma}) \cong \{v \in \mathcal{L} : (v, t_0) = -1\}$ which last is isometric (via $v \mapsto v - t_0$) to the unit sphere in $\langle t_0 \rangle^{\perp}$. Thus $\mathbb{P}(\mathcal{L}) \cong S^n$ *qua* conformal manifolds.

Moreover, it is clear that the orthogonal group $\mathrm{O}(n+1,1)$ acts transitively and conformally on $\mathbb{P}(\mathcal{L})$. In this way, it actually double covers the Möbius group of global conformal diffeomorphisms on S^n.

For more details, see [22, Chapter 1].

In the sequel, we will have much to do with $\mathrm{O}(n+1,1)$-connections and so forms with values in the Lie algebra $\mathfrak{o}(n+1,1)$. We shall make repeated (and silent!) use of the isomorphism $\wedge^2\mathbb{R}^{n+1,1} \cong o(\mathbb{R}^{n+1,1})$ given by $u \wedge v(w) := (u,w)v - (v,w)u$, for $u, v, w \in \mathbb{R}^{n+1,1}$.

2.1.2. Conformally immersed surfaces and the central sphere congruence

A map $f : \Sigma \to \mathbb{P}(W)$ of a manifold to a projective space is the same as a line subbundle of the trivial bundle $\underline{W} = \Sigma \times W$: we identify f with the bundle whose fibre at x is $f(x)$. In particular, we will usually view a map $\Lambda : \Sigma \to \mathbb{P}(\mathcal{L})$ as a null line subbundle of $\underline{\mathbb{R}}^{n+1,1}$. From this point of view, sections of Λ are simply lifts of Λ to maps $\Sigma \to \mathbb{R}^{n+1,1}$.

Given such a Λ, we define the derived bundle

$$\Lambda^{(1)} := \langle \sigma, d\sigma(T\Sigma) \rangle \leq \Lambda^{\perp},$$

for σ an arbitrary lift of Λ. Note that Λ is an immersion if and only if the bundle $\Lambda^{(1)}$ has rank $\dim\Sigma + 1$ and then every lift is also an immersion $\sigma : \Sigma \to \mathbb{R}^{n+1,1}$.

Henceforth, we will take Σ to be a Riemann surface, thus an oriented, conformal manifold of real dimension 2 or, equivalently, a complex manifold of dimension 1. We denote the complex structure of Σ by J^{Σ}. The type decomposition of $T\Sigma$ now induces a decomposition of $\Lambda^{(1)}/\Lambda$: set

$$\Lambda^{1,0} = \Lambda \oplus d\sigma(T^{1,0}\Sigma) \quad \Lambda^{0,1} = \Lambda \oplus d\sigma(T^{0,1}\Sigma),$$

for some, hence any, lift σ. Then

$$\Lambda^{1,0} + \Lambda^{0,1} = \Lambda^{(1)},$$

$$\Lambda^{1,0} \cap \Lambda^{0,1} = \Lambda,$$
$$\Lambda^{1,0} = \overline{\Lambda^{0,1}}.$$

Convention.   Here and below, we do not distinguish notationally between a subbundle and its complexification. In fact, we shall usually take all subbundles to be complex subbundles of $\underline{\mathbb{C}}^{n+2} = (\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}}$ and view such a subbundle as real if it is stable under the complex conjugation on $\underline{\mathbb{C}}^{n+2}$ with fixed set $\underline{\mathbb{R}}^{n+1,1}$.

An immersion Λ is conformal if and only if each lift is a conformal immersion $\Sigma \to \mathbb{R}^{n+1,1}$ if and only if $\Lambda^{1,0}$ (and hence also $\Lambda^{0,1}$) is isotropic with respect to the (complex bilinear extension of) $(\ ,\ )$.

A key tool in conformal submanifold geometry is the *central sphere congruence* or *conformal Gauss map* [1, 7, 18, 21, 35] V of a conformal immersion Λ. For us, this is the bundle $V \le \underline{\mathbb{R}}^{n+1,1}$ of $(3,1)$-planes given by

$$\Lambda^{(1)} \oplus \langle \Delta\sigma \rangle,$$

for σ any lift of Λ and Δ the Laplacian of g_σ.

From V, we get an orthogonal decomposition of the trivial bundle

$$\underline{\mathbb{R}}^{n+1,1} = V \oplus V^\perp$$

and a corresponding reduction of the trivial connection:

$$d = \mathcal{D}_V + \mathcal{N}_V,$$

where $V, V^\perp$ are $\mathcal{D}_V$-parallel and $\mathcal{N}_V$ takes values in $V \wedge V^\perp \subset \mathfrak{o}(\underline{\mathbb{R}}^{n+1,1})$. Alternatively, viewing V as a map $\Sigma \to \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$ to the Grassmannian of $(3,1)$-planes, $\mathcal{N}_V$ may be canonically identified with dV.

We note that since $\Lambda^{(1)} \le V$, $\mathcal{N}_V(\Lambda) = 0$ and, similarly, $\Delta\sigma \in \Gamma V$ is equivalent to $\mathcal{N}_V^{1,0}(\Lambda^{0,1}) = 0$.

2.2. Constrained Willmore surfaces and holomorphic quadric differentials

Let Λ be a conformal immersion of a compact Riemann surface Σ with central sphere congruence V. The *Willmore energy* $W(\Lambda)$ of Λ is given by

$$W(\Lambda) = \tfrac{1}{2} \int_\Sigma |\mathrm{II}_0|^2,$$

where II_0 is the trace-free second fundamental form of Λ (computed against any representative metric on S^n and independent of that choice). This is known [1, 7, 10, 13, 18] to coincide with the harmonic map energy of V:

$$(2.1) \qquad W(\Lambda) = E(V) = \tfrac{1}{2} \int_\Sigma (\mathcal{N}_V \circ J^\Sigma \wedge \mathcal{N}_V).$$

A conformal immersion is *Willmore* if it extremizes the Willmore functional and *constrained Willmore* if it extremizes the Willmore functional with respect to infinitesimally conformal variations. It follows at once from (2.1) that a surface is Willmore when its central sphere congruence is harmonic and it is well known [1, 7, 35] that the converse holds. Thus, a surface is Willmore if and only if $d^{\mathcal{D}} * \mathcal{N} = 0$. More generally, we have the following reformulation of a theorem of Bohle–Peters–Pinkall [4]:

Theorem 2.1 [13, Section 14]. Λ *is a constrained Willmore surface if and only if there exists a real form* $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ *with*

$$(2.2a) \qquad d^{\mathcal{D}_V} q = 0,$$

such that

$$(2.2b) \qquad d^{\mathcal{D}_V} * \mathcal{N}_V = 2[q \wedge *\mathcal{N}_V].$$

Such a form q *is said to be a* (Lagrange) multiplier *for* Λ. *Thus Willmore surfaces are constrained Willmore surfaces with zero multiplier.*

When Σ is non-compact, we define constrained Willmore surfaces to be the solutions of (2.2).

The 1-form q is a bundle-valued avatar of a more familiar geometric object: a holomorphic quadratic differential on Σ. Indeed, given $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$, we define a 2-tensor Q on M by setting

$$(2.3) \qquad Q(X, Y)\sigma := q_X(d_Y\sigma),$$

for $\sigma \in \Gamma\Lambda$. We then have

Lemma 2.2 [13, Section 12]. *Let* $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ *and define* Q *by* (2.3). *Then* $d^{\mathcal{D}_V} q = 0$ *if and only if* Q *is symmetric and trace-free, thus* $Q^{1,1} = 0$, *and* $Q^{2,0}$ *is holomorphic.*

In this case, $q^{1,0} \in \Omega^1(\Lambda \wedge \Lambda^{0,1})$.

It is the holomorphic quadratic differential Q that appears in the Bohle–Peters–Pinkall version of (2.2b).

Remark 2.3. *In general, for a given constrained Willmore surface Λ, the multiplier q is unique. The exception is when Λ is in addition isothermic: if $d^{\mathcal{D}_V} * \mathcal{N}_V = 2[q_1 \wedge *\mathcal{N}_V] = 2[q_2 \wedge *\mathcal{N}_V]$ with $d^{\mathcal{D}_V} q_i = 0$ then $\eta := *q_2 - *q_1$ solves $d^{\mathcal{D}_V}\eta = [\mathcal{N}_V \wedge \eta] = 0$ or, equivalently, $d\eta = 0$. This last condition amounts to the existence of a holomorphic quadratic differential that commutes with the trace-free second fundamental form, that is, that Λ is isothermic. In this situation, $q_1 + *s\eta$ is a multiplier for Λ for each $s \in \mathbb{R}$.*

The isothermic Willmore surfaces in S^3 are precisely the minimal surfaces with respect to a constant curvature metric [39] while the isothermic constrained Willmore surfaces include those of constant mean curvature [34]. See [4, 14] for more details on this topic.

2.3. Flat connections and perturbed harmonic maps

The key to the integrable systems approach to harmonic maps is the well-known observation that the harmonic map equations amount to the flatness of a family of connections [30, 40, 42, 43]. In particular, this gives a zero-curvature characterization of Willmore surfaces but more is true: the constrained Willmore equations also admit a spectral deformation [14] and hence a zero-curvature representation:

Theorem 2.4 [13]. *A conformal immersion Λ with central sphere congruence V is constrained Willmore if and only if there exists a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ such that*

$$(2.4) \qquad d^\lambda := \mathcal{D}_V + \lambda \mathcal{N}_V^{1,0} + \lambda^{-1}\mathcal{N}_V^{0,1} + (\lambda^2 - 1)q^{1,0} + (\lambda^{-2} - 1)q^{0,1}$$

is flat for all $\lambda \in \mathbb{C}^\times$.

Let us abstract the main properties enjoyed by V and the family of connections d^λ. For this, we take $V \leq \underline{\mathbb{R}}^{n+1,1}$ to be any bundle of $(3, 1)$-planes and consider the corresponding decomposition of the trivial connection: $d = \mathcal{D}_V + \mathcal{N}_V$. Let ρ_V denote reflection across V.

Definition 2.5. For $k \in \mathbb{N}$, say that V is *k-perturbed harmonic* if there is a family of flat metric connections d^λ, $\lambda \in \mathbb{C}^\times$, on $\underline{\mathbb{C}}^{n+2}$, with the following

properties:

$$(2.5\text{a}) \qquad\qquad d^1 = d \qquad\qquad\qquad \text{(normalization)},$$

$$(2.5\text{b}) \qquad\qquad \rho_V \cdot d^\lambda = d^{-\lambda} \qquad\qquad (\rho_V\text{-twisted}),$$

$$(2.5\text{c}) \qquad\qquad d^{1/\bar\lambda} = \overline{d^\lambda} \qquad\qquad \text{(reality)}$$

and $\lambda \mapsto (d^\lambda)^{1,0}$ is holomorphic on $\mathbb{C}$ with a pole of order k at ∞ (whence, thanks to (2.5c), $\lambda \mapsto (d^\lambda)^{0,1}$ is holomorphic on $\mathbb{C}^\times \cup \{\infty\}$ with a pole of order k at 0). In this case, we say that (V, d^λ) is a k-perturbed harmonic bundle.[1]

In this situation, we may write

$$d^\lambda = \sum_{|i| \le k} \lambda^i A_i$$

with A_0 a real metric connection preserving V and $A_i \in \Omega^1(\mathfrak{o}(\underline{\mathbb{C}}^{n+2}))$, $i \ne 0$, satisfying $A_i^{1,0} = 0$, for $i < 0$; $A_{-i} = \overline{A_i}$; $\mathrm{Ad}_{\rho_V} A_i = (-1)^i A_i$ and

$$\mathcal{D}_V = \sum_i A_{2i} \quad \mathcal{N}_V = \sum_i A_{2i+1}.$$

It follows that V is 1-perturbed harmonic if and only if it is harmonic ($A_0 = \mathcal{D}_V$ and $A_1 = \mathcal{N}_V^{1,0}$).

Notation. *For $\lambda \in \mathbb{C} \cup \{\infty\}$, set $\lambda^* = 1/\bar\lambda$ so that the reality condition (2.5c) reads*

$$d^{\lambda^*} = \overline{d^\lambda}.$$

With these notions in hand, we reformulate Theorem 2.4 as follows:

Theorem 2.6. *A conformal immersion Λ with central sphere congruence V is constrained Willmore if and only if V is 2-perturbed harmonic with d^λ satisfying $A_2 + A_{-2} \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$.*

Remark 2.7. *The condition of k-perturbed harmonicity makes sense for maps into an arbitrary Grassmannian of non-degenerate subspaces of an inner product space $\mathbb{R}^{p,q}$, or, indeed, an arbitrary pseudo-Riemannian symmetric space. Much of what we say below applies in this more general context.*

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[1]A little circumspection is required here as we have seen in Remark 2.3 that a given bundle V can be k-perturbed harmonic in more than one way.

3. Transformations of k-perturbed harmonic bundles and constrained Willmore surfaces

Our k-perturbed harmonic bundles come equipped with a family of flat connections d^λ. Such connections, at least on simply connected open subsets of Σ, are gauge equivalent to the trivial connection. These, and other, gauge transformations can be exploited to produce new k-perturbed harmonic bundles in various ways. We begin with the simplest of these transformations.

3.1. Spectral deformation

Let (V, d^λ) be k-perturbed harmonic. For $\mu \in S^1$, we have from (2.5c) that d^μ is a real, flat, metric connection and so there is, at least locally, a gauge transformation $\Phi_\mu \in \Gamma(\mathrm{O}(\mathbb{R}^{n+1,1}))$ with

$$(3.1) \qquad \Phi_\mu \cdot d^\mu = d,$$

where $\Phi_\mu \cdot d^\mu = \Phi_\mu \circ d^\mu \circ \Phi_\mu^{-1}$ is the usual action of gauge transformations on connections. We note that Φ_μ is unique up to left multiplication by a constant element of $\mathrm{O}(n+1,1)$.

Proposition 3.1. *Define $V_\mu := \Phi_\mu V$. Then V_μ is also k-perturbed harmonic with associated flat connections*

$$(3.2) \qquad d_\mu^\lambda = \Phi_\mu \cdot d^{\lambda\mu}.$$

Proof. The connections d_μ^λ are certainly flat, being gauges of the flat connections $d^{\lambda\mu}$, so we must show that they have the properties enumerated in Definition 2.5. First, (3.1) tells us that $d_\mu^1 = d$ so the family is normalized. Secondly, we have $\rho_{V_\mu} \circ \Phi_\mu = \Phi_\mu \circ \rho_V$ which, along with (2.5b), rapidly yields that d_μ^λ is ρ_{V_μ}-twisted. Similarly, $\mu = \mu^*$ and $\overline{\Phi_\mu} = \Phi_\mu$ which, together with (2.5c), gives the reality condition for d_μ^λ. Finally, $\lambda \mapsto d^{\lambda\mu}$ has the same poles at 0 and ∞ as d^λ whence $\lambda \mapsto d_\mu^\lambda$ does also. $\qquad \square$

We can iterate this construction: for $\nu \in S^1$, find a gauge transformation Φ_ν^μ for which $\Phi_\nu^\mu \cdot d_\mu^\nu = d$. Together with (3.2), this gives

$$(\Phi_\nu^\mu \Phi_\mu) \cdot d^{\mu\nu} = d,$$

so that we can take $\Phi_{\mu\nu} = \Phi_\nu^\mu \Phi_\mu$ and conclude:

Proposition 3.2. *For $\mu, \nu \in S^1$ and (V, d^λ) k-perturbed harmonic, $V_{\mu\nu} = (V_\mu)_\nu$.*

Thus we have an S^1-action on k-perturbed harmonic bundles up to congruence. The analysis applies for k-perturbed harmonic maps into any symmetric space, *mutatis mutandis*, and, for $k = 1$, recovers the well-known result of Terng (cf. [40]).

Expanding (3.2) in powers of λ gives:

Lemma 3.3. *In the situation of Proposition 3.1, write $d^\lambda = \sum A_i$ and $d^\lambda_\mu = \sum A_i^\mu$. Then*

$$(3.3a) \qquad\qquad A_0^\mu = \Phi_\mu \cdot A_0,$$
$$(3.3b) \qquad\qquad A_i^\mu = \mu^i \mathrm{Ad}_{\Phi_\mu} A_i,$$

for $i \neq 0$.

We use this to examine the special case where V is the central sphere congruence of a constrained Willmore surface Λ with multiplier q. In this case, set $\Lambda_\mu = \Phi_\mu \Lambda$ and define q_μ by

$$q_\mu = A_2^\mu + A_{-2}^\mu = \mathrm{Ad}_{\Phi_\mu}(\mu^2 q^{1,0} + \mu^{-2} q^{0,1}),$$

where the last equality is an instance of (3.3b). We now recover a result of [14] as formulated in [13]:

Proposition 3.4 [14]. *For each $\mu \in S^1$, Λ_μ is a conformal immersion and a constrained Willmore surface with multiplier q_μ.*

Proof. To see that Λ_μ is a conformal immersion, we need $\Lambda_\mu^{1,0}$ to be maximal isotropic in V_μ. For V_μ to be the central sphere congruence of Λ_μ, we need $\mathcal{N}_{V_\mu}^{1,0} \Lambda_\mu^{0,1} = 0$ and, finally, we need $q_\mu^{1,0}$ to take values in $\wedge^2 \Lambda_\mu^{0,1}$. Then Theorem 2.6 yields the result. However, since (3.3b) gives $\mathcal{N}_{V_\mu}^{1,0} = \mu \mathrm{Ad}_{\Phi_\mu} \mathcal{N}_V^{1,0}$, all of this follows as soon as we know that $\Lambda_\mu^{1,0} = \Phi_\mu \Lambda^{1,0}$. For this last, observe that $q\Lambda = \mathcal{N}\Lambda = 0$, so that the operators d^μ and d coincide on $\Gamma\Lambda$. We conclude that $\Lambda^{1,0} = \Lambda \oplus d^\mu \sigma(T^{1,0}\Sigma)$, for any $\sigma \in \Gamma\Lambda$ and, gauging by Φ_μ, we see that $\Lambda_\mu^{1,0} = \Phi_\mu \Lambda^{1,0}$ as required. $\qquad\square$

3.2. Dressing action

We are going to use a version of the Terng–Uhlenbeck dressing action [38] to construct new constrained harmonic bundles from V. The key idea here is to find λ-dependent gauge transformations that preserve the algebraic shape of d^λ (see [12, 15] for similar viewpoints on dressing). These transformations, when applied to the central sphere congruence of a constrained Willmore surface, will give rise to new such surfaces.

We begin by specifying the properties of this gauge transformation.

Definition 3.5. Let (V, d^λ) be a k-perturbed harmonic bundle. A *dressing gauge for* (V, d^λ) is a family of gauge transformations $\lambda \mapsto r(\lambda) \in \Gamma O(\mathbb{C}^{n+2})$ which is holomorphic in λ near $0, \infty \in \mathbb{P}^1$ and has the following properties:

(1) $r(-\lambda) \circ \rho_V \circ r(\lambda)^{-1}$ is independent of $\lambda \in \operatorname{dom}(r)$.

(2) For all $\lambda \in \operatorname{dom}(r)$,

$$(3.4) \qquad\qquad r(\lambda^*) = \overline{r(\lambda)} \quad \text{(reality)}.$$

(Thus we require that $\operatorname{dom}(r)$ is stable under $\lambda \mapsto \pm\lambda^*$.)

(3) The connections $r(\lambda) \cdot d^\lambda$ extend from $\operatorname{dom}(r) \setminus \{0, \infty\}$ to a holomorphic family of connections $\hat{d}^\lambda$ on $\mathbb{C}^\times$ with $\hat{d}^1 = d$.

(4)

$$(3.5) \qquad\qquad \det(r(0)^{-1} r(\infty)_{|V}) = 1.$$

For such an r, set $\hat{V} = r(0)V$ and call $(\hat{V}, \hat{d}^\lambda)$ the *dressing transform of* (V, d^λ) *by* r.

Note that item (1) now reads

$$(3.6) \qquad\qquad r(-\lambda) \circ \rho_V = \rho_{\hat{V}} \circ r(\lambda), \quad \text{(twisted)}$$

for $\lambda \in \operatorname{dom}(r)$, and then evaluating at $\lambda = \infty$ yields $\hat{V} = r(\infty)V$ also. In particular, $r(0)^{-1} r(\infty)V = V$, so that (3.5) makes sense.

It is our contention that $(\hat{V}, \hat{d}^\lambda)$ is again a k-perturbed harmonic map $\Sigma \to \operatorname{Gr}_{3,1}(\mathbb{R}^{n+1,1})$. We begin by showing that $\hat{V}$ is indeed (the complexification of) a bundle of real $(3, 1)$-planes. First, evaluate (3.4) at $\lambda = 0$, to get $r(\infty) = \overline{r(0)}$ so that $\overline{\hat{V}} = r(\infty)V = \hat{V}$. We now have:

Lemma 3.6. *Let $W \leq \mathbb{C}^{n+2}$ be a real (thus $W = \overline{W}$), non-degenerate 4-plane and $T \in \mathrm{O}(\mathbb{C}^{n+2})$. Then*

(1) *$W \cap \mathbb{R}^{n+1,1}$ is a $(3, 1)$-plane if and only if $\dim(U \cap \overline{U}) = 1$, for any maximal isotropic 2-plane $U \leq W$.*

(2) *If $W \cap \mathbb{R}^{n+1,1}$ is a $(3, 1)$-plane and $TW = \overline{TW}$ then $TW \cap \mathbb{R}^{n+1,1}$ is also a $(3, 1)$-plane if and only if $\det(T^{-1}\overline{T}) = 1$.*

Proof. For (1), note that $W_{\mathbb{R}} := W \cap \mathbb{R}^{n+1,1}$ has signature $(4, 0)$ or $(3, 1)$. Since $U \cap \overline{U} \cap \mathbb{R}^{n+1,1}$ is a real isotropic subspace of $W_{\mathbb{R}}$, we have $\dim(U \cap \overline{U}) \leq 1$ with equality forcing $W_{\mathbb{R}}$ to have signature $(3, 1)$. For the converse, if $\dim(U \cap \overline{U}) = 0$, then U and $\overline{U}$ are the $\pm\sqrt{-1}$-eigenspaces of an orthogonal complex structure on $W_{\mathbb{R}}$ and this requires that $W_{\mathbb{R}}$ be a $(4, 0)$-plane.

For (2), let $U \leq W$ be maximal isotropic so that TU is maximal isotropic in TW. We have just seen that $(TW)_{\mathbb{R}}$ is a $(3, 1)$-plane if and only if $\dim(TU \cap \overline{TU}) = 1$, or, equivalently, $\dim U \cap (T^{-1}\overline{T})\overline{U} = 1$. Now U and $\overline{U}$ define lines in the quadric in $\mathbb{P}(W)$ given by the ambient inner product and such lines intersect exactly when they lie in different rulings of that quadric. These rulings comprise different $\mathrm{SO}(W)$-orbits which are permuted by $\mathrm{O}(W) \setminus \mathrm{SO}(W)$. Thus $\det(T^{-1}\overline{T}) = 1$ if and only if $\overline{U}$ and $(T^{-1}\overline{T})\overline{U}$ lie in the same ruling if and only if $(T^{-1}\overline{T})\overline{U}$ intersects U in a single (projective) point. $\qquad\square$

Applying this last fibrewise to V with $T = r(0)$ and using (3.5) immediately yields:

Corollary 3.7. *Let $(\hat{V}, \hat{d}^{\lambda})$ be the dressing transform of (V, d^{λ}) by r. Then,*

(1) *$\overline{\hat{V}} = \hat{V}$.*

(2) *$\hat{V} \cap \mathbb{R}^{n+1,1}$ is a bundle of $(3, 1)$-planes.*

(3) *For any maximal isotropic subbundle of U of $\hat{V}$, $U \cap \overline{U}$ has rank 1.*

With these preliminaries dealt with, we have:

Theorem 3.8. *Let $(\hat{V}, \hat{d}^{\lambda})$ be the dressing transform of (V, d^{λ}) by r. Then $(\hat{V}, \hat{d}^{\lambda})$ is k-perturbed harmonic.*

Proof. This amounts to showing that the connections $\hat{d}^{\lambda}$ satisfy the conditions of Definition 2.5. By hypothesis, $\hat{d}^{\lambda}$ is normalized. For the remaining conditions that are pointwise in λ, we establish them first for $\lambda \in \mathrm{dom}(r) \setminus$

$\{0, \infty\}$ and then conclude they hold for all $\lambda \in \mathbb{C}^\times$ by unique continuation. Indeed, (3.6) together with the fact that d^λ is ρ_V-twisted combine to show that $\hat{d}^\lambda$ is $\rho_{\hat{V}}$-twisted on $\mathrm{dom}(r) \setminus \{0, \infty\}$. Similarly, the reality condition on $\hat{d}^\lambda$ follows immediately from that on d^λ and (3.4). Moreover, for $\lambda \in \mathrm{dom}(r) \setminus \{0, \infty\}$, $\hat{d}^\lambda$ is flat since it is a gauge of the flat connection d^λ.

Finally, the pole behaviour of $\hat{d}^\lambda$ at zero and infinity coincides with that of d^λ since the connections differ by the gauge transformations $r(\lambda)$ which are holomorphic in λ near those points. $\qquad\square$

To go further, we need to establish some relations between the coefficients $\hat{A}_i$ of $\hat{d}^\lambda$ and those of d^λ. First let us establish some notation: write

$$A_+ = \sum_{0 < i \le k} \lambda^i A_i, \quad A_- = \sum_{k \le i < 0} \lambda^i A_i$$

so that $d^\lambda = A_+ + A_0 + A_-$ and, similarly, $\hat{d}^\lambda = \hat{A}_+ + \hat{A}_0 + \hat{A}_-$. Moreover, define χ_0, χ_∞ by

$$\chi_0 = r^{-1} \partial r / \partial \lambda,$$
$$\chi_\infty = r^{-1} \partial r / \partial \mu,$$

where $\mu = 1/\lambda$. We note that $\chi_\infty(\lambda) = \overline{\chi_0(\lambda^*)}$, for $\lambda \in \mathbb{C}^\times$. On $\mathrm{dom}(r) \setminus \{0, \infty\}$, we have

$$(3.7) \qquad\qquad\qquad \hat{d}^\lambda = r(\lambda) \cdot d^\lambda.$$

The $(1,0)$-part of this is holomorphic at $\lambda = 0$ and differentiating with respect to λ yields

$$(3.8) \qquad \partial \hat{A}_+ / \partial \lambda = \mathrm{Ad}_r (\partial A_+ / \partial \lambda - A_0^{1,0} \chi_0 - [A_+, \chi_0]).$$

Again, we may write the $(0,1)$-part of (3.7) as

$$(3.9) \qquad\qquad \hat{A}_0^{0,1} + \hat{A}_- = r \cdot A_0^{0,1} + \mathrm{Ad}_r A_-,$$

from which we conclude that $\hat{A}_- - \mathrm{Ad}_r A_-$ is holomorphic near $\lambda = 0$.

Taking coefficients of powers of λ now yields:

Lemma 3.9. *In the situation of Theorem 3.8, with $d^\lambda = \sum_{|i|\le k} \lambda^i A_i$ and $\hat{d}^\lambda = \sum_{|i|\le k} \lambda^i \hat{A}_i$, we have:*

(3.10a)
$$\hat{A}_0^{1,0} = r(0) \cdot A_0^{1,0}, \qquad\qquad \hat{A}_0^{0,1} = r(\infty) \cdot A_0^{0,1},$$
(3.10b)
$$\hat{A}_1 = \mathrm{Ad}_{r(0)}(A_1 - A_0^{1,0}\chi_0(0)), \quad \hat{A}_{-1} = \mathrm{Ad}_{r(\infty)}(A_{-1} - A_0^{0,1}\chi_\infty(\infty)),$$
(3.10c)
$$\hat{A}_{-k} = \mathrm{Ad}_{r(0)}A_{-k}, \qquad\qquad \hat{A}_k = \mathrm{Ad}_{r(\infty)}A_k$$

and, for $k > 1$,

(3.10d)
$$\hat{A}_{-k+1} = \mathrm{Ad}_{r(0)}(A_{-k+1} + [\chi_0(0), A_{-k}])$$
$$\hat{A}_{k-1} = \mathrm{Ad}_{r(\infty)}(A_{k-1} + [\chi_\infty(\infty), A_k]).$$

Finally,

(3.10e)
$$\hat{A}_0^{1,0} = r(\infty) \cdot A_0^{1,0} + \frac{1}{k!}\frac{\partial^k}{\partial\mu^k}\Big|_{\mu=0} \mathrm{Ad}_r(A_k + \cdots + \mu^{k-1}A_1),$$

$$\hat{A}_0^{0,1} = r(0) \cdot A_0^{0,1} + \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\Big|_{\lambda=0} \mathrm{Ad}_r(A_{-k} + \cdots + \lambda^{k-1}A_{-1}).$$

Proof. Evaluate the $(1,0)$ part of (3.7) at $\lambda = 0$ to get $(3.10a)^{1,0}$ and evaluate (3.8) at $\lambda = 0$ to get $(3.10b)^{1,0}$.

Meanwhile, comparing coefficients of λ in (3.9) yields

$$\hat{A}_{-k+i} = \frac{1}{i!}\frac{\partial^i}{\partial\lambda^i}\Big|_{\lambda=0} \mathrm{Ad}_{r(\lambda)}(A_{-k} + \cdots + \lambda^{k-1}A_{-1}), \text{ for } 0 \le i < k,$$

$$\hat{A}_0^{0,1} = r(0) \cdot A_0^{0,1} + \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\Big|_{\lambda=0} \mathrm{Ad}_{r(\lambda)}(A_{-k} + \cdots + \lambda^{k-1}A_{-1}).$$

The last equation is $(3.10e)^{0,1}$ while the cases $i = 0, 1$ of the first yield $(3.10c)^{0,1}$ and $(3.10d)^{0,1}$. A similar argument at $\mu = 0$ (or an appeal to the reality of $\hat{d}^\lambda$) gives the remaining equations. $\qquad\square$

We pause for a short diversion of possibly independent interest: our dressing transformation changes a certain energy density by an *exact* 2-form. First a definition:

Definition 3.10. Let (V, d^λ) be k-perturbed harmonic. The *energy density* $e(V)$ of (V, d^λ) is the 2-form on Σ given by

$$e(V) = i \sum_{0 < j \leq k} j(A_j \wedge A_{-j}).$$

Note that, when V is harmonic (thus $k = 1$), $e(V)$ is the usual harmonic map energy density:

$$e(V) = \tfrac{1}{2}(\mathcal{N}_V \circ J^\Sigma \wedge \mathcal{N}_V).$$

We now have:

Proposition 3.11. *Let (V, d^λ) and $(\hat{V}, \hat{d}^\lambda)$ be k-perturbed harmonic bundles with $\hat{V}$ the dressing transform of V by r. Then*

$$e(\hat{V}) = e(V) - i\, d\mathrm{Res}_{\lambda=0}(\chi_0, A_-).$$

Proof. We begin by observing that our energy density is a residue:

$$\sum_{0 < j \leq k} j(A_j \wedge A_{-j}) = \mathrm{Res}_{\lambda=0}(\partial A^+ / \partial \lambda \wedge A^-).$$

Now $\hat{A}^- - \mathrm{Ad}_r A^-$ is holomorphic near $\lambda = 0$ so that (3.8) yields:

$$\mathrm{Res}_{\lambda=0}(\partial \hat{A}^+ / \partial \lambda \wedge \hat{A}^-) = \mathrm{Res}_{\lambda=0}\big(\mathrm{Ad}_r(\partial A^+ / \partial \lambda - A_0^{1,0}\chi_0 - [A_+, \chi_0]) \wedge \mathrm{Ad}_r A^-\big)$$
$$= \mathrm{Res}_{\lambda=0}\big(\partial A^+ / \partial \lambda - A_0^{1,0}\chi_0 - [A_+, \chi_0] \wedge A^-\big).$$

Since A_- is a $(0, 1)$-form, $(A_0^{1,0} \wedge A_-) = (A_0 \wedge A_-)$ and we have

$$(A_0^{1,0}\chi_0 + [A_+, \chi_0] \wedge A^-) = (A_0\chi_0 + [A_+, \chi_0] \wedge A^-)$$
$$= d(\chi_0, A_-) - (\chi_0, d^{A_0} A_-) - (\chi_0, [A_+ \wedge A_-])$$
$$= d(\chi_0, A_-) + (\chi_0, R^{A_0} + d^{A_0} A_+),$$

where, for the last equality, we have used the flatness of d^λ. Since $(\chi_0, R^{A_0} + d^{A_0} A_+)$ is holomorphic near $\lambda = 0$ and so has no residue there, we conclude:

$$\mathrm{Res}_{\lambda=0}(\partial \hat{A}^+ / \partial \lambda \wedge \hat{A}^-) = \mathrm{Res}_{\lambda=0}\big(\partial A^+ / \partial \lambda \wedge A_-\big) - d\mathrm{Res}_{\lambda=0}(\chi_0, A_-)$$

and the result follows at once. □

Corollary 3.12. *Let $V, \hat{V}$ be harmonic bundles with $\hat{V}$ a dressing transform of V. Then the harmonic map energy densities of V and $\hat{V}$ differ by an exact 1-form.*

So far, just as in Section 3.1, our analysis could be applied *mutatis mutandis* to k-perturbed harmonic maps into any symmetric space but now we specialize to the case of particular interest to us: we suppose that V is the central sphere congruence of a constrained Willmore surface Λ with multiplier q. In this case, Theorem 3.8 yields a 2-perturbed harmonic $\hat{V}$ and we are going to show that $\hat{V}$ is also the central sphere congruence of a constrained Willmore surface $\hat{\Lambda}$.

Our candidates for a new constrained Willmore surface and corresponding multiplier are

(3.11a) $\hat{\Lambda} = r(0)\Lambda^{1,0} \cap r(\infty)\Lambda^{0,1},$

(3.11b) $\hat{q} = \hat{A}_2 + \hat{A}_{-2} = \mathrm{Ad}_{r(\infty)} q^{1,0} + \mathrm{Ad}_{r(0)} q^{0,1},$

where the last identity is (3.10c). Since $\overline{r(0)\Lambda^{1,0}} = r(\infty)\Lambda^{0,1}$, Lemma 3.7(3) assures us that $\hat{\Lambda}$ is a real line subbundle of $\underline{\mathbb{C}}^{n+2}$.

We now have the main result of this section:

Theorem 3.13. *Let (V, d^λ) be the 2-perturbed harmonic central sphere congruence of a constrained Willmore surface Λ with multiplier q. Let $(\hat{V}, \hat{d}^\lambda)$ be the dressing transform of (V, d^λ) by r and define $\hat{\Lambda}$, $\hat{q}$ by (3.11).*

Then $\hat{\Lambda}$ is conformal and a constrained Willmore surface with multiplier $\hat{q}$ and central sphere congruence $\hat{V}$ on the open set where it immerses.

We therefore extend the terminology of Definition 3.5 and say that $\hat{\Lambda}$ is *the dressing transform of Λ by r.*

Proof. There are three things to prove here. For conformality, we must show that $\hat{\Lambda}^{1,0}$ is isotropic. To see that $\hat{\Lambda}$ is constrained Willmore with multiplier $\hat{q}$, we need to show that $\hat{q}^{1,0} \in \Omega^{1,0}(\wedge^2 \hat{\Lambda}^{0,1})$ so that $\hat{q}$ gives rise to a holomorphic quadratic differential and that $\mathcal{N}_{\hat{V}}^{1,0} \hat{\Lambda}^{0,1} = 0$ so that $\hat{V}$ is the central sphere congruence of $\hat{\Lambda}$. Theorem 2.6 then establishes the conclusion.

For all this, we prove that $\hat{\Lambda}^{0,1} = r(\infty)\Lambda^{0,1}$, whence, by reality, $\hat{\Lambda}^{1,0} = r(0)\Lambda^{1,0}$, and $\mathcal{N}_{\hat{V}}^{0,1} r(0)\Lambda^{1,0} = 0$. It is convenient to prove the second of these

assertions first. From (3.10d), we have

(3.12) $$\mathcal{N}_{\hat{V}}^{0,1} = \mathrm{Ad}_{r(0)}(\mathcal{N}_{V}^{0,1} + [\chi_0(0), q^{0,1}]),$$

so that we need to establish that

$$(\mathcal{N}_{V}^{0,1} + [\chi_0(0), q^{0,1}])\Lambda^{1,0} = 0.$$

However, $\mathcal{N}_{V}^{0,1}\Lambda^{1,0}$ vanishes, since V is the central sphere congruence of Λ, leaving us with the $[\chi_0(0), q^{0,1}]$ term. However, differentiating (3.6) at $\lambda = 0$ shows that $\chi_0(0)$ anti-commutes with ρ_V and so takes values in $V \wedge V^\perp$. On the other hand, $q^{0,1}$ takes values in $\wedge^2 \Lambda^{1,0}$ so that their bracket lies in $\Lambda^{1,0} \wedge V^\perp$ and so annihilates $\Lambda^{1,0}$ as required. Similarly, $\mathcal{N}_{\hat{V}}^{1,0}r(\infty)\Lambda^{0,1} = 0$ and, in particular, $\mathcal{N}_{\hat{V}}\hat{\Lambda}$ vanishes.

Again $\hat{q}^{0,1}r(0)\Lambda^{1,0} = r(0)q^{0,1}\Lambda^{1,0} = 0$ and, similarly, $\hat{q}^{1,0}r(\infty)\Lambda^{0,1}$ vanishes. In particular, $\hat{q}\hat{\Lambda} = 0$.

It follows that the operators $d^{0,1}$, $\mathcal{D}_{\hat{V}}^{0,1}$ and $(\mathcal{D}_{\hat{V}} - \hat{q})^{0,1}$ all coincide on $\Gamma\hat{\Lambda}$. However, (3.10a) reads

$$(\mathcal{D}_{\hat{V}} - \hat{q})^{0,1} = r(\infty) \cdot (\mathcal{D}_V - q)^{0,1}$$

and $\Lambda^{0,1}$ is $(\mathcal{D}_V - q)^{0,1}$-stable whence $r(\infty)\Lambda^{0,1}$ is $(\mathcal{D}_{\hat{V}} - \hat{q})^{0,1}$-stable. It follows at once that $d^{0,1} : \Gamma\hat{\Lambda} \to \Omega^{0,1}(r(\infty)\Lambda^{0,1})$ so that $\hat{\Lambda}^{0,1} = r(\infty)\Lambda^{0,1}$ and, similarly, $\hat{\Lambda}^{1,0} = r(0)\Lambda^{1,0}$, since $\hat{\Lambda}$ immerses. $\square$

We remark that q vanishes exactly when $\hat{q}$ does so that our dressing transforms when applied to Willmore surfaces give Willmore surfaces once more.

Proposition 3.11 applies in the current setting:

Corollary 3.14. *Let $\hat{\Lambda}$ be a dressing transform of a constrained Willmore surface Λ. Then the Willmore densities of Λ and $\hat{\Lambda}$ differ by an exact 1-form.*

Proof. We know from Section 2.2 that the Willmore density of a surface coincides with the harmonic map energy $\frac{1}{2}(\mathcal{N}_V \circ J^\Sigma \wedge \mathcal{N}_V)$ of its central sphere congruence V. On the other hand, in the present situation, the energy $e(V)$ of definition 3.10 is given by

$$e(V) = \frac{1}{2}(\mathcal{N}_V \circ J\Sigma \wedge \mathcal{N}_V) + (q \circ J^\Sigma \wedge q) = \frac{1}{2}(\mathcal{N}_V \circ J^\Sigma \wedge \mathcal{N}_V),$$

since q takes values in the isotropic bundle $\Lambda \wedge \Lambda^\perp$. $\square$

We conclude this analysis by showing that the holomorphic quadratic differentials that are the Lagrange multipliers for Λ and $\hat{\Lambda}$ coincide. First, we extend the scope of (2.3):

Lemma 3.15. *Let Λ be a conformal immersion with central sphere congruence V. Let $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)}$ corresponding to a holomorphic quadratic differential Q via (2.3). Then, for any $\tau \in \Gamma \Lambda^{1,0}$, we have*

$$(3.13) \qquad\qquad Q^{0,2}\tau = q^{0,1}(\mathcal{D}_V^{0,1}\tau).$$

Proof. Choose $Z \in T^{1,0}\Sigma$ and $\sigma \in \Gamma\Lambda$ with $(d_Z\sigma, d_{\bar{Z}}\sigma) = 1$. Since $\Lambda \wedge \Lambda^{1,0} = \bigwedge^2 \Lambda^{1,0}$ has rank 1, $q_{\bar{Z}}$ is a multiple of $\sigma \wedge d_Z\sigma$ and (2.3) quickly yields

$$(3.14) \qquad\qquad q_{\bar{Z}} = -Q(\bar{Z}, \bar{Z})\sigma \wedge d_Z\sigma.$$

The right-hand side of (3.13) is tensorial in τ so that, in view of (2.3), it suffices to check (3.13) with $\tau = d_Z\sigma$. But

$$\begin{aligned}
(\sigma \wedge d_Z\sigma)(\mathcal{D}_V)_{\bar{Z}}d_Z\sigma &= (\sigma, (\mathcal{D}_V)_{\bar{Z}}d_Z\sigma)d_Z\sigma - (d_Z\sigma, (\mathcal{D}_V)_{\bar{Z}}d_Z\sigma)\sigma \\
&= -(d_{\bar{Z}}\sigma, d_Z\sigma)d_Z\sigma + \tfrac{1}{2}d_{\bar{Z}}(d_Z\sigma, d_Z\sigma) \\
&= -d_Z\sigma
\end{aligned}$$

and the result follows at once from (3.14). $\square$

Proposition 3.16. *Let Λ be a constrained Willmore surface and $\hat{\Lambda}$ the dressing transform of Λ by r. Then $Q_{\hat{\Lambda}} = Q_\Lambda$.*

Proof. From Lemma 3.15, for any section $\hat{\tau}$ of $\hat{\Lambda}^{1,0}$, we have

$$Q_{\hat{\Lambda}}^{0,2}\hat{\tau} = \hat{q}^{0,1}(\mathcal{D}_{\hat{V}}^{0,1}\hat{\tau}) = \hat{q}^{0,1}((\mathcal{D}_{\hat{V}} - \hat{q})^{0,1}\hat{\tau}).$$

We write $\hat{\tau} = \mathrm{Ad}_{r(0)}\tau$, for $\tau \in \Gamma\Lambda^{1,0}$, and apply (3.10) to see that

$$\begin{aligned}
\hat{q}^{0,1}((\mathcal{D}_{\hat{V}} - \hat{q})^{0,1}\hat{\tau}) &= r(0)q^{0,1}((\mathcal{D}_V - q)^{0,1}\tau) \\
&\quad + \hat{q}^{0,1}\left(\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\Big|_{\lambda=0}\mathrm{Ad}_{r(\lambda)}(q^{0,1} + \lambda\mathcal{N}_V^{0,1})\hat{\tau}\right) \\
&= Q_\Lambda^{0,2}\hat{\tau} + \hat{q}^{0,1}\left(\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\Big|_{\lambda=0}\mathrm{Ad}_{r(\lambda)}(q^{0,1} + \lambda\mathcal{N}_V^{0,1})\hat{\tau}\right),
\end{aligned}$$

so that the issue is to show that the last term vanishes.

For this, note that

(3.15)
$$\frac{\partial^2}{\partial\lambda^2}_{|\lambda=0}\mathrm{Ad}_{r(\lambda)}(q^{0,1} + \lambda\mathcal{N}_V^{0,1}) = \mathrm{Ad}_{r(0)}((\partial/\partial\lambda + \mathrm{ad}\chi_0)^2(q^{0,1} + \lambda\mathcal{N}_V^{0,1})_{\lambda=0}),$$

so that it suffices to show that $C := (\partial/\partial\lambda + \mathrm{ad}\chi_0)^2(q^{0,1} + \lambda\mathcal{N}_V^{0,1})_{\lambda=0}$ preserves $\Lambda^{1,0}$. We may write (3.15) as

$$\mathrm{Ad}_{r(0)}C = \frac{\partial^2}{\partial\lambda^2}_{|\lambda=0}g(\lambda),$$

where $g(\lambda) = \mathrm{Ad}_{r(\lambda)}(q^{0,1} + \lambda\mathcal{N}_V^{0,1})$. From (3.6), we have $\mathrm{Ad}_{\rho_V}g(\lambda) = g(-\lambda)$ and, differentiating this twice, we learn that C commutes with ρ_V and so preserves V. Since $\Lambda^{1,0}$ is maximal isotropic in V, we are reduced to showing that $(C\tau_1, \tau_2)$ vanishes for all $\tau_1, \tau_2 \in \Lambda^{1,0}$. Now set $B_\lambda = (\partial/\partial\lambda + \mathrm{ad}\chi_0)(q^{0,1} + \lambda\mathcal{N}_V^{0,1})$ so that $C = (\partial/\partial\lambda + \mathrm{ad}\chi_0)_{|\lambda=0}B_\lambda$. We observe that

$$B_0 = \mathcal{N}_V^{0,1} + [\chi_0(0), q^{0,1}],$$

which, from (3.2), annihilates $\Lambda^{1,0}$. We compute:

$$\begin{aligned}(C\tau_1, \tau_2) &= (\partial/\partial\lambda_{|\lambda=0}B_\lambda\tau_1, \tau_2) + ([\chi_0(0), B_0]\tau_1, \tau_2) \\ &= (\partial/\partial\lambda_{|\lambda=0}B_\lambda\tau_1, \tau_2) - (B_0\tau_1, \chi_0(0)\tau_2) + (\chi_0(0)\tau_1, B_0\tau_2) \\ &= \partial/\partial\lambda_{|\lambda=0}(B_\lambda\tau_1, \tau_2).\end{aligned}$$

However, $B_\lambda = (\partial/\partial\lambda + \mathrm{ad}\chi_0)A_\lambda$ for $A_\lambda = q^{0,1} + \lambda\mathcal{N}_V^{0,1}$ which last annihilates $\Lambda^{1,0}$, for all λ, so that repeating the last computation with B_λ gives

$$(B_\lambda\tau_1, \tau_2) = \partial/\partial\lambda(A_\lambda\tau_1, \tau_2),$$

which vanishes identically. $\qquad\square$

3.3. Bäcklund transformation

Let us now remedy a lack in the analysis of the last section by providing examples of dressing gauges. For this, we follow Terng–Uhlenbeck [38] and contemplate dressing by *simple factors*.

Here is the basic building block of the construction: for L^+, L^- null line subbundles of $\underline{\mathbb{C}}^{n+2}$ which are *complementary* in the sense that L_x^+ and L_x^-

are not orthogonal, for all $x \in \Sigma$, define $\Gamma^{L^+}_{L^-} : \mathbb{C}^\times \to \Gamma(\mathrm{O}(\underline{\mathbb{C}}^{n+2}))$ by

$$\Gamma^{L^+}_{L^-}(\lambda) = \begin{cases} \lambda & \text{on } L^+; \\ 1 & \text{on } (L^+ \oplus L^-)^\perp; \\ \lambda^{-1} & \text{on } L^-. \end{cases}$$

The key property enjoyed by these gauge transformations is that $\mathrm{Ad}\Gamma^{L^+}_{L^-}(\lambda)$ is semisimple with eigenvalues λ, 1 and λ^{-1} only.

Our simple factors will be constructed by precomposing these gauge transformations with a linear fractional transformation of λ so the following simple lemma will be important for us:

Lemma 3.17. *Let $\lambda \mapsto d^\lambda$, $\lambda \in \mathbb{C}^\times$, be a family of metric connections on $\underline{\mathbb{C}}^{n+2}$, holomorphic in λ on an open subset of $\mathbb{P}^1$. Let $\alpha, \beta \in \mathbb{P}^1$ and $\psi^\alpha_\beta : \mathbb{P}^1 \to \mathbb{P}^1$ a linear fractional transformation with a zero at α and a pole at β. If d^λ is holomorphic at α, then the gauged family of connections*

$$\Gamma^{L^+}_{L^-}(\psi^\alpha_\beta(\lambda)) \cdot d^\lambda$$

extend holomorphically across the singularity at α if and only if L^+ is d^α-parallel. Similarly, if d^λ is holomorphic at β, the connections extend holomorphically at β if and only if L^- is d^β-parallel.

Proof. Write $W = (L^+ \oplus L^-)^\perp$, so that we have a decomposition $\underline{\mathbb{C}}^{n+2} = L^- \oplus W \oplus L^+$. We have a corresponding decomposition of d^α:

$$d^\alpha = D + \beta^+ + \beta^-$$

where D preserves $L^\pm$ and $\beta^\pm \in \Omega^1(L^\pm \wedge W)$. We note that $L^\pm \wedge W$ are the $\lambda^{\pm 1}$-eigenbundles of $\mathrm{Ad}\Gamma^{L^+}_{L^-}(\lambda)$.

Now write $d^\lambda = d^\alpha + (\lambda - \alpha)B(\lambda)$ (we assume that $\alpha \neq \infty$, otherwise we work with $1/\lambda$). Then, with $\Gamma(\lambda) = \Gamma^{L^+}_{L^-}(\psi(\lambda))$

$$\Gamma(\lambda) \cdot d^\lambda = \Gamma(\lambda) \cdot d^\alpha + (\lambda - \alpha)\mathrm{Ad}_{\Gamma(\lambda)}B(\lambda).$$

The second term is holomorphic near α since $\mathrm{Ad}_{\Gamma(\lambda)}$ introduces at most a simple pole at α. As for the first term,

$$\begin{aligned} \Gamma(\lambda) \cdot d^\alpha &= \Gamma(\lambda) \cdot D + \mathrm{Ad}_{\Gamma(\lambda)}(\beta^+ + \beta^-) \\ &= D + \psi(\lambda)\beta^+ + \psi(\lambda)^{-1}\beta^-, \end{aligned}$$

which is holomorphic near α if and only if β^- vanishes. However, this is the case if and only if L^+ is d^α-parallel.

The case at β now follows after noting that $\Gamma_{L^-}^{L^+}(\psi_\beta^\alpha(\lambda)) = \Gamma_{L^+}^{L^-}(1/\psi_\beta^\alpha(\lambda))$ to which the first case applies. $\square$

Simple factors have only two poles so, in order to take care of both the twisting and reality conditions[2] that Definition 3.5 imposes, we shall need a product of two simple factors (cf. [25, 27, 29]) and have recourse to a Bianchi permutability result to account for the non-commutativity of that product. For this and related results, the following lemma will be useful.

Lemma 3.18. *Let ℓ^+, ℓ^- and $\hat\ell^+, \hat\ell^-$ be two pairs of complementary null lines in $\mathbb{C}^{n+2}$. Let $\psi_\beta^\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ be a linear fractional transformation with zero at α and pole at β. Finally, let $E : \lambda \mapsto E(\lambda) \in \mathrm{O}(\mathbb{C}^{n+2})$ be holomorphic near α and β. Then*

$$(3.16) \qquad \lambda \mapsto \Gamma_{\hat\ell_-}^{\hat\ell^+}(\psi_\beta^\alpha(\lambda)) E(\lambda) \big(\Gamma_{\ell_-}^{\ell^+}(\psi_\beta^\alpha(\lambda))\big)^{-1}$$

is holomorphic at α if and only if $E(\alpha)\ell^+ = \hat\ell^+$ and holomorphic at β if and only if $E(\beta)\ell^- = \hat\ell^-$.

Proof. Consider first the case where $\psi_\beta^\alpha(\lambda) = \lambda$. Here, holomorphicity of (3.16) at 0 is precisely the statement of [8, Lemma 4.10] while holomorphicity at ∞ follows by swapping the roles of $\ell^\pm$ and replacing λ with $1/\lambda$.

The case of arbitrary α, β can now be reduced to the first case by pre-composing (3.16) with $(\psi_\beta^\alpha)^{-1}$. $\square$

Remark 3.19. *Lemma 3.18 can be viewed as a discrete analogue of Lemma 3.17 and, indeed, for flat d^λ, implies Lemma 3.17 by letting $E(\lambda)$ be a gauge transform relating d^λ and d.*

With all this in hand, let V be a non-degenerate subbundle of $\underline{\mathbb{C}}^{n+2}$, $\alpha \in \mathbb{C}^\times$ and L a null, line subbundle such that L and $\rho_V L$ are complementary on an open set. From these data, define gauge transformations $p_{\alpha,L}^V(\lambda)$ on

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[2]These conditions mean that if α is a pole, so is $-\alpha$ and $\pm 1/\bar\alpha$.

that open set by

$$p^V_{\alpha,L}(\lambda) = \Gamma^L_{\rho_V L}\big(\tfrac{(1+\alpha)(\lambda-\alpha)}{(1-\alpha)(\lambda+\alpha)}\big).$$

We observe:

(3.17a) $$p^V_{\alpha,L}(1) = 1,$$

(3.17b) $$p^V_{\alpha,L}(-\lambda) \circ \rho_V \circ p^V_{\alpha,L}(\lambda)^{-1} = \rho_{V'},$$

where $V' = p^V_{\alpha,L}(0)V = p^V_{\alpha,L}(\infty)V$,

(3.17c) $$p^V_{\alpha,L}(\lambda^*) = \overline{p^{\overline{V}}_{\alpha^*,\overline{L}}(\lambda)}.$$

Moreover, $\lambda \mapsto p^V_{\alpha,L}(\lambda)$ is holomorphic on $\mathbb{P}^1 \setminus \{\pm\alpha\}$. We have

(3.17d) $$(p^V_{\alpha,L}(0))^{-1} p^V_{\alpha,L}(\infty) = \Gamma^L_{\rho_V L}(-1),$$

the restriction of which to V is reflection across $(L \oplus \rho_V L)^\perp \cap V$ which has codimension 1 in V (otherwise $L = \rho_V L$) so that

(3.17e) $$\det\big(\Gamma^L_{\rho_V L}(-1)\big) = -1.$$

We now have the main result of this section:

Theorem 3.20. *Let (V, d^λ) be k-perturbed harmonic, $\alpha \in \mathbb{C}^\times \setminus S^1$ and L a d^α-parallel null line subbundle with $L, \rho_V L$ complementary. Moreover, set $L' = p^V_{\alpha,L}(\alpha^*)\overline{L}$ and $V' = p^V_{\alpha,L}(0)V$. Assume also that $L', \rho_{V'}L'$ are complementary. Then $r := p^{V'}_{\alpha^*,L'} p^V_{\alpha,L}$ is a dressing gauge for (V, d^λ).*

Thus, Theorem 3.8 applies with r so defined to give a dressing transform of V which, by Theorem 3.13, induces a transformation of constrained Willmore surfaces when V is the central sphere congruence of such a surface.

Proof. First r is rational in λ on $\mathbb{P}^1$ and holomorphic on $\mathbb{P}^1 \setminus \{\pm\alpha, \pm\alpha^*\}$.

Since d^λ is ρ_V-twisted, we see that $\rho_V L$ is $d^{-\alpha}$-parallel so that, by Lemma 3.17, $p^V_{\alpha,L}(\lambda) \cdot d^\lambda$ extends from $\mathbb{C}^\times \setminus \{\pm\alpha\}$ to a holomorphic family of connections $\tilde{d}^\lambda$ on $\mathbb{C}^\times$. By the reality (2.5c) of d^λ, $\overline{L}$ is d^{α^*}-parallel so that L' is $\tilde{d}^{\alpha^*} = p^V_{\alpha,L}(\alpha^*) \cdot d^{\alpha^*}$-parallel[3]. Moreover, we use (3.17b) to see that $\tilde{d}^\lambda$ is $\rho_{V'}$-twisted so that $\rho_{V'}L'$ is $\tilde{d}^{-\alpha^*}$-parallel. We therefore apply

---

[3]Here we use our hypothesis that $\alpha \notin S^1$.

Lemma 3.17 again to see that $\hat{d}^\lambda = r(\lambda) \cdot d^\lambda = p^{V'}_{\alpha^*, \hat{L}}(\lambda) \cdot \tilde{d}^\lambda$ is holomorphic on $\mathbb{C}^\times$ with $\hat{d}^1 = d$ since $r(1) = 1$.

Two applications of (3.17b) show that

$$
\begin{aligned}
r(-\lambda) \circ \rho_V \circ r(\lambda)^{-1} &= p^{V'}_{\alpha^*, L'}(-\lambda) \circ (p^V_{\alpha, L}(-\lambda) \circ \rho_V \circ p^V_{\alpha, L}(\lambda)^{-1}) \circ p^{V'}_{\alpha^*, L'}(\lambda)^{-1} \\
&= p^{V'}_{\alpha^*, L'}(-\lambda) \circ \rho_{V'} \circ p^{V'}_{\alpha^*, L'}(\lambda)^{-1}
\end{aligned}
$$

is independent of λ so that r satisfies (3.6).

Meanwhile, from (3.17d), we have

$$
\begin{aligned}
r(0)^{-1} r(\infty) &= (p^{V'}_{\alpha^*, L'}(0) p^V_{\alpha, L}(0))^{-1} p^{V'}_{\alpha^*, L'}(\infty) p^V_{\alpha, L}(\infty) \\
&= p^V_{\alpha, L}(0)^{-1} \Gamma^{L'}_{\rho_{V'} L'}(-1) p^V_{\alpha, L}(\infty) \\
&= \Gamma^{L''}_{\rho_V L''}(-1) \Gamma^{L}_{\rho_V L}(-1),
\end{aligned}
$$

where $L'' = p^V_{\alpha, L}(0)^{-1} L'$. Equation (3.17e) now tells us that $\det(r(0)^{-1} r(\infty)_{|V}) = 1$.

We are therefore left to deal with the reality condition (3.4). For this, it suffices to prove that $\lambda \mapsto \overline{r(\lambda^*)} r(\lambda)^{-1}$ is holomorphic on $\mathbb{P}^1$ and so independent of λ. The conclusion then follows since $r(1) = 1$. To stop the decorations piling up, let us temporarily write $M = L', U = V'$ and $\beta = \alpha^*$ so that, using (3.17c),

$$
\text{(3.18)} \qquad \overline{r(1/\bar\lambda)} r(\lambda)^{-1} = p^{\bar{U}}_{\alpha, \bar{M}}(\lambda) p^V_{\beta, \bar{L}}(\lambda) p^V_{\alpha, L}(\lambda)^{-1} p^U_{\beta, M}(\lambda)^{-1}.
$$

This is holomorphic in λ on $\mathbb{P}^1 \setminus \{\pm\alpha, \pm\beta\}$. However, Lemma 3.18 tells us that $p^V_{\beta, \bar{L}} p^V_{\alpha, L}{}^{-1} p^U_{\beta, M}{}^{-1}$ is holomorphic at β since $\bar{L} = p^V_{\alpha, L}(\beta)^{-1} M$ and holomorphic at $-\beta$ since $\rho_V \bar{L} = p^V_{\alpha, L}(-\beta)^{-1} \rho_{V'} M$. Since $p^{\bar{U}}_{\alpha, \bar{M}}$ is also holomorphic at $\pm\beta$, we see that $\overline{r(\lambda^*)} r(\lambda)^{-1}$ is holomorphic there also.

A similar argument with $p^{\bar{U}}_{\alpha, \bar{M}} p^V_{\beta, \bar{L}} p^V_{\alpha, L}{}^{-1}$ using $\bar{M} = p^V_{\beta, \bar{L}}(\alpha) L$ establishes holomorphicity at $\pm\alpha$ and we are done. $\square$

Remark 3.21. *To construct bundles L that satisfy the hypotheses of Theorem 3.20 it suffices to choose an initial condition L_x, some $x \in \Sigma$, that satisfies the (open) complementarity conditions and extend to an open set by parallel translation since d^α is flat. We observe that the complementarity conditions are non-empty since they are satisfied by real, null L_x transverse to V_x and $V_x^\perp$. Indeed, in this case, $L'_x = p^V_{\alpha, L_x}(\alpha^*) L_x = L_x$ and $\rho_{V'_x} L'_x = \rho_V L_x$ since L_x is an eigenspace of $p^V_{\alpha, L_x}(\lambda)$, for all λ.*

By analogy with the Bianchi–Bäcklund transform of constant Gauss curvature surfaces (cf. [29]), we make the following

Definition 3.22. The dressing transform by the dressing gauge r of Theorem 3.20 is the *Bäcklund transform with parameters α, L*.

Remark 3.23. *In the classical literature, Bäcklund transforms are specified by a choice of spectral parameter α and an initial condition which locates the transformed surface in space at a base-point $x \in \Sigma$. In our setting, this last amounts to choosing the fibre L_x from which our bundle L can be recovered by parallel translation with respect to the flat connection d^α.*

3.4. Bianchi permutability for Bäcklund transforms

Our terminology in Definition 3.22 may be justified by the fact that there is a Bianchi permutability theorem available for our Bäcklund transform.

Let us recall in outline the statement of Bianchi permutability: we start with a k-perturbed harmonic (V, d^λ) and construct two Bäcklund transforms (V_1, d_1^λ), (V_2, d_2^λ) with parameters α_1, L_1 and α_2, L_2, respectively. The theorem then asserts the existence of a fourth k-perturbed harmonic (V_{12}, d_{12}) which is simultaneously a Bäcklund transform of V_1 with parameter α_2 and of V_2 with parameter α_1.

This amounts to finding the right line bundles L_2^1 and L_1^2 with L_2^1 $d_1^{\alpha_2}$-parallel and L_1^2 $d_2^{\alpha_1}$-parallel. However, we have natural candidates close at hand: by construction, we have gauge transformations r_1, r_2 with $r_j(\lambda) \cdot d^\lambda = d_j^\lambda$ and so

$$(3.19\text{a}) \qquad L_2^1 := r_1(\alpha_2)L_2,$$
$$(3.19\text{b}) \qquad L_1^2 := r_2(\alpha_1)L_1$$

fit the bill. It remains to verify that the Bäcklund transformations of V_j with parameters α_k, L_k^j, $j \neq k \in \{1, 2\}$, coincide and it is to this that we now turn.

We begin with an effort to keep the notation under control: as already indicated, we denote by r_j the dressing gauge that implements the Bäcklund transform of (V, d^λ) with parameters α_j, L_j. Similarly, let r_j^k be the dressing gauge that implements the Bäcklund transform of V_k with parameters α_j, L_j^k. Thus, for example,

$$r_1(\lambda) = p_{\alpha_1^*, L_1'}^* p_{\alpha_1, L_1}^V,$$

where $*$ denotes a bundle (in fact $p^V_{\alpha_1, L_1}(0)V$) that we will not have to keep track of. We have $d^\lambda_1 = r_1(\lambda) \cdot d^\lambda$, off a divisor, $V_1 = r_1(0)V = r_1(\infty)V$, and, in case that V is the central sphere congruence of a constrained Willmore Λ, $\Lambda_1 = r_1(0)\Lambda^{1,0} \cap r_1(\infty)\Lambda^{0,1}$.

The key fact is contained in the following lemma:

Lemma 3.24. *Suppose that $\alpha_2 \notin \{\pm\alpha_1, \pm\alpha_1^*\}$. Then*

$$(3.20) \qquad\qquad r^1_2 r_1 = r^2_1 r_2.$$

Proof. We shall argue as in the proof of Theorem 3.20 (itself in part a permutability theorem for p's) and show that $\Pi := r^2_1 r_2 (r_1)^{-1}(r^1_2)^{-1}$ is holomorphic on $\mathbb{P}^1$ and so constant. Then (3.20) follows by evaluating at $\lambda = 1$.

First we consider holomorphicity at α_1: we know that r^1_2 is holomorphic and invertible at α_1 so the issue is with $r^2_1 r_2 (r_1)^{-1}$ which reads

$$p^*_{\alpha_1^*, \hat{L}^2_1} p^{V_2}_{\alpha_1, L^2_1} r_2 p^{V}_{\alpha_1, L_1}{}^{-1} p^*_{\alpha_1^*, \hat{L}_1}{}^{-1}.$$

Here, the outer p^*-terms are holomorphic at α_1 (recall that $\alpha_1 \notin S^2$!) and what is left is holomorphic thanks to Lemma 3.18 and (3.19b). Thus, our product Π is holomorphic and $O(\mathbb{C}^{n+2})$-valued at α_1. The same argument, with the roles of α_1 and α_2 swopped shows that $r^1_2 r_1 (r_2)^{-1}$ and so $r_2 (r_1)^{-1}(r^1_2)^{-1}$ is holomorphic and invertible at α_2.

For the remaining potential singularities at $-\alpha_j, \pm\alpha_j^*$, we observe that, first, Π is real: $\overline{\Pi(\lambda)} = \Pi(\lambda^*)$ and then that, with ρ^2_1 being the reflection across $\Pi^2_1(0)V_2$ and similarly for ρ^1_2, we have

$$\rho^1_2 \Pi(\lambda) = \Pi(-\lambda)\rho^2_1.$$

Consequently, holomorphicity at $-\alpha_j, \pm\alpha_j^*$ follows from that at α_j and Π is holomorphic on $\mathbb{P}^1$ as required. $\qquad\square$

With this in hand, the permutability theorem follows quickly: with r the common value in (3.20), set $V_{12} = r(0)V = r(\infty)V$, $d^\lambda_{12} = r(\lambda) \cdot d^\lambda$ and, in case we are working with a constrained Willmore surface $\Lambda_{12} = r(0)\Lambda^{1,0} \cap r(\infty)\Lambda^{0,1}$. Then

$$\begin{aligned}
V_{12} &= r^1_2 r_1(0)V = r^1_2(0)V_1, \\
d^\lambda_{12} &= r^1_2(\lambda) \cdot (r_1(\lambda) \cdot d^\lambda) = r^1_2(\lambda) \cdot d^\lambda_1, \\
\Lambda_{12} &= \left(r^1_2 r_1(0)\Lambda^{1,0}\right) \cap \left(r^1_2 r_1(\infty)\Lambda^{0,1}\right) = r^1_2(0)\Lambda^{1,0}_1 \cap r^1_2(\infty)\Lambda^{0,1}_1,
\end{aligned}$$

so that (V_{12}, d_{12}^λ) (respectively Λ_{12}) is the Bäcklund transform of (V_1, d_1^λ) (respectively Λ_1) with parameters α_2, L_2^1. Similarly, these data are the Bäcklund transform of (V_2, d_2^λ) (respectively Λ_2) with parameters α_1, L_1^2. To summarize:

Theorem 3.25. *Let (V, d^λ) be k-perturbed harmonic and $\alpha_1, \alpha_2 \in \mathbb{C}^\times \setminus S^1$ with $\alpha_2 \notin \{\pm\alpha_1, \pm\alpha_1^*\}$. Let (V_1, d_1^λ), (V_2, d_2^λ) be the Bäcklund transforms of (V, d^λ) with parameters α_1, L_1 and α_2, L_2 respectively. Then, with L_j^k defined by (3.19), the Bäcklund transform of (V_1, d_1^λ) with parameters α_2, L_2^1 coincides with the Bäcklund transform of (V_2, d_2^λ) with parameters α_1, L_1^2.*

Moreover, if V is the central sphere congruence of a constrained Willmore surface Λ, the corresponding iterated transforms of Λ coincide also.

3.5. Spectral deformation versus Dressing transform

We conclude our analysis by comparing the spectral and dressing transformations. In an appropriate sense, these commute. For this, begin with a k-perturbed harmonic map (V, d^λ). Recall from Section 3.1 that, for $\mu \in S^1$, we have k-perturbed harmonic maps (V_μ, d_μ^λ) given by $V_\mu = \Phi_\mu V$ and $d_\mu^\lambda = \Phi_\mu \cdot d^{\lambda\mu}$ where $\Phi_\mu \in \Gamma O(\mathbb{R}^{n+1,1})$ solves $\Phi_\mu \cdot d_\mu = d$. Now let $(\hat{V}, \hat{d}^\lambda)$ be a dressing transform of (V, d^λ) via a dressing gauge r. Then $(\hat{V}, \hat{d}^\lambda)$ has a spectral deformation $(\hat{V}_\mu, \hat{d}_\mu^\lambda)$ given by $\hat{V}_\mu = \hat{\Phi}_\mu \hat{V}$, $\hat{d}_\mu^\lambda = \hat{\Phi}_\mu \cdot \hat{d}^{\lambda\mu}$, where $\hat{\Phi}_\mu$ solves $\hat{\Phi}_\mu \cdot \hat{d}^\mu = d$.

For $\lambda \in \text{dom}(r) \setminus \{0, \infty\}$, we have $\hat{d}^\lambda = r(\lambda) \cdot d^\lambda$ so that, defining $r_\mu(\lambda) := \hat{\Phi}_\mu r(\lambda\mu)(\Phi_\mu)^{-1}$, we have

$$\hat{d}_\mu^\lambda = r_\mu(\lambda) \cdot d_\mu^\lambda, \quad \lambda \in \text{dom}(r_\mu) \setminus \{0, \infty\},$$
$$\hat{V}_\mu = r_\mu(0) V_\mu.$$

It is now a simple matter to check that r_μ is a dressing gauge for V_μ and we conclude that $\hat{V}_\mu$ is a dressing transform of V_μ. Thus:

Proposition 3.26. *Let $(\hat{V}, \hat{d}^\lambda)$ be a dressing transform of a k-perturbed harmonic (V, d^λ). For $\mu \in S^1$, the spectral deformation $(\hat{V}_\mu, \hat{d}_\mu^\lambda)$ of $(\hat{V}, \hat{d}^\lambda)$ is a dressing transform of the spectral deformation (V_μ, d_μ^λ) of (V, d^λ).*

In case that $(\hat{V}, \hat{d}^\lambda)$ is a Bäcklund transform of (V, d^λ), we can do a little better. First observe that, in this case, $\mu \in \text{dom}(r) \setminus \{0, \infty\}$ so that $\hat{d}^\mu = r(\mu) \cdot d^\mu = (r(\mu)\Phi_\mu^{-1}) \cdot d$ and we may therefore take $\hat{\Phi}_\mu = \Phi_\mu r(\mu)^{-1}$. Further, suppose that the Bäcklund transformation of (V, d^λ) has parameters

α, L so that L is a d^α-parallel, null line subbundle. Then $\Phi_\mu L$ is $d_\mu^{\mu/\alpha}$-parallel which prompts the following result:

Proposition 3.27. *Let $(\hat{V}, \hat{d}^\lambda)$ be the Bäcklund transform of (V, d^λ) with parameters α, L. For $\mu \in S^1$, the spectral deformation $(\hat{V}_\mu, \hat{d}_\mu^\lambda)$ is the Bäcklund transform with parameters α/μ, $\Phi_\mu L$ of the spectral deformation (V_μ, d_μ^λ).*

Proof. In this case, we have $r(\lambda) = p_{\alpha^*, L'}^{V'}(\lambda) p_{\alpha, L}^V(\lambda)$. We set $L_\mu = \Phi_\mu L$. Then, in view of the discussion above, our result amounts to the identity

$$\Phi_\mu r(\mu)^{-1} r(\lambda\mu) \Phi_\mu^{-1} = p_{\alpha^*/\mu, L'_\mu}^{V'_\mu}(\lambda) p_{\alpha/\mu, L_\mu}^{V_\mu}(\lambda).$$

For this, we use the, by now familiar, permutability argument. Rearranging all terms onto the left-hand side yields a product which is holomorphic in λ except possibly at $\pm\alpha/\mu$ and $\pm\alpha^*/\mu$. The part with possible singularities at $\pm\alpha/\mu$ is

$$\Gamma_{\rho_V L}^L\left(\tfrac{\lambda\mu-\alpha}{\lambda\mu+\alpha}\right) \Phi_\mu^{-1} \left(\Gamma_{\rho_{V_\mu} L_\mu}^{L_\mu}\left(\tfrac{\lambda-\alpha/\mu}{\lambda+\alpha/\mu}\right)\right)^{-1}$$

which is readily seen to reduce to Φ_μ^{-1} and so is independent of λ. The reality condition now establishes holomorphicity at $\pm\alpha^*/\mu$ so that the whole product is independent of λ and we are done. $\qquad\square$

In particular, we see that any Bäcklund transform can be obtained as a combination of spectral deformations and a Bäcklund transform with $\alpha \in \mathbb{R}$.

4. Codimension 2

Conformal surface geometry in S^4 has a distinctive flavour of its own. This manifests itself in various ways such as a well-developed twistor theory [6, 17, 20] and the quaternionic formalism of the Berlin school [2, 10, 19]. However, for us, the key feature of this setting is that the space of oriented 2-spheres is a complex manifold (in fact, a pseudo-Hermitian symmetric space). We will show that this structure allows us both to simplify the preceding theory and to relate it to a construction of Willmore surfaces in [10].

4.1. Additional structure in codimension 2

We consider maps of the Riemann surface Σ into the space $\tilde{G}_{3,1}(\mathbb{R}^{5,1})$ of *oriented* 2-spheres in S^4, or, equivalently, oriented bundles V of $(3, 1)$-planes in

$\underline{\mathbb{C}}^6$. Note that if V is the central sphere congruence of a conformal immersion Λ, then $\sigma \wedge d_X \sigma \wedge d_{J^\Sigma X} \sigma \wedge \Delta \sigma$, $X \in T\Sigma$, $\sigma \in \Gamma\Lambda$, gives an orientation of V which is independent of choices.

In this setting, $V^\perp$ is a real bundle of oriented $(2,0)$-planes and so splits as a sum of complex conjugate, $\mathcal{D}^V$-parallel, null line subbundles:

$$V^\perp = V_+^\perp \oplus V_-^\perp.$$

As a consequence, we have a family of $\mathcal{D}^V$-parallel gauge transformations $\tau_V(\lambda) := \Gamma_{V_-^\perp}^{V_+^\perp}(\lambda)$ and, in particular, an almost complex structure J^V on $V \wedge V^\perp$ given by $J^V = \tau_V(\sqrt{-1})_{|V \wedge V^\perp}$. We may therefore split $\mathcal{N}_V$ into its holomorphic and anti-holomorphic parts:

$$(4.1) \qquad \mathcal{N}_V = \mathcal{A}_V + \mathcal{Q}_V,$$

with $\mathcal{A}_V \circ J^\Sigma = J^V \circ \mathcal{A}_V$ and $\mathcal{Q}_V \circ J^\Sigma = -J^V \circ \mathcal{Q}_V$. Thus $\mathcal{A}_V^{1,0} \in \Omega^{1,0}(V \wedge V_+^\perp)$ while $\mathcal{Q}_V^{1,0} \in \Omega^{1,0}(V \wedge V_-^\perp)$.

We note the case where $\mathcal{A} = 0$ (respectively, $\mathcal{Q} = 0$): this means that V is a (anti-)holomorphic map into the space of oriented 2-spheres or, equivalently, $V_+^\perp$ (respectively, $V_-^\perp$) is a holomorphic subbundle of $\underline{\mathbb{C}}^6$. If, in addition, V is the central sphere congruence of a conformal immersion Λ, then this condition obtains if and only if Λ is *twistor holomorphic*, that is, Λ has a holomorphic twistor lift to $\mathbb{C}P^3$ [10, Section 8.1 and Lemma 22]. Such Λ are, of course, Willmore surfaces.

4.2. Untwisted family of flat connections

Let (V, d^λ) be a k-perturbed harmonic map into $\tilde{G}_{3,1}(\mathbb{R}^{5,1})$. Since $\lambda \mapsto \tau_V(\lambda)$ is a group homomorphism $\mathbb{C}^\times \to O(\underline{\mathbb{C}}^6)$ and $\tau_V(-1) = \rho_V$, we readily conclude from (2.5b) that

$$(4.2) \qquad \tau_V(\lambda) \cdot d^\lambda = \tau_V(-\lambda) \cdot d^{-\lambda}.$$

Thus there is a second family of connections ∇^μ, $\mu \in \mathbb{C}^\times$, defined by

$$(4.3) \qquad \nabla^{\lambda^2} = \tau_V(\lambda) \cdot d^\lambda,$$

for $\lambda \in \mathbb{C}^\times$.

Proposition 4.1. *Let (V, d^λ) be a k-perturbed harmonic map into $\tilde{G}_{3,1}(\mathbb{R}^{5,1})$ with $d^\lambda = \sum_{|i| \le k} A_k$. Define ∇^μ, $\mu \in \mathbb{C}^\times$, by (4.3). Then:*

(4.4a)	$\nabla^1 = d$	(normalization)
(4.4b)	$\nabla^{\mu^*} = \overline{\nabla^\mu}$	(reality)

and $\mu \mapsto (\nabla^\mu)^{1,0}$ is holomorphic on $\mathbb{C}$ with a pole of order l at ∞ (whence, by (4.4b), $\mu \mapsto (\nabla^\mu)^{0,1}$ is holomorphic on $\mathbb{C}^\times \cup \{\infty\}$ with a pole of order l at 0). Here

$$
(4.5) \qquad l = \begin{cases} (k-1)/2 & \text{if } k \text{ is odd and } A_k V_-^\perp = 0; \\ k/2 & \text{if } k \text{ is even}; \\ (k+1)/2 & \text{otherwise}. \end{cases}
$$

Moreover $V_-^\perp$ is $(\nabla^0)^{1,0}$-stable (whence $V_+^\perp$ is $(\nabla^\infty)^{0,1}$-stable).

Proof. We have $\tau_V(1) = 1$ and $\tau_V(\lambda^*) = \overline{\tau_V(\lambda)}$ which, together with (2.5), readily yields (4.4).

For $A \in \Omega^1(\wedge^2 \mathbb{C}^6)$, decompose $A = A^+ + A^0 + A^-$ according to the eigenspace decomposition of $\mathrm{Ad}_{\tau_V(\lambda)}$:

$$
\wedge^2 \underline{\mathbb{C}}^6 = (V \wedge V_+^\perp) \oplus (\wedge^2 V \oplus \wedge^2 V^\perp) \oplus (V \wedge V_-^\perp).
$$

Then

$$
(\nabla^{\lambda^2})^{1,0} = (A_0 + A_1^-) + \lambda^2 (A_1^+ + A_2 + A_3^-) + \cdots,
$$

which establishes the holomorphicity of $\mu \mapsto (\nabla^\mu)^{1,0}$ on $\mathbb{C}$. Moreover, as A_0 preserves $V_\pm^\perp$ while A_1^- annihilates $V_-^\perp$, we conclude that $V_-^\perp$ is $(\nabla^0)^{1,0}$-stable.

Finally, we contemplate the leading term in λ of $(\nabla^{\lambda^2})^{1,0}$: if k is even, this is $\lambda^k(A_k + A_{k-1}^+)$, while, for odd k, it is either $\lambda^{k+1} A_k^+$, if $A_k^+ \ne 0$, or $\lambda^{k-1}(A_{k-1} + A_k^-)$ when $A_k^+ = 0$, or, equivalently, $A_k V_-^\perp = 0$. $\qquad\square$

Thus, these connections share many properties of the family d^λ but are not ρ_V-twisted and have poles of half the order. In view of the first property, we call ∇^μ the *untwisted family of connections* associated to (V, d^λ).

We can reverse this line of argument and start from an untwisted family of flat connections:

Theorem 4.2. *Let V be a bundle of $(3, 1)$-planes in $\mathbb{R}^{5,1}$ and $\nabla^\mu = \sum_{|j| \le l} \mu^j B_j$, $\mu \in \mathbb{C}^\times$, a family of flat connections satisfying the conclusions of Proposition 4.1.*

Define connections d^λ by

$$(4.6) \qquad\qquad d^\lambda = \tau_V(\lambda)^{-1} \cdot \nabla^{\lambda^2},$$

$\lambda \in \mathbb{C}^\times$. *Then d^λ satisfies the conditions of Definition 2.5, so that (V, d^λ) is a k-perturbed harmonic bundle where*

$$(4.7) \qquad\qquad k = \begin{cases} 2l-1 & \text{if } B_l^- = B_l^0 = 0; \\ 2l & \text{if } B_l^- = 0 \text{ but } B_l^0 \neq 0; \\ 2l+1 & \text{otherwise.} \end{cases}$$

Proof. It is clear that, with d^λ defined by (4.6), we have $d^1 = d$ and that the reality condition (2.5c) holds. Moreover, we readily compute:

$$d^{-\lambda} = \tau_V(-\lambda)^{-1} \cdot \nabla^{\lambda^2} = (\rho_V \tau_V(\lambda)^{-1}) \cdot \nabla^{\lambda^2} = \rho_V \cdot d^\lambda,$$

so that the twisting condition (2.5b) holds also.

As for the dependence in λ of d^λ, it is clear that $\lambda \mapsto d^\lambda$ is holomorphic on $\mathbb{C}^\times$ while the argument of Lemma 3.17 can be easily adapted to treat the family of partial connections $\lambda \mapsto (\nabla^{\lambda^2})^{1,0}$ and we deduce that $(d^\lambda)^{1,0}$ is holomorphic at 0 since $V_-^\perp$ is $(\nabla^0)^{1,0}$-stable. Finally, the leading terms in λ of d^λ are

$$\lambda^{2k+1} B_l^- + \lambda^{2k} B_l^0 + \lambda^{2k-1}(B_l^+ + B_{l-1}^-) + \cdots$$

from which (4.7) follows at once. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We conclude that the families d^λ and ∇^μ carry the same information and therefore refer to a k-perturbed harmonic bundle in codimension 2 as either (V, d^λ) or (V, ∇^μ), where d^λ and ∇^μ are related via (4.3) and (4.6).

We list some examples:

(1) $\nabla^\mu \equiv d$ satisfies the conditions of Proposition 4.1 if and only if $V_+^\perp$ is holomorphic. This is the case $l = 0$ of Theorem 4.2.

(2) V is harmonic if and only if $\nabla^\mu = d + (\mu - 1)\mathcal{A}^{1,0} + (\mu^{-1} - 1)\mathcal{A}^{0,1}$ is flat, for all $\mu \in \mathbb{C}^\times$. A similar result is true for maps into any (pseudo-) Hermitian symmetric space.

(3) Let V be the central sphere congruence of Λ. Then Λ is constrained Willmore with multiplier q if and only if $\nabla^\mu = d + (\mu - 1)(q + \mathcal{A})^{1,0} + (\mu^{-1} - 1)(q + \mathcal{A})^{0,1}$ is flat, for all $\mu \in \mathbb{C}^\times$ [2]. This is the case $l = 1$ of Theorem 4.2 with $B_1^- = 0$ and the additional constraint that B_1^0 take values in $\Lambda \wedge \Lambda^{(1)}$.

4.3. Untwisted dressing gauges

The untwisted family of connections has a simpler algebraic structure. It is therefore unsurprising that we may dress such families with algebraically simpler gauge transformations:

Definition 4.3. Let (V, ∇^μ) be a k-perturbed harmonic bundle in $\underline{\mathbb{C}}^6$. An *untwisted dressing gauge* for (V, ∇^μ) is a family of gauge transformations $\mu \mapsto R(\mu) \in \Gamma O(\underline{\mathbb{C}}^6)$ which is holomorphic in μ near $0, \infty \in \mathbb{P}^1$ and has the following properties:

  (a) For all $\mu \in \mathrm{dom}(R)$,

(4.8) $$R(\mu^*) = \overline{R(\mu)} \quad \text{(reality)}.$$

  (b) The connections $R(\mu) \cdot \nabla^\mu$ extend from $\mathrm{dom}(R) \setminus \{0, \infty\}$ to a holomorphic family of connections $\hat{\nabla}^\mu$ on $\mathbb{C}^\times$ with $\hat{\nabla}^1 = d$.

  (c) The line bundles $R(0)V_-^\perp$ and $R(\infty)V_+^\perp$ are complementary.

For such an R, set $\hat{V} = (R(0)V_-^\perp \oplus R(\infty)V_+^\perp)^\perp$, oriented so that

(4.9) $$\hat{V}_-^\perp = R(0)V_-^\perp, \qquad\qquad \hat{V}_+^\perp = R(\infty)V_+^\perp,$$

and call $(\hat{V}, \hat{\nabla}^\mu)$ the *untwisted dressing transform* of (V, ∇^μ) by R.

    We note that $\hat{V}$ is a bundle of $(3, 1)$-planes since $\hat{V}^\perp$ is a sum of complex conjugate null lines.

    It is our contention that $(\hat{V}, \hat{\nabla}^\mu)$ is again k-perturbed harmonic. This will follow at once from the next proposition which reduces the situation to the twisted setup of Section 3.2.

Proposition 4.4. *Let $(\hat{V}, \hat{\nabla}^\mu)$ be the untwisted dressing transform of (V, ∇^μ) by R. Set*

(4.10) $$r(\lambda) = \tau_{\hat{V}}(\lambda)^{-1} \circ R(\lambda^2) \circ \tau_V(\lambda).$$

Then r is a dressing gauge for (V, d^λ) and, with $\hat{d}^\lambda = \tau_{\hat{V}}(\lambda)^{-1} \cdot \hat{\nabla}^{\lambda^2}$, $(\hat{V}, \hat{d}^\lambda)$ is the dressing transform of (V, d^λ) by r.

    In particular, (V, d^λ) is k-perturbed harmonic by Theorem 3.8.

Proof. We check that r defined by (4.10) satisfies the conditions of Definition 3.5. For this, first note that Lemma 3.18 along with (4.9), shows that

$\lambda \mapsto r(\lambda)$ is holomorphic near 0 and ∞. Now, for item (1) of Definition 3.5, a short calculation gives

$$(4.11) \qquad r(-\lambda) \circ \rho_V \circ r(\lambda)^{-1} = \rho_{\hat{V}},$$

which is certainly independent of λ. Item (2) is immediate from the corresponding reality conditions on $R, \tau_V, \tau_{\hat{V}}$. In particular, evaluating at $\lambda = 0$ gives $r(\infty) = \overline{r(0)}$. Again, with $d^\lambda = \tau_V(\lambda)^{-1} \cdot \nabla^{\lambda^2}$, we have, from item (b) of Definition 4.3, that $\hat{d}^\lambda = r(\lambda) \cdot d^\lambda$ on $\mathrm{dom}(r) \setminus \{0, \infty\}$. However, it is clear from its definition that $\lambda \mapsto \hat{d}^\lambda$ is holomorphic on $\mathbb{C}^\times$ and that $\hat{d}^1 = d$. This settles item (3). Finally, for item (4), evaluate (4.11) at $\lambda = 0, \infty$ to deduce that $\hat{V} = r(0)V = r(\infty)V$. We have already noted that $\hat{V}$ is a bundle of $(3, 1)$-planes and now Lemma 3.6(2) together with $r(\infty) = \overline{r(0)}$ immediately yields (3.5).

We therefore conclude that r is indeed a dressing gauge for (V, d^λ). Moreover, since $r(\lambda) \cdot d^\lambda = \hat{d}^\lambda$, for $\lambda \in \mathrm{dom}(r) \setminus \{0, \infty\}$, and $\hat{V} = r(0)V$, we see that $(\hat{V}, \hat{d}^\lambda)$ is the dressing transform of (V, d^λ) by r. $\qquad \square$

With an eye to dressing constrained Willmore surfaces, we compute $r(0)$ and $r(\infty)$:

Lemma 4.5. *With $V, \hat{V}$ and r, R as in Proposition 4.4, define projections $\pi_i, \hat{\pi}_i$ by*

$$\tau_V(\lambda) = \lambda \pi_1 + \pi_0 + \lambda^{-1} \pi_{-1},$$
$$\tau_{\hat{V}}(\lambda) = \lambda \hat{\pi}_1 + \hat{\pi}_0 + \lambda^{-1} \hat{\pi}_{-1}.$$

Then:

$$(4.12) \qquad r(0) = \sum_{|i| \leq 1} \hat{\pi}_i R(0) \pi_i \quad r(\infty) = \sum_{|i| \leq 1} \hat{\pi}_i R(\infty) \pi_i.$$

Proof. We have already seen that $r(0)V = r(\infty)V = \hat{V}$ but more is true: rearrange (4.10) to give

$$R(\lambda^2) = \tau_{\hat{V}}(\lambda) \circ r(\lambda) \circ \tau_V(\lambda)^{-1},$$

so that the holomorphicity of $\lambda \mapsto R(\lambda^2)$ near 0 and ∞, along with Lemma 3.18, tells us that $r(0)V_+^\perp = \hat{V}_+^\perp$ and $r(\infty)V_-^\perp = \hat{V}_-^\perp$. Moreover, since $r(0)$ is an

isometry and $V_-^\perp$ is maximal isotropic in $V^\perp$, we have that $r(0)V_-^\perp$ is maximal isotropic in $\hat{V}^\perp$ and so must be $\hat{V}_-^\perp$. Thus

$$r(0) = \sum_{|i|\leq 1} \hat{\pi}_i r(0)\pi_i$$

and similarly for $r(\infty)$.

However,

$$\sum_{|i|\leq 1} \hat{\pi}_i r(\lambda)\pi_i = \sum_{|i|\leq 1} \hat{\pi}_i \tau_{\hat{V}}(\lambda)^{-1}R(\lambda^2)\tau_V(\lambda)\pi_i = \sum_{|i|\leq 1} \hat{\pi}_i R(\lambda^2)\pi_i$$

and evaluating this last at $\lambda = 0, \infty$ establishes (4.12). $\qquad\square$

Putting all this together with Theorem 3.13 yields:

Theorem 4.6. *Let (V, ∇^μ) be the 2-perturbed harmonic central sphere congruence of a constrained Willmore surface Λ with multiplier q. Let $(\hat{V}, \hat{\nabla}^\mu)$ be the untwisted dressing transform of (V, ∇^μ) by R.*
Define:

(4.13a) $\qquad\qquad \hat{\Lambda} = (\pi_{\hat{V}} R(0)\Lambda^{1,0}) \cap (\pi_{\hat{V}} R(\infty)\Lambda^{0,1}),$

(4.13b) $\qquad\qquad \hat{q} = \mathrm{Ad}_{\pi_{\hat{V}} R(\infty)} q^{1,0} + \mathrm{Ad}_{\pi_{\hat{V}} R(0)} q^{0,1}.$

Then $\hat{\Lambda}$ is a constrained Willmore surface with multiplier $\hat{q}$ and central sphere congruence $\hat{V}$ on the open set where it immerses.

Definition 4.7. We say that $\hat{\Lambda}$ is the *untwisted dressing transform* of Λ by R.

We shall see below in Proposition 4.12 and Corollary 4.13 that a quaternionic formalism provides cleaner formulae in which the projections $\pi_{\hat{V}}$ do not intervene.

Remark 4.8. *Which dressing gauges r arise from untwisted dressing gauges R via (4.10)? Lemma 4.5 provides a necessary condition: $r(0)V_\pm^\perp = \hat{V}_\pm^\perp$ and $r(\infty)V_\pm^\perp = \hat{V}_\pm^\perp$. It is not difficult to see that this condition is also sufficient by arguing as in Proposition 4.4.*
Any dressing gauge has $r(0)V^\perp = \hat{V}^\perp$ and $r(0)V_\pm^\perp$ maximal isotropic in $\hat{V}^\perp$, so that the remaining possibility is that $r(0)V_\pm^\perp = \hat{V}_\mp^\perp$ and $r(\infty)V_\pm^\perp = \hat{V}_\mp^\perp$. This amounts to switching the orientation of $\hat{V}^\perp$ or, equivalently, replacing

$\tau_{\hat{V}}$ *by its inverse. We leave it as an exercise to the interested reader to work out the analogue of our analysis in this case.*

4.4. Untwisted simple factors

The lack of ρ_V-twisting allows the possibility of untwisted dressing gauges with just two poles so that, in codimension 2, we may dress by simple factors. For this, let $(V, d^\lambda) = (V, \nabla^\mu)$ be a k-perturbed harmonic bundle. For $\nu \in \mathbb{C}^\times \setminus S^1$, let M be a null, line subbundle such that M and $\overline{M}$ are complementary on an open set[4]. Use this data to define gauge transformations $P_{\nu,M}(\mu)$ on that set by

$$P_{\nu,M}(\mu) = \Gamma \frac{M}{\overline{M}} \left(\frac{(1-\nu^*)(\mu-\nu)}{(1-\nu)(\mu-\nu^*)} \right).$$

We have:

(4.14a) $$P_{\nu,M}(1) = 1,$$

(4.14b) $$P_{\nu,M}(\mu^*) = \overline{P_{\nu,M}(\mu)},$$

and that $\mu \mapsto P_{\nu,M}(\mu)$ is rational on $\mathbb{P}^1$ and holomorphic on $\mathbb{P}^1 \setminus \{\nu, \nu^*\}$.

Theorem 4.9. *Let* (V, ∇^μ) *be a k-perturbed harmonic bundle in* $\underline{\mathbb{C}}^6$, $\nu \in \mathbb{C}^\times \setminus S^1$ *and M a ∇^ν-parallel null line subbundle of* $\underline{\mathbb{C}}^6$. *Assume that $M, \overline{M}$ are complementary and then that $P_{\nu,M}(\infty)V_+^\perp$ and $P_{\nu,M}(0)V_-^\perp$ are complementary. Then $P_{\nu,M}$ is an untwisted dressing gauge for* (V, ∇^μ).

Proof. We have already seen that $\mu \mapsto P_{\nu,M}(\mu)$ is holomorphic on $\mathbb{P}^1 \setminus \{\nu, \nu^*\}$ and so, in particular, near 0 and ∞. Moreover, (4.14b) is precisely the reality condition (4.8) of Definition 4.3.

We have that M is ∇^ν-parallel, when, thanks to (4.14b), $\overline{M}$ is ∇^{ν^*}-parallel. It is now immediate from Lemma 3.17 that $\mu \mapsto P_{\nu,M}(\mu) \cdot \nabla^\mu$ extends holomorphically to $\mathbb{C}^\times$ settling Definition 4.3(b).

Finally, Definition 4.3(c) holds by hypothesis. $\qquad\square$

In fact, the dressing transformation induced by these simple factor dressing gauges are generically Bäcklund transforms as in Section 3.3. Indeed, if M is ∇^ν-parallel and $\alpha = \pm\sqrt{\nu}$, then (4.6) tells us that $L := \tau_V(\alpha)^{-1}M$ is d^α-parallel. Thus α, L are parameters for a Bäcklund transform of (V, d^λ) which turns out to have the same effect as dressing by $P_{\nu,M}$. In more detail:

---

[4]This is the open set on which M is not real.

Proposition 4.10. *Let (V, ∇^μ) be a k-perturbed harmonic bundle in $\mathbb{C}^6$, $\nu \in \mathbb{C}^\times \setminus S^1$ and M a ∇^ν-parallel null line subbundle satisfying the hypotheses of Theorem 4.9. Let $(\hat{V}, \hat{\nabla}^\mu)$ be the dressing transform of (V, ∇^μ) by $P_{\nu,M}$.*

Let $\alpha = \pm\sqrt{\nu}$ and set $L = \tau_V(\alpha)^{-1}M$. Assume that L satisfies the hypotheses of Theorem 3.20: thus $L, \rho_V L$ are complementary and, with $L' = p^V_{\alpha,L}(\alpha^*)\bar{L}$ and $V' = p^V_{\alpha,L}(0)V$, $L', \rho_{V'}L'$ are also complementary.

Then

$$(4.15) \qquad p^{V'}_{\alpha^*,L'}(\lambda)p^V_{\alpha,L}(\lambda) = \tau_{\hat{V}}(\lambda)^{-1}P_{\nu,M}(\lambda^2)\tau_V(\lambda),$$

for all λ.

Thus, Proposition 4.4 tell us that $(\hat{V}, \hat{d}^\lambda)$ is the Bäcklund transform of (V, d^λ) with parameters α, L.

Proof. Set

$$L(\lambda) := p^{V'}_{\alpha^*,L'}(\lambda)p^V_{\alpha,L}(\lambda),$$
$$R(\lambda) := \tau_{\hat{V}}(\lambda)^{-1}P_{\nu,M}(\lambda^2)\tau_V(\lambda).$$

We follow what should now be a familiar strategy: we use Lemma 3.18 to see that $\lambda \mapsto R(\lambda)L(\lambda)^{-1}$ is holomorphic on $\mathbb{P}^1$ and so constant.

Both $L(\lambda)$ and $R(\lambda)$ are holomorphic on $\mathbb{P}^1 \setminus \pm\alpha, \pm\alpha^*$, $L(\lambda)$ by construction and $R(\lambda)$ by Proposition 4.4. Moreover, both R, L satisfy the reality condition so it suffices to show holomorphicity of RL^{-1} at $\pm\alpha$. Now the part of this product with poles at $\pm\alpha$ is

(4.16a)
$$\Gamma^M_M\left(\tfrac{\lambda^2 - \alpha^2}{\lambda^2 - 1/\bar{\alpha}^2}\right)\tau_V(\lambda)\Gamma^L_{\rho_V L}\left(\tfrac{\lambda - \alpha}{\lambda + \alpha}\right)^{-1} = \Gamma^M_M\left(\tfrac{\lambda + \alpha}{\lambda + \alpha^*}\right)\Gamma^M_M\left(\tfrac{\lambda - \alpha}{\lambda - \alpha^*}\right)\tau_V(\lambda)\Gamma^L_{\rho_V L}\left(\tfrac{\lambda - \alpha}{\lambda + \alpha}\right)^{-1}$$

(4.16b)
$$= \Gamma^M_M\left(\tfrac{\lambda - \alpha}{\lambda - \alpha^*}\right)\Gamma^M_M\left(\tfrac{\lambda + \alpha}{\lambda + \alpha^*}\right)\tau_V(\lambda)\Gamma^{\rho_V L}_L\left(\tfrac{\lambda + \alpha}{\lambda - \alpha}\right)^{-1}.$$

We have $M = \tau_V(\alpha)L$ and this, together with Lemma 3.18 applied to the right hand side of (4.16a), gives holomorphicity at α. Again, $\tau_V(-\alpha)\rho_V L = \tau_V(\alpha)L = M$, so that Lemma 3.18 applies to (4.16b) to give holomorphicity at $-\alpha$. $\qquad\square$

4.5. Quaternionic formalism

The Klein correspondence offers another viewpoint on the conformal geometry of S^4 which has been heavily exploited by Pedit, Pinkall and their

collaborators [2, 3, 10, 19]. We rehearse the basics of this viewpoint with a view to extending the transformation theory of [10, Chapter 12] and relating it to our dressing transformations in Section 4.6.

We begin with a 2-dimensional quaternionic vector space which we view as $\mathbb{C}^4$ equipped with a quaternionic structure j: thus $j : \mathbb{C}^4 \to \mathbb{C}^4$ is anti-linear and $j^2 = -1$. Now fix $\det \in \wedge^2(\mathbb{C}^4)^*$ with

$$j^* \det = \overline{\det}, \quad \det(v_1 \wedge v_2 \wedge jv_1 \wedge jv_2) > 0,$$

when $v_1 \wedge v_2 \wedge jv_1 \wedge jv_2 \neq 0$ $v_i \in \mathbb{C}^4$. These data equip $\mathbb{C}^6 := \wedge^2 \mathbb{C}^4$ with a real structure $\overline{v_1 \wedge v_2} := jv_1 \wedge jv_2$ and an inner product $(\xi, \eta) := \det(\xi \wedge \eta)$ of signature $(5, 1)$.

Set $\mathrm{SL}(\mathbb{H}^2) = \{g \in \mathrm{SL}(\mathbb{C}^4) : jg = gj\}$. The action of $\mathrm{SL}(\mathbb{H}^2)$ on $\wedge^2 \mathbb{C}^4$ induces a map $\mathrm{SL}(\mathbb{H}^2) \to \mathrm{SO}(5, 1)$ which is a double cover. Differentiating gives an isomorphism of Lie algebras $\mathfrak{sl}(\mathbb{H}^2) \cong \mathfrak{o}(\mathbb{R}^{5,1})$ where $\mathfrak{sl}(\mathbb{H}^2) = \{A \in \mathfrak{sl}(\mathbb{C}^4) : [A, j] = 0\}$.

The Klein correspondence identifies the Grassmannian of 2-planes in $\mathbb{C}^4$ with the quadric in $\mathbb{P}(\mathbb{C}^6)$ defined by our inner product: $W \mapsto \wedge^2 W$. Clearly, j-stable 2-planes (thus one-dimensional quaternionic subspaces) are identified with points of $\mathbb{P}(\mathcal{L}) = S^4$ yielding the celebrated isomorphism $\mathbb{H}P^1 \cong S^4$.

Under this correspondence, oriented 2-spheres are identified with 2-planes $S^+ \leq \mathbb{C}^4$ which are not j-stable: for such an S^+, set $S^- = jS^+$ and then $\mathbb{C}^4 = S^+ \oplus S^-$ and the corresponding $(3, 1)$-plane V in $\mathbb{R}^{5,1}$ is given by:

$$V = S^+ \wedge S^-, \quad V_\pm^\perp = \wedge^2 S^\pm.$$

Equivalently, such S^+ correspond bijectively to $S \in \mathrm{SL}(\mathbb{H}^2)$ with $S^2 = -1$ via $S = \pm i$ on $S^\pm$.

Thus, a bundle V of oriented $(3, 1)$-planes amounts to $S \in \Gamma\mathrm{SL}(\underline{\mathbb{H}}^2)$ with $S^2 = -1$ and $\pm i$-eigenbundles $S^\pm$ so that $V_\pm^\perp = \wedge^2 S^\pm$. The corresponding decomposition $\underline{\mathbb{C}}^4 = S^+ \oplus S^{-1}$ induces a decomposition of the flat connection

$$d = \mathcal{D}_S + \mathcal{N}_S$$

with $\mathcal{D}_S S = 0$ and $\mathcal{N}_S$ anti-commuting with S: $\{\mathcal{N}_S, S\} = 0$. Using $V = S^+ \wedge S^-$, we readily see that $\mathcal{D}_S$ induces the connection $\mathcal{D}_V$ on $\underline{\mathbb{C}}^6 = \wedge^2 \mathbb{C}^4$ and that $\mathcal{N}_S$ coincides with $\mathcal{N}_V$ under the isomorphism $\mathfrak{sl}(\mathbb{H}^2) \cong \mathfrak{o}(5, 1)$. Again, the subspace $V \wedge V^\perp$ of $\mathfrak{o}(5, 1)$ corresponds to $\{A \in \mathfrak{sl}(\mathbb{H}^2) : \{A, S\} = 0\}$ with the almost complex structure J_V corresponding to post-composition

with S. Thus the decomposition (4.1) of $\mathcal{N}_V$ is identified with the decomposition

$$\mathcal{N}_S = \mathcal{A}_S + \mathcal{Q}_S$$

of [10, Section 5.1] where

(4.17a) $*\mathcal{A}_S = S\mathcal{A}_S = -\mathcal{A}_S S,$

(4.17b) $*\mathcal{Q}_S = -S\mathcal{Q}_S = \mathcal{Q}_S S.$

Here, and below, we follow [10] by setting $*\alpha = \alpha \circ J^\Sigma$, for $\alpha \in \Omega^1_\Sigma$. Thus our $*$ is minus that of Hodge.

We note the following simple consequences of this analysis [10, Equations (5.2) and (5.11)]:

(4.18a) $dS = 2(*\mathcal{Q}_S - *\mathcal{A}_S),$

(4.18b) $0 = \mathcal{A}_S \wedge \mathcal{Q}_S,$

(4.18c) $0 = \mathcal{Q}_S \wedge \mathcal{A}_S,$

where coefficients in the wedge products are multiplied using composition in $\mathrm{End}(\mathbb{C}^4)$.

Now let Λ be a conformal immersion with oriented central sphere congruence V. Then Λ corresponds to a quaternionic line subbundle of $\mathbb{C}^4$, thus a j-stable, rank 2 bundle $L \leq \mathbb{C}^4$ with $\wedge^2 L = \Lambda$. To the central sphere congruence V corresponds $S \in \Gamma\mathrm{SL}(\mathbb{H}^2)$ characterized by the following conditions [10, Section 5.2, Theorem 2]:

(4.19a) $SL = L, \quad dSL \leq T^*\Sigma \otimes L,$

(4.19b) $*\delta = S\delta = \delta S,$

(4.19c) $\mathcal{Q}_S L = 0,$

where $\delta = \pi \circ d$ for $\pi : \mathbb{C}^4 \to \mathbb{C}^4/L$ the projection away from L.

From (4.19a), we see that $L = L^+ \oplus L^-$ where $L^\pm = L \cap S^\pm$ and $L^- = jL^+$. Thus $\Lambda = L^+ \wedge L^-$. We remark that $L^\pm$, viewed as maps $\Sigma \to \mathbb{C}P^3$ are the two twistor lifts of Λ. Moreover, from (4.19b), we readily compute that:

$$\Lambda^{1,0} = S^+ \wedge L^-, \quad \Lambda^{0,1} = L^+ \wedge S^-.$$

We have therefore established a dictionary between subbundles of $\mathbb{C}^4$ and $\mathbb{C}^6 = \wedge^2 \mathbb{C}^4$ which we shall use without further comment. Thus we can (and will!) talk of conformal immersions L with central sphere congruence S.

With a view to working with constrained Willmore surfaces in this formalism, we list the algebraic properties of a Lagrange multiplier q when viewed as an $\mathfrak{sl}(\mathbb{H}^2)$-valued 1-form.

Lemma 4.11. *Let $\Lambda = \wedge^2 L$ be a conformal immersion with central sphere congruence S. Let $q \in \Omega^1 \otimes \mathfrak{o}(5,1)$. Then $q^{1,0}$ takes values in $\Lambda \wedge \Lambda^{0,1}$ if and only if, when viewed as a $\mathfrak{sl}(\mathbb{H}^2)$-valued 1-form, we have*

(4.20a) $$q \in \Omega^1(\operatorname{End}(\mathbb{H}^2/L, L))$$
(4.20b) $$*q = Sq = qS.$$

Proof. Suppose that $q^{1,0}$ takes values in $\Lambda \wedge \Lambda^{0,1}$, or, equivalently, q preserves V and annihilates $V^\perp$ while $q^{1,0}$ additionally annihilates $\Lambda^{0,1}$. The first two of these imply that $qS = Sq$ so that q preserves $S^\pm$. Moreover, $qV^\perp = 0$ means that $q \wedge^2 S^\pm = 0$, so that $\operatorname{trace}(q_{|S^\pm}) = 0$.

Let $Z \in T^{1,0}\Sigma$, $\sigma^+ \in L^+$ and $s^- \in S^-$. Then

$$q_Z(\sigma^+ \wedge s^-) = (q_Z\sigma^+) \wedge s^- + \sigma^+ \wedge (q_Z s^-) = 0.$$

It follows at once that L^+ is an eigenspace of q_Z with eigenvalue λ, say, and then that $q_Z = -\lambda$ on S^-. Since q_Z is trace-free on S^-, we immediately get that q_Z annihilates S^- (so that $*q = qS$) and also L^+ and so L. Finally, choose $s^+ \in S^+$ so that $\sigma^+ \wedge s^+ \neq 0$. Then $0 = q_Z(\sigma^+ \wedge s^+) = \sigma^+ \wedge (q_Z s^+)$ so that $q_Z S^+ \leq L^+$. This establishes Equations (4.20). The converse is straightforward. □

Let R be a dressing gauge for a k-perturbed harmonic (S, ∇^μ). We wish to describe its effect in quaternionic terms for which we need a little notation: for $g \in O(\mathbb{C}^6)$, write $\widetilde{g}$ for a preimage of g under the double covering $SL(\mathbb{C}^4) \to O(\mathbb{C}^6)$. Thus

$$g(v \wedge w) = (\widetilde{g}v) \wedge (\widetilde{g}w),$$

for $v, w \in \mathbb{C}^4$. Clearly $\widetilde{g}$ is determined up to sign by g and has an unambiguous projective action on $\mathbb{C}^4$.

With this in hand, we have:

Proposition 4.12. *Let (S, ∇^μ) be k-perturbed harmonic in $\underline{\mathbb{C}}^6$. Let R be an untwisted dressing gauge for (S, ∇^μ) and $(\hat{S}, \hat{\nabla}^\mu)$ the untwisted dressing*

transform by R. Then

$$\text{(4.21a)} \qquad \hat{S}^+ = \widetilde{R}(\infty)S^+ \quad \hat{S}^- = \widetilde{R}(0)S^-.$$

Moreover, if S is the central sphere congruence of a constrained Willmore surface L with multiplier q, then the dressing transform $\hat{S}$ and its multiplier $\hat{q}$ are given by:

$$\text{(4.21b)} \qquad \hat{L}^+ = \widetilde{R}(\infty)L^+ \quad \hat{L}^- = \widetilde{R}(0)S^-,$$

$$\text{(4.21c)} \qquad \hat{q}_{|\hat{S}^+} = \widetilde{R}(\infty)q\widetilde{R}(\infty)^{-1}_{|\hat{S}^+} \quad \hat{q}_{|\hat{S}^-} = \widetilde{R}(0)q\widetilde{R}(0)^{-1}_{|\hat{S}^-}.$$

Proof. First note that the right members of (4.21) follow at once from the left members because $R(0) = \overline{R}(\infty)$ so that we may take $\widetilde{R}(0)$ to be $j \circ \widetilde{R}(\infty) \circ j^{-1}$.

For (4.21a), first note that (4.9) amounts to $\wedge^2 \hat{S}^+ = R(\infty)\wedge^2 S^+$, which immediately yields $\hat{S}^+ = \widetilde{R}(\infty)S^+$.

For the rest, let $\pi^\pm$, $\hat{\pi}^\pm$ be the projections corresponding to the decompositions $\underline{\mathbb{C}}^4 = S^+ \oplus S^-$ and $\underline{\mathbb{C}}^4 = \hat{S}^+ \oplus \hat{S}^-$. It is easy to deduce from Lemma 4.5 that

$$\widetilde{r}(\infty) = \hat{\pi}^+\widetilde{R}(\infty)\pi^+ + \hat{\pi}^-\widetilde{R}(\infty)\pi^- = \widetilde{R}(\infty)\pi^+ + \hat{\pi}^-\widetilde{R}(\infty)\pi^-$$
$$\widetilde{r}(0) = \hat{\pi}^+\widetilde{R}(0)\pi^+ + \hat{\pi}^-\widetilde{R}(0)\pi^- = \hat{\pi}^+\widetilde{R}(0)\pi^+ + \widetilde{R}(0)\pi^-.$$

In particular,

$$r(\infty)\Lambda^{0,1} = \left(\widetilde{R}(\infty)L^+\right) \wedge \left(\hat{\pi}^-\widetilde{R}(\infty)S^-\right) = \left(\widetilde{R}(\infty)L^+\right) \wedge \hat{S}^-,$$

where, for the last equality, we note that $\hat{\pi}^-\widetilde{R}(\infty)$ has kernel S^+ and so injects (therefore surjects) when restricted to S^-. Complex conjugation now yields $r(0)\Lambda^{1,0} = \hat{S}^+ \wedge \widetilde{R}(0)L^-$ and then (3.11a) gives

$$\hat{\Lambda} = (\hat{S}^+ \wedge \widetilde{R}(0)L^-) \cap (\widetilde{R}(\infty)L^+ \wedge \hat{S}^-) = \widetilde{R}(\infty)L^+ \wedge \widetilde{R}(0)L^-.$$

In particular, $\hat{L}^+ = \widetilde{R}(\infty)L^+$ and $\hat{L}^- = \widetilde{R}(0)L^-$ settling (4.21b).

Finally, we consider q. From (4.20b), we see that q preserves $S^\pm$ while $q^{1,0}_{|S^-} = 0$ and similarly for $\hat{q}$. On the other hand, from (3.11b), we have:

$$\hat{q}^{1,0} = \text{Ad}_{r(\infty)}q^{1,0} = \widetilde{r}(\infty)q^{1,0}\widetilde{r}(\infty)^{-1}.$$

Since $\widetilde{r}(\infty)_{|S^+} = \widetilde{R}(\infty)_{|S^+}$ with image $\hat{S}^+$, this immediately yields

$$\hat{q}^{1,0}_{|\hat{S}^+} = \widetilde{R}(\infty)q^{1,0}\widetilde{R}(\infty)^{-1}_{|\hat{S}^+}.$$

On the other hand, both $\hat{q}^{0,1}$ and $\widetilde{R}(\infty)q^{0,1}\widetilde{R}(\infty)^{-1}$ vanish on $\hat{S}^+$ and so (4.21c) is established. □

We summarize the development in the following corollary:

Corollary 4.13. *Let R be an untwisted dressing gauge of a k-perturbed harmonic (S, ∇^μ) in $\underline{\mathbb{C}}^6$ and define $T_0 \in \Gamma\mathrm{GL}(\underline{\mathbb{H}}^2)$ by*

$$T_0 = \widetilde{R}(\infty)\pi^+ + \widetilde{R}(0)\pi^-.$$

Then the dressing transform $\hat{S}$ of S by R is given by

$$\hat{S} = T_0 S T_0^{-1}.$$

Moreover, if S is the central sphere congruence of a constrained Willmore surface L with multiplier q, then the dressing transform $\hat{L}$ and its multiplier $\hat{q}$ are given by:

$$\hat{L} = T_0 L,$$
$$\hat{q} = T_0 q T_0^{-1}.$$

Proof. The only thing to check here is that T_0 so defined is indeed an isomorphism. But the image of T_0 is $\hat{S}^+ + \hat{S}^- = \underline{\mathbb{C}}^4$ thanks to (4.21). □

Clearly there is some gauge freedom here: we could precompose T_0 with any gauge transformation, which is a scalar multiple of the identity on each of $S^\pm$. We shall exploit this below.

4.6. Darboux transforms and Riccati equations

A transformation of Willmore surfaces in S^4 via solutions of a Riccati equation is described in [10, Chapter 12] while a related transform is derived by Leschke in [26]. We now show that these transforms all amount to untwisted dressing by simple factors as described in Section 4.4. Along the way, we extend the theory to constrained Willmore surfaces.

For this, we specialize the considerations of the last section to the case where the dressing gauge R is a simple factor $P_{\nu,M}$. Recall that here $\nu \in \mathbb{C}^\times \setminus S^1$ and M is a ∇^ν-parallel null line subbundle with $M, \overline{M}$ complementary. Equivalently, in the quaternionic formalism, $M = \wedge^2 W$ for W a ∇^ν-parallel complex subbundle of $\underline{\mathbb{C}}^4$ such that $\underline{\mathbb{C}}^4 = W \oplus jW$ (so that $\mathrm{rank}\, W = 2$). If

we now define $\Gamma_{jW}^{W}(\mu) \in \Gamma\mathrm{End}(\mathbb{C}^4)$ by

$$\Gamma_{jW}^{W}(\mu) = \begin{cases} \mu & \text{on } W \\ 1/\mu & \text{on } jW, \end{cases}$$

then

$$\widetilde{P_{\nu,M}}(\mu) = \Gamma_{jW}^{W}\left(\sqrt{\tfrac{(1-\nu^*)(\mu-\nu)}{(1-\nu)(\mu-\nu^*)}}\right).$$

We now have the following improvement on Corollary 4.13 in case that R is a simple factor:

Theorem 4.14. *Let (S, ∇^μ) be k-perturbed harmonic in $\mathbb{C}^6$. Fix $\nu \in \mathbb{C}^\times \setminus S^1$ and let $W \leq \mathbb{C}^4$ be a ∇^ν-parallel subbundle such that $W \oplus jW = \mathbb{C}^4$. Define $\mu_0, \mu_1 \in \mathbb{C}^\times$ by*

$$(4.22) \qquad \mu_0 = i\frac{\nu + 1}{\nu - 1}, \qquad \mu_1 = i\frac{\nu^* + 1}{\nu^* - 1}.$$

Further, define $X \in \Gamma\mathfrak{gl}(\mathbb{H}^2)$ by

$$X = \begin{cases} \mu_0 & \text{on } W; \\ \mu_1 & \text{on } jW. \end{cases}$$

and then $T \in \Gamma\mathfrak{gl}(\mathbb{H}^2)$ by

$$(4.23) \qquad\qquad\qquad T = X - S.$$

Then $P_{\nu,\wedge^2 W}$ is an untwisted dressing gauge for (S, ∇^μ) on the open set on which T is an isomorphism and the dressing transform of S by $P_{\nu,\wedge^2 W}$ is given by

$$(4.24a) \qquad\qquad\qquad \hat{S} = TST^{-1}$$

In particular, $\hat{S}$ is k-perturbed harmonic.

Moreover, when S is the central sphere congruence of a constrained Willmore surface L with multiplier q, then the untwisted dressing transform of L by $P_{\nu,\wedge^2 W}$ is given by

$$(4.24b) \qquad\qquad\qquad \hat{L} = TL,$$
$$(4.24c) \qquad\qquad\qquad \hat{q} = TqT^{-1}.$$

Proof. Set $R = P_{\nu, \wedge^2 W}$. We claim that there are constants $\lambda_0, \lambda_1 \in \mathbb{C}^\times$ such that:

$$(4.25) \qquad\qquad \widetilde{R}(0) = \lambda_0 (X + i) \quad \widetilde{R}(\infty) = \lambda_1 (X - i).$$

Given the claim, we immediately deduce that:

$$\widetilde{R}(0)_{|S^-} = \lambda_0 (X + i)_{|S^-} = \lambda_0 T_{|S^-},$$
$$\widetilde{R}(\infty)_{|S^+} = \lambda_1 (X - i)_{|S^+} = \lambda_1 T_{|S^+}.$$

We note that $\mu_1 = -\overline{\mu_0}$ so that X and so T are $\mathfrak{gl}(\mathbb{H}^2)$-valued. Moreover, T is an isomorphism exactly when $\widetilde{R}(0)S^- \cap \widetilde{R}(\infty)S^+ = \{0\}$ or, equivalently, when $R(0)V_-^\perp$ and $R(\infty)V_+^\perp$ are complementary, that is, when R is an untwisted dressing gauge. The rest of the theorem now follows at once from Proposition 4.12.

For the claim, note that both sides of each equation in (4.25) have W and jW as eigenspaces and so we must simply equate eigenvalues and require that

$$\sqrt{\frac{\nu(1-\nu^*)}{\nu^*(1-\nu)}} = \lambda_0(\mu_0 + i) \quad \sqrt{\frac{1-\nu^*}{1-\nu}} = \lambda_1(\mu_0 - i),$$

$$\sqrt{\frac{\nu^*(1-\nu)}{\nu(1-\nu^*)}} = \lambda_0(\mu_1 + i) \quad \sqrt{\frac{1-\nu}{1-\nu^*}} = \lambda_1(\mu_i - i).$$

These amount to linear equations for the μ_i which are solved by (4.22) and then the λ_i are given by

$$\lambda_0 = -\frac{i}{2}\sqrt{\frac{(\nu-1)(\nu^*-1)}{\nu\nu^*}} \quad \lambda_1 = -\frac{i}{2}\sqrt{(\nu-1)(\nu^*-1)}.$$

$\square$

Remark 4.15. *For Willmore L and, more generally, harmonic S, the construction in Theorem 4.14 of $\hat{S}$ and $\hat{L}$ from a ∇^ν-parallel W coincides with Leschke's ν-Darboux transform [26, Theorems 4.2 and 4.4]. She conjectured that this procedure should amount to a dressing transform as we have just affirmed.*

Theorem 4.14 offers a different perspective on untwisted dressing by simple factors by focusing on the field of endomorphisms X rather than

the dressing gauge $P_{\nu,\wedge^2 W}$. The key conditions on X, that it have constant eigenvalues and ∇^ν or ∇^{ν^*}-parallel eigenspaces W, jW, are encapsulated in a Riccati equation with a conserved quantity:

Proposition 4.16. *Let (S, ∇^μ) be k-perturbed harmonic, $X \in \Gamma\mathfrak{gl}(\underline{\mathbb{H}}^2)$ and $\mu_0 \in \mathbb{C} \setminus \mathbb{R}$. Then X has eigenvalue μ_0 with rank 2 ∇^ν-parallel eigenbundle if and only if*

(4.26a)
$$(\mu_1 - \mu_0)dX = X(\beta^* - \beta)X + (\mu_0\beta - \mu_1\beta^*)X + X(\mu_1\beta - \mu_0\beta^*) + \mu_0\mu_1(\beta^* - \beta)$$
(4.26b)
$$X^2 - (\mu_0 + \mu_1)X + \mu_0\mu_1 = 0,$$

where $\mu_1 = \overline{\mu_0}$, $\nabla^\nu = d + \beta$ and $\nabla^{\nu^*} = d + \beta^*$.

Proof. First note that if X has rank 2 μ_0-eigenbundle W then it also has μ_1-eigenbundle jW with $\underline{\mathbb{C}}^4 = W \oplus jW$ so that the minimal polynomial of X is $X^2 - (\mu_0 + \mu_1)X + \mu_0\mu_1$. Conversely, if X solves (4.26b) with μ_0 non-real, then, since $[X, j] = 0$, X is not a scalar multiple of the identity and so has $X^2 - (\mu_0 + \mu_1)X + \mu_0\mu_1$ as its minimal polynomial. It follows at once that X has eigenvalues μ_0, μ_1 with rank 2 eigenbundles.

Thus, we may assume that X has rank 2 μ_0-eigenbundle W and the only remaining issue is whether W is ∇^ν-parallel, or equivalently, jW is ∇^{ν^*}-parallel. This is easily seen to be equivalent to the demand that

$$(X - \mu_0)\nabla^\nu(X - \mu_1) = 0,$$
$$(X - \mu_1)\nabla^{\nu^*}(X - \mu_0) = 0,$$

or, since these last have image in different eigenbundles,

$$(X - \mu_0)\nabla^\nu(X - \mu_1) = (X - \mu_1)\nabla^{\nu^*}(X - \mu_0).$$

However, writing this out in terms of β and β^* promptly yields (4.26a). $\square$

Remark 4.17. *In the situation of Theorem 4.14 with μ_0 given by (4.22), we see that the excluded case that $\mu_0 \in \mathbb{R}$ amounts to the already excluded case of $\nu \in S^1$.*

Observe that the coefficients in the Riccati equation (4.26a) are all pure imaginary so that solutions with initial condition in $\mathfrak{gl}(\underline{\mathbb{H}}^2)$ remain in $\mathfrak{gl}(\underline{\mathbb{H}}^2)$. In fact, much more is true: the Riccati equation is completely integrable and admits (4.26b) as a first integral. All this uses almost nothing about the

specifics of the situation beyond the flatness of the connections ∇^ν and ∇^{ν^*}. Indeed, we have:

Proposition 4.18. *Let* A *be a complex finite-dimensional associative algebra with unit,* $\mu_0 \neq \mu_1 \in \mathbb{C}$ *and* β, β^* A*-valued 1-forms on* Σ *defining flat connections:*

$$d\beta + \beta \wedge \beta = d\beta^* + \beta^* \wedge \beta^* = 0.$$

Then the Riccati equation (4.26a)

$$\begin{aligned}(\mu_1 - \mu_0)dX = X(\beta^* - \beta)X + (\mu_0\beta - \mu_1\beta^*)X \\ + X(\mu_1\beta - \mu_0\beta^*) + \mu_0\mu_1(\beta^* - \beta)\end{aligned}$$

for $X : \Sigma \to \mathsf{A}$ *is completely integrable.*

Moreover, if X *is a solution and* $I = X^2 - (\mu_0 + \mu_1)X + \mu_0\mu_1$, *then* I *solves the linear equation*

$$(\mu_1 - \mu_0)dI = I(\beta^* - \beta)X + X(\beta^* - \beta)I + I(\mu_1\beta - \mu_0\beta^*) + (\mu_0\beta - \mu_1\beta^*)I$$

and so, in particular, vanishes identically if it vanishes at a single point.

Proof. One can prove complete integrability by direct computation but we offer a more conceptual and, perhaps, instructive proof based on the well known classical observation that Riccati equations are the affine expression of linear equations in homogeneous coordinates. So consider a general Riccati equation

$$dX = XAX + BX + XC + D,$$

with A-valued 1-forms A, B, C, D. Write $X = PQ^{-1}$ for $P, Q : \Sigma \to \mathsf{A}$ (we can always do this near some initial point $p_0 \in \Sigma$ by taking $Q(p_0) = 1$). Then the equation becomes

$$dP - BP - DQ - PQ^{-1}(dQ + AP + CQ) = 0,$$

so that integrability is guaranteed by flatness of the connection on the trivial $\mathsf{A} \oplus \mathsf{A}$ bundle given by

$$d + \begin{pmatrix} -B & -D \\ A & C \end{pmatrix}.$$

In the case at hand, this connection reads

$$d + \frac{1}{\mu_1 - \mu_0}\begin{pmatrix} \mu_1\beta^* - \mu_0\beta & \mu_0\mu_1(\beta - \beta^*) \\ \beta^* - \beta & \mu_1\beta - \mu_0\beta^* \end{pmatrix} = \begin{pmatrix} \mu_0 & \mu_1 \\ 1 & 1 \end{pmatrix} \cdot \left(d + \begin{pmatrix} \beta & 0 \\ 0 & \beta^* \end{pmatrix} \right),$$

which is clearly flat.

The derivation of the linear equation for I is a straightforward compu-
tation which we leave to the interested reader.   □

Let us summarize this development which amounts to a construction of
new k-perturbed harmonic bundles from old via solving a Riccati equation:

Corollary 4.19. *Let (S, ∇^μ) be k-perturbed harmonic in $\mathbb{C}^6$ and fix $\nu \in
\mathbb{C}^\times \setminus S^1$. Define $\mu_0, \mu_1 \in \mathbb{C}^\times$ by (4.22). Fix a base-point $x_o \in \Sigma$ and $X_o \in
\mathfrak{gl}(\mathbb{H}^2)$ satisfying $X_o^2 - (\mu_0 + \mu_1)X_o + \mu_0\mu_1 = 0$.*
There is (locally) $X \in \Gamma\mathfrak{gl}(\underline{\mathbb{H}}^2)$ solving:

$$(\mu_1 - \mu_0)dX = X(\beta^* - \beta)X + (\mu_0\beta - \mu_1\beta^*)X$$
$$+ X(\mu_1\beta - \mu_0\beta^*) + \mu_0\mu_1(\beta^* - \beta)$$
$$X(x_o) = X_o,$$

*where $\nabla^\nu = d + \beta$ and $\nabla^\mu = d + \beta^*$. Then, with $T = X - S$, $\hat{S} = TST^{-1}$ is
k-perturbed harmonic on the open set on which T is invertible.*

*Moreover, if S is the central sphere congruence of a constrained Willmore
surface L, then $\hat{S}$ is the central sphere congruence of a constrained Willmore
surface TL.*

Of course, our Riccati equation for X can be phrased as a Riccati equa-
tion for T the form of which, for general ν, is not very edifying. However,
when ν is real, matters simplify considerably and we not only recover the
Darboux transforms of Willmore surfaces described in [10, Section 12.3] but
generalize them to the constrained Willmore case.

So let L be a constrained Willmore surface with multiplier q and central
sphere congruence S. We then have $\nabla^\mu = d + (\mu - 1)(\mathcal{A}_S + q)^{1,0} + (\mu^{-1} -
1)(\mathcal{A}_S + q)^{0,1}$. Fix $\nu \in \mathbb{R} \setminus \{\pm 1\}$ so that $\nu^* = 1/\nu$ and then

$$\mu_0 = i\frac{\nu + 1}{\nu - 1} = -\mu_1.$$

Set

(4.27) $$\rho = -\frac{(\nu - 1)^2}{4\nu} = \frac{2 - \nu - \nu^*}{4}$$

so that

$$\rho^{-1} - 1 = \mu_0^2 = \mu_1^2.$$

We may now use (4.18a) along with (4.17) and (4.20b) to write the Riccati equation (4.26a) for X as the following Riccati equation for $T = X - S$:

$$(4.28) \qquad dT = \rho T\big(2 * (\mathcal{A}_S + q)\big)T - 4\rho T q - 2 * (\mathcal{Q}_S + q)$$

with first integral $(T + S)^2 - (\rho^{-1} - 1)$.

In this situation, $\rho^{-1} < 1$ and, conversely, if $\rho^{-1} < 1$, we can rearrange (4.27) and recover ν, ν^* as the two real roots of $\mu^2 + 2(2\rho - 1)\mu + 1 = 0$. We therefore have:

Theorem 4.20. *Let L be a constrained Willmore surface in S^4 with multiplier q and central sphere congruence S. Let $\rho \in \mathbb{R}$ with $\rho^{-1} < 1$ and let $T \in \mathfrak{gl}(\mathbb{H}^2)$ be a solution of the integrable Riccati equation (4.28) with $(T + S)^2 - (\rho^{-1} - 1) = 0$ at one, and hence every, point.*

Then $\hat{L} = TL$ is also a constrained Willmore surface with multiplier TqT^{-1} and central sphere congruence TST^{-1} on the open set on which T is invertible.

Moreover, $\hat{L}$ is an untwisted dressing transform of L by a simple factor.

For Willmore surfaces (thus $q = 0$), this construction of $\hat{L}$ is the Darboux transform of [10, Section 12.3] with the caveat that their Riccati equation swops the roles of $\mathcal{A}_S$ and $\mathcal{Q}_S$ and their T is the inverse of ours. This is, of course, a well-known symmetry of the Riccati equation.

Acknowledgments

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