

Hyperbolic volume and Heegaard distance

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We prove (Theorem 1.5) that there exists a constant $\Lambda > 0$ so that if M is a (μ, d) -generic complete hyperbolic 3-manifold of volume $\text{Vol}(M) < \infty$ and $\Sigma \subset M$ is a Heegaard surface of genus $g(\Sigma) > \Lambda \text{Vol}(M)$, then $d(\Sigma) \leq 2$, where $d(\Sigma)$ denotes the distance of Σ as defined by Hempel. The term (μ, d) -generic is described precisely in Definition 1.3; see also Remark 1.4.

The key for the proof of Theorem 1.5 is Theorem 1.8 which is of independent interest. There we prove that if M is a compact 3-manifold that can be triangulated using at most t tetrahedra (possibly with missing or truncated vertices), and Σ is a Heegaard surface for M with $g(\Sigma) \geq 76t + 26$, then $d(\Sigma) \leq 2$.

1. Introduction

All the manifolds considered in this paper are three-dimensional, compact, connected and orientable. By *hyperbolic manifold* we mean a manifold whose interior admits a complete finite volume Riemannian metric locally isometric to hyperbolic 3-space \mathbb{H}^3 .

It is generally agreed that the volume of a hyperbolic manifold M , $\text{Vol}(M)$, is a good measure of the complexity of M . As evidence of that, arguments of M. Gromov, T. Jørgensen and W. Thurston show that the hyperbolic volume is linearly equivalent to the number of tetrahedra needed to triangulate a link exterior in M . The argument is based on Thurston's notes [31], for a detailed presentation see [13]; throughout this paper, we follow the notation and definitions given in that paper. For a precise statement, let $t_c(M)$ be the smallest number of tetrahedra needed to triangulate $M \setminus N(L)$, where the minimum is taken over all links $L \subset M$ (possibly, $L = \emptyset$) and all possible triangulations.

Theorem 1.1 Gromov, Jørgensen, Thurston. *There exist constants $A, B > 0$ so that for any hyperbolic manifold M the following holds:*

$$At_c(M) < \text{Vol}(M) < Bt_c(M).$$

Remark 1.2. In the proof of Theorem 1.1 given in [13] it is shown that given $\mu > 0$, a Margulis constant for \mathbb{H}^3 , and $d > 0$, there exists $A > 0$, so that $N_d(M_{\geq \mu})$ can be triangulated using at most $\frac{1}{A}\text{Vol}(M)$ tetrahedra; here $N_d(M_{\geq \mu})$ denotes the closed d -neighborhood of the μ -thick part of M . We note that A depends on μ and d .

Theorem 1.1 implies that manifolds of low volume admit Heegaard splittings of low genus: let M be a hyperbolic manifold, $L \subset M$ a link, and \mathcal{T} a triangulation of $M \setminus N(L)$ that realizes $t_c(M)$. Let Γ be $\mathcal{T}^{(1)} \cup \partial(M \setminus N(L))$, where $\mathcal{T}^{(1)}$ denotes the 1-skeleton of \mathcal{T} . It is easy to see that $\partial N(\Gamma)$ is a Heegaard surface for $M \setminus N(L)$ and its genus is at most the number of tetrahedra plus one, that is, $t_c(M) + 1$. Since the Heegaard genus does not increase after Dehn filling we get:

$$g(M) \leq t_c(M) + 1 \leq 2t_c(M) < \frac{2}{A}\text{Vol}(M).$$

Here and throughout this paper, $g(M)$ denotes the Heegaard genus of M . The converse is false: it is easy to construct hyperbolic manifolds of arbitrarily high volume and Heegaard genus two (for example, consider Dehn fillings of 2-bridge knots; see [29]).

Our goal is to show that any Heegaard surface for a generic hyperbolic manifold M is “simple”. This is described precisely in Theorem 1.5, and we informally explain it here. In [7] J. Hempel defined a complexity of Heegaard surfaces which we will call the *distance*, denoted by $d(\Sigma)$ (the distance, which is based on Kobayashi’s idea of *height of loops* [11], is defined in Section 4). We say that a Heegaard surface Σ is *simple* if either $g(\Sigma)$ is low (in terms of the volume) or $d(\Sigma) \leq 2$. A. Casson and C. Gordon constructed a hyperbolic manifold admitting infinitely many Heegaard surfaces, and showed that these surfaces all have distance at least two (in their language, are strongly irreducible). They further showed that there is no upper bound on the genera of these surfaces; hence this result is best possible.

We now explain what a *generic* hyperbolic manifold is. Let X be a compact 3-manifold (not necessarily hyperbolic) so that ∂X consists of tori, say T_1, \dots, T_n . Let W be a manifold obtained from X by Dehn filling some of its boundary components, say T_1, \dots, T_m , $m \leq n$. Note that $X \subset W$ and

any Heegaard surface for X is a Heegaard surface for W . Rieck and E. Sedgwick [18–20] prove that on each T_i there is a finite set of slopes, denoted by B_i , so that if the slope filled on each T_i intersects every slope of B_i more than once, then any Heegaard surface for W is a Heegaard surface for X (after isotopy if necessary). In that case, we say that W is a *generic* Dehn filling of X . With this in mind, we define:

Definition 1.3. Let μ be a Margulis constant for \mathbb{H}^3 and fix $d > 0$. Let M be a complete hyperbolic manifold of finite volume. Let $N_d(M_{\geq\mu})$ be the closed d -neighborhood of the μ -thick part of M ; for a discussion see [13], where it was observed that M is obtained from $N_d(M_{\geq\mu})$ by Dehn filling. We say that M is (μ, d) -*generic* if M is a generic Dehn filling of $N_d(M_{\geq\mu})$.

Remark 1.4. In an effort to justify the term “generic” we now sketch an argument that shows that for any μ and d , there are indeed many (μ, d) -generic manifold. Fix $V > 0$. By Remark 1.2 there are only finitely many topological types for the manifolds $N_d(M_{\geq\mu})$, where M ranges over all hyperbolic manifolds of volume less than V . Let X be one of these manifolds and denote the components of ∂X by T_1, \dots, T_n . Then for each i there is a finite set of slopes of T_i , say F_i , with the following property: as above let W be a manifold obtained from X by Dehn filling some of its boundary components, say T_1, \dots, T_m , $m \leq n$, so that slope filled is not in F_i for all i . Then W is hyperbolic, the short geodesics in W coincide with the cores of the attached solid tori, and each short geodesic has a neighborhood of radius greater than d . Thus $N_d(W_{\geq\mu}) = X$. We conclude that if W is obtained by filling X along slopes that are not in F_i and intersect every slope in B_i more than once (where B_i was defined in the paragraph preceding Definition 1.3), then W is (μ, d) -generic. This shows that if V is at least the volume of the figure eight knot exterior (so that there are infinitely many hyperbolic manifolds of volume less than V), then there are infinitely many (μ, d) -generic manifolds that have volume less than V .

In this paper we prove that (μ, d) -generic manifolds enjoy the following property:

Theorem 1.5. *Let $\mu > 0$ be a Margulis constant for \mathbb{H}^3 and fix $d > 0$. Then there exists $\Lambda > 0$ so that for any complete finite volume (μ, d) -generic hyperbolic manifold M and any Heegaard surface Σ for M the following holds:*

If $g(\Sigma) > \Lambda \text{Vol}(M)$, then $d(\Sigma) \leq 2$.

Remark 1.6. Fix a hyperbolic manifold M . It is easy to see that if d is sufficiently large or μ sufficiently small, then M is diffeomorphic to $N_d(M_{\geq \mu})$, and in particular M is (μ, d) -generic. Thus, the conclusion of Theorem 1.5 holds for M . This has two consequences:

- 1) It is well known that the examples of Casson and Gordon mentioned above have distance two and arbitrarily high genus. Hence, the conclusion of Theorem 1.5 cannot be improved.
- 2) If there exists Λ as in Theorem 1.5 that is independent of μ and d , then the assumption that M is (μ, d) -generic can be removed. Unfortunately, for Λ constructed in this paper both $\lim_{d \rightarrow \infty} \Lambda = \infty$ and $\lim_{\mu \rightarrow 0} \Lambda = \infty$ hold.

Our proof of Theorem 1.5 uses Dehn filling and hence forces us to assume that M is (μ, d) -generic. However, this does not seem to be an integral part of the theory. In light of this and Remark 1.6 (2) we ask:

Question 1.7. Is the assumption that M is (μ, d) -generic necessary?

The three ingredients necessary for the proof of Theorem 1.5 are Theorem 1.1, the work of Rieck and Sedgwick, and Theorem 1.8, which represents the bulk of the work in this paper. In this theorem, we allow a flexible definition of triangulation, which we call generalized triangulation. See Definition 4.1 and Lemma 4.2 for existence.

Theorem 1.8. *Let M be a compact orientable connected 3-manifold and Σ a Heegaard surface for M . Suppose that for some (possibly empty or disconnected) compact surface $K \subset \partial M$, $M \setminus K$ admits a generalized triangulation with t generalized tetrahedra.*

If $g(\Sigma) \geq 76t + 26$, then $d(\Sigma) \leq 2$.

Remark 1.9.

- 1) Theorem 1.8 generalizes S. Schleimer's [27, Theorem 11.1], where it was shown that if M is a closed manifold and $g(\Sigma) \geq 2^{2^{16}t^2}$, then $d(\Sigma) \leq 2$.
- 2) Theorem 1.8 implies that for every manifold M , there is $g_M \geq 0$, so that if $\Sigma \subset M$ is a Heegaard surface of genus at least g_M , then $d(\Sigma) \leq 2$; this also follows from [27, Theorem 11.1].

Outline. In Section 2, we show how Theorem 1.5 follows from Theorem 1.8. In Section 3 we explain our perspective of Theorem 1.8 and list open questions related to it. In Section 4 we explain a few preliminaries. The work begins in Section 5, where we take a strongly irreducible Heegaard surface of genus at least $76t + 26$, color it, and analyze the coloring; the climax of Section 5 is Proposition 5.6, where we prove existence of a pair of pants with certain useful properties. Finally, in Section 6 we prove Theorem 1.8.

2. Proof of Theorem 1.5

We first show how Theorem 1.5 follows from Theorem 1.8. Fix the notation of Theorem 1.5. Let $\lambda = \frac{1}{A}$, where $A > 0$ is the constant given in Theorem 1.1. By Remark 1.2, for any complete finite volume hyperbolic 3-manifold M , $N_d(M_{\geq \mu})$ can be triangulated using at most $\lambda \text{Vol}(M)$ tetrahedra.

Set $\Lambda = 76\lambda + 29$. Let $\Sigma \subset M$ be a Heegaard surface of genus $g(\Sigma) \geq \Lambda \text{Vol}(M)$. Using the definition of Λ and the fact that $\text{Vol}(M) > .9$ (Gabai, Meyerhof and Milley [5]) we get:

$$\begin{aligned} g(\Sigma) &\geq \Lambda \text{Vol}(M) \\ &= (76\lambda + 29)\text{Vol}(M) \\ &= 76\lambda \text{Vol}(M) + 29\text{Vol}(M) \\ &> 76\lambda \text{Vol}(M) + 26. \end{aligned}$$

By assumption, M is a (μ, d) -generic, that is, M is obtained from $N_d(M_{\geq \mu})$ by a generic Dehn filling (recall Definition 1.3). Hence, after isotopy if necessary, Σ is a Heegaard surface for $N_d(M_{\geq \mu})$. By Remark 1.2, $N_d(M_{\geq \mu})$ can be triangulated using $t \leq \lambda \text{Vol}(M)$ tetrahedra. We see that $g(\Sigma) > 76\lambda \text{Vol}(M) + 26 \geq 76t + 26$, and by Theorem 1.8 (applied to Σ as a Heegaard surface of $N_d(M_{\geq \mu})$), $d(\Sigma) \leq 2$. It is elementary to see that distance never increases under Dehn filling, and we conclude that $\Sigma \subset M$ is a Heegaard surface of distance at most 2, completing the proof of Theorem 1.5.

3. Open questions

Theorem 1.8 is a constraint on the distance of surfaces of genus $76t + 26$ or more. There are other constraints on the distance known, and by far the most important is Casson and Gordon's theorem [3] that says that no Heegaard surface of an irreducible, non-Haken 3-manifold has distance exactly one. Other examples include W. Haken's theorem that says that any Heegaard

surface of a reducible 3-manifold has distance zero, and T. Li's theorem [16] that says that a non-Haken 3-manifold admits only finitely many Heegaard surfaces of positive distance. Another constraint is [26, Corollary 3.5], where M. Scharlemann and M. Tomova prove that if Σ_1 and Σ_2 are non isotopic Heegaard surfaces of a closed manifold so that $d(\Sigma_2) > 2g(\Sigma_1)$, then $d(\Sigma_1) = 0$ (in fact, they show that Σ_1 is obtained from Σ_2 by stabilization).

On the positive side, all but finitely many of the surfaces constructed by Casson and Gordon have distance exactly two (Casson and Gordon's work show that the distance is at least 2 and Theorem 1.5 provides a new proof that the distance is at most 2). Hempel [7], using a construction of Kobayashi [11], shows that for any $g \geq 2$ there exists a sequence of 3-manifolds M_n and Heegaard splittings Σ_n for M_n , so that $g(\Sigma_n) = g$ and $\lim_{n \rightarrow \infty} d(\Sigma_n) = \infty$. T. Evans [4] improved this by constructing, given $g \geq 2$ and $d \geq 0$, a Heegaard splitting of genus g with distance at least d . Recently, Qiu et al. [22] and, independently, Ido et al. [9], constructed, given $g \geq 2$ and $d \geq 1$, a compact manifold with Heegaard splitting of genus g and distance exactly d . In [32], Yoshizawa shows that when d is even, a Heegaard splitting of distance exactly d can be obtained by applying high powers of a single Dehn twist.

However, the answers to the following questions are not known in general:

- Questions 3.1.** 1) Given $g_i \geq 2$ and $d_i > 0$ ($i = 1, 2$), does there exist a 3-manifold admitting distinct Heegaard surfaces Σ_1, Σ_2 , so that $g(\Sigma_i) = g_i$ and $d(\Sigma_i) = d_i$?
- 2) Given $d_i > 0$ ($i = 1, 2$), does there exist a 3-manifold admitting distinct Heegaard surfaces Σ_1, Σ_2 , so that $d(\Sigma_i) = d_i$?

Questions (1) and (2) above can naturally be generalized to more than two surfaces by setting $i = 1, \dots, n$, for some chosen n . The word "distinct" in the questions above can be interpreted as "distinct up to isotopy" or "distinct up to homeomorphism"; both yield interesting questions.

The answer for Question 3.1 (2) is known only in the following cases:

- $d_1 = d_2 = 2$: As mentioned above, there are examples of Casson and Gordon of 3-manifolds admitting infinitely many Heegaard surfaces of unbounded genera and of distance invariant two. Other examples follow from S. Beiler and Y. Moriah [2] (see also K. Morimoto and M. Sakuma [17]). They show that there exist 2-bridge knots K admitting more than one minimal genus Heegaard surface (up to homeomorphism). Let Σ be one of these surfaces. It is easy to see that $d(\Sigma) = 2$:

first, since $g(\Sigma) = 2$, it is easy to see that $d(\Sigma) \geq 2$. Next, Σ is constructed by viewing K as a torus 1-bridge knot (that is, there exists a genus 1 Heegaard splitting $T_1 \cup T_2$ so that K intersects each T_i in a single unknotted arc) and tubing once. Meridian disks for T_i which are disjoint from K and the tube, are also disjoint from the core of the tube, showing that $d(\Sigma) \leq 2$.

- $d_1 = d_2 = 1$: Let S be a 4-punctured sphere and $M = S \times S^1$. J. Schultens [28] showed that $g(M) = 3$. We note that M admits two minimal genus Heegaard splittings, say Σ_1 and Σ_2 , such that Σ_1 is obtained by tubing three boundary parallel tori, and Σ_2 is obtained by tubing two boundary parallel tori, with an extra tube that wraps around a third boundary component. Since Σ_1 and Σ_2 induce boundary partitions with distinct numbers of components, they are distinct up to homeomorphism. By construction, $d(\Sigma_1) = d(\Sigma_2) = 1$.
- $d_1 = 1, d_2 = 2$: In [12] the authors constructed a 3-manifold M admitting minimal genus Heegaard splittings Σ_1, Σ_2 , with $d(\Sigma_2) = 2$ and $d(\Sigma_1) = 1$. In this example, $g(M) = g(\Sigma_1) = g(\Sigma_2) = 3$.
- $d_1 = d_2 = 3$: Scharlemann [25], based on a preprint by Berge [1], shows that there exists a closed manifold M admitting two Heegaard splittings Σ_1 and Σ_2 , distinct up-to homeomorphism, so that $g(M) = g(\Sigma_1) = g(\Sigma_2) = 2$ and $d(\Sigma_1) = d(\Sigma_2) = 3$.

We see that much is known when $d_1, d_2 \leq 3$. By contrast, the answers to the following basic questions are unknown:

- Questions 3.2.**
- 1) Does there exist a 3-manifold admitting two (or more) distinct Heegaard surfaces with distance four or more?
 - 2) Does there exist a 3-manifold admitting a Heegaard surface of distance three or more that is not of minimal genus?

4. Preliminaries

By *manifold* we mean compact, connected, orientable 3-manifold. We assume familiarity with the basic notions of 3-manifold topology (see, for example, [8] or [10]) and the basic facts about Heegaard splittings (see, for example, [24] or [23]). We use the notation $N(\)$ for open normal neighborhood, ∂ for boundary, and $| \ |$ for the number of components. We define:

- Definitions 4.1.** 1) Let T be a tetrahedron. A *generalized tetrahedron* is obtained by fixing two disjoint sets of vertices of T , denoted V_1, V_2 , and then removing V_1 and truncating V_2 ; that is, a generalized tetrahedron T' has the form $T' = T \setminus (V_1 \cup N(V_2))$. T' has exactly four faces (resp. exactly six edges, at most four vertices), which are the intersection of the faces (resp. edges, vertices) of T with T' . In particular, the components of $\partial N(V_2) \cap T'$ are not considered faces. Important special cases are when $V_2 = \emptyset$, then T' is called *semi-ideal*, and when V_1 consists of all four vertices, then T' is called *ideal*.
- 2) A *generalized triangulation* is obtained by gluing together finitely many generalized tetrahedra, where the gluings are done by identifying faces, edges and vertices. Self-gluings (that is, gluing a tetrahedron to itself) are allowed, as are multiple gluings (that is, gluing two tetrahedra along more than one face). We refer the reader to [6] for a detailed description in the special case when only tetrahedra are used, known there as Δ complexes. If all the generalized tetrahedra are ideal (resp. semi ideal), then the generalized triangulation is called an *ideal* (resp. *semi ideal*) triangulation. If the quotient obtained is homeomorphic to a given manifold M it is said to be a *generalized triangulation of M* .

We refer the reader to, for example, [15, Section 2] for a detailed discussion of generalized tetrahedra. It is well known that a very large class of 3-manifolds admits generalized triangulations, including all compact 3-manifolds. We outline the proof here. Let W be a compact manifold and $K_i \subset \partial W$ ($i = 1, \dots, n$) a disjoint, closed, connected subsurfaces. By crushing each K_i to a point p_i , we obtain a 3-complex X . We can triangulate X so that each p_i is a vertex of the triangulation. Removing p_i we obtain a semi-ideal triangulation of $N \setminus (\cup_i K_i)$. We conclude that (with K corresponding to $\cup_i K_i$):

Lemma 4.2. *Let M be a compact manifold and $K \subset \partial M$ a (not necessarily connected) closed subsurface. Then $M \setminus K$ admits a generalized triangulation.*

In [7] Hempel defined the *distance* of a Heegaard splitting:

Definition 4.3. Let $V_1 \cup_\Sigma V_2$ be a Heegaard splitting and $d \geq 0$ an integer. We say that the distance of Σ is d , denoted by $d(\Sigma) = d$, if d is the smallest integer so that there exist meridian disks $D_1 \subset V_1$ and $D_2 \subset V_2$, and essential

curves $\alpha_i \subset \Sigma$ ($i = 0, \dots, d$), so that $\alpha_0 = \partial D_1$, $\alpha_d = \partial D_2$, and $\alpha_{i-1} \cap \alpha_i = \emptyset$ (for $1 \leq i \leq d$).

The following lemma is easy and well known (see, for example [27, Remark 2.6]):

Lemma 4.4. *Let $V_1 \cup_\Sigma V_2$ be a Heegaard splitting. Suppose that one of the following holds:*

- 1) *for $i = 1, 2$, there exists a properly embedded, non-boundary parallel annulus $A_i \subset V_i$, and there exists an essential curve $\alpha \subset \Sigma$ so that $\alpha \subset A_1 \cap A_2$ (that is to say, A_1 and A_2 have an essential common boundary component), or:*
- 2) *there exists a meridian disk $D_1 \subset V_1$ and a properly embedded non-boundary parallel annulus $A_2 \subset V_2$, so that D_1 is disjoint from at least one component of ∂A_2 that is essential in Σ .*

Then $d(\Sigma) \leq 2$.

5. Coloring Σ and constructing the pair of pants X

Fix M as in the statement of Theorem 1.8 and let $V_1 \cup_\Sigma V_2$ be a Heegaard splitting for M with $g(\Sigma) \geq 76t + 26$. Let \mathcal{T} be a generalized triangulation of $M \setminus K$ (where $K \subset \partial M$ is a closed subsurface) with t generalized tetrahedra.

If Σ weakly reduces, then $d(\Sigma) \leq 1$; we assume as we may that Σ is strongly irreducible. Rubinstein [21] (see also Stocking [30] and Lackenby [14, 15] when M is not closed) show that Σ is isotopic to an *almost normal* surface, that is, after isotopy the intersection of Σ with the generalized tetrahedra of \mathcal{T} consists of *normal faces*, of which there are two types:

- 1) normal disks (normal triangles and normal quadrilaterals);
- 2) an exceptional component, which is either an octagonal disk or an annulus obtained by tubing together two normal disks; at most one normal face of Σ is an exceptional component.

Let N be a regular neighborhood of $\mathcal{T}^{(1)}$, the 1-skeleton of \mathcal{T} . For each $v \in \mathcal{T}^{(1)} \cap \Sigma$, let D_v be the component of $\Sigma \cap N$ containing v . Then D_v is a disk properly embedded in N , called the *vertex disk corresponding to v* . Let \widehat{F} be a normal face contained in a generalized tetrahedron T . Then $F = \widehat{F} \setminus \text{int}N$ is obtained from \widehat{F} by removing a neighborhood of the vertices

of \widehat{F} . F is called a *truncated normal face*. For the remainder of this paper, by a *face* we mean a truncated normal face or a vertex disk.

Remark 5.1. The union of the boundaries of the faces forms a 3-valent graph in Σ .

Let $v, v' \in \mathcal{T}^{(1)} \cap \Sigma$ be two vertices and $D_v, D_{v'}$ the corresponding vertex disks. Then D_v and $D_{v'}$ are called *I-adjacent* if v and v' are contained in the same edge $e \in \mathcal{T}^{(1)}$ and v is adjacent to v' along e . Note that D_v is *I-adjacent* to $D_{v'}$ if and only if v and v' are contained in the same edge $e \in \mathcal{T}^{(1)}$ and there exists an *I*-bundle over D^2 with total space $Q \subset N$, so that $\partial Q \setminus (D_v \cup D_{v'}) \subset \partial N$, $Q \cap \Sigma = D_v \cup D_{v'}$, and $D_v \cup D_{v'}$ is the associated ∂I -bundle.

Let F and F' be truncated normal faces. Then F and F' are called *I-adjacent* if the corresponding normal faces are parallel and there is no normal face between the two. Note that F and F' are *I-adjacent* if and only if they are contained in the same generalized tetrahedron T , and there exists an *I*-bundle with total space $Q \subset T \setminus \text{int}N$, so that $\partial Q \setminus (F \cup F') \subset \partial(T \setminus \text{int}N)$ and is disjoint from the vertices, truncated vertices, and missing vertices, $Q \cap \Sigma = F \cup F'$, and $F \cup F'$ is the associated ∂I -bundle.

Clearly *I*-adjacency is symmetric but not, in general, transitive. The equivalence relation generated by *I*-adjacency is called *I-equivalence*, and its equivalence classes are called *I-equivalent families*. For example, suppose that a tetrahedron contains four quadrilaterals and denote the corresponding truncated normal faces by q_1, q_2, q_3, q_4 (listed in order). If there is a truncated exceptional piece between q_2 and q_3 , then the truncated quadrilaterals form exactly two *I*-equivalent families: $\{q_1, q_2\}$ and $\{q_3, q_4\}$.

Let \mathcal{F} be an *I*-equivalent family. Then *I*-adjacency induces a linear ordering on the faces in \mathcal{F} , ordered as F_1, \dots, F_n , so that F_i is *I*-adjacent to F_{i+1} ($i = 1, \dots, n-1$). This order is unique up-to reversing. We color the faces of \mathcal{F} as follows:

- 1) F_1, F_2, F_{n-1} , and F_n are colored red.
- 2) If $n \geq 5$, then F_3, \dots, F_{n-2} are colored blue and yellow alternately. Note that this leaves us the freedom to exchange the blue and yellow colors of the faces of \mathcal{F} .

Remark. For most of our work it suffices to color red F_1 and F_n . We need to color F_2 and F_{n-1} red as well for the last case of the proof of Theorem 1.8, where a further refinement of the colors will be given.

By construction, any yellow or blue face is I -adjacent to two distinct faces.

Remark 5.2. Let D_v be a red vertex disk. By construction, D_v is outermost or next to outermost along an edge of $\mathcal{T}^{(1)}$. Therefore all the truncated normal faces that intersect D_v are red as well.

Lemma 5.3. *Let $f_{r,t}$ denote the number of the red truncated triangles and $f_{r,q}$ the number of the red truncated quadrilaterals. Then one of the following holds:*

- 1) $f_{r,t} \leq 16t$ and $f_{r,q} \leq 4t + 4$.
- 2) $f_{r,t} \leq 16t + 4$ and $f_{r,q} \leq 4t$.

Proof. A generalized tetrahedron not containing the exceptional component admits at most four I -equivalent families of truncated triangles and one I -equivalent family of truncated quadrilaterals. If there is an exceptional component, the generalized tetrahedron containing it admits at most five I -equivalent families of truncated triangles and one I -equivalent family of truncated quadrilaterals, or at most four I -equivalent families of truncated triangles and two I -equivalent families of truncated quadrilaterals. Each family contains at most four red faces. The lemma follows. \square

Let B (resp. Y , R) denote the union of the blue (resp. yellow, red) faces; note that faces are closed, so R , Y , and B are compact and may intersect along their boundaries. By Remark 5.1, B , Y , R and $B \cup Y$ are subsurfaces of Σ .

Lemma 5.4. $\chi(B \cup Y) \leq -(108t + 38)$.

Proof. We first show that $\chi(R) \geq -(44t + 12)$; for that, we order the red faces as $F_0, F_1, \dots, F_k, F_{k+1}, \dots, F_n$ (for some k, n), so that F_0 is the exceptional piece (if there is one, $F_0 = \emptyset$ otherwise), F_1, \dots, F_k are the red truncated normal faces, and F_{k+1}, \dots, F_n are red vertex disks. Note that $\chi(F_0) = 0$ or $\chi(F_0) = 1$, so the worst case scenario is 0. By Remark 5.1, for $0 \leq i \leq k$, the possibilities for $F_i \cap (\cup_{j=1}^{i-1} F_j)$ are: \emptyset , S^1 , or a number of segments, each homeomorphic to I . Since a truncated normal triangle

(respectively, quadrilateral) is a hexagon (respectively octagon), the number of segments is at most 3 (respectively 4). We see that

$$\chi(\cup_{j=1}^i F_j) \geq \chi(\cup_{j=1}^{i-1} F_j) - 2,$$

when F_i is a truncated normal triangle and

$$\chi(\cup_{j=1}^i F_j) \geq \chi(\cup_{j=1}^{i-1} F_j) - 3,$$

when F_i is a truncated normal quadrilateral. By Remark 5.2, for $i \geq k + 1$, F_i caps a hole of $\cup_{j=1}^{i-1} F_j$; hence

$$\chi(\cup_{j=1}^i F_j) = \chi(\cup_{j=1}^{i-1} F_j) + 1$$

in that case. Recall that $f_{r,t}$ and $f_{r,q}$ were defined and bounded in Lemma 5.3. Adding the contributions of the exceptional component (at worst 0), the triangles (at worst $-2f_{r,t}$), the quadrilaterals (at worst $-3f_{r,q}$), and ignoring the positive contribution of the vertex disks, Lemma 5.3 gives:

$$\begin{aligned} \chi(R) &\geq 0 - 2f_{r,t} - 3f_{r,q} \\ &\geq 0 - 2(16t) - 3(4t + 4) \\ &= -(44t + 12). \end{aligned}$$

Since R and $B \cup Y$ are subsurfaces, $\Sigma = R \cup (B \cup Y)$ and $R \cap (B \cup Y) = \partial R = \partial(B \cup Y)$ consists of circles, we have that $\chi(B \cup Y) = \chi(\Sigma) - \chi(R)$. By assumption $g(\Sigma) \geq 76t + 26$, or equivalently $\chi(\Sigma) \leq 2 - 2(76t + 26)$. Hence:

$$\begin{aligned} \chi(B \cup Y) &= \chi(\Sigma) - \chi(R) \\ &\leq [2 - 2(76t + 26)] + [44t + 12] \\ &= -(108t + 38). \end{aligned}$$

□

Lemma 5.5. $|\partial(B \cup Y)| \leq 44t + 14$.

Proof. By construction $\partial(B \cup Y) = \partial R$. Bounding $|\partial R|$ is similar to the proof of the previous lemma and we only paraphrase it here: we order the red faces as F_0, \dots, F_n as in the proof of the previous lemma. It is easy to

see that $|\partial F_0|$ is at most 2, and (similar to the Euler characteristic count on the previous lemma) for $1 \leq i \leq k$,

$$|\partial(\cup_{j=1}^i F_j)| \leq |\partial(\cup_{j=1}^{i-1} F_j)| + 2,$$

when F_i is a truncated normal triangle, and

$$|\partial(\cup_{j=1}^i F_j)| \leq |\partial(\cup_{j=1}^{i-1} F_j)| + 3,$$

when F_i is a truncated normal quadrilateral. By Remark 5.2, for $i \geq k + 1$,

$$|\partial(\cup_{j=1}^i F_j)| \leq |\partial(\cup_{j=1}^{i-1} F_j)| - 1.$$

Adding up the contributions of the truncated normal faces and ignoring the negative contribution of the vertex disks, Lemma 5.3 gives:

$$\begin{aligned} |\partial(B \cup Y)| &= |\partial R| \\ &\leq 2 + 2f_{r,t} + 3f_{r,q} \\ &\leq 2 + 2(16t) + 3(4t + 4) \\ &= 44t + 14. \end{aligned} \quad \square$$

By Remark 5.1, $B \cap Y$ is a compact 1-manifold properly embedded in $B \cup Y$. Let $\Gamma \subset B \cup Y$ be the union of the arc components of $B \cap Y$. Endpoints of Γ are the vertices of Σ where red, blue and yellow faces meet. By Remark 5.2 around any vertex of Σ that is on the boundary of a red vertex disk all the colors are red; therefore the vertex disk at an endpoints of Γ is yellow or blue.

Let \mathcal{V} be the set of vertices of red truncated normal faces. We subdivide \mathcal{V} into three disjoint sets as follows: \mathcal{V}_0 are vertices that are on the boundary of at least two red faces; \mathcal{V}_+ are vertices that are on the boundary of three faces so that one is red, one is yellow, and one is blue; \mathcal{V}_- are vertices that are on the boundary of three faces so that one is red and two are yellow, or one is red and two are blue. By construction, \mathcal{V}_+ is exactly the set of endpoints of Γ .

By construction, at every vertex exactly one face is a vertex disk. We exchange the colors of the blue vertex disks with the colors of the yellow vertex disks; let R' , B' , Y' and Γ' be defined as above, using the new coloring. By Remark 5.2, \mathcal{V}_- is exactly the set of endpoints of Γ' (we emphasize that \mathcal{V}_- is the set of vertices defined above using the original coloring). Hence, by exchanging colors if necessary, we may assume that the number of endpoints

of Γ is at most $\frac{1}{2}|\mathcal{V}|$. Since every arc of Γ has two distinct endpoints and Γ has at most $\frac{1}{2}|\mathcal{V}|$ endpoints, we get that $|\Gamma| \leq \frac{1}{4}|\mathcal{V}|$.

There are at most 16 vertices in \mathcal{V} from the truncated exceptional component, at most 6 from each truncated red triangle, and at most 8 from each truncated red quadrilateral. By Lemma 5.3 we get:

$$\begin{aligned} |\mathcal{V}| &\leq 16 + 6f_{r,t} + 8f_{r,q} \\ &\leq 16 + 6(16t) + 8(4t + 4) \\ &\leq 128t + 48. \end{aligned}$$

Hence:

$$|\Gamma| \leq \frac{1}{4}|\mathcal{V}| \leq 32t + 12.$$

Let F_1, \dots, F_k be the components of $B \cup Y$ cut open along Γ (note that F_1, \dots, F_k are not, in general, faces). Cutting along Γ increases the Euler characteristic by $|\Gamma|$ and increases the number of boundary components by at most $|\Gamma|$. Using Lemma 5.4 we get:

$$\begin{aligned} \sum_{i=1}^k \chi(F_i) &= \chi(\cup_{i=1}^k F_i) \\ &= \chi(B \cup Y) + |\Gamma| \\ &\leq -(108t + 38) + (32t + 12) \\ &= -(76t + 26). \end{aligned}$$

And using Lemma 5.5 we get:

$$\begin{aligned} \sum_{i=1}^k |\partial F_i| &= |\partial \cup_{i=1}^k F_i| \\ &\leq |\partial(B \cup Y)| + |\Gamma| \\ &\leq (44t + 14) + (32t + 12) \\ &= 76t + 26. \end{aligned}$$

Combining these inequalities we get:

$$(1) \quad \sum_{i=1}^k \chi(F_i) \leq -(\sum_{i=1}^k |\partial F_i|).$$

Proposition 5.6. *There exists a pair of pants $X \subset \Sigma$ with the following two properties:*

- 1) *Either $X \subset \text{int}(B)$ or $X \subset \text{int}(Y)$ (say the former).*
- 2) *The components of ∂X , denoted by α , β , and γ , are essential in Σ .*

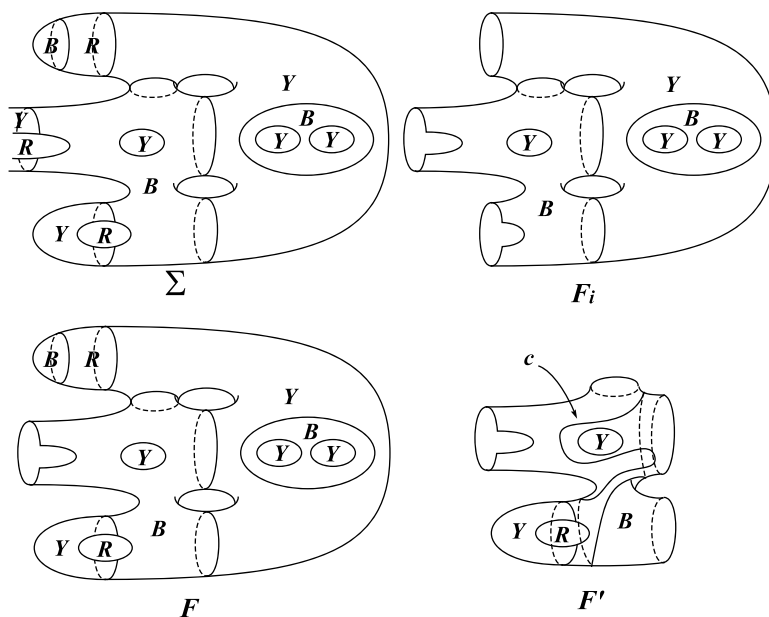


Figure 1: A process for obtaining F'

Proof. By Inequality (1) above, for some i , $\chi(F_i) \leq -|\partial F_i|$; equivalently, $g(F_i) \geq 1$. Fix such i . By construction, $(B \cap Y) \cap \text{int}(F_i)$ consists of simple closed curves; see figure 1. Let \mathcal{E} (resp. \mathcal{I}) denote the curves of $(B \cap Y) \cap \text{int}F_i$ that are essential (resp. inessential) in Σ . Let Δ be the union of the components of $\text{cl}(\Sigma \setminus F_i)$ that are disks (possibly, $\Delta = \emptyset$). Let $F = F_i \cup \Delta$. By construction, every component of ∂F is essential in Σ (possibly, $\partial F = \emptyset$). Thus, a closed curve of F is essential in Σ if and only if it is essential in F . Since $g(F) = g(F_i) > 0$, if $\partial F \neq \emptyset$ then $\chi(F) < 0$; if, on the other hand, $\partial F = \emptyset$, then $F = \Sigma$ and in particular, $g(F) = g(\Sigma) \geq 76t + 26 > 1$; we conclude that in either case $\chi(F) < 0$. Thus some component of F cut open along \mathcal{E} , denoted by F' , has $\chi(F') < 0$. Note that every curve of $\partial F'$ is essential in Σ . By construction, $(B \cap Y) \cap \text{int}F' \subset \mathcal{I}$. Let Δ' be the union of the disks bounded by outermost curves of $\mathcal{I} \cap F'$ and the disks $\Delta \cap F'$. Note that $\Delta' \subset \text{int}F'$ consists of disks, and $F' \setminus \Delta'$ is entirely blue or yellow; in figure 1, Δ' consists of two disks, one of each kind.

Assume first that $\partial F' \neq \emptyset$. Let $c \subset F'$ be a curve, parallel to a component of $\partial F'$, that decomposes F' as $A'' \cup_c F''$, where A'' is an annulus. By isotopy of c in F' we may assume that $\Delta' \subset A''$. We see that F'' is entirely blue or yellow, $\chi(F'') = \chi(F') < 0$, and $\partial F''$ is essential in Σ .

Next assume that $\partial F' = \emptyset$ (that is, $F' = \Sigma$). Let c be a separating, essential curve in F' . By isotopy of c we may assume that Δ' is contained in one component of F' cut open along c . Let F'' be the other component. We conclude that in this case too, F'' is entirely blue or yellow, $\chi(F'') < 0$, and $\partial F''$ is essential in Σ .

Let α, β and $\gamma \subset \text{int}(F'')$ be three curves that are essential in F'' (and hence in Σ) and co-bound a pair of pants, denoted by X , in F'' . It is easy to see that X, α, β and γ have the properties listed in Proposition 5.6. \square

Since $X \subset \text{int}(B)$ it is on the boundary of the total space of an I -bundle in V_i ($i = 1, 2$). The other component of the associated ∂I -bundle is a pair of pants denoted by X_i . The components of ∂X_i are denoted by α_i, β_i , and γ_i so that α_i is parallel to α , β_i is parallel to β , and γ_i is parallel to γ . Since $X \subset \text{int}(B)$, every point of X_i is yellow or red; we conclude that $X \cap X_i = \emptyset$. Hence the I -bundle in V_i is trivial. The annulus extended from α to α_i (resp. β to β_i , γ to γ_i) in V_i is denoted by A_i (resp. B_i, C_i). By construction, these annuli are embedded. Note that $X_1 \cap X_2 \neq \emptyset$ is possible.

Lemma 5.7. *One of the following holds:*

- 1) *After renaming if necessary, $A_1 \subset V_1$ and $B_2 \subset V_2$ are not boundary parallel, and $A_2 \subset V_2$, $B_1 \subset V_1$ and $C_1 \subset V_1$ are boundary parallel.*
- 2) $d(\Sigma) \leq 2$.

Proof. We claim that one of A_i, B_i or C_i is not boundary parallel in V_i ($i = 1, 2$). Suppose, for a contradiction, that A_i, B_i, C_i are all boundary parallel. Let $\tilde{A}_i \subset V_i$ be the annulus that A_i is parallel to. Since X is an essential pair of pants it is not contained in \tilde{A}_i ; it is easy to see that the intersection of the region of parallelism between A_i and \tilde{A}_i and the trivial I -bundle in V_i is exactly A_i ; similarly we treat B_i and C_i . We see that V_i is homeomorphic to the trivial I -bundle, and hence is a genus 2 handlebody. This contradicts our assumption that $g(\Sigma) \geq 76t + 26 > 2$.

Therefore one of A_1, B_1 or C_1 is not boundary parallel, and after renaming if necessary we may assume it is A_1 . We may assume A_2 is boundary parallel, for otherwise $d(\Sigma) \leq 2$ by Lemma 4.4 (1). Similarly, one of A_2, B_2 or C_2 is not boundary parallel, after renaming if necessary we may assume it is B_2 , while B_1 is boundary parallel. Finally by Lemma 4.4 (1) we may assume that C_1 or C_2 is boundary parallel, say C_1 . \square

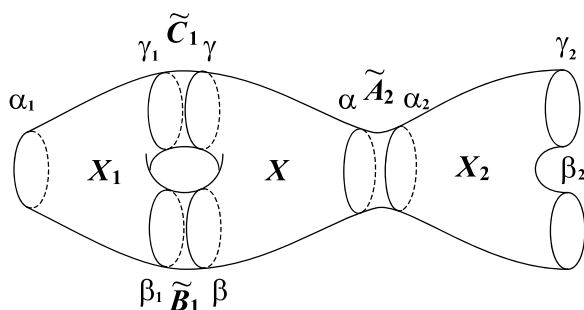


Figure 2: Configuration of X , X_1 , and X_2 in Σ

Lemma 5.8. *One of the following holds:*

- 1) α_1 , β_2 and γ_2 are essential in Σ , and α is not isotopic in Σ to α_1 , β or γ .
- 2) $d(\Sigma) \leq 2$.

Proof. We may assume that Conclusion (1) of Lemma 5.7 holds; thus A_2 , B_1 and C_1 are boundary parallel. We denote by $\tilde{A}_2, \tilde{B}_1, \tilde{C}_1 \subset \Sigma$ the annuli to which A_2 , B_1 , and C_1 are parallel (respectively). See figure 2, where $X_1 \cap X_2 = \emptyset$, but this need not be the case.

If α_1 is inessential in Σ , then we may cap A_1 off, and after a small isotopy we obtain a meridian disk $D_1 \subset V_1$ with $\partial D_1 = \alpha$. Using D_1 and B_2 , Lemma 4.4 (2) shows that $d(\Sigma) \leq 2$. Similarly if β_2 (resp. γ_2) is inessential in Σ then β (resp. γ) bounds a meridian disk $D_2 \subset V_2$. Using D_2 and A_1 , Lemma 4.4 (2) shows that $d(\Sigma) \leq 2$.

If α is isotopic to α_1 in Σ then either the annulus connecting the two contains X or $g(\Sigma) = 2$. The former is impossible since X is an essential pair of pants and the latter contradicts the assumption $g(\Sigma) \geq 76t + 26 > 2$.

Let $c \subset \Sigma$ be a closed curve constructed by pasting together four arcs, the first connecting β to γ in X , the second connecting γ to γ_1 in \tilde{C}_1 , the third connecting γ_1 to β_1 in X_1 , and the final arc connecting β_1 to β in \tilde{B}_1 . Since $X \cap X_1 = \emptyset$, we have $|c \cap \beta| = |c \cap \gamma| = 1$. By construction $|c \cap \alpha| = 0$. Therefore, α is not isotopic in Σ to either β or γ . \square

6. Proof of Theorem 1.8

With notation as in Section 5 we assume, as we may by Lemma 5.7, that A_1 and B_2 are not boundary parallel and that A_2 , B_1 and C_1 are boundary

Table 1: Colors of I -adjacent points.

p	p_1, p_2
Blue	Yellow or light red
Yellow	Blue or light red
Light red	One is dark red and the other can be any color

parallel. We assume, as we may by Lemma 5.8, that α_1, β_2 and γ_2 are essential in Σ and α is not isotopic in Σ to α_1, β or γ .

The proof is divided into the following two cases:

Case One. α_1 can be isotoped to be disjoint from X_2 . Let $\tilde{A}_2, \tilde{B}_1,$ and \tilde{C}_1 be as in Lemma 5.8. Let $T \subset \Sigma$ be the twice punctured torus $X \cup \tilde{B}_1 \cup \tilde{C}_1 \cup X_1$. Isotope α_1 so that $\alpha_1 \cap X_2 = \emptyset$. After this isotopy, $X_2 \cap (\alpha_1 \cup X) = \emptyset$. Hence either $X_2 \subset (X_1 \cup \tilde{B}_1 \cup \tilde{C}_1)$ or $X_2 \cap T = \emptyset$. In the former case, $\alpha_2 \subset (X_1 \cup \tilde{B}_1 \cup \tilde{C}_1)$. Since α is isotopic to α_2 in Σ , α is isotopic into $X_1 \cup \tilde{B}_1 \cup \tilde{C}_1$. By Proposition 5.6(2) α is essential, and hence α is isotopic to a component of $\partial(X_1 \cup \tilde{B}_1 \cup \tilde{C}_1) = \alpha_1 \cup \beta \cup \gamma$, contradicting our assumptions.

Hence, we may assume that $X_2 \cap T = \emptyset$. Let $D_1 \subset V_1$ be a meridian disk obtained by compressing or boundary compressing A_1 . After a small isotopy we may assume that $\partial D_1 \cap \partial A_1 = \partial D_1 \cap (\alpha \cup \alpha_1) = \emptyset$, and hence either $\partial D_1 \subset T$ (hence $\partial D_1 \cap \beta_2 = \emptyset$) or $\partial D_1 \cap T = \emptyset$ (hence $\partial D_1 \cap \beta = \emptyset$). Thus D_1 is disjoint from at least one component of ∂B_2 ; by Lemma 4.4 (2), $d(\Sigma) \leq 2$, proving Theorem 1.8 in Case One.

Before proceeding to Case Two we refine our colorings. Let \mathcal{F} be an I -equivalent family of faces, ordered as F_1, \dots, F_n so that F_i is I -adjacent to F_{i+1} ($i = 1, \dots, n - 1$). Then the red faces are F_1, F_2, F_{n-1} , and F_n . We color F_1 and F_n dark red. If $n \geq 3$ we color F_2 and F_{n-1} light red.

Clearly, a face is I -adjacent to two distinct faces if and only if it is colored blue, yellow, or light red. Let p be a point on such a face. Then p is on the boundary of two I -fibers, on the V_1 and V_2 sides. Denote the other endpoints of these fibers by p_1 and p_2 . By construction we see that the colors at p, p_1 and p_2 fulfill the conditions in Table 1.

Notation 6.1. Every light red face is I -equivalent to a dark red face on one side. On the other side, it is I -equivalent to a face that may be blue, yellow, light red or dark red. This decomposes the set of light red points into four disjoint subsets. We label a light red face that is I -equivalent to a blue (resp. yellow) face by $\mathbf{lr}[\mathbf{b}]$ (resp. $\mathbf{lr}[\mathbf{y}]$).

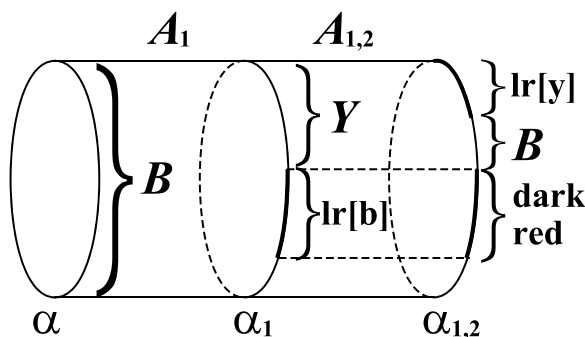


Figure 3: Coloring of the I -bundles adjacent to α_1

Case Two. α_1 cannot be isotoped to be disjoint from X_2 . Since $\alpha \subset \text{int}(B)$, each point of α_1 is yellow or light red. Hence α_1 bounds I -bundles on both sides. Let $A_{1,2}$ be the (possibly immersed) I -bundle obtained by extending α_1 into V_2 , and denote $\partial A_{1,2} \setminus \alpha_1$ by $\alpha_{1,2}$; see figure 3. Since every point of α_1 is yellow or light red and labeled $\text{lr}[\mathbf{b}]$, every point of $\alpha_{1,2}$ is blue, light red and labeled $\text{lr}[\mathbf{y}]$, or dark red (see Table 1 and Notation 6.1). Thus $\alpha_1 \cap \alpha_{1,2} = \emptyset$, and we see that $A_{1,2}$ is trivial I -bundle, that is, an embedded annulus.

Since X_2 and X co-bound an I -bundle, every point of X_2 is yellow or light red and labeled $\text{lr}[\mathbf{b}]$. Thus $\alpha_{1,2} \cap X_2 = \emptyset$. By assumption α_1 cannot be isotoped off X_2 . Hence α_1 is not isotopic to $\alpha_{1,2}$; this implies that $A_{1,2}$ is not boundary parallel. By assumption A_1 is not boundary parallel and α_1 is essential in Σ . Applying Lemma 4.4 (1) to A_1 , $A_{1,2}$, and α_1 we conclude that $d(\Sigma) \leq 2$, completing the proof of Theorem 1.8.

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