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An energy approach to the problem of uniqueness for the Ricci flow

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We revisit the problem of uniqueness for the Ricci flow and give a short, direct proof, based on the consideration of a simple energy quantity, of Hamilton/Chen-Zhu's theorem on the uniqueness of complete solutions of uniformly bounded curvature. With a variation of this technique, we prove a further uniqueness theorem for subsolutions to a general class of mixed differential inequalities and obtain an extension of Chen-Zhu's result to solutions (and initial data) of potentially unbounded curvature.

Let $M = M^n$ be a smooth manifold and g_0 a Riemannian metric on M. We are interested in the question of uniqueness of solutions to the initial value problem

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t)), \quad g(0) = g_0, \tag{0.1}$$

associated to the Ricci flow on M. The broadest category in which uniqueness is currently known to hold in every dimension is that of complete solutions of uniformly bounded curvature.

Theorem 1 (Hamilton [H1]; Chen-Zhu [CZ]). *Suppose g_0 is a complete metric and $g(t)$ and $\tilde{g}(t)$ are solutions to the initial value problem* (0.1) *satisfying*

$$\sup_{M\times[0,T]} |\,\mathrm{Rm}\,|_{g(t)}, \ \sup_{M\times[0,T]} |\widetilde{\mathrm{Rm}}|_{\tilde{g}(t)} \leq K_0.$$

Then $g(t) = \tilde{g}(t)$ for all $t \in [0, T]$.

The uniqueness of solutions to the Ricci flow is not an automatic consequence of the theory of parabolic equations, as the system (0.1) is only weakly-parabolic. For compact M, there are two basic arguments, both due to Hamilton. The first is a byproduct of the proof of the short-time existence of solutions in Hamilton's orginal paper [H1] and is based on a Nash-Moser-type inverse function theorem. The second, given in [H2], effectively reduces the question of uniqueness to that for the strictly parabolic Ricci-DeTurck flow. The basis of this argument is the observation that the DeTurck diffeomorphisms, which are generally obtained as solutions to a system of ordinary differential equations (ODE) depending on a given solution to the Ricci-DeTurck flow, can also be represented as the solutions to a certain parabolic PDE — specifically, a harmonic map heat flow — which depends on the associated solution to the Ricci flow. As DeTurck's method is applicable to many other geometric evolution equations with gauge-based degeneracies, this second argument of Hamilton's gives rise to an elegant and flexible general prescription in which one exchanges the problem of uniqueness for one weakly parabolic system for the (separate) problems of existence and uniqueness for one or more auxiliary strictly parabolic systems.

This latter prescription of Hamilton is also the basis of Chen-Zhu's proof [CZ] in the complete non-compact case, however its components are far from straightforward to assemble in this setting. Given the lack of general theory for the harmonic map flow into arbitrary target manifolds, the authors in particular had to solve the problem of short-time existence for their specific variant of this flow (and verify the crucial accompanying estimates) effectively from scratch. This they accomplished with the combination of a clever conformal transformation of the initial metric and a series of intricate a priori estimates — their approach producing, as independently useful byproducts, well-controlled solutions to both the harmonic map and Ricci-DeTurck flows associated to a given solution to the Ricci flow.

In this paper, however, we demonstrate that, if one is interested solely in the uniqueness of solutions to the Ricci flow, it is possible to eliminate the passage through the harmonic map and Ricci-DeTurck-flows and avoid these delicate issues of existence entirely. Given two solutions $g(t)$ and $\tilde{g}(t)$ to the initial value problem (0.1), our strategy is to consider a quantity of the form

$$\mathcal{E}(t) = \int_M \left(t^{-\alpha}|g - \tilde{g}|^2_{g(t)} + t^{-\beta}|\Gamma - \widetilde{\Gamma}|^2_{g(t)} + |\mathrm{Rm} - \widetilde{\mathrm{Rm}}|^2_{g(t)} \right) \Phi \, d\mu_{g(t)}$$

for a suitable choice of the constants α and β and weight function $\Phi = \Phi(x, t)$, and to argue from the differential inequality it satisfies that it must vanish identically. This functional $\mathcal{E}$ is something of a compromise between the two perhaps most "obvious" candidates for such a quantity: the L^2-norms of the differences $g - \tilde{g}$ and $\mathrm{Rm} - \widetilde{\mathrm{Rm}}$. Although neither $g - \tilde{g}$ nor $\mathrm{Rm} - \widetilde{\mathrm{Rm}}$ satisfy a strictly parabolic equation, they fail to do so in such a way that their evolution equations, together with that of $\Gamma - \widetilde{\Gamma}$, can still be organized in a closed and (for our purposes) virtually parabolic system of inequalities to which the energy method may be applied.

When M is compact, e.g., it is not hard to show that with $\alpha = \beta = 0$, and $\Phi \equiv 1$, we have $\mathcal{E}'(t) \leq C\mathcal{E}$, and hence, since $\mathcal{E}(0) = 0$, that $\mathcal{E} \equiv 0$. When M is non-compact and the curvature of $g(t)$ and $\tilde{g}(t)$ is assumed to be uniformly bounded, we can take $\alpha = 1$, $\beta \in (0, 1)$ and Φ of some sufficiently rapid decay in space in order to draw the same conclusion with much the same argument. With some minor modifications and a bit more careful estimation, we further obtain the following rather inexpensive extension of Theorem 1, which says, essentially, that uniqueness holds in the class of solutions whose curvature at times $t > 0$ has at most quadratic growth in the initial distance. Below $r_0(x) \doteqdot \mathrm{dist}_{g_0}(x, x_0)$ is the distance with respect to the metric g_0 from some fixed $x_0 \in M$.

Theorem 2. *Suppose (M, g_0) is a complete noncompact Riemannian manifold satisfying the volume growth condition*

$$\mathrm{vol}_{g_0}(B_{g_0}(x_0, r)) \leq V_0 e^{V_0 r^2} \tag{0.2}$$

for some constant V_0 and all $r > 0$. If $g(t)$ and $\tilde{g}(t)$ are smooth solutions to (0.1) on $M \times [0, T]$ for which

$$\gamma^{-1} g_0(x) \leq g(x, t), \quad \tilde{g}(x, t) \leq \gamma g_0(x), \tag{0.3}$$

and

$$|\,\mathrm{Rm}(x, t)|_{g(t)} + |\widetilde{\mathrm{Rm}}(x, t)|_{\tilde{g}(t)} \leq \frac{K_0}{t^{\delta}}(r_0^2(x) + 1), \tag{0.4}$$

on $M \times (0, T]$ for some constants γ, K_0 and $\delta \in (0, 1/2)$, then $g(t) = \tilde{g}(t)$ for all $t \in [0, T]$.

Note that, in particular, we do not impose any condition on the curvature tensor of the initial metric, although the volume condition (0.2) is implied, e.g., by a bound of the form $\mathrm{Rc}(g_0) \geq -C(r_0^2 + 1)g_0$. Also observe that the uniform equivalence (0.3) is automatic in the case that the curvature tensors of $g(t)$ and $\tilde{g}(t)$ are assumed bounded on the time-slices $M \times \{t\}$ for $t > 0$ (but blow-up at a rate no greater than $t^{-\delta}$ as $t \searrow 0$).

Corollary 3. *If g_0 is a complete metric satisfying* (0.2) *and $g(t)$ and $\tilde{g}(t)$ are smooth solutions to* (0.1) *satisfying*

$$|\operatorname{Rm}(x,t)|_{g(t)} + |\widetilde{\operatorname{Rm}}(x,t)|_{\tilde{g}(t)} \leq \frac{K_0}{t^\delta},$$

on $M \times (0,T]$ for some K_0 and $\delta \in (0,1/2)$, then $g(t) = \tilde{g}(t)$ for all $t \in [0,T]$.

With Shi's existence theorem [S], we also have have the following special case.

Corollary 4. *If g_0 is complete and of bounded curvature, then any solution $g(t)$ to the initial value problem* (0.1) *which remains uniformly equivalent to g_0 and obeys the bound $|\operatorname{Rm}(x,t)| \leq K_0(r_0^2(x)+1)$ on $M \times (0,T]$ must have bounded curvature tensor.*

Other extensions of Theorem 1 and related results can be found in the papers [C, CY, F, GT, Hs, LT, T]. For example, in [C], Chen proves that if (M^3, g_0) is complete and has bounded non-negative sectional curvature, then any two complete solutions to the Ricci flow with this initial data must agree identically; for surfaces, he proves the same result for initial data with Gaussian curvature of arbitrary sign. It is an interesting question whether his result may be extended to higher dimensions and to initial metrics with some curvature growth. Since we have only used rather classical estimates in this paper and have made no fine use of the non-linear reaction terms in the evolution of the curvature, we expect that the combination of conditions (0.3) and (0.4) in Theorem 2 can be relaxed further. Perhaps the general technique we describe may be of use to such future extensions as an alternative strategy to bypass the gauge-degeneracy of the equation and possibly also of use for the quantitative comparison of solutions that "agree" in some limiting sense as $t \to T \in [-\infty, \infty)$.

1. Preliminaries

Going forward, we assume that $g(t)$ and $\tilde{g}(t)$ are complete solutions to (0.1). We will select one of the metrics, $g = g(t)$, as our reference metric and use the notation $V * W$ below to represent a linear combination of contractions of the tensors V and W and g in which the coefficients and the total number of terms are bounded by some constant depending only on the dimension. In the estimates, we will use $C = C(n)$ to such denote such a constant, which may change from line to line. To further reduce clutter in our expressions we will often use $|\cdot| \doteqdot |\cdot|_g$ to denote the norms induced on $T^k_l(M)$ by g (distinguishing other norms by a subscript), and simply write $R = \mathrm{Rm}$ and $\widetilde{R} = \widetilde{\mathrm{Rm}}$ for the $(3,1)$ curvature tensors associated with g and $\tilde{g}$. (In the few instances in which it occurs, we will use $\mathrm{scal}(g)$ for the scalar curvature.)

It will be convenient also to use index notation in some of our calculations, but to potential confusion, we will only "raise" or "lower" indices explicitly to identify the metric involved. We will only violate this (and then only slightly) in some of the schematic equations below, where, according to our convention above, we would simply write, e.g., $g^{ik}\tilde{g}^{jl}T^k_{ij}U_k$ as $\tilde{g}^{-1} * T * U$, and $g_{ij}g^{kl}T^j_{kl}$ as T. For similar reasons, we will only use the summation convention in its most restricted sense (summing over repeated upper and lower indices from 1 to n), when it describes a metric independent contraction; e.g., we can write $R_{jk} = R^l_{ljk}$ and $\tilde{R}_{jk} = \tilde{R}^l_{ljk}$ without ambiguity for the coordinate representations of Rc and $\widetilde{\mathrm{Rc}}$. Finally, we introduce the notation $T^* \doteqdot \max\{T, 1\}$.

1.1. Evolution equations and inequalities

Define

$$h \doteqdot g - \tilde{g}, \quad A \doteqdot \nabla - \widetilde{\nabla}, \quad S \doteqdot R - \widetilde{R},$$

that is, $A^k_{ij} \doteqdot \Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}$, and $S^l_{ijk} \doteqdot R^l_{ijk} - \widetilde{R}^l_{ijk}$. We begin by organizing the evolution equations satisfied by these tensors in such a way that every term contains either a factor of (some contraction of) one of the group or ∇S. We note for later that we can write

$$g^{ij} - \tilde{g}^{ij} = -g^{ik}\tilde{g}^{jl}h_{kl} \quad \text{and} \quad \nabla_k \tilde{g}^{ij} = \tilde{g}^{aj}A^i_{ka} + \tilde{g}^{ia}A^j_{ka}, \tag{1.1}$$

or, according to our convention,

$$g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * h \quad \text{and} \qquad \nabla\tilde{g}^{-1} = \tilde{g}^{-1} * A.$$

Now, on $M \times [0, T]$, the tensors h and A satisfy

$$\frac{\partial}{\partial t} h_{ij} = -2(R_{ij} - \tilde{R}_{ij}) = -2S^l_{lij}$$

and

$$\frac{\partial}{\partial t}A_{ij}^{k} = \tilde{g}^{mk}\left(\widetilde{\nabla}_i\tilde{R}_{jm} + \widetilde{\nabla}_j\tilde{R}_{im} - \widetilde{\nabla}_m\tilde{R}_{ij}\right) - g^{mk}\left(\nabla_i R_{jm} + \nabla_j R_{im} - \nabla_m R_{ij}\right), \tag{1.2}$$

respectively. Since

$$\widetilde{\nabla}_i\tilde{R}_{jk} - \nabla_i\tilde{R}_{jk} = A_{ij}^{p}\tilde{R}_{pk} + A_{ik}^{p}\tilde{R}_{jp}, \tag{1.3}$$

and $\nabla_i(R_{jk} - \widetilde{R}_{jk}) = \nabla_i S_{ljk}^{l}$, we can use the first equation in (1.1) to put the equation for $\frac{\partial}{\partial t}A$ in the schematic form

$$\frac{\partial}{\partial t}A = \tilde{g}^{-1} * h * \widetilde{\nabla}\widetilde{R} + A * \tilde{R} + \nabla S, \tag{1.4}$$

where according to our convention, the last term denotes a constant number of terms of contractions of ∇S (in this case, of the form $\nabla_i S_{ljk}^{l}$). Similarly, using

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk}^{l} =& \Delta R_{ijk}^{l} + g^{pq}\left(R_{ijp}^{r}R_{rqk}^{l} - 2R_{pik}^{r}R_{jqr}^{l} + 2R_{pir}^{l}R_{jqk}^{r}\right)\\ &- g^{pq}\left(R_{ip}R_{qjk}^{l} + R_{jp}R_{iqk}^{l} + R_{kp}R_{ijq}^{l}\right) + g^{pl}R_{pq}R_{ijk}^{q},\end{aligned} \tag{1.5}$$

together with the generalization $(\nabla - \widetilde{\nabla})W = A * W$ of (1.3) to arbitrary tensors W, we can express the evolution equation of S in the schematic form

$$\begin{aligned}\frac{\partial}{\partial t}S =& \nabla_a(g^{ab}\nabla_b R - \tilde{g}^{ab}\widetilde{\nabla}_b\tilde{R}) + \tilde{g}^{-1} * A * \widetilde{\nabla}\tilde{R}\\ &+ \tilde{g}^{-1} * h * \widetilde{R} * \widetilde{R} + S * R + S * \widetilde{R}.\end{aligned} \tag{1.6}$$

Here the shorthand $\nabla_a(g^{ab}\nabla_b R - \tilde{g}^{ab}\widetilde{\nabla}_b\tilde{R})$ represents $\nabla_a(g^{ab}\nabla_b R_{ijk}^{l} - \tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}_{ijk}^{l})$, and we have obtained (1.6) from (1.1) and (1.5) via the following explicit representation of the $\tilde{g}(t)$-Laplacian of $\widetilde{R}$ as

$$\begin{aligned}\widetilde{\Delta}\widetilde{R}_{ijk}^{l} &= \tilde{g}^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b\widetilde{R}_{ijk}^{l} = \widetilde{\nabla}_a\left(\tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}_{ijk}^{l}\right)\\ &= \nabla_a\left(\tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}_{ijk}^{l}\right) - A_{ap}^{a}\tilde{g}^{pb}\widetilde{\nabla}_b\widetilde{R}_{ijk}^{l} - A_{ap}^{l}\tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}_{ijk}^{p}\\ &\quad + A_{ai}^{p}\tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}_{pjk}^{l} + A_{aj}^{p}\tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}_{ipk}^{l} + A_{ak}^{p}\tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}_{ijp}^{l}.\end{aligned}$$

Now using the Cauchy-Schwarz inequality together with the above representations, we obtain that

$$\left|\frac{\partial}{\partial t}h\right| \leq C|S|, \quad \left|\frac{\partial}{\partial t}A\right| \leq C\left(|\tilde{g}^{-1}||\widetilde{\nabla}\tilde{R}||h| + |\tilde{R}||A| + |\nabla S|\right), \tag{1.7}$$

and

$$\begin{aligned}\left|\frac{\partial S}{\partial t} - \Delta S - \operatorname{div} U\right| & \\ \leq C\Big(|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{R}||A| &+ |\tilde{g}^{-1}||\widetilde{R}|^2|h| + (|R| + |\widetilde{R}|)|S|\Big).\end{aligned} \tag{1.8}$$

where $U \doteqdot g^{ab}\nabla_b\widetilde{R} - \tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}$ is the section of $T_3^2(M)$ given in local coordinates by

$$\begin{aligned}U_{ijk}^{al} &= g^{ab}\nabla_b\widetilde{R}_{ijk}^l - \tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}_{ijk}^l \\ &= (g^{ab} - \tilde{g}^{ab})\widetilde{\nabla}_b\widetilde{R}_{ijk}^l + g^{ab}(\nabla_b\widetilde{R}_{ijk}^l - \widetilde{\nabla}_b\widetilde{R}_{ijk}^l) \\ &= -g^{ak}\tilde{g}^{bl}h_{kl}\widetilde{\nabla}_b\widetilde{R}_{ijk}^l + g^{ab}\left(A_{bp}^l\widetilde{R}_{ijk}^p - A_{bi}^p\widetilde{R}_{pjk}^l - A_{bj}^p\widetilde{R}_{ipk}^l - A_{bk}^p\widetilde{R}_{ijp}^l\right),\end{aligned} \tag{1.9}$$

and by $\operatorname{div} U$ we mean the section of $T_3^1(M)$ given by $(\operatorname{div} U)_{ijk}^l = \nabla_a U_{ijk}^{al}$. Observe that U satisfies

$$|U| \leq C(|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{R}||h| + |A||\widetilde{R}|). \tag{1.10}$$

We leave these evolution inequalities in the above rather raw form for the moment and return to simplify them later in forms specialized to the specific assumptions of Theorems 1 and 2.

1.2. A function of rapid decay

In order that our energy quantity $\mathcal{E}$ be well-defined and that the differentiations and integrations-by-parts we wish to perform be valid, we will need to select a weight function which decays suitably rapidly.

Lemma 5. *Suppose $\bar{g}(t)$ is a smooth family of complete metrics on $M \times [0,T]$ satisfying $\gamma^{-1}\bar{g} \leq g(t)$, where $\bar{g} = \bar{g}(0)$ and $x_0 \in M$. Define $\bar{r}(x) = \operatorname{dist}_{\bar{g}(0)}(x, x_0)$. Then, for any positive constants L_1 and L_2, there exists a positive constant $T' = T'(n, \gamma, L_1, L_2, T)$, and a function $\eta : M \times [0, T'] \to \mathbb{R}$*

that is smooth in t*, Lipschitz (and smooth* $d\mu_{g(t)} - a.e.$*) on each* $M \times \{t\}$*, and that simultaneously satisfies the conditions*

$$-\frac{\partial \eta}{\partial t} + L_1 |\eta|^2_{\bar{g}(t)} \le 0, \quad \text{and} \quad \mathrm{e}^{-\eta} \le e^{-L_2 \bar{r}^2(x)},$$

on $M \times [0, \tau]$ *whenever* $0 < \tau \le T'$*.*

Proof. We follow the construction in Chapter 12 of [CRF], (cf. also [KL], [LY]), and define $\eta(x,t) \doteqdot \eta_{B,\tau}(x,t) \doteqdot B\bar{r}^2(x)/(4(2\tau - t))$ for on $M \times [0, \tau]$. The function $\bar{r}$ is continuous and smooth off of the $\bar{g}$-cut locus of x_0, where it satisfies $|\nabla \bar{r}|^2_{\bar{g}(0)} = 1$, and hence $|\nabla \bar{r}|^2_{g(t)} \le \gamma$. It follows that that $\bar{r}$ is Lipschitz and smooth $d\mu_{\bar{g}(t)}$-a .e. for each $t \in [0, \tau]$. For each $L_1 > 0$, we have

$$-\frac{\partial \eta}{\partial t} + L_1 |\nabla \eta|^2_{\bar{g}(t)} \le -B(1 - BL_1\gamma)\frac{\bar{r}^2(x)}{4(2\tau - t)^2},$$

and we can guarantee the first condition provided $B < 1/(\gamma L_1)$. Also, for any $t \in [0, \tau]$, we have

$$B\bar{r}^2(x)/(8\tau^2) \le \eta(x,t) \le B\bar{r}^2(x)/(4\tau^2),$$

so, given $L_2 > 0$, we can ensure the second condition on $M \times [0, \tau]$ provided $0 < \tau \le T' \doteqdot \min\{(B/(8L_2))^{1/2}, T\}$. □

2. The case of uniformly bounded curvature

Although Theorem 2 is strictly stronger than Theorem 1, the proof can be substantially simplified in the case of bounded curvature, and thus we give a separate argument here to demonstrate the technique. We will consider only the case of non-compact M, as the proof for that of compact M is nearly identical, but less involved, and can be easily reconstructed from the argument below. For the remainder of this section, we will assume that g_0, $g(t)$ and $\tilde{g}(t)$ satisfy the assumptions of Theorem 1, and that $\beta \in (0,1)$ is a fixed constant.

2.1. Derivative and decay estimates

We first recall that the global estimates of Bando [B] and Shi [S] imply that there exists a constant $N = N(n, K_0, T^*)$ such that

(2.1)

$$|R|_{g(t)} + |\widetilde{R}|_{g(t)} + \sqrt{t}|\nabla R|_{g(t)} + \sqrt{t}|\widetilde{\nabla}\widetilde{R}|_{\tilde{g}(t)} + t|\nabla\nabla R|_{g(t)} + t|\widetilde{\nabla}\widetilde{\nabla}\widetilde{R}|_{\tilde{g}(t)} \le N.$$

on $M \times [0,T]$ It is a standard argument (cf., e.g., Theorem 14.1 in [H1]) that the uniform curvature bounds on $g(t)$ and $\tilde{g}(t)$ imply that the metrics $g(t)$, $\tilde{g}(t)$ and g_0 remain uniformly equivalent. Thus, the estimates above hold (for some potentially larger N) when the norms are replaced by any one of $|\cdot| \doteqdot |\cdot|_{g(t)}$, $|\cdot|_{\tilde{g}(t)}$ and $|\cdot|_{g_0}$. In what follows, we will use N to denote a series of constants depending only on n, K_0 and T^* which may vary from one inequality to the next.

We begin by noting that the same argument that yields the uniform equivalence of the metrics can be used to produce simple estimates on the decay of h and A as $t \searrow 0$ that imply, in particular, that $t^{-1}|h|^2$ and $t^{-\beta}|A|^2$ tend to zero uniformly as $t \searrow 0$ on M and can be continuously extended to $M \times [0,T]$.

Lemma 6. *Under the assumptions of Theorem 1, we have $|h(p,t)| \leq Nt$ and $|A(p,t)| \leq N\sqrt{t}$ on $M \times [0,T]$ for some constant $N = N(n, K_0, T^*)$.*

Proof. At an arbitrary p in M, we have

$$\begin{aligned} |h(p,t)| \leq N|h(p,t)|_{g_0} &\leq N\int_0^t |S(p,s)|_{g_0}\, ds \\ &\leq N\int_0^t \left(|R(p,s)|_{g_0} + |\widetilde{R}(p,s)|_{g_0}\right) ds \leq Nt, \end{aligned}$$

and similarly, using (1.2), for any $0 < \epsilon < t$, we have

$$\begin{aligned} |A(p,t) - A(p,\epsilon)| &\leq N|A(p,t) - A(p,\epsilon)|_{g_0} \\ &\leq N\int_\epsilon^t \left(|\nabla R(p,s)|_{g_0} + |\widetilde{\nabla}\widetilde{R}(p,s)|_{g_0}\right) ds \\ &\leq N\int_\epsilon^t s^{-1/2}\, ds \leq N(\sqrt{t} - \sqrt{\epsilon}). \end{aligned}$$

Sending $\epsilon \to 0$ completes the proof. □

2.2. Definition and differentiability of $\mathcal{E}$

Next, since the uniform curvature bound on $g(0)$ implies a lower bound on $\mathrm{Rc}(g(0))$, the Bishop–Gromov volume comparison theorem implies that

$$\mathrm{vol}_{g(0)}(B_{g(0)}(x_0, r)) \leq Ne^{Nr}$$

for some constant N and all $r > 0$. Since the metrics $g(t)$ and g_0 are uniformly equivalent, we thus also have

$$\mathrm{vol}_{g(t)}(B_{g_0}(x_0, r)) \leq N e^{Nr}$$

for some N. With any choice of $B > 0$ (and independent of T), the function $\eta = \eta_{B,T}$ of Lemma 5 (with $\bar{g}(t) = g(t)$) will satisfy $\mathrm{e}^{-\eta} \leq e^{-Br_0^2/(8T)}$ on $M \times [0, T]$, so if we choose $B > 0$ sufficiently small to ensure, say, that

$$\frac{\partial \eta}{\partial t} - 3|\nabla \eta|^2 \geq 0,$$

it will still follow that any continuous function of at most (sub-quadratic) exponential growth in $r_0(x)$ at t will be $\mathrm{e}^{-\eta}\, d\mu$-integrable. So $\frac{\partial \eta}{\partial t}$ and $|\nabla \eta|^2$, being of quadratic growth in $r_0(x)$, are $\mathrm{e}^{-\eta}\, d\mu$-integrable for any $t \in [0, T]$ (where, here and elsewhere, we write $d\mu \doteqdot d\mu_{g(t)}$), as are the uniformly bounded quantities $t^{-1}|h|^2$, $t^{-\beta}|A|^2$ and $|S|^2$. Moreover, since

$$\nabla \widetilde{R} = \widetilde{\nabla} \widetilde{R} + A * \widetilde{R}, \quad \text{and} \quad \nabla \widetilde{\nabla} \widetilde{R} = \widetilde{\nabla} \widetilde{\nabla} \widetilde{R} + A * \widetilde{\nabla} \widetilde{R},$$

it follows from equations (1.4), (1.6), (1.7) and (2.1) that $|\nabla S|$ and $|\nabla_a (g^{ab} \nabla_b R - \tilde{g}^{ab} \widetilde{\nabla}_b \widetilde{R})|$, and hence $\frac{\partial}{\partial t}|h|^2$, $\frac{\partial}{\partial t}|A|^2$ and $\frac{\partial}{\partial t}|S|^2$, are uniformly bounded on $M \times [\epsilon, T]$ for any $\epsilon > 0$, and consequently $\mathrm{e}^{-\eta}\, d\mu$-integrable for each $t \in (0, T]$ (although they need not be bounded as $t \to 0$).

Since $\frac{\partial}{\partial t} d\mu = -\operatorname{scal}(g(t))\, d\mu$, and the scalar curvature of $g(t)$ is bounded by assumption, these observations imply that, for fixed $\beta \in (0, 1)$ and η as above, the quantity

$$\mathcal{E}(t) \doteqdot \int_M \left(t^{-1}|h|^2 + t^{-\beta}|A|^2 + |S|^2 \right) \mathrm{e}^{-\eta}\, d\mu$$

is differentiable on $(0, T]$, and (with the dominated convergence theorem) satisfies $\lim_{t \searrow 0} \mathcal{E}(t) = 0$.

2.3. Vanishing of $\mathcal{E}$

We claim in fact that $\mathcal{E}$ vanishes identically on $[0, T]$. This is a consequence of iterating the following result.

Proposition 7. *There exists $N_0 = N_0(n, K_0, T^*) > 0$ and $T_0 = T_0(n, \beta) \in (0, T]$ such that $\mathcal{E}'(t) \leq N_0 \mathcal{E}(t)$ for all $t \in (0, T_0]$. Hence $\mathcal{E} \equiv 0$ on $(0, T_0]$.*

Proof. For $t \in (0,T]$ and $\alpha \in (0,1)$, define

$$\mathcal{G}(t) \doteqdot \int_M |S|^2 \, \mathrm{e}^{-\eta} \, d\mu, \quad \mathcal{H}(t) \doteqdot t^{-1} \int_M |h|^2 \, \mathrm{e}^{-\eta} \, d\mu,$$
$$\mathcal{I}(t) \doteqdot t^{-\beta} \int_M |A|^2 \, \mathrm{e}^{-\eta} \, d\mu \quad \text{and} \quad \mathcal{J}(t) \doteqdot \int_M |\nabla S|^2 \, \mathrm{e}^{-\eta} \, d\mu,$$

so $\mathcal{E}(t) = \mathcal{G}(t) + \mathcal{H}(t) + \mathcal{I}(t)$. In view of the discussion in Section 2.2, we may freely differentiate under the integral sign and integrate by parts about any $0 < t \leq T$. As before, in the estimates below, C will denote a series of constants depending only on n, and N a series of constants depending at most on n, β, K_0 and T^*. Taking into account the time dependency of the norms $|\cdot| = |\cdot|_{g(t)}$ and the measure $d\mu = d\mu_{g(t)}$, and using (1.8) together with (2.1) we have

$$\begin{aligned}
\mathcal{G}' &\leq N\mathcal{G} + \int_M \left(2 \left\langle \frac{\partial S}{\partial t}, S \right\rangle - \frac{\partial \eta}{\partial t} |S|^2 \right) \mathrm{e}^{-\eta} \, d\mu \\
&\leq N\mathcal{G} + \int_M \Big(2 \left\langle \Delta S + \operatorname{div} U, S \right\rangle + C|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{R}||A||S| \\
&\qquad + C|\tilde{g}^{-1}||\widetilde{R}|^2|h||S| + (|R| + |\widetilde{R}|)|S|^2 - \frac{\partial \eta}{\partial t}|S|^2 \Big) \mathrm{e}^{-\eta} \, d\mu \\
&\leq N\mathcal{G} + \int_M \Big(2 \left\langle \Delta S + \operatorname{div} U, S \right\rangle + N t^{-1/2}|A||S| \\
&\qquad + N|h||S| - \frac{\partial \eta}{\partial t}|S|^2 \Big) \mathrm{e}^{-\eta} \, d\mu \\
&\leq N\mathcal{G} + t\mathcal{H} + t^{\beta-1}\mathcal{I} + \int_M \left(2 \left\langle \Delta S + \operatorname{div} U, S \right\rangle - \frac{\partial \eta}{\partial t}|S|^2 \right) \mathrm{e}^{-\eta} \, d\mu,
\end{aligned}$$

for $t > 0$, where we have estimated

$$N t^{-1/2}|A||S| \leq t^{\beta-1}(t^{-\beta}|A|^2) + N|S|^2 \quad \text{and} \quad N|h||S| \leq t(t^{-1})|h|^2 + N|S|^2,$$

to obtain the third line. We can then integrate by parts in the last integral to obtain

$$\begin{aligned}
&\int_M \left(2 \left\langle \Delta S + \operatorname{div} U, S \right\rangle - \frac{\partial \eta}{\partial t}|S|^2 \right) \mathrm{e}^{-\eta} \, d\mu \\
&\quad \leq -2\mathcal{J} + \int_M \Big(2|\nabla \eta||\nabla S||S| + 2|\nabla S||U|
\end{aligned}$$

$$+ 2|\nabla\eta||U||S| - \frac{\partial\eta}{\partial t}|S|^2\Big) \, e^{-\eta}\, d\mu$$
$$\leq -\mathcal{J} + \int_M \left(\left(3|\nabla\eta|^2 - \frac{\partial\eta}{\partial t}\right)|S|^2 + 3|U|^2\right) e^{-\eta}\, d\mu.$$

Here we have estimated

$$2|\nabla\eta||\nabla S||S| + 2|\nabla S||U| \leq |\nabla S|^2 + 2|\nabla\eta|^2|S|^2 + 2|U|^2,$$

and

$$2|\nabla\eta||U||S| \leq |\nabla\eta|^2|S|^2 + |U|^2.$$

Since $|U|^2 \leq Nt^{-1}|h|^2 + N|A|^2$ by (1.10) and (2.1), and $\frac{\partial\eta}{\partial t} \geq 3|\nabla\eta|^2$ by assumption, putting everything together, we have

$$\begin{aligned} \mathcal{G}' &\leq N\mathcal{G} + (t+N)\mathcal{H} + (t^{\beta-1} + Nt^{\beta})\mathcal{I} - \mathcal{J} \\ &\leq N\mathcal{G} + N\mathcal{H} + (t^{\beta-1} + N)\mathcal{I} - \mathcal{J}, \end{aligned} \tag{2.2}$$

for any $t \in (0, T]$, using $t \leq T^*$, and $t^\beta \leq (T^*)^\beta$.

Similarly, with (1.7) and (2.1), we compute that

$$\begin{aligned} \mathcal{H}' &\leq (N - t^{-1})\mathcal{H} + t^{-1}\int_M \left(2\left\langle \frac{\partial h}{\partial t}, h\right\rangle - \frac{\partial\eta}{\partial t}\right) e^{-\eta}\, d\mu \\ &\leq (N - t^{-1})\mathcal{H} + \int_M Ct^{-1}|S||h|\, e^{-\eta}\, d\mu \\ &\leq (N - (1/2)t^{-1})\mathcal{H} + C\mathcal{G} \end{aligned} \tag{2.3}$$

where we have used that $\frac{\partial\eta}{\partial t} \geq 0$ and $Ct^{-1}|S||h| \leq (1/2)t^{-2}|h|^2 + C|S|^2$

Finally, using (1.7) and (2.1) again, we have

$$\begin{aligned} \mathcal{I}' &\leq (N - \beta t^{-1})\mathcal{I} + t^{-\beta}\int_M \left(2\left\langle \frac{\partial A}{\partial t}, A\right\rangle - \frac{\partial\eta}{\partial t}|A|^2\right) e^{-\eta}\, d\mu \\ &\leq (N - \beta t^{-1})\mathcal{I} + t^{-\beta}\int_M C\left(|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{R}||h| + |\widetilde{R}||A| + |\nabla S|\right)|A|\, e^{-\eta}\, d\mu \\ &\leq (N - \beta t^{-1})\mathcal{I} + \int_M \left(Nt^{-1/2-\beta}|h||A| + Ct^{-\beta}|\nabla S||A|\right) e^{-\eta}\, d\mu \\ &\leq N\mathcal{H} + (N - \beta t^{-1} + Ct^{-\beta})\mathcal{I} + \mathcal{J} \end{aligned} \tag{2.4}$$

where we have again used that $\frac{\partial\eta}{\partial t} \geq 0$ and have estimated

$$Nt^{-1/2-\beta}|h||A| + Ct^{-\beta}|\nabla S||A| \leq Nt^{-1}|h|^2 + Ct^{-2\beta}|A|^2 + |\nabla S|^2.$$

Combining (2.2), (2.3) and (2.4), we then obtain that, for any $t \in (0, T]$,

$$\mathcal{E}'(t) \leq N\mathcal{E}(t) - (1/2)t^{-1}\mathcal{H}(t) - t^{-1}(\beta - t^{\beta} - Ct^{1-\beta})\mathcal{I}(t).$$

Thus, for T_0 sufficiently small depending only on β and $C = C(n)$, and for some $N_0 = N_0(n, K_0, \beta, T^*)$ sufficiently large, we have $\mathcal{E}'(t) \leq N_0\mathcal{E}(t)$ on $(0, T_0]$. Since $\lim_{t\searrow 0}\mathcal{E}(t) = 0$, it follows from Gronwall's inequality that $\mathcal{E} \equiv 0$ on $[0, T_0]$. $\square$

3. The case of potentially unbounded curvature

In this section, we reduce Theorem 2 to a special case of a general result, Theorem 13, in the following section. The strategy is essentially the same as that of the bounded curvature setting, but here we will need to organize our estimates more carefully in order to "squeeze" the differential inequality satisfied by our energy quantity sufficiently to absorb the growth of coefficients that we were able to regard as uniformly bounded in our previous computations.

3.1. Derivative estimates and consequences

First, we recall the following refined form of Shi's local first derivative estimate (due to Hamilton, [H2]) in which the dependencies of the bound on the local curvature bound, the radius of the ball and the elapsed time are explicit.

Theorem 8 (Shi/Hamilton). *Suppose that (M, g_0) is an open Riemannian manifold in which, for a given $x_0 \in M$ and $r > 0$, the closure of $B_{g_0}(x_0, r)$ is compactly contained. Then, if $g(t)$ is a solution to (0.1) on $M \times [0, T]$ with*

$$\sup_{B_{g_0}(x_0,r)\times[0,T]} |R|_{g(t)} \leq K_0,$$

there exists a constant $C = C(n)$ such that

$$|\nabla R|_{g(t)} \leq CK_0\left(\frac{1}{r^2} + \frac{1}{t} + K_0\right)^{1/2},$$

on $B_{g_0}(x_0, r/2) \times (0, T]$.

We can use this result on a series of domains to obtain a derivative bound for solutions satisfying assumptions of the general form of those in Theorem 2.

Corollary 9. *Suppose* (M, g_0) *is a complete noncompact Riemannian manifold. If* $g(t)$ *is a smooth family of complete metrics solving* (0.1) *and satisfying*

$$t^{\delta}|R|_{g(t)}(x,t) \leq K_0(r_0^2(x)+1)$$

on $M \times [0,T]$ *for some constant* K *and* $\delta \in [0,1]$, *where* $r_0(x) = \operatorname{dist}_{g_0}(x, x_0)$ *for some* $x_0 \in M$, *then there exists a constant* $K = K(n, \delta, K_0, T^*)$ *such that*

$$t^{\delta+1/2}|\nabla R|_{g(t)}(x,t) \leq K(r_0^2(x)+1)^{3/2}. \tag{3.1}$$

Proof. Let $t_0 \in (0,T]$ and $\epsilon \doteqdot t_0/2$. For any $r > 0$, we have

$$|R(x,t)| \leq \epsilon^{-\delta} K_0(r^2+1)$$

on $B_{g_0}(x_0, r) \times [\epsilon, T]$, and so by Theorem 8, there is a $C = C(n)$ such that

$$t^{\delta+1/2}|\nabla R|(x,t) \leq C\left(\frac{t}{\epsilon}\right)^{\delta} K_0(r^2+1)\left(\frac{t}{r^2}+\frac{t}{t-\epsilon}+\frac{t}{\epsilon^{\delta}}K_0(r^2+1)\right)^{1/2},$$

on $B_{g_0}(x_0, r/2) \times (\epsilon, T]$. Thus, for any $r \geq 1$, we have

$$\begin{aligned} t_0^{\delta+1/2}|\nabla R|(x,t_0) &\leq C2^{\delta}K_0(r^2+1)\left(t_0+2+t_0^{1-\delta}2^{\delta}K_0(r^2+1)\right)^{1/2} \\ &\leq K'(r^2+1)^{3/2}, \end{aligned}$$

for some $K' = K'(n, \delta, K_0, T^*)$ on $B_{g_0}(x_0, r/2)$. Since $t_0 \in (0,T]$ and $r \geq 1$ were arbitrary, this implies that

$$\sup_{B_{g_0}(x_0,r)\times[0,T]} t^{\delta+1/2}|\nabla R|(x,t) \leq 8K'(r^2+1)^{3/2}$$

for any $r \geq 1$.

For the pointwise estimate, we note that if $x \in B_{g_0}(x_0, 1)$, we have

$$\frac{t^{\delta+1/2}}{(r_0^2(x)+1)^{3/2}}|\nabla R|(x,t) \le t^{\delta+1/2}|\nabla R|(x,t) \le 16K',$$

and if $x \in M \setminus B_{g_0}(x_0, 1)$, we have

$$\begin{aligned}\frac{t^{\delta+1/2}}{(r_0^2(x)+1)^{3/2}}|\nabla R|(x,t) &\le \frac{t^{\delta+1/2}}{(r_0^2(x)+1)^{3/2}}\left\{\sup_{y\in B_{g_0}(x_0,2r_0(x))}|\nabla R|(y,t)\right\}\\ &\le 8K'\left(\frac{4r_0^2(x)+1}{r_0^2(x)+1}\right)^{3/2} \le 64K'.\end{aligned}$$

This verifies (3.1) with $K = 64K'$. $\square$

In Theorem 2, we assume that $g(t)$ and $\tilde{g}(t)$ are uniformly equivalent to g_0, and so, effectively, that both $|R(x,t)|_{g_0}$ and $|\widetilde{R}(x,t)|_{g_0}$ have at most quadratic growth in $r_0(x)$. With an argument exactly analogous to Lemma 6, and using (3.1), we could then obtain global bounds of the form $|h(x,t)| \le Nt^{1-\delta}(r_0^2(x)+1)$ and $|A(x,t)| \le Nt^{1/2-\delta}(r_0^2(x)+1)^{3/2}$ on the decay of h and A as $t \searrow 0$. In fact, since the proof of Theorem 13 below is based on localized energy quantities, we will not need global estimates on either h or A, and instead we just note that (since $g(t)$ and $\tilde{g}(t)$ are assumed to be smooth solutions which agree at $t = 0$) we have naive estimates of the form $|h(x,t)| \le Pt$ and $|A(x,t)| \le Pt$ on $\Omega \times [0,T]$ for some $P = P(\Omega, g, \tilde{g})$ and any compact $\Omega \subset M$.

Lemma 10. *For any $r > 0$, there exists a constant P depending on γ and the maximum values of $|R|_{g_0}$, $|\widetilde{R}|_{g_0}$, $|\nabla R|_{g_0}$ and $|\widetilde{\nabla}\widetilde{R}|_{g_0}$ on $\overline{B_{g_0}(x_0,r)} \times [0,T]$ such that*

$$\sup_{x\in B_{g_0}(x_0,r)}(|h(x,t)| + |A(x,t)|) \le Pt. \tag{3.2}$$

Proof. Just define

$$\tilde{P}(r) \doteqdot \sup_{\overline{B_{g_0}(x_0,r)}\times[0,T]}\left(|R|_{g_0} + |\widetilde{R}|_{g_0} + |g^{-1}|_{g_0}|\nabla\,\mathrm{Rc}\,|_{g_0} + |\tilde{g}^{-1}|_{g_0}|\widetilde{\nabla}\widetilde{\mathrm{Rc}}|_{g_0}\right).$$

Then, by (1.2), for any $r > 0$ and $x \in B_{g_0}(x_0, r)$, we have $|\frac{\partial h}{\partial t}|_{g_0} \le 2\sqrt{n}\tilde{P}(r)$ and $|\frac{\partial A}{\partial t}|_{g_0} \le 3\tilde{P}(r)$ and $A(x,0) = 0$, so $|h(x,t)| \le \gamma|h(x,t)|_{g_0} \le 2\sqrt{n}\gamma\tilde{P}t$, and $|A(x,t)| \le \gamma^{3/2}|A(x,t)|_{g_0} \le 3\gamma^{3/2}\tilde{P}t$ on $B_{g_0}(x_0,r)$. $\square$

3.2. Evolution inequalities for h, A and S revisited

We next organize the inequalities satisfied by the time-derivatives of $t^{-\alpha}|h|^2$, $t^{-\beta}|A|^2$ and $|S|^2$ so that the coefficients $|R|$, $|\widetilde{R}|$, $|\nabla R|$ and $|\widetilde{\nabla}\widetilde{R}|$ are distributed across the totality of terms in a way that their growth can be adequately absorbed. Going forward, we will write

$$\rho(x) \doteqdot r_0^2(x) + 1.$$

Lemma 11. *For any solutions $g(t)$, $\tilde{g}(t)$ to the Ricci flow on $M \times [0,T]$ and any α, β, δ, $\sigma \in \mathbb{R}$, there exists a constant $C = C(n)$ such that, on $M \times (0,T]$,*

$$\frac{\partial}{\partial t}\left(\frac{\rho^2}{t^\alpha}|h|^2\right) \leq \frac{C\rho^2}{t^\alpha}\left(|R| + \frac{\rho}{t^\sigma} - \frac{\alpha}{t}\right)|h|^2 + \frac{C\rho}{t^{\alpha-\sigma}}|S|^2, \tag{3.3}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\rho}{t^\beta}|A|^2\right) \leq \frac{C\rho}{t^\beta}&\left(\left(|\tilde{g}^{-1}||\widetilde{R}| + |R|\right) + \frac{\rho}{t^\delta}|\tilde{g}^{-1}|^2 + \frac{\rho}{t^\beta} - \frac{\beta}{t}\right)|A|^2 \\ &+ \frac{C}{t^{\beta-\delta}}|\widetilde{\nabla}\widetilde{R}|^2|h|^2 + \frac{1}{4}|\nabla S|^2,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\left\langle \frac{\partial S}{\partial t} - \Delta S - \operatorname{div} U, S\right\rangle \leq\ & C|\tilde{g}^{-1}|^2|\widetilde{\nabla}\widetilde{R}|^{4/3}|A|^2 + C|\tilde{g}^{-1}|^2|\widetilde{R}|^3|h|^2 \\ &+ C\left(|\widetilde{\nabla}\widetilde{R}|^{2/3} + \left(|R| + |\widetilde{R}|\right)\right)|S|^2,\end{aligned} \tag{3.5}$$

and

$$|U|^2 \leq C\left(|\tilde{g}^{-1}|^2|\widetilde{\nabla}\widetilde{R}|^2|h|^2 + |\widetilde{R}|^2|A|^2\right), \tag{3.6}$$

where U is as in (1.9).

Proof. Since $\frac{\partial}{\partial t}|h|^2 \leq C|R||h|^2 + 2\langle\frac{\partial h}{\partial t}, h\rangle$, we find that

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\rho^2}{t^\alpha}|h|^2\right) &\leq \frac{C\rho^2}{t^\alpha}\left(|R| - \frac{\alpha}{t}\right)|h|^2 + \frac{2\rho^2}{t^\alpha}\left\langle\frac{\partial h}{\partial t}, h\right\rangle \\ &\leq \frac{C\rho^2}{t^\alpha}\left(|R| - \frac{\alpha}{t}\right)|h|^2 + \frac{C\rho^2}{t^\alpha}|S||h| \\ &\leq \frac{C\rho^2}{t^\alpha}\left(|R| + \frac{\rho}{t^\sigma} - \frac{\alpha}{t}\right)|h|^2 + \frac{C\rho}{t^{\alpha-\sigma}}|S|^2\end{aligned}$$

where we have estimated

$$\frac{\rho^2}{t^\alpha}|S||h| \leq \frac{C\rho}{t^\alpha}\left(\frac{\rho^2}{t^\sigma}|h|^2 + t^\sigma|S|^2\right).$$

Likewise, from (1.7), we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\rho}{t^\beta}|A|^2\right) \leq \frac{C\rho}{t^\beta}\left(|R| - \frac{\beta}{t}\right)|A|^2 + |A|\frac{C\rho}{t^\beta}\Big(&|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{R}||h| \\ &+ |\widetilde{R}||A| + |\nabla S|\Big), \\ \leq \frac{C\rho}{t^\beta}&\left(\left(|R| + |\widetilde{R}|\right) + \frac{\rho}{t^\delta}|\tilde{g}^{-1}|^2 + \frac{\rho}{t^\beta} - \frac{\beta}{t}\right)|A|^2 \\ &+ \frac{C}{t^{\beta-\delta}}|\widetilde{\nabla}\widetilde{R}|^2|h|^2 + \frac{1}{4}|\nabla S|^2,\end{aligned}$$

where we have used

$$\frac{\rho}{t^\beta}|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{R}||h||A| \leq \frac{C}{t^\beta}\left(\frac{\rho^2}{t^\delta}|\tilde{g}^{-1}|^2|A|^2 + t^\delta|\widetilde{\nabla}\widetilde{R}|^2|h|^2\right),$$

and

$$\frac{C\rho}{t^\beta}|A||\nabla S| \leq \frac{C\rho^2}{t^{2\beta}}|A|^2 + \frac{1}{4}|\nabla S|^2.$$

Next, from (1.8), we have

$$\begin{aligned}&\left\langle \frac{\partial S}{\partial t} - \Delta S - \operatorname{div} U, S\right\rangle \\ &\quad \leq C|S|\Big(|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{R}||A| + |\tilde{g}^{-1}||\widetilde{R}|^2|h| \\ &\qquad + \left(|R| + |\widetilde{R}|\right)|S|\Big) \leq C\left(|R| + |\widetilde{R}| + |\widetilde{\nabla}\widetilde{R}|^{2/3}\right)|S|^2 \\ &\qquad + C|\tilde{g}^{-1}|^2|\widetilde{\nabla}\widetilde{R}|^{4/3}|A|^2 + C|\widetilde{R}|^3|h|^2,\end{aligned}$$

since

$$|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{R}||A||S| \leq C\left(|\tilde{g}^{-1}|^2|\widetilde{\nabla}\widetilde{R}|^{4/3}|A|^2 + |\widetilde{\nabla}\widetilde{R}|^{2/3}|S|^2\right),$$

and

$$|\tilde{g}^{-1}||\widetilde{R}|^2|h||S| \leq C\left(|\tilde{g}^{-1}|^2|\widetilde{R}|^3|h|^2 + |\widetilde{R}||S|^2\right).$$

Finally, from (1.10), we have $|U|^2 \leq C\left(|g^{-1}|^2|\widetilde{\nabla}\widetilde{R}|^2|h|^2 + |\widetilde{R}|^2|A|^2\right)$. □

We now specialize to the setting of Theorem 2 (except that we continue to permit $\delta \in [0,1]$) and define

$$\alpha \doteqdot (3+\delta)/2, \quad \beta \doteqdot \alpha - 1 = (1+\delta)/2, \quad \sigma \doteqdot \alpha/2 = (3+\delta)/4, \tag{3.7}$$

(so that, in particular, $\alpha \leq 2$ and β, $\sigma \leq 1$ if $\delta \in [0,1]$). Together with the derivative estimate (3.1), we can put the above inequalities into the following simplified form.

Proposition 12. *Suppose g_0 is a complete metric satisfying the volume growth condition* (0.2), *$g(t)$ and $\tilde{g}(t)$ are solutions to* (0.1) *satisfying* (0.3) *and the curvature bound* (0.4) *for some $\delta \in [0,1]$, and α, β, σ are as in* (3.7). *Then there exists a constant $N = N(n,\gamma,\delta,K,T^*)$ such that, on $M \times (0,T]$,*

$$\frac{\partial}{\partial t}|\bar{h}|^2 \leq \frac{N\rho}{t^\sigma}\left(|\bar{h}|^2 + |S|^2\right), \tag{3.8}$$

$$\frac{\partial}{\partial t}|\bar{A}|^2 \leq \frac{N\rho}{t^\sigma}\left(|\bar{h}|^2 + |\bar{A}|^2\right) + \frac{1}{4}|\nabla S|^2, \tag{3.9}$$

$$\left\langle \frac{\partial S}{\partial t} - \Delta S - \operatorname{div} U, S\right\rangle \leq \frac{N\rho}{t^\sigma}\left(|\bar{h}|^2 + |\bar{A}|^2 + |S|^2\right), \tag{3.10}$$

and

$$|U|^2 \leq \frac{N\rho}{t^\sigma}\left(|\bar{h}|^2 + |\bar{A}|^2\right), \tag{3.11}$$

where $\bar{h} \doteqdot \rho t^{-\alpha/2} h$, $\bar{A} \doteqdot \rho^{1/2} t^{-\beta/2} A$ and $\rho(x) = r_0^2(x) + 1$ are as before.

Proof. By assumption, we have bounds on $|R|$ and $|\widetilde{R}|_{\tilde{g}}$, and from (3.1), estimates on $|\nabla R|$ and $|\widetilde{\nabla}\widetilde{R}|_{\tilde{g}}$. Since $g(t)$ and $\tilde{g}(t)$ are uniformly equivalent, we also have estimates on $|\widetilde{R}|$ and $|\widetilde{\nabla}\widetilde{R}|$, and we collect all of these estimates here with a common constant $K_1 = K_1(n,\delta,\gamma,K_0,T^*)$:

$$\sup_{M\times[0,T]} t^\delta\left(|R| + |\widetilde{R}|\right) \leq K_1\rho, \qquad \sup_{M\times[0,T]} t^{\delta+1/2}\left(|\nabla R| + |\widetilde{\nabla}\widetilde{R}|\right) \leq K_1\rho^{3/2}.$$

The assumption of uniform equivalence also implies

$$|\tilde{g}^{-1}| \leq \gamma|\tilde{g}^{-1}|_{g_0} \leq \gamma^2|\tilde{g}^{-1}|_{\tilde{g}} = \gamma^2\sqrt{n}.$$

So now it is just a matter of substituting these bounds into (3.3), (3.4), (3.5) and (3.6). In what follows we will use N to denote any constant that depends only on n, δ, γ, K_1 and T^*.

First, from (3.3), using $\sigma = (3+\delta)/4$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}|\bar{h}|^2 &\le \left(|R| + \frac{\rho}{t^{(3+\delta)/4}} - \frac{\alpha}{t}\right)|\bar{h}|^2 + \frac{\rho}{t^{\alpha-(3+\delta)/4}}|S|^2 \\
&\le \frac{\rho}{t^{(3+\delta)/4}}\left(K_1 t^{3(1-\delta)/4} + 1\right)|\bar{h}|^2 + \frac{C\rho}{t^{(3+\delta)/4}}|S|^2 \\
&\le \frac{N\rho}{t^{(3+\delta)/4}}(|\bar{h}|^2 + |S|^2)
\end{aligned}
\tag{3.12}
$$

for any $t > 0$, since $\alpha = (3+\delta)/2$ and $t^{3(1-\delta)/4} \le T^*$. Next, from (3.4), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}|\bar{A}|^2 &\le C\left(\left(|\tilde{g}^{-1}||\widetilde{R}| + |R|\right) + \frac{\rho}{t^\delta}|\tilde{g}^{-1}|^2 + \frac{\rho}{t^\beta} - \frac{\beta}{t}\right)|\bar{A}|^2 \\
&\quad + \frac{Ct^{\alpha-\beta+\delta}}{\rho^2}|\widetilde{\nabla}\widetilde{R}|^2|\bar{h}|^2 + \frac{1}{4}|\nabla S|^2 \\
&\le \frac{N\rho}{t^{(1+\delta)/2}}\left(1 + t^{(1-\delta)/2}\right)|\bar{A}|^2 + \frac{N\rho}{t^\delta}|\bar{h}|^2 + \frac{1}{4}|\nabla S|^2 \\
&\le \frac{N\rho}{t^{(1+\delta)/2}}\left(|\bar{h}|^2 + |\bar{A}|^2\right) + \frac{1}{4}|\nabla S|^2,
\end{aligned}
\tag{3.13}
$$

since $\beta - \delta = (1-\delta)/2 \ge 0$, and $\alpha - \beta + \delta - (1+2\delta) = -\delta$. Similarly,

$$
\begin{aligned}
&\left\langle \frac{\partial S}{\partial t} - \Delta S - \operatorname{div} U, S\right\rangle \\
&\le C\left\{|\tilde{g}^{-1}|^2|\widetilde{\nabla}\widetilde{R}|^{4/3}|A|^2 + |\tilde{g}^{-1}|^2|\widetilde{R}|^3|h|^2 + \left(|\widetilde{\nabla}\widetilde{R}|^{2/3} + \left(|R| + |\widetilde{R}|\right)\right)|S|^2\right\} \\
&\le \frac{N\rho}{t^{(2+4\delta)/3-\beta}}|\bar{A}|^2 + \frac{N\rho}{t^{3\delta-\alpha}}|\bar{h}|^2 + N\rho\left(\frac{1}{t^{(1+2\delta)/3}} + \frac{1}{t^\delta}\right)|S|^2 \\
&\le \frac{N\rho}{t^{(1+5\delta)/6}}|\bar{A}|^2 + \frac{N\rho}{t^{(5\delta-3)/2}}|\bar{h}|^2 + \frac{N\rho}{t^{(1+2\delta)/3}}\left(1 + t^{(1-\delta)/3}\right)|S|^2 \\
&\le \frac{N\rho}{t^{(1+2\delta)/3}}\left(|\bar{h}|^2 + |\bar{A}|^2 + |S|^2\right)
\end{aligned}
\tag{3.14}
$$

since $(2+4\delta)/3 - \beta = (1+5\delta)/6$, $3\delta - \alpha = (5\delta-3)/2$, and

$$
\frac{5\delta - 3}{2} \le \frac{1+5\delta}{6} \le \frac{1+2\delta}{3},
$$

when $\delta \in [0,1]$.

Finally, from (3.6), we have

(3.15)
$$
\begin{aligned}
|U|^2 &\leq C\left(|\tilde{g}^{-1}|^2|\widetilde{\nabla}\widetilde{R}|^2|h|^2 + |\widetilde{R}|^2|A|^2\right) \leq N\rho\left(\frac{1}{t^{1+2\delta-\alpha}}|\bar{h}|^2 + \frac{1}{t^{2\delta-\beta}}|\bar{A}|^2\right)\\
&\leq \frac{N\rho}{t^{(3\delta-1)/2}}\left(|\bar{h}|^2 + |\bar{A}|^2\right),
\end{aligned}
$$

since $1 + 2\delta - \alpha = 2\delta - \beta = (3\delta - 1)/2$. Since $\sigma = (3 + \delta)/4$ is the largest exponent of $1/t$ to appear in the coefficients of (3.12), (3.13), (3.14) and (3.15), we obtain (3.8), (3.9), (3.10) and (3.11). □

3.3. A remark on the general strategy

At this point, we could multiply $|\bar{h}|^2$, $|\bar{A}|^2$ and $|S|^2$ by suitable cutoff and decay functions and introduce localized versions of the integral quantities $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ from the last section in order to argue as before. However, the argument we will use is only dependent on the structure of the system of inequalities in Proposition 12 and this structure is not really specific to the Ricci flow.

The guiding principle behind our efforts thus far has been that, while g and $\tilde{g}$ themselves do not satisfy strictly parabolic equations, the individual curvature tensors R and $\widetilde{R}$ do, and do so with respect to elliptic operators Δ and $\widetilde{\Delta}$ whose coefficients, respectively, depend on g and $\tilde{g}$ and their derivatives (up to second-order). Since the difference of g and $\tilde{g}$ (and those of their derivatives) can be controlled in an essentially ordinary-differential way by the difference of R and $\widetilde{R}$ and those of their derivatives, by prolonging the system for $S = R - \widetilde{R}$ to include just as many of the differences of g and $\tilde{g}$ and their derivatives as are needed to control $\Delta - \widetilde{\Delta}$, we can hope to obtain a closed and nearly parabolic system of inequalities. As we have seen (thanks to an integration by parts) we need only to add h and A to obtain a system for which uniqueness can be established much as for strictly parabolic systems.

Thus the strategy is, (1) to begin with the system of equations satisfied by the difference $\mathrm{Rm} - \widetilde{\mathrm{Rm}}$ of curvature tensors, (2) to prolong the system by lower order quantities whose vanishing is logically redundant for the purposes of establishing uniqueness, but whose inclusion allows us to account for the lack of a common elliptic operator in the separate parabolic equations satisfied by Rm and $\widetilde{\mathrm{Rm}}$, and, (3) to apply the energy method to the mixed (but, in practice, nearly parabolic) system of inequalities satisfied by the aggregate of the curvature and lower-order quantities. This strategy can be used to encode uniqueness problems for other geometric evolution equations, such as the mean curvature flow, into those for similar systems of inequalities, and thus we formulate a somewhat more general uniqueness result than is necessary to prove Theorem 2 in the chance that it might be of some independent interest.

This approach to handling gauge-degeneracies in evolution equations involving curvature, is similar to that employed in [K1, K2], and has its origins in the work of Alexakis [A] on unique-continuation for the vacuum Einstein equations. (See also [WY].)

3.4. Reduction to system of mixed differential inequalities

For the application of the result of the next section to the situation of Theorem 2, we'll take (in the notation of that section) $\mathcal{X} = T^1_3(M)$, $\mathcal{Y} = T^0_2(M) \oplus T^1_2(M)$ and $X(t) = S(t) \in \mathcal{X}$, $Y(t) = \bar{h}(t) \oplus \bar{A}(t) \in \mathcal{Y}$, and take one of the metrics, $g(t)$, as our family of reference metrics. Note that, by assumption, X is smooth on $M \times [0, T]$ and satisfies $X(x, 0) \equiv 0$ and

$$|X(x,t)|^2 \leq K_1 t^{-2\delta}(r_0^2(x) + 1)^2 \leq N t^{-\sigma'} e^{N r_0^2(x)}$$

on $M \times (0, T]$ for an appropriate constant N where

$$\sigma' \doteqdot \max\{(3 + \delta)/4, 2\delta\} < 1.$$

(It is here that we need $\delta < 1/2$, as opposed to $\delta < 1$.) The family of sections $Y(t)$ is smooth in t and Lipschitz over M (smooth but for the factors of ρ) for $t > 0$, and, as noted in Lemma 10, satisfies

$$|Y(x,t)|^2 = t^{-\alpha}\rho^2(x)|h(x,t)|^2 + t^{-\beta}\rho|A(x,t)|^2 \leq t^{2-\alpha}(r^2 + 1)^2 P(r)$$

on $B_{g_0}(x_0, r) \times [0, T]$ for any $r > 0$. Thus $|Y(x, t)|$ tends to zero uniformly as $t \searrow 0$ on any compact set. (We will not need any assumptions on the behavior of $Y(t)$ at spatial infinity.) In terms of X and Y (and the norms on $\mathcal{X}$ and $\mathcal{Y}$ induced by $g(t)$), Proposition 12 implies

$$\left\langle \frac{\partial X}{\partial t} - \Delta X - \operatorname{div} U, X \right\rangle \leq \frac{N\rho}{t^{\sigma'}}(|X|^2 + |Y|^2),$$
$$\left\langle \frac{\partial Y}{\partial t}, Y \right\rangle \leq \frac{1}{4}|\nabla X|^2 + \frac{N\rho}{t^{\sigma'}}(|X|^2 + |Y|^2),$$

on $M \times (0,T]$, with $|U|^2 \leq (N\rho/t^{\sigma'})(|X|^2 + |Y|^2)$. Note that, although equations (3.8) and (3.9) only directly imply an inequality on $\frac{\partial}{\partial t}|Y|^2$, in our derivation of these inequalities, we immediately estimated the contribution of the time-derivative of $|\cdot| = |\cdot|_{g(t)}$ from above by terms proportional to $|R||Y|^2 \leq CK_1(\rho/t^{\delta})|Y|^2 \leq N(\rho/t^{\sigma'})|Y|^2$, so we in fact also have the inequalities in the above weaker (although somewhat more symmetric) form. Theorem 2 now follows at once from Theorem 13 below.

4. A uniqueness theorem for systems of virtually parabolic differential inequalities

Let (M, g_0) be a complete Riemannian manifold, satisfying

$$\mathrm{vol}(B_{g_0}(x_0, r)) \leq A_0 e^{A_0 r^2}, \tag{4.1}$$

for some $x_0 \in M$, constant A_0, and all $r > 0$. Define $r_0(x) \doteqdot \mathrm{dist}_{g_0}(x_0, x)$ and $\rho(x) \doteqdot r_0^2(x) + 1$ as before. Suppose that $g(t)$ is a smooth family of metrics on $M \times [0,T]$ such that, writing $\frac{\partial}{\partial t} g_{ij} = -2p_{ij}$, the conditions

$$\gamma^{-1} g_0 \leq g(t) \leq \gamma g_0, \quad \text{and} \quad t^{\sigma}|p(x,t)| \leq N_0 \rho(x) \tag{4.2}$$

are satisfied for some constants γ, $\sigma \in (0,1)$, and N_0, where $|\cdot| \doteqdot |\cdot|_{g(t)}$ as before.

Now let $\mathcal{X} = \bigoplus_{i=1}^{r} T^{k_i}_{l_i}(M)$ and $\mathcal{Y} = \bigoplus_{i=1}^{r'} T^{k_i'}_{l_i'}(M)$ represent tensor bundles over M equipped with the metrics and connections induced by $g(t)$ and $\nabla(t)$, the Levi-Civita connection of $g(t)$.

Theorem 13. *For any choice of a, $\sigma \in (0,1)$, $\gamma > 0$, and non-negative constants A_0, A_1, N_0 and N_1, there exists $T_0 = T_0(n, \gamma, \sigma, a, A_0, A_1, N_0, N_1) > 0$, such that, whenever $g(t)$ is a smooth family of metrics on $M \times [0, T_0]$ satisfying* (4.1) *and* (4.2), *and $X(t) \in C^{\infty}(\mathcal{X})$, $Y(t) \in C(\mathcal{Y})$ are families of sections depending smoothly on $t \in (0, T_0]$ satisfying the initial conditions*

$$\lim_{t \searrow 0} \sup_{x \in \Omega} |X(x,t)| = 0, \quad \lim_{t \searrow 0} \sup_{x \in \Omega} |Y(x,t)| = 0 \tag{4.3}$$

on every compact $\Omega \subset M$, the growth bound

$$t^{\sigma}|X(x,t)|^2 \leq A_1 e^{A_1 r_0^2(x)}, \tag{4.4}$$

on $M \times (0, T_0]$, *and the system of inequalities*

$$\left\langle \frac{\partial X}{\partial t} - \Delta X - \operatorname{div} U, X \right\rangle \leq \frac{a}{2}|\nabla X|^2 + \frac{N_1 \rho}{t^\sigma}\left(|X|^2 + |Y|^2\right), \tag{4.5}$$
$$\left\langle \frac{\partial Y}{\partial t}, Y \right\rangle \leq \frac{a}{2}|\nabla X|^2 + \frac{N_1 \rho}{t^\sigma}\left(|X|^2 + |Y|^2\right),$$

on $M \times (0, T_0]$ *for* $U(t) \in C^\infty(TM \otimes \mathcal{X})$ *satisfying*

$$|U|^2 \leq \frac{N_1 \rho}{t^\sigma}\left(|X|^2 + |Y|^2\right), \tag{4.6}$$

then $X(t) \equiv 0$, $Y(t) \equiv 0$ *for all* $t \in (0, T_0]$.

Remark 14. *Here, by* $\operatorname{div} U = \operatorname{div}_{g(t)} U$ *we mean the section of* $\mathcal{X}$ *whose value in the fiber over* $(x, t) \in M \times (0, T_0]$ *is* $\sum_{i=1}^n \nabla_{e_i} U(e_i, \cdot)$ *for a* $g(t)$*-orthonormal basis* $\{e_i\}_{i=1}^n$ *of* $T_x M$.

Remark 15. *With a simple modification of the proof below (and an appropriate reduction of the constant* a *in the statement) one can substitute for the operator* $\Delta = \Delta_{g(t)}$ *in Theorem 13 any elliptic operator of the form* $\mathcal{L} = \Lambda^{ij}\nabla_i\nabla_j$ *where* $\Lambda(t) \in C^\infty(T_0^2 M)$ *satisfies* $\lambda^{-1} g^{ij}(x,t) \leq \Lambda^{ij}(x,t) \leq \lambda g^{ij}(x,t)$ *on* $M \times (0, T_0]$ *with*

$$|\nabla \Lambda|^2 \leq N\rho/t^\sigma$$

for some constants λ *and* N.

Proof. Again, we'll assume that M is non-compact, as the argument in the compact case is very similar and less involved. For the time being we will take $0 < T_0 \leq T$ to be a small constant to be determined later. We begin by introducing a suitable cutoff function. Choose a nonincreasing $\psi \in C^\infty(\mathbb{R}, [0,1])$ satisfying

$$\begin{cases} \psi \equiv 1 & \text{on} \quad (-\infty, 1/2] \\ \psi \equiv 0 & \text{on} \quad [1, \infty) \end{cases}$$

and $(\psi')^2 \leq C\psi$. The function $\phi_r : M \to [0,1]$ defined by $\phi_r(x) = \psi(r_0(x)/r)$ then satisfies

$$\phi_r \equiv 1 \quad \text{on} \quad B_{g_0}(x_0, r/2), \quad \phi_r \equiv 0 \quad \text{on} \quad M \setminus B_{g_0}(x_0, r),$$

and is Lipschitz (smooth off of the g_0-cut locus of x_0). On account of the uniform equivalence of $g(t)$ with g_0, we have

$$|\nabla\phi_r|^2 \leq C\gamma r^{-2}\phi_r$$

off of a $d\mu_{g_0}-$ (hence $d\mu_{g(t)}-$) set of measure zero in $B_{g_0}(x_0, r) \times [0, T_0]$.

Then, for any $r > 0$ and $t > 0$, we define

$$\begin{aligned}
\mathcal{G}_r &\doteqdot \int_M |X|^2\phi_r\,\mathrm{e}^{-\eta}\,d\mu, \quad \mathcal{H}_r \doteqdot \int_M |Y|^2\phi_r\,\mathrm{e}^{-\eta}\,d\mu,\\
\mathcal{J}_r &\doteqdot \int_M |\nabla X|^2\phi_r\,\mathrm{e}^{-\eta}\,d\mu,
\end{aligned}$$

with $\mathcal{E}_r \doteqdot \mathcal{G}_r + \mathcal{H}_r$, where $\eta = \eta_{B,T_0}$ is as in Lemma 5 with $B = B(n, a, \gamma) > 0$ taken small enough to ensure that

$$\frac{\partial\eta}{\partial t} - \frac{5-2a}{2(1-a)}|\nabla\eta|^2 \geq 0 \tag{4.7}$$

on $M \times [0, T_0]$. As noted in that lemma, this can be achieved independently of our choice of T_0, and is not affected by a further reduction of T_0. Below we will continue to use $C = C(n)$ to denote a universal constant and N any constant which depends at most on n, the ranks (k_i, l_i), (k_i', l_i') from the definitions of $\mathcal{X}$ and $\mathcal{Y}$, and the constants a, γ, σ, A_0, A_1, N_0 and N_1. For convenience, we'll write $\theta \doteqdot \theta(x,t) \doteqdot \rho(x)/t^\sigma$.

Now we compute the evolution equations for $\mathcal{G}_r$ and $\mathcal{H}_r$. First, since $\frac{\partial}{\partial t}d\mu = -g^{ij}P_{ij}d\mu$, taking into account the time-dependency of $d\mu$ and the norms $|\cdot|$, it follows from (4.2) and (4.5) that

(4.8)
$$\begin{aligned}
\mathcal{G}_r'(t) &\leq \int_M \left(N\theta|X|^2 + 2\left\langle\frac{\partial X}{\partial t}, X\right\rangle - \frac{\partial\eta}{\partial t}|X|^2\right)\phi_r\,\mathrm{e}^{-\eta}\,d\mu\\
&\leq \frac{N(r^2+1)}{t^\sigma}\mathcal{G}_r + \int_M\left(2\left\langle\Delta X + \operatorname{div}U, X\right\rangle - \frac{\partial\eta}{\partial t}|X|^2\right)\phi_r\,\mathrm{e}^{-\eta}\,d\mu\\
&\quad + \int_M\left(2N_1\theta(|X|^2+|Y|^2) + a|\nabla X|^2\right)\phi_r\,\mathrm{e}^{-\eta}\,d\mu\\
&\leq a\mathcal{J}_r + \frac{N(r^2+1)}{t^\sigma}(\mathcal{G}_r + \mathcal{H}_r) - \int_M\frac{\partial\eta}{\partial t}|X|^2\phi_r\,\mathrm{e}^{-\eta}\,d\mu\\
&\quad + 2\int_M\left\langle\Delta X + \operatorname{div}U, X\right\rangle\phi_r\,\mathrm{e}^{-\eta}\,d\mu
\end{aligned}$$

on $M \times (0, T_0]$. Integrating by parts in the last term in (4.8), we find that

$$\begin{aligned}2\int_M &\langle \Delta X + \operatorname{div} U, X\rangle\, \phi_r\, \mathrm{e}^{-\eta}\, d\mu \\ &\le -2\mathcal{J}_r + 2\int_M \Big(|\nabla X||U|\phi_r + (|\nabla X||X| + |U||X|)(|\nabla \eta|\phi_r + |\nabla \phi_r|)\Big)\, \mathrm{e}^{-\eta}\, d\mu,\end{aligned}$$

and, where $\phi_r > 0$, we can estimate

$$\begin{aligned}2|\nabla X|&(|U|\phi_r + |X||\nabla\eta|\phi_r + |X||\nabla\phi_r|) \\ &\le 2(1-a)|\nabla X|^2\phi_r + \frac{3}{2(1-a)}|U|^2\phi_r + \frac{3}{2(1-a)}\left(|\nabla\eta|^2\phi_r + \frac{|\nabla\phi_r|^2}{\phi_r}\right)|X|^2,\end{aligned}$$

and

$$2|U||X|(|\nabla\eta|\phi_r + |\nabla\phi_r|) \le 2|U|^2\phi_r + \left(|\nabla\eta|^2\phi_r + \frac{|\nabla\phi_r|^2}{\phi_r}\right)|X|^2.$$

So, using (4.6) and (4.7), we have

$$\begin{aligned}\int_M &\left(2\langle \Delta X + \operatorname{div} U, X\rangle - \frac{\partial\eta}{\partial t}\right)\phi_r\, \mathrm{e}^{-\eta}\, d\mu \\ &\le -2a\mathcal{J}_r + N\int_M \left(|U|^2 + \frac{5-2a}{2(1-a)}|\nabla\eta|^2 - \frac{\partial\eta}{\partial t}\right)\phi_r\, \mathrm{e}^{-\eta} d\mu \\ &\qquad + N\int_{\operatorname{supp}\phi_r} |X|^2 \frac{|\nabla\phi_r|^2}{\phi_r}\, \mathrm{e}^{-\eta} d\mu \\ &\le -2a\mathcal{J}_r + \frac{N(r^2+1)}{t^\sigma}(\mathcal{G}_r + \mathcal{H}_r) + \frac{N}{r^2}\int_{\operatorname{supp}|\nabla\phi_r|} |X|^2\, \mathrm{e}^{-\eta} d\mu, \qquad (4.9)\end{aligned}$$

and thus, combining (4.8) and (4.9), that, for any $0 < t \le T_0$,

$$\mathcal{G}_r' \le -a\mathcal{J}_r + \frac{N(r^2+1)}{t^\sigma}\mathcal{E}_r + \frac{N}{r^2}\int_{A(x_0,r,r/2)} |X|^2\, \mathrm{e}^{-\eta} d\mu, \qquad (4.10)$$

where $A(x_0, r, r/2) \doteqdot B_{g(0)}(x_0, r) \setminus B_{g(0)}(x_0, r/2)$.

Now we examine the last term in (4.10). Since the metric $g(t)$ is uniformly equivalent to g_0, we have

$$\operatorname{vol}_{g(t)}(A(x_0, r, r/2)) \le \operatorname{vol}_{g_0}(B_{g_0}(x_0, r)) \le \gamma^{n/2} A_0 e^{A_0 r^2},$$

by the volume growth assumption on g_0. Since we also assume that the integrand $|X|^2$ satisfies the similar growth bound (4.4), by choosing $T_0' =$

$T_0'(n,\gamma,A_0,A_1,B)$ sufficiently small, we can arrange that

$$\int_{A(x_0,r,r/2)} |X|^2\,\mathrm{e}^{-\eta} d\mu \leq \frac{N}{t^\sigma}\mathrm{e}^{-\frac{\epsilon r^2}{T_0}},$$

for some $\epsilon = \epsilon(A_0, A_1, B) > 0$, provided $T_0 \leq T_0'$. So we have

$$\mathcal{G}_r'(t) \leq \frac{N(r^2+1)}{t^\sigma}(\mathcal{G}_r + \mathcal{H}_r) - a\mathcal{J}_r + \frac{N}{t^\sigma r^2}\mathrm{e}^{-\frac{\epsilon r^2}{T_0}} \tag{4.11}$$

for any $r > 0$ and $t \in (0, T_0]$, if $T_0 \leq T_0'$.

Similarly, by (4.5), we compute (using here only that $\frac{\partial\eta}{\partial t} \geq 0$) that

(4.12)
$$\begin{aligned}
\mathcal{H}_r'(t) &\leq \int_M \left(N\theta|Y|^2 + 2\left\langle\frac{\partial Y}{\partial t}, Y\right\rangle - \frac{\partial\eta}{\partial t}|Y|^2\right)\phi_r\,\mathrm{e}^{-\eta}\,d\mu\\
&\leq \frac{N(r^2+1)}{t^\sigma}\mathcal{H}_r + \int_M \left(2N_1\theta(|X|^2+|Y|^2) + a|\nabla X|^2\right)\phi_r\,\mathrm{e}^{-\eta}\,d\mu\\
&\leq a\mathcal{J}_r + \frac{N(r^2+1)}{t^\sigma}(\mathcal{G}_r + \mathcal{H}_r).
\end{aligned}$$

Combining (4.11) and (4.12), we conclude that, for all $r > 0$ and $t \in (0, T_0]$,

$$\mathcal{E}_r'(t) \leq \frac{N(r^2+1)}{t^\sigma}\mathcal{E}_r(t) + \frac{N}{t^\sigma r^2}\,\mathrm{e}^{-\frac{\epsilon r^2}{T_0}}.$$

provided $T_0 \leq T_0'$. It follows then, that for any $0 < t_0 < t \leq T_0 \leq T_0'$, we have

$$e^{-Q(r)t^{1-\sigma}}\mathcal{E}_r(t) - e^{-Q(r)t_0^{1-\sigma}}\mathcal{E}_r(t_0) \leq \frac{\mathrm{e}^{-\frac{\epsilon r^2}{T_0}}}{r^2(r^2+1)}\left(\mathrm{e}^{-Q(r)t_0^{1-\sigma}} - \mathrm{e}^{-Q(r)t^{1-\sigma}}\right),$$

where $Q(r) \doteqdot N(r^2+1)/(1-\sigma)$.

Now, since X and Y tend to zero uniformly on any compact set, we have $\lim_{t_0\searrow 0}\mathcal{E}_r(t_0) = 0$ for any fixed r. Therefore, sending $t_0 \searrow 0$, we obtain

$$\mathcal{E}_r(t) \leq \frac{\mathrm{e}^{-\frac{\epsilon r^2}{T_0}}}{r^2}\left(\mathrm{e}^{Q(r)t^{1-\sigma}} - 1\right) \leq \frac{\mathrm{e}^{\frac{NT_0^{1-\sigma}}{1-\sigma}}}{r^2}\mathrm{e}^{-\left(\frac{\epsilon}{T_0} - \frac{NT_0^{1-\sigma}}{1-\sigma}\right)r^2}.$$

If we choose T_0 smaller still, say $T_0 \leq \min\{T_0', (\epsilon(1-\sigma)/(2N))^{1/(2-\sigma)}\}$, the above inequality implies

$$\mathcal{E}_r(t) \leq \frac{\mathrm{e}^{\frac{NT_0^{1-\sigma}}{1-\sigma}}\,\mathrm{e}^{-\frac{\epsilon r^2}{2T_0}}}{r^2} \leq \frac{N}{r^2}\mathrm{e}^{-\frac{\epsilon r^2}{2T_0}}$$

for all $r > 0$ and $0 < t \leq T_0$. Fixing t in this range and sending $r \to \infty$ then finishes the argument. $\square$

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