

Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles

CHIN-YU HSIAO AND GEORGE MARINESCU

In this paper we study the asymptotic behaviour of the spectral function corresponding to the lower part of the spectrum of the Kodaira Laplacian on high tensor powers of a holomorphic line bundle. This implies a full asymptotic expansion of this function on the set where the curvature of the line bundle is non-degenerate. As application we obtain the Bergman kernel asymptotics for adjoint semi-positive line bundles over complete Kähler manifolds, on the set where the curvature is positive. We also prove the asymptotics for big line bundles endowed with singular Hermitian metrics with strictly positive curvature current. In this case, the full asymptotics holds outside the singular locus of the metric.

1. Introduction and statement of the main results	3
1.1. Statement of main results	5
2. Preliminaries	16
2.1. Some standard notations	16
2.2. Metric data	17
2.3. A self-adjoint extension of the Kodaira Laplacian	19
2.4. Schwartz kernel theorem	20
3. Szegő kernels and semi-classical Hodge decomposition	20
3.1. The Grauert tube	21
3.2. Approximate Szegő kernels	25

3.3. Semi-classical Hodge decomposition for the Kodaira Laplacian	34
4. Asymptotic expansion of the spectral function for lower energy forms	45
4.1. Asymptotic upper bounds	46
4.2. Kernel of the spectral function	49
4.3. Asymptotic expansion of the spectral function. Proof of Theorem 1.1	61
4.4. Asymptotic expansion of the Bergman kernel. Proof of Theorem 1.6	64
4.5. Calculation of the leading coefficients. Proof of Theorem 1.2	66
5. Asymptotic upper bounds near the degeneracy set	72
6. Bergman kernel asymptotic expansion for adjoint semi-positive line bundles	77
7. Singular L^2-estimates	82
8. Bergman kernel asymptotics for semi-positive line bundles	86
9. Multiplier ideal Bergman kernel asymptotics. Proof of Theorem 1.8	88
10. Further applications	91
10.1. Existence of “small” eigenvalues of the Kodaira Laplacian	91
10.2. Bouche integral condition	94
10.3. Asymptotics for arbitrary semi-positive metrics on ample line bundles	94

10.4. Expansion for Bergman kernel on forms	95
10.5. Holomorphic Morse inequalities	96
10.6. Tian’s theorem and equidistribution of zeros	99
Acknowledgments	102
References	103

1. Introduction and statement of the main results

Let L be a holomorphic line bundle over a complex manifold M and let L^k be the k -th tensor power of L . The Bergman kernel is the smooth kernel of the orthogonal projection onto the space of L^2 -integrable holomorphic sections of L^k . The study of the large k behaviour of the Bergman kernel is an active research subject in complex geometry and is closely related to topics like the structure of algebraic manifolds (e.g., [23, 58]), the existence of canonical Kähler metrics (e.g., [13–15, 27, 31, 32, 63, 64]), Berezin–Toeplitz quantization (e.g., [6, 30, 51, 52]), equidistribution of zeros of holomorphic sections (e.g., [16, 25, 59, 60]), quantum chaos and mathematical physics [29]. We refer the reader to the book [49] for a comprehensive study of the Bergman kernel and its applications and also to the survey [46].

In the case of a positive line bundle L over a compact base manifold M , Catlin [12] and Zelditch [70] established the asymptotic expansion of the Bergman kernel on the diagonal by using a fundamental result by Boutet de Monvel–Sjöstrand [10] about the asymptotics of the Szegő kernel on a strictly pseudoconvex boundary. It was already pointed out by Boutet de Monvel–Guillemin [11] that the Bergman kernel of L^k is linked to the Szegő kernel on the boundary of the unit disc bundle (Grauert tube), which is strictly pseudoconvex if L is positive.

Dai, Liu and Ma [19, 20] obtained the full off-diagonal asymptotic expansion and Agmon estimates of the Bergman kernel for a high power of positive line bundle on a compact complex manifold by using the heat kernel method. Their result holds actually for the more general Bergman kernel of the spin^c Dirac operator associated to a positive line bundle on a compact symplectic manifold. In [47, 49, 50], Ma and the second-named author proved the

asymptotic expansion for yet another generalization of the Kodaira Laplacian, namely the renormalized Bochner–Laplacian on a symplectic manifold and also showed the existence of the estimate on a large class of non-compact manifolds. The main analytic tool in [19, 20, 47, 49, 50] is the analytic localization technique in local index theory developed by Bismut–Lebeau [49].

Another proof of the existence of the full asymptotic expansion for the Bergman kernel for a high power of a positive line bundle on a compact complex manifold was obtained by Berndtsson et al. [3].

A natural generalization is the asymptotic of the kernel of the projection on the harmonic forms in the case of a line bundle with non-degenerate curvature. Berman and Sjöstrand [4] obtained these asymptotics building on the heat equation method of Menikoff–Sjöstrand [55]. More generally, the expansion in the non-degenerate case was obtained independently by Ma and the second-named author [48] for the kernel of the projection on the kernel of the spin^c Dirac operator on symplectic manifolds. The asymptotics of the Szegő kernel for forms on a compact abstract CR manifold with non-degenerate Levi form were established by the first-named author [40].

For a singular Hermitian metric h^L on L with strictly positive curvature current, Ma and the second-named author used the generalized Poincaré metric on the regular locus of h^L and a modified fibre metric on L to obtain a full asymptotic expansion for the associated Bergman kernel [49, 50]. As a corollary, they could reprove the Shiffman conjecture, asserting that Moishezon manifolds can be characterized in terms of integral Kähler currents.

Witten [68] suggested that the subcomplex of eigenforms of the Witten Laplacian, corresponding to the lower part of the spectrum, is isomorphic to the Thom–Smale complex. This was first made rigorous by Helffer–Sjöstrand [37] by means of microlocal analysis. Inspired by [68], Demailly [23] used the subcomplex of eigenforms of the Kodaira Laplacian on L^k in order to prove the holomorphic Morse inequalities (see also Bismut [5]). In this paper we give the first microlocal study of the latter complex.

The first main result of this paper is a local asymptotic expansion of the spectral function of the Kodaira Laplacian on L^k on a not necessarily compact Hermitian manifold M for states of energy less than k^{-N_0} , for $N_0 \in \mathbb{N}$ fixed, on the non-degenerate locus of L , cf. Theorem 1.1. This is a very general result since it holds without global assumptions on the manifold or the line bundle. However, the estimates obtained do not apply directly to the Bergman kernel, which is obtained by formally letting $N_0 \rightarrow \infty$ in (1.7). We then impose a very mild semiclassical local condition on the Kodaira Laplacian, namely the $O(k^{-n_0})$ small spectral gap on an open set $D \Subset M$ (see Definition 1.5). We prove that the Bergman kernel admits an asymptotic

expansion on D if the Kodaira Laplacian has $O(k^{-n_0})$ small spectral gap on D , cf. Theorem 1.6.

The distinctive feature of these asymptotics is that they work under minimal hypotheses. This allows us to apply them in situations which were up to now out of reach. We illustrate this in the study of the Bergman kernel of semi-positive or positive but singular Hermitian line bundles. We prove that if M is a complete Kähler manifold and L is semi-positive on M , then the Bergman kernel of $L^k \otimes K_M$ admits a full asymptotic expansion on the non-degenerate locus of L , cf. Theorem 1.7. Moreover, we show in Theorem 1.10 that if M is any compact complex manifold and L is semi-positive and positive at some point, then the Bergman kernel of L^k admits a full asymptotic expansion on the set where L is positive, with the possible exception of a proper analytic variety $\Sigma \subset M$.

We also consider the case of a singular Hermitian fibre metric on L . The holomorphic sections which are L^2 with respect to the singular metric turn out to be sections of L twisted with a multiplier ideal sheaf. One can naturally define the orthogonal projection on this space of sections and consider its kernel on the regular locus of the metric. We show that this kernel has an asymptotic expansion on the regular locus, if the curvature current is strictly positive and smooth outside a proper analytic set (Theorem 1.8). This yields yet another proof of the Shiffman conjecture.

We further give formulas for the first top leading terms of the asymptotic expansion of the spectral function and recover the top leading coefficients of the Bergman kernel expansion. These coefficients recently attracted a lot of attention, see the comments after Theorem 1.6.

Other applications of the main results are local and global holomorphic Morse inequalities, a local form of the expansion of the Bergman kernel on forms, examples of manifolds having very small spectral gap, Tian's convergence theorem and equidistribution of zeros of holomorphic sections in the case of singular metrics.

We now formulate the main results. We refer to Section 2 for some standard notations and terminology used here.

1.1. Statement of main results

Let (M, Θ, J) be a Hermitian manifold of complex dimension n , where Θ is a smooth positive $(1, 1)$ -form and J is the complex structure. Let $g_\Theta^{TM}(\cdot, \cdot) = \Theta(\cdot, J\cdot)$ be the Riemannian metric on TM induced by Θ and J and let $\langle \cdot, \cdot \rangle$ be the Hermitian metric on $\mathbb{C}TM$ induced by g_Θ^{TM} . The Riemannian volume form dv_M of (M, Θ) has the form $dv_M = \Theta^n/n!$.

Let (L, h^L) be a holomorphic Hermitian line bundle on M and set $L^k := L^{\otimes k}$. Let ∇^L be the holomorphic Hermitian connection on (L, h^L) with curvature $R^L = (\nabla^L)^2$. We will identify the curvature form R^L with the Hermitian matrix $\dot{R}^L \in \mathcal{C}^\infty(M, \text{End}(T^{(1,0)}M))$ satisfying for every $U, V \in T_x^{(1,0)}M$, $x \in M$,

$$(1.1) \quad \langle R^L(x), U \wedge \bar{V} \rangle = \langle \dot{R}^L(x)U, V \rangle.$$

Let $\det \dot{R}^L(x) := \mu_1(x), \dots, \mu_n(x)$, where $\{\mu_j(x)\}_{j=1}^n$ are the eigenvalues of \dot{R}^L with respect to $\langle \cdot, \cdot \rangle$. For $q \in \{0, 1, \dots, n\}$, let

$$(1.2) \quad M(q) = \{x \in M; \dot{R}^L(x) \in \text{End}(T_x^{(1,0)}M) \text{ is non-degenerate} \\ \text{and has exactly } q \text{ negative eigenvalues}\}.$$

We denote by W the subbundle of rank q of $T^{(1,0)}M|_{M(q)}$ generated by the eigenvectors corresponding to negative eigenvalues of \dot{R}^L . Then $\det \bar{W}^* := \Lambda^q \bar{W}^* \subset \Lambda^q T^{*(0,1)}M|_{M(q)}$ is a rank one subbundle, where $\Lambda^q T^{*(0,1)}M$ is the bundle of $(0, q)$ forms, \bar{W}^* is the dual bundle of the complex conjugate bundle of W and $\Lambda^q \bar{W}^*$ is the vector space of all finite sums of $v_1 \wedge \dots \wedge v_q$, $v_1, \dots, v_q \in \bar{W}^*$. We denote by $I_{\det \bar{W}^*} \in \text{End}(\Lambda^q T^{*(0,1)}M)$ the orthogonal projection from $\Lambda^q T^{*(0,1)}M$ onto $\det \bar{W}^*$.

Let (L^k, h^k) be the k -th tensor power of (L, h^L) , where $h^k := (h^L)^{\otimes k}$. Let $(\cdot, \cdot)_k$ be the inner product on $\Omega_0^{0,q}(M, L^k)$ induced by g_Θ^{TM} and h^k (see (2.3)). Let $\|\cdot\|$ be the corresponding norm and let $L_{(0,q)}^2(M, L^k)$ be the completion of $\Omega_0^{0,q}(M, L^k)$ with respect to $\|\cdot\|$. For $q = 0$, we write $L^2(M, L^k) := L_{(0,0)}^2(M, L^k)$.

Let $\square_k^{(q)}$ be the Kodaira Laplacian acting on $(0, q)$ -forms with values in L^k , cf. (2.5). We denote by the same symbol $\square_k^{(q)}$ the Gaffney extension of the Kodaira Laplacian; cf. (2.8). It is well-known that $\square_k^{(q)}$ is self-adjoint and the spectrum of $\square_k^{(q)}$ is contained in $\overline{\mathbb{R}}_+$ (see [49, Proposition 3.1.2]). For a Borel set $B \subset \mathbb{R}$ we denote by $E(B)$ the spectral projection of $\square_k^{(q)}$ corresponding to the set B , where E is the spectral measure of $\square_k^{(q)}$ (see Section 2 in Davies [21]) and for $\lambda \in \mathbb{R}$ we set $E_\lambda = E((-\infty, \lambda])$ and

$$(1.3) \quad \mathcal{E}_\lambda^q(M, L^k) = \text{Range } E_\lambda \subset L_{(0,q)}^2(M, L^k).$$

If $\lambda = 0$, then $\mathcal{E}_0^q(M, L^k) = \text{Ker } \square_k^{(q)} =: \mathcal{H}^q(M, L^k)$ is the space of global harmonic sections. For a holomorphic vector bundle E over M we have

$$H_{(2)}^0(M, E) := \{s \in L^2(M, E); \bar{\partial}_E s = 0\} = \text{Ker } \square^E,$$

where $\bar{\partial}_E$ is the Cauchy–Riemann operator with values in E and \square^E is the Kodaira Laplacian with values in E (see Section 2.3). The *spectral projection* of $\square_k^{(q)}$ is the orthogonal projection

$$(1.4) \quad P_{k,\lambda}^{(q)} : L_{(0,q)}^2(M, L^k) \rightarrow \mathcal{E}_\lambda^q(M, L^k).$$

The *spectral function* $P_{k,\lambda}^{(q)}(\cdot, \cdot) = P_k^{(q)}(\cdot, \cdot, \lambda)$ is the Schwartz kernel of $P_{k,\lambda}^{(q)}$, see (4.16) and (4.17). Since $\square_k^{(q)}$ is elliptic, it is not difficult to see that

$$P_k^{(q)}(\cdot, \cdot, \lambda) \in \mathcal{C}^\infty(M \times M, L^k \otimes (\Lambda^q T^{*(0,1)} M \boxtimes \Lambda^q T^{*(0,1)} M) \otimes (L^k)^*)$$

and $\mathcal{E}_\lambda^q(M, L^k) \subset \Omega^{0,q}(M, L^k)$. Since $L_x^k \otimes (L_x^k)^* \cong \mathbb{C}$ we can identify $P_k^{(q)}(x, x, \lambda)$ to an element of $\text{End}(\Lambda^q T_x^{*(0,1)} M)$. Then

$$(1.5) \quad M \ni x \longmapsto P_k^{(q)}(x, x, \lambda) = P_{k,\lambda}^{(q)}(x, x) \in \text{End}(\Lambda^q T_x^{*(0,1)} M)$$

is a smooth section of $\text{End}(\Lambda^q T^{*(0,1)} M)$, called *local density of states* of $\mathcal{E}_\lambda^q(M, L^k)$. The trace of $P_k^{(q)}(x, x, \lambda)$ is given by

$$\text{Tr } P_{k,\lambda}^{(q)}(x, x) = \text{Tr } P_k^{(q)}(x, x, \lambda) := \sum_{j=1}^d \langle P_k^{(q)}(x, x, \lambda) e_{J_j}(x), e_{J_j}(x) \rangle,$$

where e_{J_1}, \dots, e_{J_d} is a local orthonormal basis of $\Lambda^q T^{*(0,1)} M$ with respect to $\langle \cdot, \cdot \rangle$. The projection

$$(1.6) \quad P_k^{(q)} := P_{k,0}^{(q)} : L_{(0,q)}^2(M, L^k) \rightarrow \text{Ker } \square_k^{(q)},$$

on the lowest energy level $\lambda = 0$ is called the *Bergman projection*, its kernel $P_k^{(q)}(\cdot, \cdot)$ is called the *Bergman kernel*. The restriction to the diagonal of $P_k^{(q)}(\cdot, \cdot)$ is denoted $P_k^{(q)}(\cdot)$ and is called the *Bergman kernel form*. The function $\text{Tr } P_{k,0}^{(q)}(x, x) := \text{Tr } P_k^{(q)}(x)$ is called the *Bergman kernel function*. We notice that $\text{Tr } P_k^{(0)}(x) = P_k^{(0)}(x)$.

We introduce the notion of asymptotic expansion (see Definition 3.10). Let $D \subset M$ be an open set and $a(x, k), a_j(x) \in \mathcal{C}^\infty(M, \text{End}(\Lambda^q T^{*(0,1)} M))$, $j = 0, 1, \dots$ and $m \in \mathbb{Z}$. We say that $a(x, k)$ has an asymptotic expansion

$$a(x, k) \sim \sum_{j=0}^{\infty} a_j(x) k^{m-j} \quad \text{locally uniformly on } D,$$

if for every $N \in \mathbb{N}_0$, $\ell \in \mathbb{N}_0$ and every compact set $K \subset D$, there exists a constant $C_{N,\ell,K} > 0$ independent of k , such that for k sufficiently large

$$\left| a(x, k) - \sum_{j=0}^N k^{m-j} a_j(x) \right|_{\mathcal{C}^\ell(K)} \leq C_{N,\ell,K} k^{m-N-1}.$$

We say that $a(x, k) = O(k^{-\infty})$ locally uniformly on D if $a(x, k) \sim 0$ locally uniformly on D .

The following theorem is one of the main results. It expresses the fact that the Kodaira Laplacian acting on $\Omega^{\bullet,\bullet}(M, L^k)$ admits a *local* semi-classical Hodge decomposition. Note that there are neither global assumptions on the positivity of the bundle nor on the base manifold.

Theorem 1.1. *Let (M, Θ) be a Hermitian manifold, (L, h^L) be a holomorphic Hermitian line bundle on M . Fix $q \in \{0, 1, \dots, n\}$ and $N_0 \geq 1$. Then for every $m \in \{0, 1, \dots, n\}$ there exists a k -dependent section $b^{(m)}(x, k) \in \mathcal{C}^\infty(M(q), \text{End}(\Lambda^m T^{*(0,1)} M))$ with the following properties: for every $D \Subset M(q)$, $\ell \in \mathbb{N}_0$, there exists a constant $C_{D,\ell} > 0$ independent of k with*

$$(1.7) \quad \left| P_k^{(m)}(x, x, k^{-N_0}) - b^{(m)}(x, k) \right|_{\mathcal{C}^\ell(D)} \leq C_{D,\ell} k^{3n+2\ell-N_0},$$

$b^{(m)}(x, k) = 0$ for $m \neq q$ and $b^{(q)}(x, k)$ has an asymptotic expansion

$$(1.8) \quad b^{(q)}(x, k) \sim \sum_{j=0}^{\infty} b_j^{(q)}(x) k^{n-j} \quad \text{locally uniformly on } M(q),$$

for some $b_j^{(q)} \in \mathcal{C}^\infty(M(q), \text{End}(\Lambda^q T^{*(0,1)} M))$, $j = 0, 1, 2, \dots$. On $M(q)$ we have

$$(1.9) \quad b_0^{(q)} = (2\pi)^{-n} | \det \dot{R}^L | I_{\det \bar{W}^*}.$$

We wish to give formulas for the top coefficients of the expansion in the case $q = 0$. We introduce the geometric objects used in Theorems 1.2 and 1.7

below. Put

$$(1.10) \quad \omega := \frac{\sqrt{-1}}{2\pi} R^L.$$

On the set $M(0)$ the $(1, 1)$ -form ω is positive and induces a Riemannian metric $g_\omega^{TM}(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Let ∇_ω^{TM} be the Levi–Civita connection on (M, g_ω^{TM}) , $R_\omega^{TM} = (\nabla_\omega^{TM})^2$ its curvature (cf. (4.71)), Ric its Ricci curvature and r the scalar curvature of g_ω^{TM} (see (4.69)). We denote by $\text{Ric}_\omega = \text{Ric}(J\cdot, \cdot)$ the $(1, 1)$ -form associated to Ric (cf. (4.73)) and by Δ_ω be the complex Laplacian with respect to ω (see (4.67)). We also denote by $\langle \cdot, \cdot \rangle_\omega$ the pointwise Hermitian metric induced by g_ω^{TM} on (p, q) -forms on M and by $|\cdot|_\omega$ the corresponding norm.

Let R_Θ^{\det} denote the curvature of the canonical line bundle

$$K_M = \det T^{*(1,0)}M$$

with respect to the metric induced by Θ (see (4.70)). Put

$$\hat{r} = \Delta_\omega \log V_\Theta, \quad V_\Theta = \det(\Theta_{j,k})_{j,k=1}^n,$$

where $\Theta = \sqrt{-1} \sum_{j,k=1}^n \Theta_{j,k} dz_j \wedge d\bar{z}_k$ in local holomorphic coordinates $z = (z_1, \dots, z_n)$.

Theorem 1.2. *Let (M, Θ) and (L, h^L) be as in Theorem 1.1. The coefficients $b_1^{(0)}$ and $b_2^{(0)}$ in the expansion (1.8) for $q = 0$ have the following form:*

$$(1.11) \quad \begin{aligned} b_1^{(0)} &= (2\pi)^{-n} \det \dot{R}^L \left(\frac{1}{8\pi} r - \frac{1}{4\pi} \Delta_\omega \log \det \dot{R}^L \right) \\ &= (2\pi)^{-n} \det \dot{R}^L \left(\frac{1}{4\pi} \hat{r} - \frac{1}{8\pi} r \right), \end{aligned}$$

$$(1.12) \quad \begin{aligned} b_2^{(0)} &= (2\pi)^{-n} \det \dot{R}^L \left(\frac{1}{128\pi^2} r^2 - \frac{1}{32\pi^2} r \hat{r} + \frac{1}{32\pi^2} (\hat{r})^2 - \frac{1}{32\pi^2} \Delta_\omega \hat{r} \right. \\ &\quad \left. - \frac{1}{8\pi^2} |R_\Theta^{\det}|_\omega^2 + \frac{1}{8\pi^2} \langle \text{Ric}_\omega, R_\Theta^{\det} \rangle_\omega + \frac{1}{96\pi^2} \Delta_\omega r - \frac{1}{24\pi^2} |\text{Ric}_\omega|_\omega^2 \right. \\ &\quad \left. + \frac{1}{96\pi^2} |R_\omega^{TM}|_\omega^2 \right), \end{aligned}$$

where $|R_\omega^{TM}|_\omega^2$ is given by (4.72).

On the set where the curvature of L is degenerate we have the following behaviour.

Theorem 1.3. *Let (M, Θ) and (L, h^L) be as in Theorem 1.1. Set*

$$M_{\text{deg}} = \left\{ x \in M; \dot{R}^L \text{ is degenerate at } x \in M \right\}.$$

Then for every $x_0 \in M_{\text{deg}}$, $\varepsilon > 0$ and every $m \in \{0, 1, \dots, n\}$, there exists a neighbourhood U of x_0 and $k_0 > 0$, such that for all $k \geq k_0$ we have

$$(1.13) \quad \text{Tr } P_k^{(m)}(x, x, k^{-N_0}) \leq \varepsilon k^n, \quad x \in U.$$

As a Corollary of Theorems 1.1, 1.2 and 1.3, we obtain

Corollary 1.4 (Local holomorphic Morse inequalities). *Let (M, Θ) , (L, h^L) be as in Theorem 1.1. Let $N_0 \geq 2n + 1$. Then the spectral function of the Kodaira Laplacian has the following asymptotic behaviour:*

$$(1.14) \quad \text{Tr } P_k^{(q)}(x, x, k^{-N_0}) = k^n (2\pi)^{-n} \left| \det \dot{R}^L(x) \right| + O(k^{n-1}), \quad k \rightarrow \infty,$$

locally uniformly on $M(q)$, and if $\mathbf{1}_{M(q)}$ denotes the characteristic function of $M(q)$,

$$(1.15) \quad \lim_{k \rightarrow \infty} k^{-n} \text{Tr } P_k^{(q)}(x, x, k^{-N_0}) = (2\pi)^{-n} \left| \det \dot{R}^L(x) \right| \mathbf{1}_{M(q)}(x), \quad x \in M.$$

Moreover, for every $\varepsilon > 0$, every $D \Subset M$, there exists a $k_0 > 0$, such that for all $k \geq k_0$, we have

$$(1.16) \quad \text{Tr } P_k^{(q)}(x, x, k^{-N_0}) \leq \left(\varepsilon + (2\pi)^{-n} \left| \det \dot{R}^L(x) \right| \mathbf{1}_{M(q)}(x) \right) k^n, \quad x \in D,$$

and for $q = 0$ and $N_0 \geq 2n + 3$, we have as $k \rightarrow \infty$

$$(1.17) \quad P_k^{(0)}(x, x, k^{-N_0}) \leq k^n (2\pi)^{-n} \det \dot{R}^L(x) + k^{n-1} b_1^{(0)}(x) + k^{n-2} b_2^{(0)}(x) + O(k^{n-3}),$$

locally uniformly on $M(0)$, where $b_1^{(0)}(x)$ and $b_2^{(0)}(x)$ are as in (1.11) and (1.12) respectively.

The term local holomorphic Morse inequalities is motivated by the fact that when M is compact, integration of the inequalities from Corollary 1.4 yields the holomorphic Morse inequalities of Demailly; see Section 10.5. Berman [1] proved that

$$\limsup_{k \rightarrow \infty} k^{-n} \operatorname{Tr} P_k^{(q)}(x) \leq (2\pi)^{-n} \left| \det \dot{R}^L(x) \right| \mathbb{1}_{M(q)}(x), \quad x \in M,$$

and when M is compact, there exists a sequence $\mu_k \rightarrow 0$, as $k \rightarrow \infty$, such that

$$\lim_{k \rightarrow \infty} k^{-n} \operatorname{Tr} P_k^{(q)}(x, x, \mu_k) = (2\pi)^{-n} \left| \det \dot{R}^L(x) \right| \mathbb{1}_{M(q)}(x), \quad x \in M.$$

Corollary 1.4 refines and generalizes Berman's results.

In order to obtain precise asymptotics we combine the local asymptotics from Theorem 1.1 with a mild condition on the semiclassical behaviour of the spectrum of the Kodaira Laplacian $\square_k^{(q)}$ for k large, which we call (local) $O(k^{-n_0})$ small spectral gap.

Definition 1.5. Let $D \subset M$. We say that $\square_k^{(q)}$ has $O(k^{-n_0})$ small spectral gap on D if there exist constants $C_D > 0$, $n_0 \in \mathbb{N}$, $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ and $u \in \Omega_0^{0,q}(D, L^k)$, we have

$$\left\| (I - P_k^{(q)})u \right\| \leq C_D k^{n_0} \left\| \square_k^{(q)} u \right\|.$$

To explain this condition, assume that M is a complete Hermitian manifold. Then the operator $\square_k^{(q)}$ is essentially self-adjoint and $\Omega_0^{0,q}(D, L^k)$ is dense with respect to the graph-norm in the domain of the quadratic form of $\square_k^{(q)}$ (see e.g., [49, Section 3.3]). If $\square_k^{(q)}$ has $O(k^{-n_0})$ small spectral gap on M then $\inf \left\{ \lambda \in \operatorname{Spec}(\square_k^{(q)}); \lambda \neq 0 \right\} \geq Ck^{-n_0}$, for some $n_0 \in \mathbb{N}$ and $C > 0$ independent of k .

From Theorem 1.1, Definition 1.5 and some simple arguments (see Section 4.4), we deduce:

Theorem 1.6. Let (M, Θ) be a Hermitian manifold, (L, h^L) be a holomorphic Hermitian line bundle on M . Fix $q \in \{0, 1, \dots, n\}$ and $N_0 \geq 1$. Let $D \subset M(q)$. If $\square_k^{(q)}$ has $O(k^{-n_0})$ small spectral gap on D , then for every

$D' \Subset D$, $\ell \in \mathbb{N}_0$, there exists a constant $C_{D',\ell} > 0$ independent of k with

$$\left| P_k^{(q)}(x, x, k^{-N_0}) - P_k^{(q)}(x) \right|_{\mathcal{C}^\ell(D')} \leq C_{D',\ell} k^{3n+2\ell-N_0}.$$

In particular,

$$(1.18) \quad P_k^{(q)}(x) \sim \sum_{j=0}^{\infty} b_j^{(q)}(x) k^{n-j} \quad \text{locally uniformly on } D,$$

where $b_j^{(q)} \in \mathcal{C}^\infty(D, \text{End}(\Lambda^q T^{*(0,1)} M))$, $j = 0, 1, 2, \dots$, are as in (1.8) and $b_0^{(q)}$, $b_1^{(0)}$, $b_2^{(0)}$ are given by (1.9), (1.11), (1.12).

Note that if L is a positive line bundle on a compact manifold M , or more generally L is uniformly positive on a complete manifold (M, Θ) with $\sqrt{-1}R^{K_M^*}$ and $\partial\Theta$ bounded below, then the Kodaira Laplacian $\square_k^{(0)}$ has a “large” spectral gap on M , i.e., there exists a constant $C > 0$ such that for all k we have

$$\inf \left\{ \lambda \in \text{Spec}(\square_k^{(0)}); \lambda \neq 0 \right\} \geq Ck,$$

(see [49, Theorem 1.5.5], [49, Theorem 6.1.1, (6.1.8)]). Therefore the Bergman kernel $P_k^{(q)}$ has the asymptotic expansion (1.18) and we recover from Theorem 1.6 the asymptotic expansion of the Bergman kernel for:

- (i) compact manifolds for $q = 0$, [12], [70] (cf. also [49, Theorem 4.1.1]);
- (ii) compact manifolds for arbitrary q , [4], [48] [49, Theorem 8.2.4];
- (iii) for complete manifolds [50, Theorem 3.11], [49, Theorem 6.1.1].

In the case $q = 0$ the precise formulas (1.11), (1.12) for the coefficients of the Bergman kernel expansion (1.18) play an important role in the investigations about the relation between canonical metrics in Kähler geometry and stability in algebraic geometry, see e.g., [27, 31, 32, 34, 63, 66, 67] (cf. also [49, Section 5.2]).

The coefficients $b_1^{(0)}$, $b_2^{(0)}$ were computed by Z. Lu [44], L. Wang [66], X. Wang [67], in various degrees of generality. The method of these authors is to construct appropriate peak sections as in [63], using Hörmander’s L^2 $\bar{\partial}$ -method.

In [19, Section 5.1], Dai-Liu-Ma computed $b_1^{(0)}$ by using the heat kernel, and in [50, Section 2], [48, Section 2] (cf. also [49, Section 4.1.8, Section 8.3.4]), $b_1^{(0)}$ was computed in the symplectic case. The coefficient $b_2^{(0)}$ was

calculated in [53, Theorem 0.1] (these results include a twisting Hermitian vector bundle E). Recently, a combinatorial formula for the coefficients $b_j^{(0)}$ was obtained in [65] and the formula for $b_2^{(0)}$ was rederived in [34]. In the above mentioned results it was supposed that the curvature $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ equals the underlying metric Θ . If $\omega \neq \Theta$ formulas for $b_1^{(0)}$, $b_2^{(0)}$ were given in [49, Theorem 4.1.3], [53, Remark 0.5], [41, Theorem 1.4]. We notice that all the results mentioned above concern the coefficients of the Bergman kernel expansion and our results (1.11), (1.12) could recover the first three coefficients of the Bergman kernel expansion in the complex case.

Let M be a compact complex manifold and $L \rightarrow M$ be a holomorphic line bundle with non-degenerate curvature of signature $q \in \{0, 1, \dots, n\}$, i.e., $M(q) = M$. The coefficient $b_0^{(q)}$ given by (1.9) appeared in [48, Theorem 1.3] (the manifold M there is supposed to be symplectic). The coefficient $b_1^{(q)}$ was calculated recently by Wen Lu [45] and by Hsiao [42] for the trivial line bundle with mixed curvature over \mathbb{C}^n endowed with the Euclidean metric.

Since we allow a local $O(k^{-n_0})$ small spectral gap, we can obtain the Bergman kernel expansion under weak conditions, such as semi-positivity of the line bundle. In this case, we have to twist L^k with the canonical line bundle K_M , which we endow with the natural Hermitian metric induced by Θ . We denote by P_{k, K_M} the orthogonal projection from $L^2(M, L^k \otimes K_M)$ on $H_{(2)}^0(M, L^k \otimes K_M) = \mathcal{H}^0(M, L^k \otimes K_M)$.

Theorem 1.7. *Let (M, Θ) be a complete Kähler manifold and (L, h^L) be a semi-positive line bundle over M . Then the Bergman kernel function $P_{k, K_M}(\cdot)$ of $H_{(2)}^0(M, L^k \otimes K_M)$ has the asymptotic expansion*

$$(1.19) \quad P_{k, K_M}(x) \sim \sum_{j=0}^{\infty} k^{n-j} b_{j, K_M}^{(0)}(x) \quad \text{locally uniformly on } M(0),$$

where $b_{j, K_M}^{(0)} \in \mathcal{C}^\infty(M(0), \text{End}(K_M))$, $j = 0, 1, 2, \dots$, are given by

$$(1.20) \quad \begin{aligned} b_{0, K_M}^{(0)} &= (2\pi)^{-n} \det \dot{R}^L \text{Id}_{K_M}, \\ b_{1, K_M}^{(0)} &= (2\pi)^{-n} \det \dot{R}^L \left(-\frac{1}{8\pi} r \right) \text{Id}_{K_M}, \\ b_{2, K_M}^{(0)} &= (2\pi)^{-n} \det \dot{R}^L \left(\frac{1}{128\pi^2} r^2 + \frac{1}{96\pi^2} \Delta_\omega r - \frac{1}{24\pi^2} |\text{Ric}_\omega|_\omega^2 \right. \\ &\quad \left. + \frac{1}{96\pi^2} |R_\omega^{TM}|_\omega^2 \right) \text{Id}_{K_M}, \end{aligned}$$

where $|R_\omega^{TM}|_\omega^2$ is given by (4.72) and Id_{K_M} is the identity map on K_M .

In [2, Theorem 1.8] the expansion is proved on $M(0) \setminus B_+(L)$ for M compact, where $B_+(L)$ is the augmented base locus of L . Note that L is ample if and only if $B_+(L)$ is empty.

Let us consider a singular Hermitian holomorphic line bundle $(L, h^L) \rightarrow M$ (see, e.g., [49, Definition 2.3.1]). We assume that h^L is smooth outside a proper analytic set Σ and the curvature current of h^L is strictly positive. The metric $h = h^L$ induces singular Hermitian metrics h^k on L^k . We denote by $\mathcal{I}(h^k)$ the Nadel multiplier ideal sheaf associated to h^k and by $H^0(M, L^k \otimes \mathcal{I}(h^k)) \subset H^0(M, L^k)$ the space of global sections of the sheaf $\mathcal{O}(L^k) \otimes \mathcal{I}(h^k)$ (see (9.3)), where $H^0(M, L^k) := \{u \in \mathcal{C}^\infty(X, L^k); \bar{\partial}_k u = 0\}$. We denote by $(\cdot, \cdot)_k$ the natural inner products on $\mathcal{C}^\infty(M, L^k \otimes \mathcal{I}(h^k))$ induced by h^L and the volume form dv_M on M (see (9.2) and see also (9.1) for the precise meaning of $\mathcal{C}^\infty(M, L^k \otimes \mathcal{I}(h^k))$). Let $\{S_j^k\}_{j=1}^{d_k}$ be an orthonormal basis of $H^0(M, L^k \otimes \mathcal{I}(h^k))$ with respect to the inner product induced $(\cdot, \cdot)_k$. The (multiplier ideal) Bergman kernel function of $H^0(M, L^k \otimes \mathcal{I}(h^k))$ is defined by

$$(1.21) \quad P_{k, \mathcal{I}}^{(0)}(x) := \sum_{j=1}^{d_k} |S_j^k(x)|_{h^k}^2, \quad x \in M \setminus \Sigma.$$

Theorem 1.8. *Let (L, h^L) be a singular Hermitian holomorphic line bundle over a compact Hermitian manifold (M, Θ) . We assume that h^L is smooth outside a proper analytic set Σ and the curvature current of h^L is strictly positive. Then the Bergman kernel function $P_{k, \mathcal{I}}(\cdot)$ of $H^0(M, L^k \otimes \mathcal{I}(h^k))$ has the asymptotic expansion*

$$(1.22) \quad P_{k, \mathcal{I}}^{(0)}(x) \sim \sum_{j=0}^{\infty} k^{n-j} b_j^{(0)}(x) \quad \text{locally uniformly on } M \setminus \Sigma,$$

where $b_j^{(0)} \in \mathcal{C}^\infty(M \setminus \Sigma)$, $j = 0, 1, 2, \dots$, $b_0^{(0)} = (2\pi)^{-n} \det \dot{R}^L$ and $b_1^{(0)}$ and $b_2^{(0)}$ are given by (1.11) and (1.12), respectively.

We obtain in this way another proof of the Shiffman–Ji–Bonavero–Takayama criterion (cf. [49, Theorem 2.3.28, 2.3.30]).

Corollary 1.9. *Under the assumptions in Theorem 1.8, we have*

$$\dim H^0(M, L^k \otimes \mathcal{I}(h^k)) \geq ck^n$$

for k large, where $c > 0$ is independent of k . Therefore, L is big and M is Moishezon.

We assume that (M, Θ) is compact and we set

$$\text{Herm}(L) = \{\text{singular Hermitian metrics on } L\},$$

$$\mathcal{M}(L) = \{h^L \in \text{Herm}(L); h^L \text{ is smooth outside a proper analytic set} \\ \text{and the curvature current of } h^L \text{ is strictly positive}\}.$$

Note that by Siu's criterion [49, Theorem 2.2.27], L is big under the hypotheses of Theorem 1.10 below. By [49, Lemma 2.3.6], $\mathcal{M}(L) \neq \emptyset$. Set

$$(1.23) \quad M' := \{p \in M; \exists h^L \in \mathcal{M}(L) \text{ with } h^L \text{ smooth near } p\}.$$

Theorem 1.10. *Let (M, Θ) be a compact Hermitian manifold. Let $(L, h^L) \rightarrow M$ be a Hermitian holomorphic line bundle with smooth Hermitian metric h^L having semi-positive curvature and with $M(0) \neq \emptyset$. Then the Bergman kernel function $P_k(\cdot)$ has the asymptotic expansion*

$$P_k(x) \sim \sum_{j=0}^{\infty} k^{n-j} b_j^{(0)}(x) \quad \text{locally uniformly on } M(0) \cap M',$$

where $b_j^{(0)} \in \mathcal{C}^\infty(M(0))$, $j = 0, 1, 2, \dots$, $b_0^{(0)} = (2\pi)^{-n} \det \dot{R}^L$ and $b_1^{(0)}$ and $b_2^{(0)}$ are given in (1.11) and (1.12), respectively.

The existence of the asymptotic expansion from Theorem 1.10 was obtained by Berman [2] in the case of a projective manifold M .

Remark 1.11. (I) In Theorems 1.1, 1.6, we obtain the diagonal expansion of the kernels $P_{k, k-N_0}^{(q)}(\cdot, \cdot)$. We will prove actually more, namely the off-diagonal asymptotic expansion for $P_{k, k-N_0}^{(q)}(x, y)$ on the non-degenerate part of L , see Theorems 4.11, 4.12 and 4.14 for the details. In the same vein, the diagonal expansions of the Bergman kernels from Theorems 1.6, 1.7, 1.8 and 1.10 have off-diagonal counterparts. See Theorems 6.4, 9.2 and 8.3 for the details.

(II) Let E be a holomorphic vector bundle over M . Theorems 1.1, 1.3, 1.6, 1.8 and 1.10 and their off-diagonal counterparts can be generalized to the situation when L^k is replaced by $L^k \otimes E$. See Remark 4.13 and the discussions in the end of Sections 4.4 and 5, for the details.

The layout of this paper is as follows. In Section 2, we collect some notations, definitions and statements we use throughout (geometric set-up, self-adjoint extension of the Kodaira Laplacian, Schwartz kernel theorem). In Section 3, we exhibit a microlocal Hodge decomposition for the Kohn Laplacian on a non-degenerate CR manifold and apply this to obtain the semiclassical Hodge decomposition for the Kodaira Laplacian on a complex manifold. In Section 4, we prove the existence of the asymptotic expansion of the spectral function associated to forms of energy less than k^{-N_0} . As a consequence we obtain the expansion of the Bergman kernel if the local $O(k^{-n_0})$ spectral gap exists. In Section 5 we get an asymptotic upper bound near the degeneracy set of the curvature of L . In Section 6, we prove the expansion of the Bergman kernel on the positivity set of an adjoint semi-positive line bundle over a complete Kähler manifold. In Section 7, we prove an L^2 -estimate for the $\bar{\partial}$ for singular metrics. We use this estimate in Sections 8 and 9, to prove the existence of the Bergman kernel expansion for semi-positive line bundles and bundles endowed with a strictly positively-curved singular Hermitian metric. In Section 10, we apply the previous methods to obtain miscellaneous results, such as Bergman kernel expansion under various conditions, holomorphic Morse inequalities, Tian's convergence theorem and equidistribution of zeros of holomorphic sections.

2. Preliminaries

2.1. Some standard notations

We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers and by \mathbb{R} the set of real numbers. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We use the standard notations w^α , ∂_x^α for multi-indices $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}_0^m$, $w \in \mathbb{C}^m$, $\partial_x = (\partial_{x_1}, \dots, \partial_{x_m})$.

Let M be a complex manifold of dimension n . We always assume that M is paracompact. We denote holomorphic charts on M by (D, z) , where $z = (z_1, \dots, z_n) : D \rightarrow \mathbb{C}^n$ are local coordinates. The associated real chart is denoted by $(D, x) \cong (D, z)$, where $x = (x_1, \dots, x_{2n})$ are real coordinates on M given by $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$. For a multi-index $J = (j_1, \dots, j_q) \in \{1, \dots, n\}^q$ we set $|J| = q$. We say that J is strictly increasing if $1 \leq j_1 < j_2 < \dots < j_q \leq n$. We put $d\bar{z}^J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$. A $(0, q)$ -form f on M has

the local representation

$$f|_D = \sum'_{|J|=q} f_J(z) d\bar{z}^J,$$

where \sum' means that the summation is performed only over strictly increasing multi-indices. In this paper all multi-indices will be supposed to be strictly increasing.

Let Ω be a \mathcal{C}^∞ paracompact manifold equipped with a smooth density of integration. We let $T\Omega$ and $T^*\Omega$ denote the tangent bundle of Ω and the cotangent bundle of Ω respectively. The complexified tangent bundle of Ω and the complexified cotangent bundle of Ω will be denoted by $\mathbb{C}T\Omega$ and $\mathbb{C}T^*\Omega$ respectively. We write $\langle \cdot, \cdot \rangle$ to denote the pointwise duality between $T\Omega$ and $T^*\Omega$. We extend $\langle \cdot, \cdot \rangle$ bilinearly to $\mathbb{C}T\Omega \times \mathbb{C}T^*\Omega$. Let E be a \mathcal{C}^∞ vector bundle over Ω . We write E^* to denote the dual bundle of E . The fibre of E at $x \in \Omega$ will be denoted by E_x . We write $\text{End}(E)$ to denote the vector bundle over Ω with fibre over $x \in \Omega$ consisting of the linear maps from E_x to E_x . Let F be another vector bundle over Ω . We write $E \boxtimes F$ to denote the vector bundle over $\Omega \times \Omega$ with fibre over $(x, y) \in \Omega \times \Omega$ consisting of the linear maps from E_x to F_y . Let $Y \subset \Omega$ be an open set. From now on, the spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $\mathcal{C}^\infty(Y, E)$ and $\mathcal{D}'(Y, E)$ respectively. Let $\mathcal{E}'(Y, E)$ be the subspace of $\mathcal{D}'(Y, E)$ whose elements have compact support in Y . For $m \in \mathbb{R}$, we let $H^m(Y, E)$ denote the Sobolev space of order m of sections of E over Y . Put

$$\begin{aligned} H_{\text{loc}}^m(Y, E) &= \{u \in \mathcal{D}'(Y, E); \varphi u \in H^m(Y, E), \varphi \in \mathcal{C}_0^\infty(Y)\}, \\ H_{\text{comp}}^m(Y, E) &= H_{\text{loc}}^m(Y, E) \cap \mathcal{E}'(Y, E). \end{aligned}$$

2.2. Metric data

Let (M, Θ) be a complex manifold of dimension n , where Θ is a smooth positive $(1, 1)$ form, which induces a Hermitian metric $\langle \cdot, \cdot \rangle$ on the holomorphic tangent bundle $T^{(1,0)}M$. In local holomorphic coordinates $z = (z_1, \dots, z_n)$, if $\Theta = \sqrt{-1} \sum_{j,k=1}^n \Theta_{j,k} dz_j \wedge d\bar{z}_k$, then $\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \rangle = \Theta_{j,k}$, $j, k = 1, \dots, n$. Let $T^{(0,1)}M$ be the anti-holomorphic tangent bundle of M . We extend the Hermitian metric $\langle \cdot, \cdot \rangle$ to $\mathbb{C}TM$ in a natural way by requiring $T^{(1,0)}M$ to be orthogonal to $T^{(0,1)}M$ and $\langle u, v \rangle = \langle \bar{u}, \bar{v} \rangle$, $u, v \in T^{(0,1)}M$. Let $T^{*(1,0)}M$ be the holomorphic cotangent bundle of M and let $T^{*(0,1)}M$ be the anti-holomorphic cotangent bundle of M . For $p, q \in \mathbb{N}_0$, let $\Lambda^{p,q}T^*M = \Lambda^p T^{*(1,0)}M \otimes \Lambda^q T^{*(0,1)}M$.

$M \otimes \Lambda^q T^{*(0,1)} M$ be the bundle of (p, q) forms of M . We write $\Lambda^{0,q} T^* M = \Lambda^q T^{*(0,1)} M$. The Hermitian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{C}TM$ induces a Hermitian metric on $\Lambda^{p,q} T^* M$ also denoted by $\langle \cdot, \cdot \rangle$. Let $D \subset M$ be an open set. Let $\Omega^{p,q}(D)$ denote the space of smooth sections of $\Lambda^{p,q} T^* M$ over D . Similarly, if E is a vector bundle over D , then we let $\Omega^{p,q}(D, E)$ denote the space of smooth sections of $(\Lambda^{p,q} T^* M) \otimes E$ over D . Let $\Omega_0^{p,q}(D, E)$ be the subspace of $\Omega^{p,q}(D, E)$ whose elements have compact support in D .

If $w \in \Lambda^r T_z^{*(0,1)} M$, $r \in \mathbb{N}$, let $(w \wedge)^* : \Lambda^{q+r} T_z^{*(0,1)} M \rightarrow \Lambda^q T_z^{*(0,1)} M$, $q \geq 0$, be the adjoint of left exterior multiplication

$$w \wedge : \Lambda^q T_z^{*(0,1)} M \rightarrow \Lambda^{q+r} T_z^{*(0,1)} M.$$

That is,

$$(2.1) \quad \langle w \wedge u, v \rangle = \langle u, (w \wedge)^* v \rangle,$$

for all $u \in \Lambda^q T_z^{*(0,1)} M$, $v \in \Lambda^{q+r} T_z^{*(0,1)} M$. Notice that $(w \wedge)^*$ depends anti-linearly on w .

Let (L, h^L) be a Hermitian holomorphic line bundle over M , where the Hermitian metric on L is denoted by h^L . Until further notice, we assume that h^L is smooth. Given a local holomorphic frame s of L on an open subset $D \subset M$ we define the associated local weight of h^L by

$$(2.2) \quad |s(x)|^2 = |s(x)|_{h^L}^2 = e^{-2\phi(x)}, \quad \phi \in \mathcal{C}^\infty(D, \mathbb{R}).$$

Let $R^L = (\nabla^L)^2$ be the Chern curvature of L , where ∇^L is the Hermitian holomorphic connection. Then $R^L|_D = 2\partial\bar{\partial}\phi$. Let L^k , $k > 0$, be the k -th tensor power of the line bundle L . The Hermitian fibre metric on L induces a Hermitian fibre metric on L^k that we shall denote by h^k . If s is a local trivializing section of L then s^k is a local trivializing section of L^k . For $p, q \in \mathbb{N}_0$, the Hermitian metric $\langle \cdot, \cdot \rangle$ on $\Lambda^{p,q} T^* M$ and h^k induce a Hermitian metric on $\Lambda^{p,q} T^* M \otimes L^k$, also denoted by $\langle \cdot, \cdot \rangle$. For $f \in \Omega^{p,q}(M, L^k)$, we denote the pointwise norm $|f(x)|^2 := |f(x)|_{h^k}^2 = \langle f(x), f(x) \rangle$. We take $dv_M = dv_M(x)$ as the induced volume form on M . The L^2 -Hermitian inner products on the spaces $\Omega_0^{p,q}(M, L^k)$ and $\Omega_0^{p,q}(M)$ are given by

$$(2.3) \quad \begin{aligned} (s_1, s_2)_k &= \int_M \langle s_1(x), s_2(x) \rangle dv_M(x), & s_1, s_2 &\in \Omega_0^{p,q}(M, L^k), \\ (f_1, f_2) &= \int_M \langle f_1(x), f_2(x) \rangle dv_M(x), & f_1, f_2 &\in \Omega_0^{p,q}(M). \end{aligned}$$

We write $\|f\|^2 := \|f\|_{h^k}^2 = (f, f)_k$, $f \in \Omega_0^{p,q}(M, L^k)$. For $g \in \Omega_0^{p,q}(M)$, we also write $\|g\|^2 := (g, g)$. Let $L_{(p,q)}^2(M, L^k)$ be the completion of $\Omega_0^{p,q}(M, L^k)$ with respect to $\|\cdot\|$.

2.3. A self-adjoint extension of the Kodaira Laplacian

We denote by

$$(2.4) \quad \bar{\partial}_k : \Omega^{0,r}(M, L^k) \rightarrow \Omega^{0,r+1}(M, L^k), \quad \bar{\partial}_k^* : \Omega^{0,r+1}(M, L^k) \rightarrow \Omega^{0,r}(M, L^k)$$

the Cauchy–Riemann operator acting on sections of L^k and its formal adjoint with respect to $(\cdot, \cdot)_k$ respectively. Let

$$(2.5) \quad \square_k^{(q)} := \bar{\partial}_k \bar{\partial}_k^* + \bar{\partial}_k^* \bar{\partial}_k : \Omega^{0,q}(M, L^k) \rightarrow \Omega^{0,q}(M, L^k)$$

be the Kodaira Laplacian acting on $(0, q)$ –forms with values in L^k . We extend $\bar{\partial}_k$ to $L_{(0,r)}^2(M, L^k)$ by

$$(2.6) \quad \bar{\partial}_k : \text{Dom } \bar{\partial}_k \subset L_{(0,r)}^2(M, L^k) \rightarrow L_{(0,r+1)}^2(M, L^k),$$

where $\text{Dom } \bar{\partial}_k := \{u \in L_{(0,r)}^2(M, L^k); \bar{\partial}_k u \in L_{(0,r+1)}^2(M, L^k)\}$, where $\bar{\partial}_k u$ is defined in the sense of distributions. We also write

$$(2.7) \quad \bar{\partial}_k^* : \text{Dom } \bar{\partial}_k^* \subset L_{(0,r+1)}^2(M, L^k) \rightarrow L_{(0,r)}^2(M, L^k)$$

to denote the Hilbert space adjoint of $\bar{\partial}_k$ in the L^2 space with respect to $(\cdot, \cdot)_k$. Let $\square_k^{(q)}$ denote the Gaffney extension of the Kodaira Laplacian given by

$$(2.8) \quad \text{Dom } \square_k^{(q)} = \{s \in L_{(0,q)}^2(M, L^k); s \in \text{Dom } \bar{\partial}_k \cap \text{Dom } \bar{\partial}_k^*, \\ \bar{\partial}_k s \in \text{Dom } \bar{\partial}_k^*, \bar{\partial}_k^* s \in \text{Dom } \bar{\partial}_k\},$$

and $\square_k^{(q)} s = \bar{\partial}_k \bar{\partial}_k^* s + \bar{\partial}_k^* \bar{\partial}_k s$ for $s \in \text{Dom } \square_k^{(q)}$. By a result of Gaffney [49, Proposition 3.1.2], $\square_k^{(q)}$ is a positive self-adjoint operator. Note that if M is complete, the Kodaira Laplacian $\square_k^{(q)}$ is essentially self-adjoint [49, Corollary 3.3.4] and the Gaffney extension coincides with the Friedrichs extension of $\square_k^{(q)}$.

2.4. Schwartz kernel theorem

We recall the Schwartz kernel theorem [39, Theorem 5.2.1, 5.2.6], [62, p. 296]. Let Ω be a \mathcal{C}^∞ paracompact manifold equipped with a smooth density of integration. Let E and F be smooth vector bundles over Ω . Then any continuous linear operator $A : \mathcal{C}_0^\infty(\Omega, E) \rightarrow \mathcal{D}'(\Omega, F)$ has a Schwartz distribution kernel, denoted $K_A(x, y)$ or $A(x, y)$. Moreover, the following two statements are equivalent

- 1) A is continuous: $\mathcal{E}'(\Omega, E) \rightarrow \mathcal{C}^\infty(\Omega, F)$,
- 2) $K_A(x, y) \in \mathcal{C}^\infty(\Omega \times \Omega, E_y \boxtimes F_x)$.

If A satisfies (I) or (II), we say that A is a *smoothing operator*. Furthermore, A is smoothing if and only if $A : H_{\text{comp}}^s(\Omega, E) \rightarrow H_{\text{loc}}^{s+N}(\Omega, F)$ is continuous, for all $N \geq 0$, $s \in \mathbb{R}$. We say that A is properly supported if $\text{Supp } K_A \subset \Omega \times \Omega$ is proper. That is, the two projections: $t_x : (x, y) \in \text{Supp } K_A \rightarrow x \in \Omega$, $t_y : (x, y) \in \text{Supp } K_A \rightarrow y \in \Omega$ are proper (i.e., the inverse images of t_x and t_y of all compact subsets of Ω are compact). We say that A is smoothing away the diagonal if $\chi_1 A \chi_2$ is smoothing, for all $\chi_1, \chi_2 \in \mathcal{C}_0^\infty(\Omega)$ with $\text{Supp } \chi_1 \cap \text{Supp } \chi_2 = \emptyset$.

Let $H(x, y) \in \mathcal{D}'(\Omega \times \Omega, E_y \boxtimes F_x)$. We write H to denote the unique continuous operator $H : \mathcal{C}_0^\infty(\Omega, E) \rightarrow \mathcal{D}'(\Omega, F)$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$. Let $A, B : \mathcal{C}_0^\infty(\Omega, E) \rightarrow \mathcal{D}'(\Omega, F)$ be continuous operators. We write $A \equiv B$ or $A(x, y) \equiv B(x, y)$ if $A - B$ is a smoothing operator.

3. Szegő kernels and semi-classical Hodge decomposition

The goal of this Section is to prove the semiclassical Hodge decomposition for the Kodaira Laplacian, i.e., to find a semi-classical partial inverse and an approximate kernel for $\square_k^{(q)}$, cf. Theorem 3.11. For this purpose we reduce the analysis of the Kodaira Laplacian to the analysis of the Kohn Laplacian on the Grauert tube of the line bundle L . In Section 3.1 we recall the construction of these two objects. Section 3.2 contains a detailed study of the microlocal Hodge decomposition of the Kohn Laplacian on a non-degenerate CR manifold and especially on the Grauert tube, by following [40]. Finally, in Section 3.3 we apply this results in order to obtain the semi-classical Hodge decomposition for the Kodaira Laplacian.

3.1. The Grauert tube

Let (M, Θ) be a Hermitian manifold and (L, h^L) be a holomorphic Hermitian line bundle on M . Let (L^*, h^{L^*}) be the dual bundle of L . We denote

$$(3.1) \quad G := \{v \in L^*; |v|_{h^{L^*}} < 1\}, \quad X := \partial G = \{v \in L^*; |v|_{h^{L^*}} = 1\}.$$

The domain G is called *Grauert tube* associated to L . We denote

$$T^{(1,0)}X := T^{(1,0)}L^* \cap \mathbb{C}TX, \quad T^{(0,1)}X := T^{(0,1)}L^* \cap \mathbb{C}TX.$$

Then $(X, T^{(1,0)}X)$ is a CR manifold of dimension $2n + 1$ and the bundle $T^{(1,0)}X$ is called the holomorphic tangent bundle of X . The manifold X is equipped with a natural S^1 action. Locally X can be represented in local holomorphic coordinates (z, λ) , where λ is the fibre coordinate, as the set of all (z, λ) such that $|\lambda|^2 e^{2\phi(z)} = 1$, where ϕ is a local weight of h^L . The S^1 action on X is given by $e^{i\theta} \circ (z, \lambda) = (z, e^{i\theta}\lambda)$, $e^{i\theta} \in S^1$, $(z, \lambda) \in X$. Let Y be the global real vector field on X determined by

$$Yu(x) = \frac{\partial}{\partial \theta} u(e^{i\theta} \circ x) \Big|_{\theta=0} \quad \text{for all } u \in \mathcal{C}^\infty(X).$$

Let $\pi : X \rightarrow M$ be the natural projection. We have the bijective map:

$$\pi^* : T^{(1,0)}X \oplus T^{(0,1)}X \rightarrow T^{(1,0)}M \oplus T^{(0,1)}M, \quad W \rightarrow \pi^*W,$$

where $(\pi^*W)f = W(f \circ \pi)$, for all $f \in \mathcal{C}^\infty(M)$. We take the Hermitian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{C}TX$ so that $Y \perp (T^{(1,0)}X \oplus T^{(0,1)}X)$, $\langle Y, Y \rangle = 1$ and $\langle Z, W \rangle = \langle \pi^*Z, \pi^*W \rangle$, $Z, W \in T^{(1,0)}X \oplus T^{(0,1)}X$. The Hermitian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on the complexified cotangent bundle $\mathbb{C}T^*X$ that we shall also denote by $\langle \cdot, \cdot \rangle$. Define $T^{*(1,0)}X := (T^{(0,1)}X \oplus \mathbb{C}Y)^\perp \subset \mathbb{C}T^*X$, $T^{*(0,1)}X := (T^{(1,0)}X \oplus \mathbb{C}Y)^\perp \subset \mathbb{C}T^*X$. For $q \in \mathbb{N}$, the bundle of $(0, q)$ forms of X is given by $\Lambda^q T^{*(0,1)}X := \Lambda^q(T^{*(0,1)}X)$. The Hermitian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{C}T^*X$ induces a Hermitian metric on $\Lambda^q T^{*(0,1)}X$ also denoted by $\langle \cdot, \cdot \rangle$.

Locally there is a real one form ω_0 of length one which is pointwise orthogonal to $T^{*(0,1)}X \oplus T^{*(1,0)}X$. ω_0 is unique up to the choice of sign. We take ω_0 so that $\langle \omega_0, Y \rangle = -1$. Therefore ω_0 , so chosen, is globally defined.

The *Levi form* \mathcal{L}_p of X at $p \in X$ is the Hermitian quadratic form on $T_p^{(1,0)}X$ defined as follows:

$$(3.2) \quad \mathcal{L}_p(U, \bar{V}) = \frac{1}{2i} \langle [\mathcal{U}, \bar{\mathcal{V}}](p), \omega_0(p) \rangle, \quad U, V \in T_p^{(1,0)}X$$

where $\mathcal{U}, \mathcal{V} \in \mathcal{C}^\infty(X, T^{(1,0)}X)$ that satisfy $\mathcal{U}(p) = U, \mathcal{V}(p) = V$ and $[\mathcal{U}, \bar{\mathcal{V}}] = \mathcal{U}\bar{\mathcal{V}} - \bar{\mathcal{V}}\mathcal{U}$ denotes the commutator of \mathcal{U} and $\bar{\mathcal{V}}$.

Let $B \subset X$ be an open set. Let $\Omega^{0,q}(B)$ denote the space of smooth sections of $\Lambda^q T^{*(0,1)}X$ over B . Let $\Omega_0^{0,q}(B)$ be the subspace of $\Omega^{0,q}(B)$ whose elements have compact support in B . Let $\bar{\partial}_b : \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X)$ be the tangential Cauchy–Riemann operator. We take $dv_X = dv_X(x)$ as the induced volume form on X . Then, we get natural inner product (\cdot, \cdot) on $\Omega^{0,q}(X)$. Let $\bar{\partial}_b^* : \Omega^{0,q+1}(X) \rightarrow \Omega^{0,q}(X)$ be the formal adjoint of $\bar{\partial}_b$ with respect to (\cdot, \cdot) . The *Kohn Laplacian* on $(0, q)$ forms is given by

$$\square_b^{(q)} := \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b : \Omega^{0,q}(X) \rightarrow \Omega^{0,q}(X).$$

We introduce now a local holomorphic frame and local coordinates in terms of which we shall write down the operators explicitly. Let

- (i) s be a local trivializing section of L on an open set $D \Subset M$,
- (ii) $\phi \in \mathcal{C}^\infty(D)$ be the local weight of the metric h^L defined by $|s|_{h^L}^2 = e^{-2\phi}$.

Then $s^* := s^{-1}$ is a local trivializing section of L^* on D . We have $|s^*|_{h^{L^*}}^2 = e^{2\phi}$.

We introduce holomorphic and real coordinates on D by

$$(3.3) \quad z = (z_1, \dots, z_n), \quad x' = (x_1, \dots, x_{2n}), \quad z_j = x_{2j-1} + ix_{2j}, \quad j = 1, \dots, n.$$

We identify D with an open set of \mathbb{C}^n . We have the local diffeomorphism:

$$(3.4) \quad \tau : D \times]-\varepsilon_0, \varepsilon_0[\rightarrow X, \quad (z, \theta) \mapsto e^{-\phi(z)} s^*(z) e^{-i\theta}, \quad 0 < \varepsilon_0 \leq \pi.$$

It is convenient to work with the local coordinates (z, θ) . In terms of these coordinates, it is straightforward to see that $Y = -\frac{\partial}{\partial \theta}$. Moreover,

$$T_v^{(1,0)}X = \mathbb{C} \left\{ \frac{\partial}{\partial z_j} - i \frac{\partial \phi}{\partial z_j}(z) \frac{\partial}{\partial \theta}; j = 1, \dots, n \right\}, \quad v = e^{-\phi(z)} s^*(z) e^{-i\theta} \in X.$$

Further, let $\{\bar{Z}_j\}_{j=1}^n$ be an orthonormal basis for the holomorphic tangent bundle $T^{(1,0)}M$ and let $\{\bar{e}_j\}_{j=1}^n$ be the dual basis of $T^{*(1,0)}M$. Then,

$\{\bar{Z}_j - i\bar{Z}_j(\phi)\frac{\partial}{\partial\theta}\}_{j=1}^n$ is an orthonormal basis for $T^{(1,0)}X$ and $\{\bar{e}_j\}_{j=1}^n$ is the dual orthonormal basis for $T^{*(1,0)}X$. Furthermore, we can check that

$$(3.5) \quad \omega_0 = d\theta + \sum_{j=1}^n (-iZ_j(\phi)e_j + i\bar{Z}_j(\phi)\bar{e}_j).$$

From this, we can compute for $j, k = 1, \dots, n, p \in X$:

$$\mathcal{L}_p \left(\frac{\partial}{\partial z_j} - i \frac{\partial\phi}{\partial z_j}(\pi(p)) \frac{\partial}{\partial\theta}, \frac{\partial}{\partial \bar{z}_k} + i \frac{\partial\phi}{\partial \bar{z}_k}(\pi(p)) \frac{\partial}{\partial\theta} \right) = \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}(\pi(p)).$$

Thus, for a given point $p \in X$, we have

$$(3.6) \quad \begin{aligned} \mathcal{L}_p(U, \bar{V}) &= \langle \partial\bar{\partial}\phi(\pi(p)), \pi^*U \wedge \pi^*\bar{V} \rangle = \left\langle \frac{1}{2}R^L(\pi(p)), \pi^*U \wedge \pi^*\bar{V} \right\rangle \\ &= \left\langle \frac{1}{2}\dot{R}^L(\pi(p))\pi^*U, \pi^*V \right\rangle, \quad \forall U, V \in T_p^{(1,0)}X. \end{aligned}$$

We deduce the following:

Proposition 3.1. *Let (L, h^L) be a Hermitian holomorphic line bundle over a complex manifold M and let $X \subset L^*$ be the boundary of the Grauert tube associated to L . Let $p \in X$. If the curvature R^L has signature (n_-, n_+) at $\pi(p)$, then \mathcal{L}_p has signature (n_-, n_+) .*

We define the operators $\bar{\partial}_s, \bar{\partial}_s^*, \square_s^{(q)}$, which are the local versions of the operators $\bar{\partial}_k, \bar{\partial}_k^*, \square_k^{(q)}$ (see (2.4)–(2.8)), by the following equations:

$$(3.7) \quad \begin{aligned} \bar{\partial}_s &= \bar{\partial} + k(\bar{\partial}\phi)\wedge : \Omega^{0,q}(D) \rightarrow \Omega^{0,q+1}(D), \\ \bar{\partial}_s^* &= \bar{\partial}^* + k((\bar{\partial}\phi)\wedge)^* : \Omega^{0,q+1}(D) \rightarrow \Omega^{0,q}(D), \\ \square_s^{(q)} &= \bar{\partial}_s\bar{\partial}_s^* + \bar{\partial}_s^*\bar{\partial}_s : \Omega^{0,q}(D) \rightarrow \Omega^{0,q}(D). \end{aligned}$$

Here $\bar{\partial}^* : \Omega^{0,q+1}(D) \rightarrow \Omega^{0,q}(D)$ is the formal adjoint of $\bar{\partial}$ with respect to (\cdot, \cdot) . We have the unitary identifications:

$$(3.8) \quad \begin{aligned} \Omega^{0,q}(D, L^k) &\longleftrightarrow \Omega^{0,q}(D) \\ f = s^k g &\longleftrightarrow \widehat{f}(z) = e^{-k\phi} s^{-k} f = g(z) e^{-k\phi(z)}, \quad g \in \Omega^{0,q}(D), \\ \bar{\partial}_k &\longleftrightarrow \bar{\partial}_s, \quad \bar{\partial}_k f = s^k e^{k\phi} \bar{\partial}_s \widehat{f}, \\ \bar{\partial}_k^* &\longleftrightarrow \bar{\partial}_s^*, \quad \bar{\partial}_k^* f = s^k e^{k\phi} \bar{\partial}_s^* \widehat{f}, \\ \square_k^{(q)} &\longleftrightarrow \square_s^{(q)}, \quad \square_k^{(q)} f = s^k e^{k\phi} \square_s^{(q)} \widehat{f}. \end{aligned}$$

We continue to work with the local coordinates (z, θ) . As above, let $(Z_j)_{j=1}^n$ be an orthonormal basis for $T^{(0,1)}M$ and let $(e_j)_{j=1}^n$ be an orthonormal basis for $T^{*(0,1)}M$ which is dual to $(Z_j)_{j=1}^n$. We can check that

$$(3.9) \quad \bar{\partial}_b = \sum_{j=1}^n (e_j \wedge) \circ \left(Z_j + iZ_j(\phi) \frac{\partial}{\partial \theta} \right) + \sum_{j=1}^n ((\bar{\partial}e_j) \wedge) \circ (e_j \wedge)^*$$

and correspondingly

$$(3.10) \quad \bar{\partial}_b^* = \sum_{j=1}^n ((e_j \wedge)^*) \circ \left(Z_j^* + i\bar{Z}_j(\phi) \frac{\partial}{\partial \theta} \right) + \sum_{j=1}^n (e_j \wedge) \circ ((\bar{\partial}e_j) \wedge)^*,$$

where Z_j^* is the formal adjoint of Z_j with respect to (\cdot, \cdot) , $j = 1, \dots, n$.

Let $\bar{\partial}_s$ and $\bar{\partial}_s^*$ be as in (3.8) and (3.7). We can check that

$$(3.11) \quad \begin{aligned} \bar{\partial}_s &= \sum_{j=1}^n (e_j \wedge) \circ (Z_j + kZ_j(\phi)) + \sum_{j=1}^n ((\bar{\partial}e_j) \wedge) \circ (e_j \wedge)^*, \\ \bar{\partial}_s^* &= \sum_{j=1}^n ((e_j \wedge)^*) \circ (Z_j^* + k\bar{Z}_j(\phi)) + \sum_{j=1}^n (e_j \wedge) \circ ((\bar{\partial}e_j) \wedge)^*. \end{aligned}$$

From now on, we identify $\Lambda^q T^{*(0,1)}M$ with $\Lambda^q T^{*(0,1)}X$. From (3.8), (3.7), explicit formulas of $\bar{\partial}_s$, $\bar{\partial}_s^*$ and (3.9), (3.10), we get

$$(3.12) \quad \square_k^{(q)} f = s^k e^{k\phi} e^{ik\theta} \square_b^{(q)} (\hat{f} e^{-ik\theta}),$$

for all $f \in \Omega^{0,q}(D, L^k)$, where \hat{f} is given by (3.8).

Let $u(z, \theta) \in \Omega_0^{0,q}(D \times (-\varepsilon_0, \varepsilon_0))$. Note that

$$k \int e^{i\theta k} u(z, \theta) d\theta = \int (-i) \frac{\partial}{\partial \theta} (e^{i\theta k}) u(z, \theta) d\theta = \int e^{i\theta k} i \frac{\partial u}{\partial \theta}(z, \theta) d\theta.$$

From this observation and explicit formulas of $\bar{\partial}_b$, $\bar{\partial}_b^*$, $\bar{\partial}_s$ and $\bar{\partial}_s^*$ (see (3.9), (3.10) and (3.11)), we conclude that

$$(3.13) \quad \square_s^{(q)} \left(\int e^{i\theta k} u(z, \theta) d\theta \right) = \int e^{i\theta k} (\square_b^{(q)} u)(z, \theta) d\theta,$$

for all $u(z, \theta) \in \Omega_0^{0,q}(D \times (-\varepsilon_0, \varepsilon_0))$.

3.2. Approximate Szegő kernels

In this Section we review the results in [40] about the existence of a microlocal Hodge decomposition of the Kohn Laplacian on an open set of a CR manifold where the Levi form is non-degenerate. The approximate Szegő kernel is a Fourier integral operator with complex phase in the sense of Melin–Sjöstrand [54]. We then specialize to the case of the Grauert tube of a line bundle and give a useful formula for the phase function of the approximate Szegő kernel in Theorem 3.8.

Theorems 3.2–3.4 are proved in chapter 6, chapter 7 and chapter 8 of part I in [40]. In [40] the existence of the microlocal Hodge decomposition is stated for compact CR manifolds, but the construction and arguments used are essentially local.

Theorem 3.2. *Let X be an orientable CR manifold whose Levi form \mathcal{L} is non-degenerate of constant signature (n_-, n_+) at each point of an open set $B \Subset X$. Let $q \neq n_-, n_+$. There exists a properly supported continuous operator*

$$(3.14) \quad A : \begin{cases} H_{\text{loc}}^s(B, \Lambda^q T^{*(0,1)} X) \rightarrow H_{\text{loc}}^{s+1}(B, \Lambda^q T^{*(0,1)} X), \\ H_{\text{comp}}^s(B, \Lambda^q T^{*(0,1)} X) \rightarrow H_{\text{comp}}^{s+1}(B, \Lambda^q T^{*(0,1)} X) \end{cases}$$

for all $s \geq 0$, such that A is smoothing away the diagonal and $\square_b^{(q)} A \equiv I$.

For $m \in \mathbb{R}$ let $S_{1,0}^m$ be the Hörmander symbol space (see the book Grigis–Sjöstrand [36, Definition 1.1], for the definition of $S_{1,0}^m$). Let $p_0(x, \xi) \in \mathcal{C}^\infty(T^*X)$ be the principal symbol of $\square_b^{(q)}$. Note that $p_0(x, \xi)$ is a polynomial of degree 2 in ξ . The characteristic manifold of $\square_b^{(q)}$ is given by $\Sigma = \Sigma^+ \cup \Sigma^-$, where

$$\begin{aligned} \Sigma^+ &= \{(x, \lambda\omega_0(x)) \in T^*X; \lambda > 0\}, \\ \Sigma^- &= \{(x, \lambda\omega_0(x)) \in T^*X; \lambda < 0\}. \end{aligned}$$

Theorem 3.3. *Let X , B and (n_-, n_+) be as in Theorem 3.2. Let $q = n_-$ or n_+ . Then there exist properly supported continuous operators*

$$(3.15) \quad \begin{aligned} A : & \begin{cases} H_{\text{loc}}^s(B, \Lambda^q T^{*(0,1)} X) \rightarrow H_{\text{loc}}^{s+1}(B, \Lambda^q T^{*(0,1)} X), \\ H_{\text{comp}}^s(B, \Lambda^q T^{*(0,1)} X) \rightarrow H_{\text{comp}}^{s+1}(B, \Lambda^q T^{*(0,1)} X), \end{cases} \\ S_-, S_+ : & \begin{cases} H_{\text{loc}}^s(B, \Lambda^q T^{*(0,1)} X) \rightarrow H_{\text{loc}}^s(B, \Lambda^q T^{*(0,1)} X), \\ H_{\text{comp}}^s(B, \Lambda^q T^{*(0,1)} X) \rightarrow H_{\text{comp}}^s(B, \Lambda^q T^{*(0,1)} X), \end{cases} \end{aligned}$$

for all $s \geq 0$, such that A, S_-, S_+ are smoothing away the diagonal and

$$\begin{aligned} \square_b^{(q)} A + S_- + S_+ &= I, & \square_b^{(q)} S_- &\equiv 0, & \square_b^{(q)} S_+ &\equiv 0, \\ A &\equiv A^*, & S_- &\equiv S_-^* \equiv S_-^2, & S_+ &\equiv S_+^* \equiv S_+^2, & S_- S_+ &\equiv S_+ S_- \equiv 0, \end{aligned}$$

where A^* , S_-^* and S_+^* are the formal adjoints of A , S_- and S_+ with respect to (\cdot, \cdot) respectively and $K_{S_-}(x, y)$ satisfies

$$K_{S_-}(x, y) \equiv \int_0^\infty e^{i\varphi_-(x, y)t} s_-(x, y, t) dt$$

with a symbol

$$\begin{aligned} (3.16) \quad & s_-(x, y, t) \in S_{1,0}^n(B \times B \times]0, \infty[, \Lambda^q T_y^{*(0,1)} X \boxtimes \Lambda^q T_x^{*(0,1)} X), \\ & s_-(x, y, t) \sim \sum_{j=0}^\infty s_-^j(x, y) t^{n-j} \\ & \text{in } S_{1,0}^n(B \times B \times]0, \infty[, \Lambda^q T_y^{*(0,1)} X \boxtimes \Lambda^q T_x^{*(0,1)} X), \\ & s_-^j(x, y) \in \mathcal{C}^\infty(B \times B, \Lambda^q T_y^{*(0,1)} X \boxtimes \Lambda^q T_x^{*(0,1)} X), \quad j \in \mathbb{N}_0, \end{aligned}$$

and phase function

$$(3.17) \quad \varphi_- \in \mathcal{C}^\infty(B \times B), \quad \text{Im } \varphi_-(x, y) \geq 0, \quad \varphi_-(x, x) = 0, \quad \varphi_-(x, y) \neq 0 \\ \text{if } x \neq y,$$

$$(3.18) \quad d_x \varphi_- \neq 0, \quad d_y \varphi_- \neq 0 \quad \text{where } \text{Im } \varphi_- = 0,$$

$$(3.19) \quad d_x \varphi_-(x, y)|_{x=y} = -\omega_0(x), \quad d_y \varphi_-(x, y)|_{x=y} = \omega_0(x),$$

$$(3.20) \quad \varphi_-(x, y) = -\bar{\varphi}_-(y, x).$$

Moreover, there is a function $f \in \mathcal{C}^\infty(B \times B)$ such that

$$(3.21) \quad p_0(x, (\varphi_-)'_x(x, y)) - f(x, y) \varphi_-(x, y)$$

vanishes to infinite order at $x = y$.

Similarly,

$$K_{S_+}(x, y) \equiv \int_0^\infty e^{i\varphi_+(x, y)t} s_+(x, y, t) dt$$

with $s_+(x, y, t) \in S_{1,0}^n(B \times B \times]0, \infty[, \Lambda^q T_y^{*(0,1)} X \boxtimes \Lambda^q T_x^{*(0,1)} X)$,

$$s_+(x, y, t) \sim \sum_{j=0}^\infty s_+^j(x, y) t^{n-j}$$

in $S_{1,0}^n(B \times B \times]0, \infty[, \Lambda^q T_y^{*(0,1)} X \boxtimes \Lambda^q T_x^{*(0,1)} X)$, where

$$s_+^j(x, y) \in \mathcal{C}^\infty(B \times B, \Lambda^q T_y^{*(0,1)} X \boxtimes \Lambda^q T_x^{*(0,1)} X), \quad j \in \mathbb{N}_0,$$

and $-\bar{\varphi}_+(x, y)$ satisfies (3.18)–(3.21). Moreover, if $q \neq n_+$, then $s_+(x, y, t)$ vanishes to infinite order at $x = y$. If $q \neq n_-$, then $s_-(x, y, t)$ vanishes to infinite order at $x = y$.

The operators S_+ , S_- are called *approximate Szegő kernels*.

Proof. We only sketch the proof. For all the details, we refer the reader to Part I in [40]. We will use the heat equation method. We work with some real local coordinates $x = (x_1, \dots, x_{2n+1})$ defined on B . We will say that $a \in \mathcal{C}^\infty(\overline{\mathbb{R}_+} \times B \times \mathbb{R}^{2n+1})$ is quasi-homogeneous of degree j if $a(t, x, \lambda\eta) = \lambda^j a(\lambda t, x, \eta)$ for all $\lambda > 0$. We consider the problem

$$(3.22) \quad \begin{cases} (\partial_t + \square_b^{(q)})u(t, x) = 0 & \text{in } \mathbb{R}_+ \times B, \\ u(0, x) = v(x). \end{cases}$$

We start by a formal construction. We look for an approximate solution of (3.22) of the form $u(t, x) = A(t)v(x)$,

$$(3.23) \quad A(t)v(x) = \frac{1}{(2\pi)^{2n+1}} \int e^{i(\psi(t, x, \eta) - \langle y, \eta \rangle)} a(t, x, \eta) v(y) dy d\eta,$$

where formally

$$a(t, x, \eta) \sim \sum_{j=0}^{\infty} a_j(t, x, \eta),$$

with $a_j(t, x, \eta)$ matrix-valued quasi-homogeneous functions of degree $-j$.

The full symbol of $\square_b^{(q)}$ equals $\sum_{j=0}^2 p_j(x, \xi)$, where $p_j(x, \xi)$ is positively homogeneous of order $2 - j$ in the sense that

$$p_j(x, \lambda\eta) = \lambda^{2-j} p_j(x, \eta), \quad |\eta| \geq 1, \quad \lambda \geq 1.$$

We apply $\partial_t + \square_b^{(q)}$ formally inside the integral in (3.23) and then introduce the asymptotic expansion of $\square_b^{(q)}(ae^{i\psi})$. Set $(\partial_t + \square_b^{(q)})(ae^{i\psi}) \sim 0$ and regroup the terms according to the degree of quasi-homogeneity. The phase

$\psi(t, x, \eta)$ should solve

$$(3.24) \quad \begin{cases} \frac{\partial \psi}{\partial t} - ip_0(x, \psi'_x) = O(|\operatorname{Im} \psi|^N), & \forall N \geq 0, \\ \psi|_{t=0} = \langle x, \eta \rangle. \end{cases}$$

This equation can be solved with $\operatorname{Im} \psi(t, x, \eta) \geq 0$ and the phase $\psi(t, x, \eta)$ is quasi-homogeneous of degree 1. Moreover,

$$\begin{aligned} \psi(t, x, \eta) &= \langle x, \eta \rangle \text{ on } \Sigma, \quad d_{x, \eta}(\psi - \langle x, \eta \rangle) = 0 \text{ on } \Sigma, \\ \operatorname{Im} \psi(t, x, \eta) &\asymp \left(|\eta| \frac{t|\eta|}{1+t|\eta|} \right) \left(\operatorname{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right)^2, \quad |\eta| \geq 1. \end{aligned}$$

Furthermore, there exists $\psi(\infty, x, \eta) \in \mathcal{C}^\infty(B \times \mathbb{R}^{2n+1})$ with a uniquely determined Taylor expansion at each point of Σ such that for every compact set $K \subset B \times \mathbb{R}^{2n+1}$ there is a constant $c_K > 0$ such that

$$\operatorname{Im} \psi(\infty, x, \eta) \geq c_K |\eta| \left(\operatorname{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right)^2, \quad |\eta| \geq 1.$$

If $\lambda \in \mathcal{C}(T^*B \setminus 0)$, $\lambda > 0$ is positively homogeneous of degree 1 and $\lambda|_\Sigma < \min \lambda_j$, $\lambda_j > 0$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of the fundamental matrix of $\square_b^{(q)}$, then the solution $\psi(t, x, \eta)$ of (3.24) can be chosen so that for every compact set $K \subset B \times \mathbb{R}^{2n+1}$ and all indices α, β, γ , there is a constant $c_{\alpha, \beta, \gamma, K}$ such that

$$\left| \partial_x^\alpha \partial_\eta^\beta \partial_t^\gamma (\psi(t, x, \eta) - \psi(\infty, x, \eta)) \right| \leq c_{\alpha, \beta, \gamma, K} e^{-\lambda(x, \eta)t} \text{ on } \overline{\mathbb{R}}_+ \times K.$$

We obtain the transport equations

$$(3.25) \quad \begin{cases} T(t, x, \eta, \partial_t, \partial_x) a_0 = O(|\operatorname{Im} \psi|^N), & \forall N, \\ T(t, x, \eta, \partial_t, \partial_x) a_j + l_j(t, x, \eta, a_0, \dots, a_{j-1}) = O(|\operatorname{Im} \psi|^N), & \forall N, \quad j \in \mathbb{N}. \end{cases}$$

Following the method of Menikoff–Sjöstrand [55], we see that we can solve (3.25). Moreover, a_j decay exponentially fast in t when $q \neq n_-, n_+$, and has subexponential growth in general. We assume that $q = n_-$ or n_+ .

We use $\bar{\partial}_b \square_b^{(q)} = \square_b^{(q+1)} \bar{\partial}_b$, $\bar{\partial}_b^* \square_b^{(q)} = \square_b^{(q-1)} \bar{\partial}_b^*$ and get

$$\begin{aligned} \partial_t(\bar{\partial}_b(e^{i\psi} a)) + \square_b^{(q+1)}(\bar{\partial}_b(e^{i\psi} a)) &\sim 0, \\ \partial_t(\bar{\partial}_b^*(e^{i\psi} a)) + \square_b^{(q-1)}(\bar{\partial}_b^*(e^{i\psi} a)) &\sim 0. \end{aligned}$$

Put

$$\bar{\partial}_b(e^{i\psi} a) = e^{i\psi} \hat{a}, \quad \bar{\partial}_b^*(e^{i\psi} a) = e^{i\psi} \tilde{a}.$$

We have

$$\begin{aligned} (\partial_t + \square_b^{(q+1)})(e^{i\psi} \hat{a}) &\sim 0, \\ (\partial_t + \square_b^{(q-1)})(e^{i\psi} \tilde{a}) &\sim 0. \end{aligned}$$

The corresponding degrees of \hat{a} and \tilde{a} are $q+1$ and $q-1$. We deduce as above that \hat{a} and \tilde{a} decay exponentially fast in t . This also applies to

$$\square_b^{(q)}(ae^{i\psi}) = \bar{\partial}_b(\bar{\partial}_b^* ae^{i\psi}) + \bar{\partial}_b^*(\bar{\partial}_b ae^{i\psi}) = \bar{\partial}_b(e^{i\psi} \tilde{a}) + \bar{\partial}_b^*(e^{i\psi} \hat{a}).$$

Thus, $\partial_t(ae^{i\psi})$ decay exponentially fast in t . Since $\partial_t \psi$ decay exponentially fast in t so does $\partial_t a$. Hence, there exist positively homogeneous functions of degree $-j$

$$a_j(\infty, x, \eta) \in C^\infty(T^*B, \Lambda^q T^{*(0,1)} X \boxtimes \Lambda^q T^{*(0,1)} X), \quad j = 0, 1, 2, \dots,$$

such that $a_j(t, x, \eta)$ converges exponentially fast to $a_j(\infty, x, \eta)$, $t \rightarrow \infty$, for all $j \in \mathbb{N}_0$.

Choose $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2n+1})$ so that $\chi(\eta) = 1$ when $|\eta| < 1$ and $\chi(\eta) = 0$ when $|\eta| > 2$. We formally set

$$\begin{aligned} A = \frac{1}{(2\pi)^{2n+1}} \int \int_0^\infty &\left(e^{i(\psi(t,x,\eta) - \langle y, \eta \rangle)} a(t, x, \eta) \right. \\ &\left. - e^{i(\psi(\infty, x, \eta) - \langle y, \eta \rangle)} a(\infty, x, \eta) \right) (1 - \chi(\eta)) dt d\eta \end{aligned}$$

and

$$S = \frac{1}{(2\pi)^{2n+1}} \int (e^{i(\psi(\infty, x, \eta) - \langle y, \eta \rangle)} a(\infty, x, \eta)) d\eta.$$

We can show that A is a pseudodifferential operator of order -1 and type $(\frac{1}{2}, \frac{1}{2})$ satisfying

$$S + \square_b^{(q)} \circ A \equiv I, \quad \square_b^{(q)} \circ S \equiv 0.$$

Moreover, the stationary phase formula of Melin–Sjöstrand [54] shows that $S \equiv S_- + S_+$, where S_- , S_+ are as in Theorem 3.3. \square

The following result describes the phase function in local coordinates.

Theorem 3.4. *Let X , B and (n_-, n_+) be as in Theorem 3.2. For a given point $x_0 \in B$, let $\{W_j\}_{j=1}^n$ be an orthonormal frame of $T^{(1,0)}X$ in a neighbourhood of x_0 , such that the Levi form is diagonal at x_0 , i.e., $\mathcal{L}_{x_0}(W_j, \bar{W}_j) = \mu_j$, $j = 1, \dots, n$. We take local coordinates $x = (x_1, \dots, x_{2n+1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, defined on some neighbourhood of x_0 such that $\omega_0(x_0) = dx_{2n+1}$, $x(x_0) = 0$, and for some $c_j \in \mathbb{C}$, $j = 1, \dots, n$,*

$$W_j = \frac{\partial}{\partial z_j} - i\mu_j \bar{z}_j \frac{\partial}{\partial x_{2n+1}} - c_j x_{2n+1} \frac{\partial}{\partial x_{2n+1}} + O(|x|^2), \quad j = 1, \dots, n.$$

Set $y = (y_1, \dots, y_{2n+1})$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \dots, n$. Then, for φ_- in Theorem 3.3, we have

$$(3.26) \quad \text{Im } \varphi_-(x, y) \geq c \sum_{j=1}^{2n} |x_j - y_j|^2, \quad c > 0,$$

in some neighbourhood of $(0, 0)$ and

$$(3.27) \quad \begin{aligned} \varphi_-(x, y) = & -x_{2n+1} + y_{2n+1} + i \sum_{j=1}^{n-1} |\mu_j| |z_j - w_j|^2 \\ & + \sum_{j=1}^{n-1} \left(i\mu_j (\bar{z}_j w_j - z_j \bar{w}_j) + c_j (-z_j x_{2n+1} + w_j y_{2n+1}) \right. \\ & \left. + \bar{c}_j (-\bar{z}_j x_{2n+1} + \bar{w}_j y_{2n+1}) \right) + (x_{2n+1} - y_{2n+1}) f(x, y) \\ & + O(|(x, y)|^3), \end{aligned}$$

where f is smooth and satisfies $f(0, 0) = 0$, $f(x, y) = \bar{f}(y, x)$.

Remark 3.5. If we go through the proofs of Theorems 3.2 and 3.3 (see [40]), it is not difficult to see that Theorems 3.2 and 3.3 have straightforward generalizations to the case when the functions take values in $\Lambda^q T^{*(0,1)}X \otimes F$, for a given smooth CR vector bundle F over X . We recall that F is a CR vector bundle if its transition functions are CR.

Remark 3.6. Let $\hat{\varphi} \in \mathcal{C}^\infty(B \times B)$. We assume that $\hat{\varphi}$ satisfies (3.18)–(3.19), (3.21) and (3.26), (3.27). Then it is well-known (see [40, Section 3,7] and Menikoff–Sjöstrand [55]) that $\hat{\varphi}(x, y)t$, $t > 0$, and $\varphi_-(x, y)t$, $t > 0$, are

equivalent at each point of $\text{diag}((\Sigma^- \cap T^*B) \times (\Sigma^- \cap T^*B))$ in the sense of Melin–Sjöstrand (see Melin–Sjöstrand [54, p. 172]). We recall briefly that $\widehat{\varphi}(x, y)t$, $t > 0$, and $\varphi_-(x, y)t$, $t > 0$, are equivalent at each point of

$$\text{diag}((\Sigma^- \cap T^*B) \times (\Sigma^- \cap T^*B))$$

if for every

$$(x_0, -\lambda_0\omega_0) = (x_0, \lambda_0 d_x \varphi_-(x_0, x_0)) = (x_0, \lambda_0 d_x \widehat{\varphi}(x_0, x_0)) \in \Sigma^- \cap T^*B,$$

there is a conic neighbourhood Γ of (x_0, x_0, λ_0) , such that for every $a(x, y, t) \in S_{\text{cl}}^m(B \times B \times \mathbb{R}_+)$, $m \in \mathbb{Z}$, with support in Γ , we can find $\widehat{a}(x, y, t) \in S_{\text{cl}}^m(B \times B \times \mathbb{R}_+)$ with support in Γ , such that

$$\int_0^\infty e^{i\varphi_-(x, y)t} a(x, y, t) dt \equiv \int_0^\infty e^{i\widehat{\varphi}(x, y)t} \widehat{a}(x, y, t) dt$$

and vice versa, where S_{cl}^m denotes the classical symbol of order m (see [36, p. 38] for the definition of S_{cl}^m).

If $\omega \in T_x^{*(0,1)}X$, as (2.1), we let $(\omega \wedge)^* : \Lambda^{q+1}T_x^{*(0,1)}X \rightarrow \Lambda^q T_x^{*(0,1)}X$, $q \geq 0$, denote the adjoint of left exterior multiplication $\omega \wedge : \Lambda^q T_x^{*(0,1)}X \rightarrow \Lambda^{q+1}T_x^{*(0,1)}X$.

The following formula for the principal symbol s_-^0 on the diagonal follows from [40, Section 8], its calculation being local in nature.

Theorem 3.7. *Let $q = n_-$. For a given point $x_0 \in X$, let $\{W_j\}_{j=1}^{n_-}$ be an orthonormal frame of $T^{(1,0)}X$ near x_0 , for which the Levi form is diagonal at x_0 . Put $\mathcal{L}_{x_0}(W_j, \overline{W}_j) = \mu_j(x_0)$, $j = 1, \dots, n_-$. Let $\{T_j\}_{j=1}^{n_-}$ denote the dual basis of $T^{*(0,1)}X$, dual to $\{\overline{W}_j\}_{j=1}^{n_-}$. We assume that $\mu_j(x_0) < 0$ if $1 \leq j \leq n_-$. Then, for $s_-^0(x, y)$ in (3.16), we have*

$$s_-^0(x_0, x_0) = \frac{1}{2} |\mu_1(x_0)| \cdots |\mu_{n_-}(x_0)| \pi^{-n_-1} \prod_{j=1}^{n_-} (T_j(x_0) \wedge) \circ (T_j(x_0) \wedge)^*.$$

We return now to the situation where X is the Grauert tube of a line bundle L as in Section 3.1 and use the notations introduced there. Let (z, θ) be the coordinates as in (3.3), (3.4) on $B = D \times]-\varepsilon_0, \varepsilon_0[$, $\varepsilon_0 > 0$, $D \Subset M$. Until further notice, we work with the local coordinates $(z, \theta) = (x', x_{2n+1}) = x$. If we denote the holomorphic coordinates of D by $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \dots, n$, and by y_{2n+1} the coordinate of $]-\varepsilon_0, \varepsilon_0[$, we

also write $(w, y_{2n+1}) = (y', y_{2n+1}) = y$, $y' = (y_1, \dots, y_{2n})$. Let ξ be the dual variables of x . From (3.9) and (3.10), we can check that the principal symbol of $\square_b^{(q)}$ satisfies

$$(3.28) \quad p_0(x, \xi) = p_0(x', \xi).$$

That is, the principal symbol of $\square_b^{(q)}$ is independent of x_{2n+1} .

Using (3.19) and recalling (3.5), we have

$$d_x \varphi_-(x, x) = -dx_{2n+1} + a(x') dx', \quad a \in \mathcal{C}^\infty.$$

Thus, near a given point $(x_0, x_0) \in B \times B$, we have $\frac{\partial \varphi_-}{\partial x_{2n+1}} \neq 0$. Using the Malgrange preparation theorem [39, Theorem 7.57], we have

$$(3.29) \quad \varphi_-(x, y) = g(x, y)(-x_{2n+1} + h(x', y))$$

in some neighbourhood of (x_0, x_0) , where $g, h \in \mathcal{C}^\infty$, $g(x, x) = 1$, $h(x', x) = x_{2n+1}$. Since $\text{Im } \varphi_- \geq 0$, it is not difficult to see that $\text{Im } h \geq 0$ in some neighbourhood of (x_0, x_0) . We may take B small enough so that (3.29) holds and $\text{Im } h \geq 0$ on $B \times B$. From the global theory of Fourier integral operators [54, Theorem 4.2], we see that $\varphi_-(x, y)t$ and $(-x_{2n+1} + h(x', y))t$ are equivalent in the sense of Melin–Sjöstrand. We can replace the phase φ_- by $-x_{2n+1} + h(x', y)$. Again from (3.19), we have

$$\frac{\partial h}{\partial x'}(x', x) dx' - dx_{2n+1} = -\omega_0(x) = -dx_{2n+1} + a(x') dx'.$$

Thus, $\frac{\partial h}{\partial x'}(x', x)$ is independent of x_{2n+1} . We conclude that

$$(3.30) \quad \frac{\partial h}{\partial x'}(x', x) dx' - dx_{2n+1} = \frac{\partial h}{\partial x'}(x', x') dx' - dx_{2n+1} = -\omega_0(x).$$

Similarly, we have

$$(3.31) \quad \frac{\partial h}{\partial y}(y', y) dy = dy_{2n+1} + \frac{\partial h}{\partial y'}(y', y') dy' = \omega_0(y).$$

Put

$$\widehat{\varphi} = -x_{2n+1} + y_{2n+1} + h(x', y').$$

Note that $-x_{2n+1} + h(x', y)$ satisfies (3.21). From this and (3.28), we have

$$\begin{aligned} p_0 \left(x, \left(\frac{\partial h}{\partial x'}(x', y), -1 \right) \right) &= p_0 \left(x', \left(\frac{\partial h}{\partial x'}(x', y), -1 \right) \right) \\ &= f(x, y)(-x_{2n+1} + h(x', y)) + O(|x - y|^N) \end{aligned}$$

for all $N \in \mathbb{N}$, for some $f \in \mathcal{C}^\infty$. Hence,

$$(3.32) \quad \begin{aligned} p_0(x, \widehat{\varphi}'_x) &= p_0(x', \widehat{\varphi}'_x) \\ &= f(x, y')(-x_{2n+1} + h(x', y')) + O(|x' - y'|^N + |x_{2n+1}|^N), \end{aligned}$$

for all $N \in \mathbb{N}$. We replace x_{2n+1} by $x_{2n+1} - y_{2n+1}$ in (3.32) and get

$$(3.33) \quad p_0(x, \widehat{\varphi}'_x) = p_0(x', \widehat{\varphi}'_x) = \widehat{f}(x, y)\widehat{\varphi} + O(|x - y|^N),$$

for all $N \in \mathbb{N}$, for some $\widehat{f} \in \mathcal{C}^\infty$. Thus, $\widehat{\varphi}$ satisfies (3.21). Note that $p_0(x, \widehat{\varphi}'_x)$ is independent of x_{2n+1} . Take $x_{2n+1} = y_{2n+1} + h(x', y')$ in (3.33) and note that $h(x', x') = 0$, we conclude that

$$(3.34) \quad p_0(x, \widehat{\varphi}'_x) = O(|x' - y'|^N), \quad \forall N \in \mathbb{N}.$$

Furthermore, from (3.30) and (3.31), we see that $\widehat{\varphi}$ satisfies (3.19). Moreover, for a given point $p \in D$, we may take local coordinates $z = (z_1, \dots, z_n)$ centred at p such that

$$(3.35) \quad \begin{aligned} \Theta(p) &= \sqrt{-1} \sum_{j=1}^n dz_j \wedge d\bar{z}_j, \\ \phi(z) &= \sum_{j=1}^n \lambda_j |z_j|^2 + O(|z|^3), \quad z \text{ near } p, \quad \{\lambda_j\}_{j=1}^n \subset \mathbb{R} \setminus \{0\}. \end{aligned}$$

From (3.27) and (3.29), it is not difficult to see that

$$(3.36) \quad h(x', y') = i \sum_{j=1}^n |\lambda_j| |z_j - w_j|^2 + i \sum_{j=1}^n \lambda_j (\bar{z}_j w_j - z_j \bar{w}_j) + O(|(x', y')|^3).$$

Thus $\widehat{\varphi}$ satisfies (3.27). Formula (3.36) and the Taylor expansion of $h(x', y')$ at $x' = y'$ yield

$$\operatorname{Im} h(x', y') \geq c |x' - y'|^2, \quad c > 0.$$

Thus, $\widehat{\varphi} = 0$ if and only if $x = y$. We conclude that $\widehat{\varphi}$ satisfies (3.18)–(3.19), (3.21), (3.26) and (3.27). In view of Remark 3.6, we see that $t\varphi_-$ and $t\widehat{\varphi}$ are equivalent at each point of $\operatorname{diag}((\Sigma^- \cap T^*B) \times (\Sigma^- \cap T^*B))$ in the sense of

Melin–Sjöstrand. Since $\varphi_-(x, y) = -\overline{\varphi_-}(y, x)$, we can replace φ_- by

$$\frac{1}{2}(\widehat{\varphi}(x, y) - \overline{\widehat{\varphi}(y, x)}) = (-x_{2n+1} + y_{2n+1}) + \frac{1}{2}(h(x', y') - \overline{h}(y', x')).$$

Summing up, we get the following.

Theorem 3.8. *Let (L, h^L) be a holomorphic Hermitian line bundle over M whose curvature R^L is non-degenerate of constant signature (n_-, n_+) at each point of an open set $D \Subset M$. We assume that L is trivial on D . Let $\pi : X \rightarrow M$ be the Grauert tube of L (cf. (3.1)) and let $B = \pi^{-1}(D)$. With the notations used before, we can take the phase $\varphi_-(x, y)$ in Theorem 3.3 so that*

(3.37)

$$\varphi_-(x, y) = -x_{2n+1} + y_{2n+1} + \Psi(z, w), \Psi(x', y') = \Psi(z, w) \in \mathcal{C}^\infty,$$

$$p_0(x, \varphi'_-(x, y)) = O(|x' - y'|^N), \text{ locally uniformly on } B \times B, \text{ for all } N \in \mathbb{N},$$

where $p_0(x, \xi)$ is the principal symbol of $\square_b^{(q)}$ and Ψ satisfies

(3.38)

$$\Psi(z, w) = -\overline{\Psi}(w, z), \exists c > 0 : \operatorname{Im} \Psi \geq c|z - w|^2, \Psi(z, w) = 0 \Leftrightarrow z = w.$$

For a given point $p \in D$, let $z = (z_1, \dots, z_n)$ be local holomorphic coordinates centred at p satisfying (3.35). Then, near $(0, 0)$, we have

(3.39)

$$\Psi(z, w) = i \sum_{j=1}^n |\lambda_j| |z_j - w_j|^2 + i \sum_{j=1}^n \lambda_j (\overline{z}_j w_j - z_j \overline{w}_j) + O(|(z, w)|^3).$$

From now on, we assume that φ_- has the form (3.37).

3.3. Semi-classical Hodge decomposition for the Kodaira Laplacian

In this Section we apply the results about the Szegö kernel previously deduced in order to describe the semiclassical behaviour of the spectrum of the Kodaira Laplacian $\square_k^{(q)}$. We work locally in the following setup.

Setup 3.9. Let (M, Θ) be a Hermitian manifold, (L, h^L) be a holomorphic Hermitian line bundle on M . Assume that the curvature $\sqrt{-1}R^L$ is non-degenerate of constant signature (n_-, n_+) on the domain of a chart $(D, z) \cong (D, x) \Subset M$. Assume that $L|_D$ is trivial and let s be a frame of $L|_D$ and set $|s|_{h^L}^2 = e^{-2\phi}$.

We introduce some notations. For an open set $D \Subset M$ and any k -dependent continuous function

$$F_k : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{loc}}^{s'}(D, \Lambda^q T^{*(0,1)} M), \quad s, s' \in \mathbb{R},$$

we write

$$F_k = O(k^{n_0}) : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{loc}}^{s'}(D, \Lambda^q T^{*(0,1)} M), \quad n_0 \in \mathbb{Z},$$

if for any $\chi_0, \chi_1 \in \mathcal{C}_0^\infty(D)$, there is a positive constant c , c is independent of k , such that

$$(3.40) \quad \|(\chi_0 F_k \chi_1) u\|_{s'} \leq c k^{n_0} \|u\|_s, \quad \forall u \in H_{\text{loc}}^s(D, \Lambda^q T^{*(0,1)} M),$$

where $\|u\|_s$ is the usual Sobolev norm of order s .

A k -dependent smoothing operator $A_k : \Omega_0^{0,q}(D) \rightarrow \Omega^{0,q}(D)$ is called k -negligible if the kernel $A_k(x, y)$ of A_k satisfies $|\partial_x^\alpha \partial_y^\beta A_k(x, y)| = O(k^{-N})$ locally uniformly on every compact set in $D \times D$, for all multi-indices α, β and all $N \in \mathbb{N}$. A_k is k -negligible if and only if

$$A_k = O(k^{-N'}) : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{loc}}^{s+N}(D, \Lambda^q T^{*(0,1)} M),$$

for all $N, N' \geq 0$ and $s \in \mathbb{Z}$. Let $C_k : \Omega_0^{0,q}(D) \rightarrow \Omega^{0,q}(D)$ be another k -dependent smoothing operator. We write $A_k \equiv C_k \pmod{O(k^{-\infty})}$ or

$$A_k(x, y) \equiv C_k(x, y) \pmod{O(k^{-\infty})}$$

if $A_k - C_k$ is k -negligible.

We recall the definition of semi-classical Hörmander symbol spaces:

Definition 3.10. Let U be an open set in \mathbb{R}^N . Let $S(1; U) = S(1)$ be the set of $a \in \mathcal{C}^\infty(U)$ such that for every $\alpha \in \mathbb{N}_0^N$, there exists $C_\alpha > 0$, such that $|\partial_x^\alpha a(x)| \leq C_\alpha$ on U . If $a = a(x, k)$ depends on $k \in]1, \infty[$, we say that $a(x, k) \in S_{\text{loc}}(1)$ if $\chi(x) a(x, k)$ uniformly bounded in $S(1)$ when k varies in $]1, \infty[$, for any $\chi \in \mathcal{C}_0^\infty(U)$. For $m \in \mathbb{R}$, we put $S_{\text{loc}}^m(1) = k^m S_{\text{loc}}(1)$. If $a_j \in S_{\text{loc}}^{m_j}(1)$, $m_j \searrow -\infty$, we say that $a \sim \sum_{j=0}^\infty a_j$ in $S_{\text{loc}}^{m_0}(1)$ if $a - \sum_{j=0}^{N_0} a_j \in S_{\text{loc}}^{m_{N_0+1}}(1)$ for every N_0 . From this, we form $S_{\text{loc}}^m(1; Y, E)$ in the natural way, where Y is a smooth paracompact manifold and E is a vector bundle over Y .

Let D, s, ϕ be as in Setup 3.9. Let (z, θ) be the local coordinates as in (3.3) and (3.4) defined on $D \times]-\varepsilon_0, \varepsilon_0[$, $\pi \geq \varepsilon_0 > 0$. Let $\square_s^{(q)}$ be the

operator as in (3.8) and (3.7). Since $\partial\bar{\partial}\phi$ has constant signature (n_-, n_+) at each point of D , from (3.6), we know that the Levi form \mathcal{L} has constant signature (n_-, n_+) at each point of $D \times] - \varepsilon_0, \varepsilon_0[$.

Let $q = n_-$ or n_+ and let S_-, S_+ be the approximate Szegö kernels defined in Theorem 3.3. Define also the approximate Szegö kernel

$$(3.41) \quad S = S_- + S_+.$$

Let $\chi(\theta), \chi_1(\theta) \in \mathcal{C}_0^\infty(] - \varepsilon_0, \varepsilon_0[)$, $\chi, \chi_1 \geq 0$. We assume that $\chi_1 = 1$ on $\text{Supp } \chi$. We take χ so that $\int \chi(\theta) d\theta = 1$. Put

$$(3.42) \quad \chi_k(\theta) = e^{-ik\theta} \chi(\theta).$$

The approximate Szegö kernel was introduced in (3.41). We introduce the *localized approximate Szegö kernel* \mathcal{S}_k by

$$(3.43) \quad \begin{aligned} \mathcal{S}_k : H_{\text{loc}}^s(D, \Lambda^q T^{*(0,1)} M) &\rightarrow H_{\text{loc}}^s(D, \Lambda^q T^{*(0,1)} M), \quad \forall s \in \mathbb{N}_0, \\ u(z) &\rightarrow \int e^{i\theta k} \chi_1(\theta) S(\chi_k u)(z, \theta) d\theta. \end{aligned}$$

Let $u(z) \in H_{\text{loc}}^s(D, \Lambda^q T^{*(0,1)} M)$, $s \in \mathbb{N}_0$. We have $\chi_k(\theta)u(z) \in H_{\text{loc}}^s(D \times] - \varepsilon_0, \varepsilon_0[, \Lambda^q T^{*(0,1)} X)$. From Theorem 3.3, we know that

$$S(\chi_k u) \in H_{\text{loc}}^s(D \times] - \varepsilon_0, \varepsilon_0[, \Lambda^q T^{*(0,1)} X).$$

From this, it is not difficult to see that

$$\int e^{i\theta k} \chi_1(\theta) S(\chi_k u)(z, \theta) d\theta \in H_{\text{loc}}^s(D, \Lambda^q T^{*(0,1)} M).$$

Thus, the localization \mathcal{S}_k is well-defined. Since S is properly supported, \mathcal{S}_k is properly supported, too. Moreover, from (3.15) and (3.43), we can check that

$$(3.44) \quad \mathcal{S}_k = O(k^s) : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M),$$

for all $s \in \mathbb{N}_0$.

Let $\mathcal{S}_k^* : \mathcal{D}'(D, \Lambda^q T^{*(0,1)} M) \rightarrow \mathcal{D}'(D, \Lambda^q T^{*(0,1)} M)$ be the formal adjoint of \mathcal{S}_k with respect to (\cdot, \cdot) . Then \mathcal{S}_k^* is also properly supported and we have

$$(3.45) \quad \mathcal{S}_k^* : \mathcal{E}'(D, \Lambda^q T^{*(0,1)} M) \rightarrow \mathcal{E}'(D, \Lambda^q T^{*(0,1)} M).$$

From (3.13), we have

$$(3.46) \quad \begin{aligned} \square_s^{(q)} \circ \left(\int e^{i\theta k} \chi_1(\theta) S(\chi_k u) d\theta \right) &= \int e^{i\theta k} (\square_b^{(q)}(\chi_1 S))(\chi_k u)(z, \theta) d\theta \\ &= \int e^{i\theta k} (\square_b^{(q)}(\chi_1 S \tilde{\chi}))(\chi_k u)(z, \theta) d\theta, \end{aligned}$$

where $\tilde{\chi} \in \mathcal{C}_0^\infty(] - \varepsilon_0, \varepsilon_0[)$, $\tilde{\chi} = 1$ on $\text{Supp } \chi$ and $\chi_1 = 1$ on $\text{Supp } \tilde{\chi}$ and $u \in \Omega_0^{0,q}(D)$. Note that $\square_b^{(q)}(\chi_1 S \tilde{\chi}) = \square_b^{(q)}(S \tilde{\chi}) - \square_b^{(q)}((1 - \chi_1) S \tilde{\chi})$. From Theorem 3.3, we know that $\square_b^{(q)} S$ is smoothing and the kernel of S is smoothing away the diagonal. Thus, $(1 - \chi_1) S \tilde{\chi}$ is smoothing. It follows that $\square_b^{(q)}((1 - \chi_1) S \tilde{\chi})$ is smoothing. We conclude that $\square_b^{(q)}(\chi_1 S \tilde{\chi})$ is smoothing. Let $K((z, \theta), (w, \eta)) \in \mathcal{C}^\infty$ be the distribution kernel of $\square_b^{(q)}(\chi_1 S \tilde{\chi})$, where $w = (w_1, \dots, w_n)$ are the local holomorphic coordinates of D and η is the coordinate of $] - \varepsilon_0, \varepsilon_0[$. From (3.46) and recall the form χ_k (see (3.42)), we see that the distribution kernel of $\square_s^{(q)} \mathcal{S}_k$ is given by

$$(3.47) \quad (\square_s^{(q)} \mathcal{S}_k)(z, w) = \int e^{i(\theta - \eta)k} K((z, \theta), (w, \eta)) \chi(\eta) d\eta d\theta.$$

For $N \in \mathbb{N}$, we have

$$(3.48) \quad \begin{aligned} \left| k^N (\square_s^{(q)} \mathcal{S}_k)(z, w) \right| &= \left| \int \left(\left(i \frac{\partial}{\partial \eta} \right)^N e^{i(\theta - \eta)k} \right) K((z, \theta), (w, \eta)) \chi(\eta) d\eta d\theta \right| \\ &= \left| \int e^{i(\theta - \eta)k} \left(-i \frac{\partial}{\partial \eta} \right)^N (K((z, \theta), (w, \eta)) \chi(\eta)) d\eta d\theta \right|. \end{aligned}$$

Thus, $(\square_s^{(q)} \mathcal{S}_k)(z, w) = O(k^{-N})$, locally uniformly for all $N \in \mathbb{N}$, and similarly for the derivatives. We deduce that

$$(3.49) \quad \square_s^{(q)} \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})}.$$

Thus,

$$(3.50) \quad \mathcal{S}_k^* \square_s^{(q)} \equiv 0 \pmod{O(k^{-\infty})}.$$

Let A be the partial parametrrix of $\square_b^{(q)}$ described in Theorem 3.3. Define the *localized partial parametrrix* \mathcal{A}_k by

$$(3.51) \quad \mathcal{A}_k : H_{\text{loc}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{loc}}^{s+1}(D, \Lambda^q T^{*(0,1)} M), \quad \forall s \in \mathbb{N}_0,$$

$$u(z) \mapsto \int e^{i\theta k} \chi_1 A(\chi_k u)(z, \theta) d\theta.$$

As above, we can show that \mathcal{A}_k is well-defined. Since A is properly supported, \mathcal{A}_k is properly supported, too. Moreover, from (3.15) and (3.51), we can check that

$$(3.52) \quad \mathcal{A}_k = O(k^s) : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{comp}}^{s+1}(D, \Lambda^q T^{*(0,1)} M),$$

for all $s \in \mathbb{N}_0$.

Let $\mathcal{A}_k^* : \mathcal{D}'(D, \Lambda^q T^{*(0,1)} M) \rightarrow \mathcal{D}'(D, \Lambda^q T^{*(0,1)} M)$ be the formal adjoint of \mathcal{A}_k with respect to (\cdot, \cdot) . We can check that

$$(\mathcal{A}_k^* v)(z) = \int \overline{\chi_k(\theta)} A^*(v e^{-i\theta k} \chi_1)(z, \theta) d\theta \in \Omega_0^{0,q}(D),$$

for all $v \in \Omega_0^{0,q}(D)$. Thus, $\mathcal{A}_k^* : \Omega_0^{0,q}(D) \rightarrow \Omega_0^{0,q}(D)$. Moreover, as before, we can show that

$$(3.53) \quad \mathcal{A}_k^* = O(k^s) : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{comp}}^{s+1}(D, \Lambda^q T^{*(0,1)} M), \quad \forall s \in \mathbb{N}_0.$$

Let $u \in \Omega_0^{0,q}(D)$. From (3.13), we have

$$\square_s^{(q)}(\mathcal{A}_k u) = \square_s^{(q)} \circ \left(\int e^{i\theta k} \chi_1 A(\chi_k u) d\theta \right) = \int e^{i\theta k} (\square_b^{(q)} \chi_1 A \tilde{\chi})(\chi_k u)(z, \theta) d\theta,$$

where $\tilde{\chi}$ is as in (3.46). Note that $\square_b^{(q)}(\chi_1 A \tilde{\chi}) = \square_b^{(q)}(A \tilde{\chi}) - \square_b^{(q)}((1 - \chi_1) A \tilde{\chi})$. From Theorem 3.3, we know that $\square_b^{(q)} A + S = I$ and the kernel of A is smoothing away the diagonal. Thus, $(1 - \chi_1) A \tilde{\chi}$ is smoothing. It follows that $\square_b^{(q)}((1 - \chi_1) A \tilde{\chi})$ is smoothing. We conclude that $\square_b^{(q)}(\chi_1 A \tilde{\chi}) \equiv (I - S) \tilde{\chi}$.

From this, we get

$$\begin{aligned}
(3.54) \quad \square_s^{(q)}(\mathcal{A}_k u) &= \int e^{i\theta k} (I - S)(\chi_k u)(z, \theta) d\theta + \int e^{i\theta k} F(\chi_k u)(z, \theta) d\theta \\
&= u - \int e^{i\theta k} S(\chi_k u)(z, \theta) d\theta + \int e^{i\theta k} F(\chi_k u)(z, \theta) d\theta \\
&= (I - \mathcal{S}_k)u - \int e^{i\theta k} (1 - \chi_1)S(\chi_k u)(z, \theta) d\theta \\
&\quad + \int e^{i\theta k} F(\chi_k u)(z, \theta) d\theta,
\end{aligned}$$

where F is a smoothing operator. We can repeat the procedure as in (3.48) and conclude that the operator

$$u \rightarrow \int e^{i\theta k} F(\chi_k u)(z, \theta) d\theta, \quad u \in \Omega_0^{0,q}(D),$$

is k -negligible. Similarly, since $(1 - \chi_1)S\chi$ is smoothing, the operator

$$u \rightarrow \int e^{i\theta k} (1 - \chi_1)S(\chi_k u)(z, \theta) d\theta, \quad u \in \Omega_0^{0,q}(D),$$

is also k -negligible. Summing up, we obtain

$$(3.55) \quad \square_s^{(q)} \mathcal{A}_k + \mathcal{S}_k \equiv I \pmod{O(k^{-\infty})}.$$

We may replace \mathcal{S}_k by $I - \square_s^{(q)} \mathcal{A}_k$ and we have $\square_s^{(q)} \mathcal{A}_k + \mathcal{S}_k = I$ and hence $\mathcal{A}_k^* \square_s^{(q)} + \mathcal{S}_k^* = I$. Thus,

$$(3.56) \quad \mathcal{S}_k = (\mathcal{A}_k^* \square_s^{(q)} + \mathcal{S}_k^*) \mathcal{S}_k = \mathcal{A}_k^* \square_s^{(q)} \mathcal{S}_k + \mathcal{S}_k^* \mathcal{S}_k.$$

From (3.49) and (3.53), we see that

$$\mathcal{A}_k^* \square_s^{(q)} \mathcal{S}_k = O(k^{-N'}) : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{comp}}^{s+N}(D, \Lambda^q T^{*(0,1)} M'),$$

for all $s \in \mathbb{Z}$ and $N', N \in \mathbb{N}$. Thus, $\mathcal{A}_k^* \square_s^{(q)} \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})}$. From this and (3.56), we get

$$(3.57) \quad \mathcal{S}_k^* \mathcal{S}_k \equiv \mathcal{S}_k \pmod{O(k^{-\infty})}.$$

It follows that

$$(3.58) \quad \mathcal{S}_k \equiv \mathcal{S}_k^* \pmod{O(k^{-\infty})}, \quad \mathcal{S}_k^2 \equiv \mathcal{S}_k \pmod{O(k^{-\infty})}.$$

From (3.44), (3.49), (3.50), (3.52), (3.53), (3.55), (3.57) and (3.58), we get our main technical result:

Theorem 3.11. *In the situation of Setup 3.9 let $q = n_-$ or n_+ and let \mathcal{S}_k be the localized approximate Szegö kernel (3.43) and \mathcal{A}_k the localized partial parametrix (3.51). Then,*

$$(3.59) \quad \begin{aligned} \mathcal{S}_k = O(k^s) &: H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M), \quad \forall s \in \mathbb{N}_0, \\ \mathcal{A}_k = O(k^s) &: H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{comp}}^{s+1}(D, \Lambda^q T^{*(0,1)} M), \quad \forall s \in \mathbb{N}_0, \end{aligned}$$

and we have

$$(3.60) \quad \square_s^{(q)} \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})}, \quad \mathcal{S}_k^* \square_s^{(q)} \equiv 0 \pmod{O(k^{-\infty})},$$

$$(3.61) \quad \mathcal{S}_k \equiv \mathcal{S}_k^* \pmod{O(k^{-\infty})}, \quad \mathcal{S}_k \equiv \mathcal{S}_k^2 \equiv \mathcal{S}_k^* \mathcal{S}_k \pmod{O(k^{-\infty})},$$

$$(3.62) \quad \mathcal{S}_k^* + \mathcal{A}_k^* \square_s^{(q)} \equiv I \pmod{O(k^{-\infty})}, \quad \mathcal{S}_k + \square_s^{(q)} \mathcal{A}_k \equiv I \pmod{O(k^{-\infty})},$$

where \mathcal{S}_k^* and \mathcal{A}_k^* are the formal adjoints of \mathcal{S}_k and \mathcal{A}_k with respect to (\cdot, \cdot) respectively and $\square_s^{(q)}$ is given by (3.8) and (3.7).

We notice that $\mathcal{S}_k, \mathcal{S}_k^*, \mathcal{A}_k, \mathcal{A}_k^*$ are all properly supported. We need

Theorem 3.12. *The localized approximate Szegö kernel \mathcal{S}_k given by (3.43) is a smoothing operator. Moreover, if $q = n_-$, then the kernel of \mathcal{S}_k satisfies*

$$(3.63) \quad \mathcal{S}_k(z, w) \equiv e^{ik\Psi(z, w)} b(z, w, k) \pmod{O(k^{-\infty})},$$

with

$$(3.64) \quad \begin{aligned} b(z, w, k) &\in S_{\text{loc}}^n(1; D \times D, \Lambda^q T_w^{*(0,1)} M \boxtimes \Lambda^q T_z^{*(0,1)} M), \\ b(z, w, k) &\sim \sum_{j=0}^{\infty} b_j(z, w) k^{n-j} \text{ in } S_{\text{loc}}^n(1; D \times D, \Lambda^q T_w^{*(0,1)} M \boxtimes \Lambda^q T_z^{*(0,1)} M), \\ b_j(z, w) &\in \mathcal{C}^\infty(D \times D, \Lambda^q T_w^{*(0,1)} M \boxtimes \Lambda^q T_z^{*(0,1)} M), \quad j = 0, 1, 2, \dots, \end{aligned}$$

and $\Psi(z, w)$ is as in Theorem 3.8.

If $q = n_+$, $n_- \neq n_+$, then

$$(3.65) \quad \mathcal{S}_k(z, w) \equiv 0 \pmod{O(k^{-\infty})}.$$

Proof. Theorem 3.12 essentially follows from the stationary phase formula of Melin–Sjöstrand [54]. Let D, s, ϕ be as in Setup 3.9. Let $q = n_-$ or n_+ . Let $(z, \theta) = x = (x', x_{2n+1})$ be the local coordinates as in (3.3), (3.4) defined on $D \times] - \varepsilon_0, \varepsilon_0[$. We identify x' with $(x', 0)$. If we denote the holomorphic coordinates of D by $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \dots, n$, and by η the coordinate of $] - \varepsilon_0, \varepsilon_0[$, we also write

$$(w, \eta) = (y', \eta) = y, \quad y' = (y_1, \dots, y_{2n}).$$

From Definition (3.43) of \mathcal{S}_k and Theorem 3.3, we see that the distribution kernel of \mathcal{S}_k is given by

$$\begin{aligned} \mathcal{S}_k(x', y') &\equiv \int_{t \geq 0} e^{i\varphi_-(x,y)t + i\theta k - i\eta k} s_-(x, y, t) \chi_1(\theta) \chi(\eta) d\theta dt d\eta \\ (3.66) \quad &+ \int_{t \geq 0} e^{i\varphi_+(x,y)t + i\theta k - i\eta k} s_+(x, y, t) \chi_1(\theta) \chi(\eta) d\theta dt d\eta \quad \text{mod } O(k^{-\infty}) \\ &\equiv I_0(x', y') + I_1(x', y') \quad \text{mod } O(k^{-\infty}), \end{aligned}$$

where the integrals above are defined as oscillatory integrals. First, we study the kernel

$$I_1(x', y') = \int_{t \geq 0} e^{i\varphi_+(x,y)t + i\theta k - i\eta k} s_+(x, y, t) \chi_1(\theta) \chi(\eta) d\theta dt d\eta.$$

By the change of variables $t = k\sigma$ we get

$$I_1(x', y') = \int_{\sigma \geq 0} e^{ik(\varphi_+(x,y)\sigma + \theta - \eta)} k s_+(x, y, k\sigma) \chi_1(\theta) \chi(\eta) d\theta d\sigma d\eta.$$

Note that $d_x \varphi_+(x, x) = \omega_0(x)$. Taking into account the form $\omega_0(x)$ (cf. (3.5)), we see that $\frac{\partial \varphi_+}{\partial \theta}(x, x) = 1$. In view of Theorem 3.3, we see that $\varphi_+(x, y) = 0$ if and only if $x = y$. We conclude that

$$(d_\sigma(\varphi_+(x, y)\sigma + \theta - \eta), d_\theta(\varphi_+(x, y)\sigma + \theta - \eta)) \neq (0, 0), \quad \sigma \geq 0.$$

Thus, we can integrate by parts in σ and θ and conclude that I_1 is smoothing and

$$(3.67) \quad I_1 \equiv 0 \quad \text{mod } O(k^{-\infty}).$$

Now, we study the kernel

$$I_0(x', y') = \int_{t \geq 0} e^{i\varphi_-(x,y)t + i\theta k - i\eta k} s_-(x, y, t) \chi_1(\theta) \chi(\eta) d\theta dt d\eta.$$

As before, by letting $t = k\sigma$, we get

$$(3.68) \quad I_0(x', y') = \int_{\sigma \geq 0} e^{ik(\varphi_-(x,y)\sigma + \theta - \eta)} k s_-(x, y, k\sigma) \chi_1(\theta) \chi(\eta) d\theta d\sigma d\eta.$$

In view of (3.37), we see that

$$(3.69) \quad \varphi_-(x, y) = \Psi(x', y') + \eta - \theta, \quad \text{Im } \Psi(x', y') \geq 0.$$

Put

$$(3.70) \quad \Psi(x, y, \sigma) = (\Psi(x', y') + \eta - \theta)\sigma + \theta - \eta.$$

Let $\varphi(\sigma) \in \mathcal{C}_0^\infty(\mathbb{R}_+)$ with $\varphi(\sigma) = 1$ in some small neighbourhood of 1. We introduce the cut-off functions $\varphi(\sigma)$ and $1 - \varphi(\sigma)$ in the integral (3.68):

$$(3.71) \quad I_0^0(x', y') := \int_{\sigma \geq 0} e^{ik\Psi(x,y,\sigma)} \varphi(\sigma) k s_-(x, y, k\sigma) \chi_1(\theta) \chi(\eta) d\theta d\sigma d\eta,$$

$$(3.72) \quad I_0^1(x', y') := \int_{\sigma \geq 0} e^{ik\Psi(x,y,\sigma)} (1 - \varphi(\sigma)) k s_-(x, y, k\sigma) \chi_1(\theta) \chi(\eta) d\theta d\sigma d\eta,$$

so that

$$I_0(x', y') = I_0^0(x', y') + I_0^1(x', y').$$

First, we study $I_0^1(x', y')$. Note that when $\sigma \neq 1$, $d_\theta \Psi(x, y, \sigma) = 1 - \sigma \neq 0$. Thus, we can integrate by parts and get that I_0^1 is smoothing and $I_0^1(x', y') \equiv 0 \pmod{O(k^{-\infty})}$.

Next, we study the kernel $I_0^0(x', y')$. First, we assume that $q = n_+$, $n_+ \neq n_-$. In view of Theorem 3.3, we see that $s_-(x, y, t)$ vanishes to infinite order at $x = y$. From this observation, it is straightforward to see that $I_0^0 \equiv 0 \pmod{O(k^{-\infty})}$. Therefore, we get (3.65).

Now, we assume that $q = n_-$. Since the integral (3.71) converges, we have

$$(3.73) \quad \begin{aligned} I_0^0(x', y') &= \int H(x', y) \chi(\eta) d\eta, \\ H(x', y) &= \int_{\sigma \geq 0} e^{ik\Psi(x,y,\sigma)} \varphi(\sigma) k s_-(x, y, k\sigma) \chi_1(\theta) d\theta d\sigma. \end{aligned}$$

Recalling the form of $\Psi(x, y, \sigma)$, we have $\text{Im } \Psi(x, y, \sigma) \geq 0$, $d_\sigma \Psi(x, y, \sigma) = 0$ if and only if $x = y$ and $d_\theta \Psi(x, y, \sigma)|_{x=y} = 1 - \sigma$. Thus, $x = y$ and $\sigma = 1$ are

real critical points. Moreover, we can check that the Hessian of $\Psi(x, y, \sigma)$ at $x = y, \sigma = 1$, is given by

$$\begin{pmatrix} \Psi''_{\sigma\sigma}(x, x, 1) & \Psi''_{\theta\sigma}(x, x, 1) \\ \Psi''_{\sigma\theta}(x, x, 1) & \Psi''_{\theta\theta}(x, x, 1) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Thus, $\Psi(x, y, \sigma)$ is a non-degenerate complex valued phase function in the sense of Melin–Sjöstrand [54]. Let

$$\tilde{\Psi}(\tilde{x}, \tilde{y}, \tilde{\sigma}) := (\tilde{\Psi}(\tilde{x}', \tilde{y}') + (\tilde{\eta} - \tilde{\theta}))\tilde{\sigma} + \tilde{\theta} - \tilde{\eta}$$

be an almost analytic extension of $\Psi(x, y, \sigma)$, where $\tilde{\Psi}(\tilde{x}', \tilde{y}')$ is an almost analytic extension of $\Psi(x', y')$ (with $\Psi(x', y')$ as in (3.69)) and similarly for $\tilde{\eta}, \tilde{\theta}$ and $\tilde{\sigma}$ (see [54, Section 2] for the precise meaning of the almost analytic extension). We can check that given η and (x', y') , $\theta = \eta + \Psi(x', y')$, $\tilde{\sigma} = 1$ are the solutions of

$$\frac{\partial \tilde{\Psi}}{\partial \tilde{\sigma}} = 0, \quad \frac{\partial \tilde{\Psi}}{\partial \tilde{\theta}} = 0.$$

From this and by the stationary phase formula of Melin–Sjöstrand [54], we get

$$(3.74) \quad H(x', y) \equiv e^{ik\Psi(x', y')} a(x', y, k) \pmod{O(k^{-\infty})},$$

where $a(x', y, k) \in \mathcal{C}^\infty(D \times (D \times] - \varepsilon_0, \varepsilon_0[), \Lambda^q T^{*(0,1)} M \boxtimes \Lambda^q T^{*(0,1)} M)$,

$$\begin{aligned} a(x', y, k) &\sim \sum_{j=0}^{\infty} k^{n-j} a_j(x', y) \\ &\text{in } S_{\text{loc}}^n(1; D \times (D \times] - \varepsilon_0, \varepsilon_0[), \Lambda^q T^{*(0,1)} M \boxtimes \Lambda^q T^{*(0,1)} M), \end{aligned}$$

$$a_j(x', y) \in \mathcal{C}^\infty(D \times (D \times] - \varepsilon_0, \varepsilon_0[), \Lambda^q T^{*(0,1)} M \boxtimes \Lambda^q T^{*(0,1)} M), \quad j \in \mathbb{N}_0,$$

and

$$(3.75) \quad a_0(x', y) = 2\pi \tilde{s}_-^0((x', \eta + \Psi(x', y')), y),$$

where \tilde{s}_-^0 is an almost analytic extension of s_-^0 , s_-^0 is as in (3.16). From (3.73) and (3.74) we get

$$(3.76) \quad I_0^0(x', y') \equiv e^{ik\Psi(x', y')} b(x', y', k) \pmod{O(k^{-\infty})},$$

where

$$b(x', y', k) \sim \sum_{j=0}^{\infty} k^{n-j} b_j(x', y') \text{ in } S_{\text{loc}}^n(1; D \times D, \Lambda^q T^{*(0,1)} M \boxtimes \Lambda^q T^{*(0,1)} M),$$

with

$$(3.77) \quad \begin{aligned} & b_j(x', y') \\ &= \int a_j(x', y) \chi(\eta) d\eta \in \mathcal{C}^\infty(D \times D, \Lambda^q T^{*(0,1)} M \boxtimes \Lambda^q T^{*(0,1)} M), \quad j \in \mathbb{N}_0. \end{aligned}$$

Theorem 3.12 follows. \square

Let D , s , ϕ be as in Setup 3.9. In view of Theorem 3.11 and (3.65), we see that when $q = n_+$, $n_+ \neq n_-$, we have

$$(3.78) \quad \square_s^{(q)} \mathcal{A}_k \equiv I \pmod{O(k^{-\infty})},$$

where \mathcal{A}_k is as in Theorem 3.11.

Now, we assume that $q \neq n_-, n_+$. Using Theorem 3.2 and repeating the proof of Theorem 3.11 we conclude that there exists a properly supported continuous operator

$$\mathcal{A}_k = O(k^s) : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{comp}}^{s+1}(D, \Lambda^q T^{*(0,1)} M), \quad \forall s \in \mathbb{N}_0,$$

such that

$$(3.79) \quad \square_s^{(q)} \mathcal{A}_k \equiv I \pmod{O(k^{-\infty})}.$$

Summing up, we obtain

Theorem 3.13. *In the situation of Setup 3.9 let $q \neq n_-$. Then, there exists a properly supported continuous operator*

$$\mathcal{A}_k = O(k^s) : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{comp}}^{s+1}(D, \Lambda^q T^{*(0,1)} M), \quad \forall s \in \mathbb{N}_0,$$

such that

$$\square_s^{(q)} \mathcal{A}_k \equiv I \pmod{O(k^{-\infty})}.$$

Remark 3.14. From Remark 3.5, we can generalize Theorem 3.11 and Theorem 3.13 with essentially the same proofs to the case when the forms take values in $L^k \otimes E$, for a given holomorphic vector bundle E over M .

We have the following

Theorem 3.15. *In the situation of Setup 3.9 let $q = n_-$. For a given point $p \in D$, let V_1, \dots, V_n be an orthonormal frame of $T^{(1,0)}M$ in a neighbourhood of p , for which \dot{R}^L is diagonalized at p , namely,*

$$\begin{aligned} \dot{R}^L(p)V_j(p) &= \lambda_j(p)V_j(p), \quad j = 1, \dots, n, \\ \lambda_j(p) &< 0, \quad j = 1, \dots, q, \\ \lambda_j(p) &> 0, \quad j = q + 1, \dots, n. \end{aligned}$$

Let $(T_j)_{j=1}^n$ denote the basis of $T^{*(0,1)}M$, which is dual to $(\bar{V}_j)_{j=1}^n$. Then,

$$\begin{aligned} (3.80) \quad b_0(p, p) &= (2\pi)^{-n} \left| \det \dot{R}^L(p) \right| \prod_{j=1}^q (T_j(p) \wedge) \circ (T_j(p) \wedge)^* \\ &= (2\pi)^{-n} \left| \det \dot{R}^L(p) \right| I_{\det \bar{W}^*}, \end{aligned}$$

where $I_{\det \bar{W}^*} \in \text{End}(\Lambda^q T^{*(0,1)}M)$ is as in the discussion after (1.2).

Proof. We use the same notations as in the proof of Theorem 3.12. From (3.75) and (3.77), we have

$$(3.81) \quad b_0(x', x') = 2\pi \int s_-^0((x', \eta), (x', \eta)) \chi(\eta) d\eta.$$

In view of Theorem 3.7, we know that

$$(3.82) \quad s_-^0((x', \eta), (x', \eta)) = \frac{1}{2} |\mu_1(x')| \cdots |\mu_n(x')| \pi^{-n-1} \prod_{j=1}^{n_-} (T_j(x') \wedge) \circ (T_j(x') \wedge)^*,$$

where $\{\mu_j(x')\}_{j=1}^n$ are the eigenvalues of $\mathcal{L}_{x'}$ and $\{T_j(x')\}_{j=1}^n$ are as in Theorem 3.7. Here, we identify $x' \in D$ with $(x', 0) \in X$. Note that

$$(3.83) \quad |\mu_1(p)| \cdots |\mu_n(p)| = 2^{-n} |\lambda_1(p)| \cdots |\lambda_n(p)| = 2^{-n} \left| \det \dot{R}^L(p) \right|.$$

Now, (3.6), (3.81), (3.82) and (3.83) yield (3.80). \square

4. Asymptotic expansion of the spectral function for lower energy forms

Let (M, Θ) be a Hermitian manifold and let (L, h^L) be a Hermitian holomorphic line bundle on M . We recall that (cf. (1.3)) $\mathcal{E}_k^q(M, L^k)$ denote the

spectral space of $\square_k^{(q)}$ corresponding to energy less than k^{-N_0} . In the present Section we study the asymptotic expansion of the spectral function associated to $\mathcal{E}_{k^{-N_0}}^q(M, L^k)$. In Section 4.1 we prove pointwise upper bounds for the eigenforms of the spectral spaces $\mathcal{E}_{k^{-N_0}}^q(M, L^k)$ in terms of their L^2 -norm (Theorem 4.3). In Section 4.2 we compare the localized spectral projection with the localized approximate Szegő projection \mathcal{S}_k . In Section 4.3, we apply this results to prove the asymptotic expansion of the spectral function and thus give the proof of Theorem 1.1. In Section 4.4, we exhibit the asymptotic expansion of the Bergman kernel and prove Theorem 1.6. Finally, in Section 4.5, we calculate the coefficients b_1^0 and b_2^0 and thus prove Theorem 1.2.

4.1. Asymptotic upper bounds

Fix $N_0 \geq 1$. In this Section we will give pointwise upper bounds for u and $\partial^\alpha u$, where $u \in \mathcal{E}_{k^{-N_0}}^q(M, L^k)$.

Let $D \Subset M$ be a chart domain such that $L|_D$ is trivial. Let s be a local frame of L on D and set $|s|_{h^L}^2 = e^{-2\phi}$. Let $(\cdot, \cdot)_{k\phi}$ be the inner product on the space $\Omega_0^{0,q}(D)$ defined as follows:

$$(f, g)_{k\phi} = \int_D \langle f, g \rangle e^{-2k\phi} dv_M(x), \quad f, g \in \Omega_0^{0,q}(D).$$

Let $\bar{\partial}^{*,k\phi} : \Omega^{0,q+1}(D) \rightarrow \Omega^{0,q}(D)$ be the formal adjoint of $\bar{\partial}$ with respect to $(\cdot, \cdot)_{k\phi}$. Put $\square_{k\phi}^{(q)} = \bar{\partial} \bar{\partial}^{*,k\phi} + \bar{\partial}^{*,k\phi} \bar{\partial} : \Omega^{0,q}(D) \rightarrow \Omega^{0,q}(D)$. Let $u \in \Omega^{0,q}(D, L^k)$. On D , we write $u = s^k \tilde{u}$, $\tilde{u} \in \Omega^{0,q}(D)$. We have

$$(4.1) \quad \square_k^{(q)} u = s^k \square_{k\phi}^{(q)} \tilde{u}.$$

Fix $p \in D$ and consider local coordinates $(D, z) \cong (D, x)$, such that $x(p) = z(p) = 0$ and $\phi(z) = O(|z|^2)$ near p . Let $F_k(z) := \frac{z}{\sqrt{k}}$ be the scaling map. For $r > 0$, let $D_r = \{x; |x_j| < r, j = 1, \dots, 2n\}$. Let $f \in \Omega^{0,q}(D_{\frac{\log k}{\sqrt{k}}})$, $f = \sum'_{|J|=q} f_J d\bar{z}^J$. We define the scaled form $F_k^* f \in \Omega^{0,q}(D_{\log k})$ by

$$F_k^* f = \sum'_{|J|=q} f_J (k^{-1/2} z) d\bar{z}^J \in \Omega^{0,q}(D_{\log k}).$$

Let $\square_{k\phi, (k)}^{(q)} : \Omega^{0,q}(D_{\log k}) \rightarrow \Omega^{0,q}(D_{\log k})$ be the scaled Laplacian defined by

$$(4.2) \quad \square_{k\phi, (k)}^{(q)} (F_k^* u) = \frac{1}{k} F_k^* (\square_{k\phi}^{(q)} u), \quad u \in \Omega^{0,q}(D_{\frac{\log k}{\sqrt{k}}}).$$

By Berman [1, Section 2] and Hsiao–Marinescu [43, Section 2] it is known all the derivatives of the coefficients of the operator $\square_{k\phi, (k)}^{(q)}$ are uniformly bounded in k on $D_{\log k}$. Let $D_r \subset D_{\log k}$ and let $W_{kF_k^*\phi}^s(D_r, \Lambda^q T^{*(0,1)} M)$, $s \in \mathbb{N}_0$, denote the Sobolev space of order s of sections of $\Lambda^q T^{*(0,1)} M$ over D_r with respect to the weight $e^{-2kF_k^*\phi}$. The Sobolev norm on this space is given by

$$\|u\|_{kF_k^*\phi, s, D_r}^2 = \sum'_{\alpha \in \mathbb{N}_0^{2n}, |\alpha| \leq s, |J|=q} \int_{D_r} |\partial_x^\alpha u_J|^2 e^{-2kF_k^*\phi} (F_k^* m)(x) dx,$$

where $u = \sum'_{|J|=q} u_J d\bar{z}^J \in W_{kF_k^*\phi}^s(D_r, \Lambda^q T^{*(0,1)} M)$ and $m(x)dx$ is the volume form. If $s = 0$, we write $\|\cdot\|_{kF_k^*\phi, D_r}$ to denote $\|\cdot\|_{kF_k^*\phi, 0, D_r}$.

Lemma 4.1. *For every $r > 0$ with $D_{2r} \subset D_{\log k}$ and $s \in \mathbb{N}_0$, there is a constant $C_{r,s} > 0$ independent of k , such that for every $u \in \Omega^{0,q}(D_{\log k})$,*

$$(4.3) \quad \|u\|_{kF_k^*\phi, 2s, D_r}^2 \leq C_{r,s} \left(\|u\|_{kF_k^*\phi, D_{2r}}^2 + \sum_{m=1}^s \|(\square_{k\phi, (k)}^{(q)})^m u\|_{kF_k^*\phi, D_{2r}}^2 \right).$$

Proof. Since $\square_{k\phi, (k)}^{(q)}$ is elliptic, we conclude from Gårding's inequality that for every $r > 0$, $D_{2r} \subset D_{\log k}$ and $s \in \mathbb{N}_0$, we have for every $u \in \Omega^{0,q}(D_{\log k})$,

$$(4.4) \quad \|u\|_{kF_k^*\phi, s+2, D_r}^2 \leq \tilde{C}_{r',s} \left(\|u\|_{kF_k^*\phi, D_{r'}}^2 + \|\square_{k\phi, (k)}^{(q)} u\|_{kF_k^*\phi, s, D_{r'}}^2 \right),$$

for some $r' > r$. Since all the derivatives of the coefficients of the operator $\square_{k\phi, (k)}^{(q)}$ are uniformly bounded in k , it is straightforward to see that $\tilde{C}_{r',s}$ can be taken to be independent of k . (See Proposition 2.4 and Remark 2.5 in Hsiao–Marinescu [43].) From (4.4) and using induction, we get (4.3). \square

Lemma 4.2. *For k large and for every $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $C_\alpha > 0$ independent of k , such that*

$$(4.5) \quad |(\partial_x^\alpha u)(0)| \leq C_\alpha,$$

where $u \in \Omega^{0,q}(D_{\log k})$, $\|u\|_{kF_k^*\phi, D_{\log k}} \leq 1$ and $\|(\square_{k\phi, (k)}^{(q)})^m u\|_{kF_k^*\phi, D_{\log k}} \leq k^{-m}$, $\forall m \in \mathbb{N}_0$.

Proof. Let $u \in \Omega^{0,q}(D_{\log k})$, $\|u\|_{kF_k^*\phi, D_{\log k}} \leq 1$, $\|(\square_{k\phi, (k)}^{(q)})^m u\|_{kF_k^*\phi, D_{\log k}} \leq k^{-m}$, $\forall m \in \mathbb{N}_0$. By using Fourier transform, it is easy to see that (cf.

Lemma 2.6 in [43])

$$(4.6) \quad |(\partial_x^\alpha u)(0)| \leq C \|u\|_{kF_k^* \phi, n+1+|\alpha|, D_r},$$

for some $r > 0$, where $C > 0$ only depends on the dimension and the length of α . From (4.3), we see that

$$(4.7) \quad \begin{aligned} \|u\|_{kF_k^* \phi, n+1+|\alpha|, D_r}^2 &\leq C_{r,\alpha} \left(\|u\|_{kF_k^* \phi, D_{2r}}^2 + \sum_{m=1}^N \left\| (\square_{k\phi, (k)}^{(q)})^m u \right\|_{kF_k^* \phi, D_{2r}} \right) \\ &\leq C_{r,\alpha} \left(1 + \sum_{m=1}^{\infty} k^{-m} \right) \leq \tilde{C}_\alpha \end{aligned}$$

if k large, where $2N \geq n + 1 + |\alpha|$ and $\tilde{C}_\alpha > 0$ is independent of k . Combining (4.6) with (4.7), (4.5) follows. \square

Now, we can prove

Theorem 4.3. *For k large and for every $\alpha \in \mathbb{N}_0^{2n}$, $D' \Subset D$, there is a constant $C_{\alpha, D'} > 0$ independent of k , such that*

$$(4.8) \quad \left| (\partial_x^\alpha (\tilde{u} e^{-k\phi}))(x) \right| \leq C_{\alpha, D'} k^{\frac{n}{2} + |\alpha|} \|u\|, \quad \forall x \in D',$$

where $u \in \mathcal{E}_{k-N_0}^q(M, L^k)$, $N_0 \geq 1$, $u|_D = s^k \tilde{u}$, $\tilde{u} \in \Omega^{0,q}(D)$.

Remark 4.4. Let s_1 be another local frame of L on D , $|s_1|^2 = e^{-2\phi_1}$. We have $s_1 = gs$ for some holomorphic function $g \in \mathcal{C}^\infty(D)$, $g \neq 0$ on D . Let $u \in \Omega^{0,q}(D, L^k)$. On D , we write $u = s^k \tilde{u} = s_1^k \tilde{v}$. Then, we can check that

$$(4.9) \quad \tilde{v} e^{-k\phi_1} = \tilde{u} (\bar{g}^{1/2} g^{-1/2})^k e^{-k\phi}.$$

From (4.9), it is easy to see that if \tilde{u} satisfies (4.8), then \tilde{v} also satisfies (4.8). Thus, the conclusion of Theorem 4.3 makes sense.

Proof of Theorem 4.3. We may assume that $0 \in D'$. Let $u \in \mathcal{E}_{k-N_0}^q(M, L^k)$, $N_0 \geq 1$, $u|_D = s^k \tilde{u}$, $\tilde{u} \in \Omega^{0,q}(D)$. We may assume that $D_{\frac{\log k}{\sqrt{k}}} \subset D$ and consider $\tilde{u}|_{D_{\frac{\log k}{\sqrt{k}}}}$. Set $\beta_k := k^{-\frac{n}{2}} F_k^* \tilde{u} = k^{-\frac{n}{2}} \tilde{u}(\frac{x}{\sqrt{k}}) \in \Omega^{0,q}(D_{\log k})$. We can check

that

$$(4.10) \quad \|\beta_k\|_{kF_k^*\phi, D_{\log k}} \leq \|u\|.$$

Since $u \in \mathcal{E}_{k^{-N_0}}^q(M, L^k)$, we have $\left\| (\square_k^{(q)})^m u \right\| \leq k^{-mN_0} \|u\|$ for all $m \in \mathbb{N}$. From this observation and (4.2), we have

$$(4.11) \quad \begin{aligned} \left\| (\square_{k\phi, (k)}^{(q)})^m \beta_k \right\|_{kF_k^*\phi, D_{\log k}} &= \frac{1}{k^{m+\frac{n}{2}}} \left\| F_k^* \left((\square_{k\phi}^{(q)})^m \tilde{u} \right) \right\|_{kF_k^*\phi, D_{\log k}} \\ &\leq \frac{1}{k^m} \left\| (\square_k^{(q)})^m u \right\| \leq k^{-mN_0-m} \|u\|. \end{aligned}$$

From (4.10), (4.11) and Lemma 4.2, we conclude that for every $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $\tilde{C}_\alpha > 0$ independent of k , such that

$$\left| k^{-\frac{n}{2} - \frac{|\alpha|}{2}} (\partial_x^\alpha \tilde{u})(0) \right| = |(\partial_x^\alpha \beta_k)(0)| \leq \tilde{C}_\alpha \|u\|.$$

Thus, for every $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $C_\alpha > 0$ independent of k , such that

$$\left| (\partial_x^\alpha (\tilde{u} e^{-k\phi}))(0) \right| \leq C_\alpha k^{\frac{n}{2} + |\alpha|} \|u\|.$$

Let x_0 be another point of D' . We can repeat the procedure above and conclude that for every $\alpha \in \mathbb{N}_0^{2n}$, there is a $C_\alpha(x_0) > 0$ independent of k , such that

$$\left| (\partial_x^\alpha (\tilde{u} e^{-k\phi}))(x_0) \right| \leq C_\alpha(x_0) k^{\frac{n}{2} + |\alpha|} \|u\|.$$

It is straightforward to see that the constant $C_\alpha(x_0)$ depends continuously on ϕ and the coefficients of $\square_{k\phi, (k)}^{(q)}$ in $\mathcal{C}^m(D)$ topology, for some $m \in \mathbb{N}_0$. (See Remark 2.5 and Theorem 2.7 in [43], for the details.) Since $\overline{D'} \subset D$ is compact, $C_\alpha(x_0)$ can be taken to be independent of the point x_0 . The theorem follows. \square

4.2. Kernel of the spectral function

As in (1.4), let

$$P_{k, k^{-N_0}}^{(q)} : L_{(0, q)}^2(M, L^k) \rightarrow \mathcal{E}_{k^{-N_0}}^q(M, L^k)$$

be the spectral projection on the spectral space of $\square_k^{(q)}$ corresponding to energy less than k^{-N_0} . The goal of this Section is to compare the localized

spectral projection $\widehat{P}_{k,k-N_0,s}^{(q)}$ (see (4.18)) to the localized approximate Szegő projection \mathcal{S}_k defined in (3.43). This will be achieved in Proposition 4.10.

We introduce some notations. Let (e_1, \dots, e_n) be a smooth local orthonormal frame of $T_x^{*(0,1)}M$ over an open set $D \Subset M$. Then $(e^J := e_{j_1} \wedge \dots \wedge e_{j_q})_{1 \leq j_1 < j_2 < \dots < j_q \leq n}$ is an orthonormal frame of $\Lambda^q T_x^{*(0,1)}M$ over D . For $f \in \Omega^{0,q}(D)$, we may write $f = \sum'_{|J|=q} f_J e^J$, with $f_J = \langle f, e^J \rangle \in \mathcal{C}^\infty(D)$. We call f_J the component of f along e^J . Let $A : \Omega_0^{0,q}(D) \rightarrow \Omega^{0,q}(D)$ be a continuous operator with smooth kernel. We write

$$(4.12) \quad A(x, y) = \sum'_{|I|=q, |J|=q} e^I(x) A_{I,J}(x, y) e^J(y), \quad A_{I,J} \in \mathcal{C}^\infty(D \times D).$$

We have

$$(4.13) \quad (Au)(x) = \sum'_{|I|=q, |J|=q} e^I(x) \int_D A_{I,J}(x, y) u_J(y) dv_M(y),$$

for all $u = \sum'_{|J|=q} u_J e^J \in \Omega_0^{0,q}(D)$. Let A^* be the formal adjoint of A with respect to (\cdot, \cdot) . We can check that

$$(4.14) \quad A^*(x, y) = \sum'_{|I|=q, |J|=q} e^I(x) A_{I,J}^*(x, y) e^J(y), \quad A_{I,J}^*(x, y) = \overline{A_{J,I}(y, x)},$$

Let

$$\begin{aligned} B : \Omega^{0,q}(D) &\rightarrow \Omega^{0,q}(D), \quad \Omega_0^{0,q}(D) \rightarrow \Omega_0^{0,q}(D), \\ B(x, y) &= \sum'_{|I|=q, |J|=q} e^I(x) B_{I,J}(x, y) e^J(y), \end{aligned}$$

be a properly supported smoothing operator. We write

$$(B \circ A)(x, y) = \sum'_{|I|=q, |J|=q} e^I(x) (B \circ A)_{I,J}(x, y) e^J(y)$$

in the sense of (4.13). It is not difficult to see that

$$(4.15) \quad (B \circ A)_{I,J}(x, y) = \sum'_{|K|=q} \int_D B_{I,K}(x, z) A_{K,J}(z, y) dv_M(z).$$

Now, we return to our situation. Let

$$P_{k,\lambda}^{(q)}(x, y) \in \mathcal{C}^\infty(M \times M, (\Lambda^q T_y^{*(0,1)}M \otimes L_y^k) \boxtimes (\Lambda^q T_x^{*(0,1)}M \otimes L_x^k))$$

be the spectral function, i.e., the Schwartz kernel of $P_{k,\lambda}^{(q)}$:

$$(4.16) \quad (P_{k,\lambda}^{(q)}u)(x) = \int_M P_{k,\lambda}^{(q)}(x,y)u(y)dv_M(y), \quad u \in L^2_{(0,q)}(M, L^k).$$

Let s be a local frame of L over D , where $D \subset M$. Then on $D \times D$ we can write

$$P_{k,\lambda}^{(q)}(x,y) = s(x)^k P_{k,\lambda,s}^{(q)}(x,y) s^*(y)^k,$$

where $P_{k,\lambda,s}^{(q)}(x,y)$ is smooth on $D \times D$, so that for $x \in D$, $u \in \Omega_0^{0,q}(D, L^k)$,

$$(4.17) \quad \begin{aligned} (P_{k,\lambda}^{(q)}u)(x) &= s(x)^k \int_M P_{k,\lambda,s}^{(q)}(x,y) \langle u(y), s^*(y)^k \rangle dv_M(y) \\ &= s(x)^k \int_M P_{k,\lambda,s}^{(q)}(x,y) \tilde{u}(y) dv_M(y), \quad u = s^k \tilde{u}, \quad \tilde{u} \in \Omega_0^{0,q}(D). \end{aligned}$$

For $x = y$, we can check that $P_{k,\lambda,s}^{(q)}(x,x) \in \mathcal{C}^\infty(D, \text{End}(\Lambda^q T^{*(0,1)} M))$ is independent of the choices of local frame s .

Let $D \Subset M$ be an open set. Assume that $L|_D$ is trivial and let s be a local frame of L on D and set $|s|_{h^L}^2 = e^{-2\phi}$. Let $(D, z) \cong (D, x)$ be local coordinates of D . Fix $N_0 \geq 1$. We define the *localized spectral projection* (with respect to the trivializing section s) by

$$(4.18) \quad \begin{aligned} \widehat{P}_{k,k-N_0,s}^{(q)} : L^2_{(0,q)}(D) \cap \mathcal{E}^l(D, \Lambda^q T^{*(0,1)} M) &\rightarrow \Omega_0^{0,q}(D), \\ u &\rightarrow e^{-k\phi} s^{-k} P_{k,k-N_0,s}^{(q)}(s^k e^{k\phi} u). \end{aligned}$$

That is, if $P_{k,k-N_0,s}^{(q)}(s^k e^{k\phi} u) = s^k v$ on D , then $\widehat{P}_{k,k-N_0,s}^{(q)} u = e^{-k\phi} v$. We notice that

$$(4.19) \quad \widehat{P}_{k,k-N_0,s}^{(q)}(x,y) = e^{-k\phi(x)} P_{k,k-N_0,s}^{(q)}(x,y) e^{k\phi(y)},$$

where $\widehat{P}_{k,k-N_0,s}^{(q)}(x,y)$ is the kernel of $\widehat{P}_{k,k-N_0,s}^{(q)}$ with respect to (\cdot, \cdot) and $P_{k,k-N_0,s}^{(q)}(x,y)$ is as in (4.17). We write

$$(4.20) \quad \widehat{P}_{k,k-N_0,s}^{(q)}(x,y) = \sum'_{|I|=q, |J|=q} e^I(x) \widehat{P}_{k,k-N_0,s,I,J}^{(q)}(x,y) e^J(y)$$

in the sense of (4.13), where $P_{k,k^{-N_0},s,I,J}^{(q)} \in \mathcal{C}^\infty(D \times D)$. Since $P_{k,k^{-N_0}}^{(q)}$ is self-adjoint, we have

$$(4.21) \quad \widehat{P}_{k,k^{-N_0},s,I,J}^{(q)}(x, y) = \overline{\widehat{P}_{k,k^{-N_0},s,J,I}^{(q)}(y, x)},$$

for all strictly increasing I, J with $|I| = |J| = q$.

Let $\{f_j\}_{j=1}^{d_k} \subset \Omega^{0,q}(M, L^k)$ be an orthonormal frame for $\mathcal{E}_{k^{-N_0}}^q(M, L^k)$, $d_k \in \mathbb{N}_0 \cup \{\infty\}$. For each j , we write

$$f_j|_D = \sum'_{|J|=q} f_{j,J}(x)e^J(x), \quad f_{j,J} \in \mathcal{C}^\infty(D, L^k).$$

For $j = 1, \dots, d_k$ and strictly increasing J with $|J| = q$ we define $\widetilde{f}_{j,J} \in \mathcal{C}^\infty(D)$ and $\widetilde{f}_j \in \Omega^{0,q}(D)$ by

$$f_{j,J} = s^k \widetilde{f}_{j,J}, \quad \widetilde{f}_j = \sum'_{|J|=q} \widetilde{f}_{j,J}(x)e^J(x).$$

Then, $f_j|_D = s^k \widetilde{f}_j$, $j = 1, \dots, d_k$, and it is not difficult to see that

$$(4.22) \quad \widehat{P}_{k,k^{-N_0},s,I,J}^{(q)}(x, y) = \sum_{j=1}^{d_k} \widetilde{f}_{j,I}(x) \overline{\widetilde{f}_{j,J}(y)} e^{-k(\phi(x)+\phi(y))},$$

for all strictly increasing I, J with $|I| = |J| = q$. Since $\widehat{P}_{k,k^{-N_0},s,I,J}^{(q)}$ are smooth for all strictly increasing I, J , $|I| = |J| = q$, we conclude that for all $\alpha \in \mathbb{N}_0^{2n}$,

$$(4.23) \quad \sum_{j=1}^{d_k} \left| (\partial_x^\alpha (\widetilde{f}_j e^{-k\phi}))(x) \right|^2 \text{ converges at each point of } x \in D.$$

Similarly, if $F : \mathcal{E}'(D, \Lambda^q T^{*(0,1)} M) \rightarrow \mathcal{E}'(D, \Lambda^q T^{*(0,1)} M)$ is a properly supported continuous operator such that for all $s \in \mathbb{N}_0$,

$$F : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{comp}}^{s+s_0}(D, \Lambda^q T^{*(0,1)} M)$$

is continuous, for some $s_0 \in \mathbb{R}$. Then, we can check that

$$(4.24) \quad \sum_{j=1}^{d_k} \left| (F(\widetilde{f}_j e^{-k\phi}))(x) \right|^2 \text{ converges at each point of } x \in D.$$

Proposition 4.5. *With the notations used above, for every $\alpha \in \mathbb{N}_0^{2n}$, $D' \Subset D$, there is a constant $C_{\alpha, D'} > 0$ independent of k , such that*

$$(4.25) \quad \sum_{j=1}^{d_k} \left| (\partial_x^\alpha (\tilde{f}_j e^{-k\phi}))(x) \right|^2 \leq C_{\alpha, D'} k^{n+2|\alpha|}, \quad \forall x \in D'.$$

Proof. Fix $\alpha \in \mathbb{N}_0^{2n}$ and $p \in D'$. We may assume that

$$\sum_{j=1}^{d_k} \left| (\partial_x^\alpha (\tilde{f}_j e^{-k\phi}))(p) \right|^2 \neq 0.$$

Set

$$u(x) = \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left| (\partial_x^\alpha (\tilde{f}_j e^{-k\phi}))(p) \right|^2}} \sum_{j=1}^{d_k} \tilde{f}_j(x) \overline{(\partial_x^\alpha (\tilde{f}_j e^{-k\phi}))(p)}.$$

Since $\sum_{j=1}^{d_k} \left| (\partial_x^\alpha (\tilde{f}_j e^{-k\phi}))(p) \right|^2$ converges, it is easy to check that

$$u \in \mathcal{E}_{k-N_0}^q(M, L^k), \quad \|u\| = 1.$$

On D , we write $u = s^k \tilde{u}$, $\tilde{u} \in \Omega^{0,q}(D)$. We can check that

$$(4.26) \quad \tilde{u} = \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left| (\partial_x^\alpha (\tilde{f}_j e^{-k\phi}))(p) \right|^2}} \sum_{j=1}^{d_k} \tilde{f}_j(x) \overline{(\partial_x^\alpha (\tilde{f}_j e^{-k\phi}))(p)}.$$

In view of Theorem 4.3, we see that $\left| (\partial_x^\alpha (\tilde{u} e^{-k\phi}))(p) \right| \leq C_\alpha k^{\frac{n}{2} + |\alpha|}$, with $C_\alpha > 0$ independent of k and of the point p . From (4.26), it is straightforward to see that

$$\left| (\partial_x^\alpha (\tilde{u} e^{-k\phi}))(p) \right| = \sqrt{\sum_{j=1}^{d_k} \left| (\partial_x^\alpha (\tilde{f}_j e^{-k\phi}))(p) \right|^2} \leq C_\alpha k^{\frac{n}{2} + |\alpha|}.$$

The proposition follows. \square

Now, assume that $\partial\bar{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) at each point of D and let $q = n_-$. Let $\mathcal{S}_k, \mathcal{A}_k$ be as in Theorem 3.11 and let

$\square_s^{(q)}$ be as in (3.8), (3.7). If we replace \mathcal{S}_k by $I - \square_s^{(q)}\mathcal{A}_k$, then $\square_s^{(q)}\mathcal{A}_k + \mathcal{S}_k = I = \mathcal{A}_k^*\square_s^{(q)} + \mathcal{S}_k^*$ on $\mathcal{D}'(D, \Lambda^q T^{*(0,1)}M)$. Now,

$$(4.27) \quad \widehat{P}_{k,k^{-N_0},s}^{(q)} = (\mathcal{A}_k^*\square_s^{(q)} + \mathcal{S}_k^*)\widehat{P}_{k,k^{-N_0},s}^{(q)} = R + \mathcal{S}_k^*\widehat{P}_{k,k^{-N_0},s}^{(q)} \text{ on } \mathcal{E}'(D, \Lambda^q T^{*(0,1)}M),$$

where we denote

$$R = \mathcal{A}_k^*\square_s^{(q)}\widehat{P}_{k,k^{-N_0},s}^{(q)}.$$

We write

$$R(x, y) = \sum_{|I|=q, |J|=q} e^I(x)R_{I,J}(x, y)e^J(y), \quad R_{I,J} \in \mathcal{C}^\infty(D \times D),$$

in the sense of (4.13). From (4.22), it is straightforward to see that

$$(4.28) \quad \begin{aligned} R_{I,J}(x, y) &= \sum_{j=1}^{d_k} \widetilde{g}_{j,I}(x) \overline{\widetilde{f}_{j,J}(y)} e^{-k\phi(y)}, \\ \widetilde{g}_j &= \mathcal{A}_k^*\square_s^{(q)}(\widetilde{f}_j e^{-k\phi})(x), \quad \widetilde{g}_j(x) = \sum_{|I|=q} \widetilde{g}_{j,I}(x) e^I(x), \quad j = 1, \dots, d_k, \end{aligned}$$

for all strictly increasing I, J , $|I| = |J| = q$. From (4.24), we see that for all $\alpha \in \mathbb{N}_0^{2n}$,

$$\sum_{j=1}^{d_k} |(\partial_x^\alpha \widetilde{g}_j)(x)|^2 \text{ converges at each point of } x \in D.$$

To estimate $R_{I,J}(x, y)$, we first need

Lemma 4.6. *With the notations used above, for every $D' \Subset D$, $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha, D'} > 0$ independent of k , such that for all $u \in \mathcal{E}_{k^{-N_0}}^q(M, L^k)$, $\|u\| = 1$, $u|_D = s^k \widetilde{u}$, $\widetilde{u} \in \Omega^{0,q}(D)$, if we set $\widetilde{v}(x) = \mathcal{A}_k^*\square_s^{(q)}(\widetilde{u}e^{-k\phi})$, then*

$$|(\partial_x^\alpha \widetilde{v})(x)| \leq C_{\alpha, D'} k^{\frac{5n}{2} + 2|\alpha| - N_0}, \quad \forall x \in D'.$$

Proof. Let $u \in \mathcal{E}_{k^{-N_0}}^q(M, L^k)$, $\|u\| = 1$, $u|_D = s^k \widetilde{u}$, $\widetilde{u} \in \Omega^{0,q}(D)$. Set $\widetilde{v}(x) = \mathcal{A}_k^*\square_s^{(q)}(\widetilde{u}e^{-k\phi})$. We recall that

$$(4.29) \quad \mathcal{A}_k^* : O(k^s) : H_{\text{comp}}^s(D, \Lambda^q T^{*(0,1)}M) \rightarrow H_{\text{comp}}^{s+1}(D, \Lambda^q T^{*(0,1)}M), \quad \forall s \in \mathbb{N}_0.$$

Let $D' \Subset D'' \Subset D$. By using Fourier transforms, we see that for all $x \in D'$, we have

$$|(\partial_x^\alpha \tilde{v})(x)| \leq C_\alpha \|\tilde{v}\|_{n+1+|\alpha|, D''},$$

where C_α only depends on the dimension and the length of α . Here $\|\cdot\|_{s, D''}$ denotes the usual Sobolev norm of order s on D'' . From this observation and (4.29), we see that

$$(4.30) \quad |(\partial_x^\alpha \tilde{v})(x)| \leq C_\alpha \|\tilde{v}\|_{n+1+|\alpha|, D''} \leq C'_\alpha k^{n+|\alpha|} \left\| \square_s^{(q)}(\tilde{u}e^{-k\phi}) \right\|_{n+|\alpha|, D''},$$

where $C'_\alpha > 0$ is independent of k . Let $\square_k^{(q)}u = f$, $f|_D = s^k \tilde{f}$, $\tilde{f} \in \Omega^{0,q}(D)$. We can check that $f \in \mathcal{E}_{k^{-N_0}}^q(M, L^k)$ and $\|f\| \leq k^{-N_0}$. From (3.8), we see that

$$(4.31) \quad \square_s^{(q)}(e^{-k\phi}\tilde{u}) = e^{-k\phi}\tilde{f}.$$

In view of Theorem 4.3, we know that for all $\beta \in \mathbb{N}_0^{2n}$,

$$\left| \partial_x^\beta (\square_s^{(q)}(e^{-k\phi}\tilde{u})) \right| = \left| \partial_x^\beta (e^{-k\phi}\tilde{f}) \right| \leq C_\beta k^{\frac{n}{2}+|\beta|} \|f\| \leq C_\beta k^{\frac{n}{2}+|\beta|-N_0} \quad \text{on } D'',$$

where $C_\beta > 0$ is independent of k . Thus,

$$(4.32) \quad \left\| \square_s^{(q)}(e^{-k\phi}\tilde{u}) \right\|_{n+|\alpha|, D''} \leq \tilde{C}_\alpha k^{\frac{3n}{2}+|\alpha|-N_0},$$

where $\tilde{C}_\alpha > 0$ is independent of k . Combining (4.32) with (4.30), the lemma follows. \square

Lemma 4.7. *Let $\tilde{g}_j(x) \in \Omega^{0,q}(D)$, $j = 1, \dots, d_k$, be as in (4.28). For every $D' \Subset D$, $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $C_\alpha > 0$ independent of k , such that for all $x \in D'$*

$$\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(x)|^2 \leq C_\alpha k^{5n+4|\alpha|-2N_0}.$$

Proof. Fix $\alpha \in \mathbb{N}_0^{2n}$ and $p \in D'$. We may assume that $\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2 \neq 0$. Set

$$h(x) = \frac{1}{\sqrt{\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2}} \sum_{j=1}^{d_k} f_j(x) \overline{(\partial_x^\alpha \tilde{g}_j)(p)}.$$

Since $\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2$ converges, we can check that $h \in \mathcal{E}_{k-N_0}^q(M, L^k)$, $\|h\| = 1$. On D , we write $h = s^k \tilde{h}$. We can check that

$$\mathcal{A}_k^* \square_s^{(q)}(\tilde{h} e^{-k\phi}) = \frac{1}{\sqrt{\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2}} \sum_{j=1}^{d_k} \tilde{g}_j(x) \overline{(\partial_x^\alpha \tilde{g}_j)(p)}.$$

In view of Lemma 4.6, we see that

$$\left| \partial_x^\alpha (\mathcal{A}_k^* \square_s^{(q)}(\tilde{h} e^{-k\phi}))(p) \right| = \sqrt{\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2} \leq C_\alpha k^{\frac{5n}{2} + 2|\alpha| - N_0},$$

where $C_\alpha > 0$ is independent of k and the point p . The lemma follows. \square

Now, we can prove

Proposition 4.8. *With the notations used above, for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha, \beta} > 0$ independent of k , such that*

$$(4.33) \quad \left| (\partial_x^\alpha \partial_y^\beta R_{I, J})(x, y) \right| \leq C_{\alpha, \beta} k^{3n + 2|\alpha| + |\beta| - N_0}, \quad \forall (x, y) \in D' \times D',$$

for all strictly increasing I, J , $|I| = |J| = q$, where $R_{I, J}(x, y)$ is as in (4.28).

Proof. Fix $p \in D'$ and J strictly increasing, $|J| = q$. Let $\alpha, \beta \in \mathbb{N}_0^{2n}$. We may assume that $\sum_{j=1}^{d_k} \left| (\partial_y^\beta (\tilde{f}_{j, J} e^{-k\phi}))(p) \right|^2 \neq 0$. Put

$$(4.34) \quad u(x) = \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left| (\partial_y^\beta (\tilde{f}_{j, J} e^{-k\phi}))(p) \right|^2}} \sum_{j=1}^{d_k} f_j(x) \overline{(\partial_y^\beta (\tilde{f}_{j, J} e^{-k\phi}))(p)}.$$

Then, $u \in \mathcal{E}_{k-N_0}^q(M, L^k)$, $\|u\| = 1$. On D , we write $u = s^k \tilde{u}$, $\tilde{u} = \sum'_{|I|=q} \tilde{u}_I e^I$. Put $\tilde{v} = \mathcal{A}_k^* \square_s^{(q)}(\tilde{u} e^{-k\phi}) = \sum'_{|I|=q} \tilde{v}_I e^I \in \Omega^{0, q}(D)$. It is not difficult to check that

$$\tilde{v} = \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left| (\partial_y^\beta (\tilde{f}_{j, J} e^{-k\phi}))(p) \right|^2}} \sum_{j=1}^{d_k} \tilde{g}_j \overline{(\partial_y^\beta (\tilde{f}_{j, J} e^{-k\phi}))(p)},$$

where $\{\tilde{g}_j\}_{j=1}^{d_k}$ are as in (4.28). In view of Lemma 4.6, there exists $C_\alpha > 0$ independent of k and the point p such that $|(\partial_x^\alpha \tilde{v})(x)| \leq C_\alpha k^{\frac{5n}{2} + 2|\alpha| - N_0}$, for

all $x \in D'$. In particular,

$$(4.35) \quad \begin{aligned} |(\partial_x^\alpha \tilde{v}_I)(x)| &= \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left| (\partial_y^\beta (\tilde{f}_{j,J} e^{-k\phi}))(p) \right|^2}} \left| \sum_{j=1}^{d_k} (\partial_x^\alpha \tilde{g}_{j,I})(x) \overline{(\partial_y^\beta (\tilde{f}_{j,J} e^{-k\phi}))(p)} \right| \\ &\leq C_\alpha k^{\frac{5n}{2} + 2|\alpha| - N_0}, \quad \forall x \in D', \end{aligned}$$

for all strictly increasing I , $|I| = q$. In view of Proposition 4.5, we see that

$$\sum_{j=1}^{d_k} \left| (\partial_y^\beta (\tilde{f}_{j,J} e^{-k\phi}))(p) \right|^2 \leq C_\beta k^{n+2|\beta|},$$

where $C_\beta > 0$ is independent of k and the point p . From this and (4.35), we conclude the existence of a constant $C_{\alpha,\beta} > 0$ independent of k and the point p with

$$\begin{aligned} \left| (\partial_x^\alpha \partial_y^\beta R_{I,J})(x, p) \right| &= \sqrt{\sum_{j=1}^{d_k} \left| (\partial_y^\beta (\tilde{f}_{j,J} e^{-k\phi}))(p) \right|^2} |(\partial_x^\alpha \tilde{v}_I)(x)| \\ &\leq C_{\alpha,\beta} k^{3n+2|\alpha|+|\beta|-N_0}, \end{aligned}$$

for all $x \in D'$, all strictly increasing I, J with $|I| = |J| = q$. The proposition follows. \square

From (4.27) and Proposition 4.8, we know that

$$\widehat{P}_{k,k^{-N_0},s}^{(q)} = R + \mathcal{S}_k^* \widehat{P}_{k,k^{-N_0},s}^{(q)},$$

where $R(x, y)$ satisfies (4.33). We have

$$(4.36) \quad \widehat{P}_{k,k^{-N_0},s}^{(q)} \mathcal{S}_k = (R + \mathcal{S}_k^* \widehat{P}_{k,k^{-N_0},s}^{(q)}) \mathcal{S}_k = R \mathcal{S}_k + \mathcal{S}_k^* \widehat{P}_{k,k^{-N_0},s}^{(q)} \mathcal{S}_k.$$

Let R^* be the formal adjoint R with respect to (\cdot, \cdot) . Then,

$$(4.37) \quad \widehat{P}_{k,k^{-N_0},s}^{(q)} = R^* + \widehat{P}_{k,k^{-N_0},s}^{(q)} \mathcal{S}_k.$$

From (4.37) and (4.36), we get

$$(4.38) \quad \widehat{P}_{k,k^{-N_0},s}^{(q)} = R^* + R \mathcal{S}_k + \mathcal{S}_k^* \widehat{P}_{k,k^{-N_0},s}^{(q)} \mathcal{S}_k.$$

We also write

$$R^*(x, y) = \sum'_{|I|=q, |J|=q} e^I(x) R_{I,J}^*(x, y) e^J(y).$$

Note that $R_{I,J}^*(x, y) = \overline{R_{J,I}(y, x)}$, for all strictly increasing I, J , $|I| = |J| = q$. Combining this observation with (4.33), we conclude that for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha, \beta} > 0$ independent of k , such that

$$\left| (\partial_x^\alpha \partial_y^\beta R_{I,J}^*)(x, y) \right| \leq C_{\alpha, \beta} k^{3n+2|\beta|+|\alpha|-N_0}, \quad \forall (x, y) \in D' \times D',$$

for all strictly increasing I, J , $|I| = |J| = q$.

Now, we study the kernel of $R\mathcal{S}_k$. We write

$$(R\mathcal{S}_k)(x, y) = \sum'_{|I|=q, |J|=q} e^I(x) (R\mathcal{S}_k)_{I,J}(x, y) e^J(y).$$

From (4.15), we know that for all strictly increasing I, J , $|I| = |J| = q$,

$$(4.39) \quad (R\mathcal{S}_k)_{I,J}(x, y) = \sum'_{|K|=q} \int_D R_{I,K}(x, z) \mathcal{S}_{kK,J}(z, y) dv_M(z).$$

Lemma 4.9. *For every $D' \Subset D$, $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $C_\alpha > 0$ independent of k , such that for all strictly increasing I , $|I| = q$, we have*

$$(4.40) \quad \sum'_{|K|=q} \int_D |(\partial_x^\alpha R_{I,K})(x, z)|^2 dv_M(z) \leq C_\alpha k^{5n+4|\alpha|-2N_0}, \quad x \in D'.$$

Proof. From (4.28), we see that for $\alpha \in \mathbb{N}_0^{2n}$ we have

$$(4.41) \quad (\partial_x^\alpha R_{I,K})(x, y) = \sum_{j=1}^{d_k} (\partial_x^\alpha \tilde{g}_{j,I})(x) \overline{\tilde{f}_{j,K}(y)} e^{-k\phi(y)}.$$

We claim that

$$(4.42) \quad \sum'_{|K|=q} \int_D |(\partial_x^\alpha R_{I,K})(x, y)|^2 dv_M(y) \leq \sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_{j,I})(x)|^2,$$

for all $x \in D$, strictly increasing I , $|I| = q$. Fix such I and $p \in D$. We may assume that $\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_{j,I})(p)|^2 \neq 0$. Put

$$u(x) = \frac{1}{\sqrt{\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_{j,I})(p)|^2}} \sum_{j=1}^{d_k} \overline{(\partial_x^\alpha \tilde{g}_{j,I})(p)} f_j(x) \in \mathcal{E}_{k-N_0}^q(M, L^k).$$

We see that $\|u\| = 1$. Thus, $\int_D |u|^2 \leq 1$. On D , we can check that

$$(4.43) \quad \int_D |u|^2 = \frac{1}{\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_{j,I})(p)|^2} \sum'_{|K|=q} \int_D \left| \sum_{j=1}^{d_k} (\partial_x^\alpha \tilde{g}_{j,I})(p) \overline{f_{j,K}(y)} \right|^2 e^{-2k\phi(y)} dv_M(y) \leq 1.$$

From (4.41) and (4.43), we see that

$$\sum'_{|K|=q} \int_D |(\partial_x^\alpha R_{I,K})(p, y)|^2 dv_M(y) \leq \sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_{j,I})(p)|^2.$$

(4.42) follows. From (4.42) and Lemma 4.7, the lemma follows. \square

From (4.39), for all strictly increasing I, J , $|I| = |J| = q$, we have

$$(4.44) \quad \begin{aligned} & \left| \partial_x^\alpha \partial_y^\beta ((R\mathcal{S}_k)_{I,J})(x, y) \right| \\ &= \left| \sum'_{|K|=q} \int_D (\partial_x^\alpha R_{I,K})(x, z) (\partial_y^\beta \mathcal{S}_{kK,J})(z, y) dv_M(z) \right| \\ &\leq \sum'_{|K|=q} \left(\int_D |(\partial_x^\alpha R_{I,K})(x, z)|^2 dv_M(z) \right)^{\frac{1}{2}} \left(\int_D |(\partial_y^\beta \mathcal{S}_{kK,J})(z, y)|^2 dv_M(z) \right)^{\frac{1}{2}} \\ &\leq \sqrt{\sum'_{|K|=q} \int_D |(\partial_x^\alpha R_{I,K})(x, z)|^2 dv_M(z)} \sqrt{\sum'_{|K|=q} \int_D |(\partial_y^\beta \mathcal{S}_{kK,J})(z, y)|^2 dv_M(z)}. \end{aligned}$$

Note that

$$\begin{aligned}
(4.45) \quad & \sum'_{|K|=q} \int_D \left| (\partial_y^\beta \mathcal{S}_{kK,J})(z, y) \right|^2 dv_M(z) \\
&= \sum'_{|K|=q} \int_D (\partial_x^\beta \mathcal{S}_{k,J,K}^*)(y, z) (\partial_y^\beta \mathcal{S}_{kK,J})(z, y) dv_M(z) \\
&= (\partial_x^\beta \partial_y^\beta (\mathcal{S}_k^* \mathcal{S}_k)_{J,J})(y, y).
\end{aligned}$$

We notice that $\mathcal{S}_k^* \mathcal{S}_k \equiv \mathcal{S}_k \pmod{O(k^{-\infty})}$. From this observation and the explicit formula of the kernel of \mathcal{S}_k (see (3.63)), we conclude that

$$(4.46) \quad \left| (\partial_x^\beta \partial_y^\beta (\mathcal{S}_k^* \mathcal{S}_k)_{J,J})(y, y) \right| \leq C_\beta k^{n+2|\beta|},$$

locally uniformly on D , for all strictly increasing J , $|J| = q$, where $C_\beta > 0$ is independent of k . From (4.46), (4.45), (4.44) and Lemma 4.9, we conclude that for all strictly increasing I, J , $|I| = |J| = q$,

$$\left| (\partial_x^\alpha \partial_y^\beta (R\mathcal{S}_k)_{I,J})(x, y) \right| \leq C_{\alpha,\beta} k^{3n+2|\alpha|+|\beta|-N_0},$$

locally uniformly on D , where $C_{\alpha,\beta} > 0$ is independent of k . Put

$$T = R^* + R\mathcal{S}_k.$$

We write

$$T(x, y) = \sum'_{|I|=q, |J|=q} e^I(x) T_{I,J}(x, y) e^J(y)$$

in the sense of (4.13). From (4.38), we know that

$$(4.47) \quad \widehat{P}_{k,k^{-N_0},s}^{(q)} = T + \mathcal{S}_k^* \widehat{P}_{k,k^{-N_0},s}^{(q)} \mathcal{S}_k.$$

From the discussion above, we know that for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, every strictly increasing I, J , $|I| = |J| = q$, there is a constant $C_{\alpha,\beta} > 0$ independent of k such that

$$(4.48) \quad \left| (\partial_x^\alpha \partial_y^\beta T_{I,J})(x, y) \right| \leq C_{\alpha,\beta} (k^{3n+2|\alpha|+|\beta|-N_0} + k^{3n+|\alpha|+2|\beta|-N_0}),$$

for every $(x, y) \in D' \times D'$.

Summing up, we get the following.

Proposition 4.10. *In the situation of Setup 3.9 let $q = n_-$. Fix $N_0 \geq 1$. Let \mathcal{S}_k be the localized approximate Szegő kernel (3.43) and let $\widehat{P}_{k,k-N_0,s}^{(q)}$ be the localized spectral projection (4.18). Then,*

$$\widehat{P}_{k,k-N_0,s}^{(q)} = T + \mathcal{S}_k^* \widehat{P}_{k,k-N_0,s}^{(q)} \mathcal{S}_k,$$

where T is smoothing and the distribution kernel of T satisfies (4.48).

4.3. Asymptotic expansion of the spectral function. Proof of Theorem 1.1

Consider $\lambda \geq 0$ and denote by $\mathcal{E}_{>\lambda}^q(M, L^k) \subset L_{(0,q)}^2(M, L^k)$ the spectral space given by the range of $E((\lambda, \infty))$, where E is the spectral measure of $\square_k^{(q)}$. Let

$$P_{k,>\lambda}^{(q)} : L_{(0,q)}^2(M, L^k) \rightarrow \mathcal{E}_{>\lambda}^q(M, L^k)$$

be the orthogonal projection. As before, let s be a local frame of L on an open set $D \Subset M$ and $|s|_{h^L}^2 = e^{-2\phi}$. Consider the localization

$$(4.49) \quad \widehat{P}_{k,>\lambda,s}^{(q)} : L_{(0,q)}^2(D) \cap \mathcal{E}'(D, \Lambda^q T^{*(0,1)} M) \rightarrow L_{(0,q)}^2(D),$$

$$u \mapsto e^{-k\phi} s^{-k} P_{k,>\lambda}^{(q)}(s^k e^{k\phi} u).$$

Fix $N_0 \geq 1$. It is well-known that (see Davies [21, Section 2])

$$L_{(0,q)}^2(M) = \mathcal{E}_{k-N_0}^q(M, L^k) \oplus \mathcal{E}_{>k-N_0}^q(M, L^k)$$

and

$$(4.50) \quad \|u\| \leq k^{N_0} \left\| \square_k^{(q)} u \right\|, \quad \forall u \in \mathcal{E}_{>k-N_0}^q(M, L^k) \cap \text{Dom } \square_k^{(q)}.$$

We have the decomposition

$$(4.51) \quad u = \widehat{P}_{k,k-N_0,s}^{(q)} u + \widehat{P}_{k,>k-N_0,s}^{(q)} u, \quad u \in \Omega_0^{0,q}(D).$$

Now, assume that $\partial\bar{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) at each point of D and let $q = n_-$. Let \mathcal{S}_k be the localized approximate Szegő

kernel (3.43). From the explicit formula of the kernel of \mathcal{S}_k (see (3.63)), we can check that

$$(4.52) \quad \mathcal{S}_k^*, \mathcal{S}_k = O(k^{n+|s_1|+|s|}) : H_{\text{loc}}^{s_1}(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{loc}}^s(D, \Lambda^q T^{*(0,1)} M),$$

locally uniformly on D , for all $s, s_1 \in \mathbb{Z}$, $s_1 \leq 0$, $s \geq 0$.

Let $u \in H_{\text{comp}}^{s_1}(D, \Lambda^q T^{*(0,1)} M)$, $s_1 \leq 0$, $s_1 \in \mathbb{Z}$. From (4.51), we have

$$(4.53) \quad \mathcal{S}_k u = \widehat{P}_{k, k^{-N_0}, s}^{(q)} \mathcal{S}_k u + \widehat{P}_{k, >k^{-N_0}, s}^{(q)} \mathcal{S}_k u.$$

From (4.49) and (4.50), we can check that

$$(4.54) \quad \begin{aligned} \left\| \widehat{P}_{k, >k^{-N_0}, s}^{(q)} \mathcal{S}_k u \right\|_D &\leq \left\| P_{k, >k^{-N_0}}^{(q)}(s^k e^{k\phi}(\mathcal{S}_k u)) \right\| \\ &\leq k^{N_0} \left\| \square_k^{(q)} P_{k, >k^{-N_0}}^{(q)}(s^k e^{k\phi}(\mathcal{S}_k u)) \right\| \\ &\leq k^{N_0} \left\| \square_k^{(q)}(s^k e^{k\phi}(\mathcal{S}_k u)) \right\| = k^{N_0} \left\| \square_s^{(q)}(\mathcal{S}_k u) \right\|. \end{aligned}$$

Here we have used (3.8). In view of Theorem 3.11, we see that $\square_s^{(q)} \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})}$. From this observation and (4.54), we conclude that

$$(4.55) \quad \widehat{P}_{k, >k^{-N_0}, s}^{(q)} \mathcal{S}_k = O(k^{-N}) : H_{\text{comp}}^{s_1}(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{loc}}^0(D, \Lambda^q T^{*(0,1)} M),$$

locally uniformly on D , for all $N \geq 0$, $s_1 \in \mathbb{Z}$, $s_1 \leq 0$. From (4.52) and (4.55), we conclude that

$$(4.56) \quad \mathcal{S}_k^* \widehat{P}_{k, >k^{-N_0}, s}^{(q)} \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})}.$$

Combining (4.56) with (4.53) and using that $\mathcal{S}_k^* \mathcal{S}_k \equiv \mathcal{S}_k \pmod{O(k^{-\infty})}$, we get

$$(4.57) \quad \mathcal{S}_k \equiv \mathcal{S}_k^* \widehat{P}_{k, k^{-N_0}, s}^{(q)} \mathcal{S}_k \pmod{O(k^{-\infty})}.$$

From (4.57) and Proposition 4.10, Theorem 3.12 and Theorem 3.15, we get one of the main results of this work:

Theorem 4.11. *In the situation of Setup 3.9 let $q = n_-$, fix $N_0 \geq 1$ and let $\widehat{P}_{k, k^{-N_0}, s}^{(q)}$ be the localized spectral projection (4.18) and let $\widehat{P}_{k, k^{-N_0}, s}^{(q)}(\cdot, \cdot)$ be its*

distribution kernel. Then, for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha, \beta} > 0$ independent of k , such that

$$(4.58) \quad \left| \partial_x^\alpha \partial_y^\beta (\widehat{P}_{k, k^{-N_0}, s}^{(q)}(x, y) - \mathcal{S}_k(x, y)) \right| \leq C_{\alpha, \beta} (k^{3n+2|\alpha|+|\beta|-N_0} + k^{3n+|\alpha|+2|\beta|-N_0})$$

holds on $D' \times D'$, where

$$\mathcal{S}_k(x, y) = \mathcal{S}_k(z, w) \equiv e^{ik\Psi(z, w)} b(z, w, k) \pmod{O(k^{-\infty})},$$

with

$$\begin{aligned} b(z, w, k) &\in S_{\text{loc}}^n(1; D \times D, \Lambda^q T_w^{*(0,1)} M \boxtimes \Lambda^q T_z^{*(0,1)} M), \\ b(z, w, k) &\sim \sum_{j=0}^{\infty} b_j(z, w) k^{n-j} \text{ in } S_{\text{loc}}^n(1; D \times D, \Lambda^q T_w^{*(0,1)} M \boxtimes \Lambda^q T_z^{*(0,1)} M), \\ b_j(z, w) &\in \mathcal{C}^\infty(D \times D, \Lambda^q T_w^{*(0,1)} M \boxtimes \Lambda^q T_z^{*(0,1)} M), \quad j = 0, 1, 2, \dots, \\ b_0(z, z) &\text{ is given by (3.80),} \end{aligned}$$

and $\Psi \in \mathcal{C}^\infty(D \times D)$ satisfying (3.38) and for a given point $p \in D$, consider local holomorphic coordinates $z = (z_1, \dots, z_n)$ centred at p as in (3.35). Then Ψ has the form (3.39) near $(0, 0)$. Moreover, let $\{Z_j\}_{j=1}^n$ be a smooth orthonormal frame of $T^{(0,1)}M$ over D . Then,

$$(4.59) \quad \sum_{j=1}^n \left((iZ_j \Psi)(z, w) + (Z_j \phi)(z) \right) \left((-i\bar{Z}_j \Psi)(z, w) + (\bar{Z}_j \phi)(z) \right) = O(|z - w|^N),$$

locally uniformly on $D \times D$, for all $N \in \mathbb{N}$.

When $q \neq n_-$, we use Theorem 3.13 and repeat the proof of Theorem 4.11 to conclude that

Theorem 4.12. *In the situation of Setup 3.9 let $q \neq n_-$, fix $N_0 \geq 1$. With the notations used in Theorem 4.11. Then, for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha, \beta} > 0$ independent of k , such that*

$$(4.60) \quad \left| \partial_x^\alpha \partial_y^\beta (\widehat{P}_{k, k^{-N_0}, s}^{(q)}(x, y)) \right| \leq C_{\alpha, \beta} (k^{3n+2|\alpha|+|\beta|-N_0} + k^{3n+|\alpha|+2|\beta|-N_0})$$

on $D' \times D'$.

Proof of Theorem 1.1. Combining Theorems 4.11 and 4.12, we get (1.7), (1.8) and (1.9). \square

Remark 4.13. In view of Remark 3.14, we can generalize Theorem 4.11 and Theorem 4.12 with essentially the same proofs to the case when the forms take values in $L^k \otimes E$, for a given holomorphic vector bundle E over M .

4.4. Asymptotic expansion of the Bergman kernel. Proof of Theorem 1.6

We are now ready to prove Theorem 1.6. In the situation of Setup 3.9 let $q = n_-$. Define the *localized Bergman projection* (with respect to s) by

$$(4.61) \quad \widehat{P}_{k,s}^{(q)} : L_{(0,q)}^2(D) \cap \mathcal{E}'(D, \Lambda^q T^{*(0,1)} M) \rightarrow \Omega^{0,q}(D),$$

$$u \mapsto e^{-k\phi} s^{-k} P_k^{(q)}(s^k e^{k\phi} u).$$

Let $\widehat{P}_{k,s}^{(q)}(x, y)$ be the distribution kernel of $\widehat{P}_{k,s}^{(q)}$. We have the following

Theorem 4.14. *With the assumptions and notations above, fix $N_0 \geq 1$ and assume that $\square_k^{(q)}$ has $O(k^{-N_0})$ small spectral gap on D . Then for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha,\beta} > 0$ independent of k , such that*

$$(4.62) \quad \left| \partial_x^\alpha \partial_y^\beta (\widehat{P}_{k,k^{-N_0},s}^{(q)}(x, y) - \widehat{P}_{k,s}^{(q)}(x, y)) \right|$$

$$\leq C_{\alpha,\beta} (k^{3n+2|\alpha|+|\beta|-N_0} + k^{3n+|\alpha|+2|\beta|-N_0}) \quad \text{on } D' \times D',$$

where $\widehat{P}_{k,k^{-N_0},s}^{(q)}$ is as in Theorem 4.11. In particular,

$$\widehat{P}_{k,s}^{(q)} \equiv \mathcal{S}_k \quad \text{mod } O(k^{-\infty})$$

locally uniformly on D , where \mathcal{S}_k is as in Theorem 4.11.

Proof. Let \mathcal{S}_k be the operator as in Theorem 4.11. We can repeat the proof of Proposition 4.10 and conclude that

$$(4.63) \quad \widehat{P}_{k,k^{-N_0},s}^{(q)} - \widehat{P}_{k,s}^{(q)} = T + \mathcal{S}_k^* \left(\widehat{P}_{k,k^{-N_0},s}^{(q)} - \widehat{P}_{k,s}^{(q)} \right) \mathcal{S}_k,$$

where T is smoothing and the Schwartz distribution kernel $T(x, y)$ of T satisfies (4.48). Let

$$u \in H_{\text{comp}}^m(D, \Lambda^q T^{*(0,1)} M), \quad m \leq 0, \quad m \in \mathbb{Z}.$$

We consider

$$v = s^k e^{k\phi} \mathcal{S}_k u - P_k^{(q)}(s^k e^{k\phi} \mathcal{S}_k u).$$

Since \mathcal{S}_k is a smoothing operator, $v \in \mathcal{C}^\infty(M, L^k)$. Moreover, it is easy to see that $v \perp \mathcal{H}^0(M, L^k)$. We have

$$(4.64) \quad \square_k^{(q)} v = s^k e^{k\phi} \square_s^{(q)} \mathcal{S}_k u.$$

From Theorem 3.11, we see that $\square_s^{(q)} \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})}$. Combining this with (4.64), we obtain

$$\left\| \square_k^{(q)} v \right\| \leq C_N k^{-N} \|u\|_m,$$

for every $N > 0$, where $C_N > 0$ is independent of k . Since $v \perp \mathcal{H}^0(M, L^k)$, from Definition 1.5 we conclude that

$$\|v\| \leq \tilde{C}_N k^{-N} \|u\|_m,$$

for every $N > 0$, where $\tilde{C}_N > 0$ is independent of k . Thus,

$$\mathcal{S}_k - \widehat{P}_{k,s}^{(q)} \mathcal{S}_k = O(k^{-N}) : H_{\text{comp}}^m(D, \Lambda^q T^{*(0,1)} M) \rightarrow L^2(D, \Lambda^q T^{*(0,1)} M),$$

for all $N > 0$, $m \in \mathbb{Z}$, $m \leq 0$, and hence

$$\begin{aligned} & \mathcal{S}_k^* \mathcal{S}_k - \mathcal{S}_k^* \widehat{P}_{k,s}^{(q)} \mathcal{S}_k \\ &= O(k^{-N}) : H_{\text{comp}}^m(D, \Lambda^q T^{*(0,1)} M) \rightarrow H_{\text{loc}}^{m+N_1}(D, \Lambda^q T^{*(0,1)} M), \end{aligned}$$

for all $N, N_1 > 0$, $m \in \mathbb{Z}$. We conclude that

$$\mathcal{S}_k^* \mathcal{S}_k \equiv \mathcal{S}_k^* \widehat{P}_{k,s}^{(q)} \mathcal{S}_k \pmod{O(k^{-\infty})}.$$

From this, (3.61) and (4.58), we obtain

$$\widehat{P}_{k,k^{-N_0},s}^{(q)} = \tilde{T} + \mathcal{S}_k^* \widehat{P}_{k,s}^{(q)} \mathcal{S}_k,$$

where \tilde{T} is smoothing and the Schwartz distribution kernel $\tilde{T}(x, y)$ of \tilde{T} satisfies (4.48). From this and Proposition 4.10, we conclude that the distribution kernel of $\mathcal{S}_k^* (\widehat{P}_{k,k^{-N_0},s}^{(q)} - \widehat{P}_{k,s}^{(q)}) \mathcal{S}_k$ satisfies (4.48). Combining this with (4.63) and (4.62) follows. \square

Since Theorems 3.11 and 4.11 hold in the case when the forms take values in $L^k \otimes E$, for a given holomorphic vector bundle E over M , we can generalize Theorem 4.14 with the same proof to the case when the forms take values in $L^k \otimes E$.

4.5. Calculation of the leading coefficients. Proof of Theorem 1.2

Now, we prove (1.11) and (1.12). In this section, we assume that $q = 0$. First, let us review the necessary definitions from Riemannian geometry. We will use the same notations as in the discussion after (1.9).

Consider the Kähler metric $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ introduced in (1.10). Let $\langle \cdot, \cdot \rangle_\omega$ be the Hermitian metric on $\mathbb{C}TM$ induced by ω . In local holomorphic coordinates $z = (z_1, \dots, z_n)$, put

$$(4.65) \quad \omega = \sqrt{-1} \sum_{j,k=1}^n \omega_{j,k} dz_j \wedge d\bar{z}_k, \quad \Theta = \sqrt{-1} \sum_{j,k=1}^n \Theta_{j,k} dz_j \wedge d\bar{z}_k,$$

where $\Theta_{j,k} = \langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \rangle$, $\omega_{j,k} = \langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k} \rangle_\omega$, $j, k = 1, \dots, n$. Put

$$(4.66) \quad h = (h_{j,k})_{j,k=1}^n, \quad h_{j,k} = \omega_{k,j}, \quad j, k = 1, \dots, n,$$

and $h^{-1} = (h^{j,k})_{j,k=1}^n$, h^{-1} is the inverse matrix of h . The complex Laplacian with respect to ω is given by

$$(4.67) \quad \Delta_\omega = (-2) \sum_{j,k=1}^n h^{j,k} \frac{\partial^2}{\partial z_j \partial \bar{z}_k}.$$

We note that $h^{j,k} = \langle dz_j, dz_k \rangle_\omega$, $j, k = 1, \dots, n$. Put

$$(4.68) \quad V_\omega := \det(\omega_{j,k})_{j,k=1}^n, \quad V_\Theta := \det(\Theta_{j,k})_{j,k=1}^n$$

and set

$$(4.69) \quad r = \Delta_\omega \log V_\omega, \quad \hat{r} = \Delta_\omega \log V_\Theta.$$

Then r is the scalar curvature of g_ω^{TX} . Let R_Θ^{\det} be the curvature of the canonical line bundle $K_M = \det T^{*(1,0)}M$ with respect to the real two form

Θ . We recall that

$$(4.70) \quad R_{\Theta}^{\det} = -\bar{\partial}\partial \log V_{\Theta}.$$

Let h be as in (4.66). The connection matrix of the Chern connection on $T^{(1,0)}M$ is given by $\theta = h^{-1}\partial h = (\theta_{j,k})_{j,k=1}^n$, $\theta_{j,k} \in T^{*(1,0)}M$, $j, k = 1, \dots, n$. θ is the Chern connection matrix with respect to ω . The Chern curvature with respect to ω is given by

$$(4.71) \quad \begin{aligned} R_{\omega}^{TM} &= \bar{\partial}\theta = (\bar{\partial}\theta_{j,k})_{j,k=1}^n = (\mathcal{R}_{j,k})_{j,k=1}^n \\ &\in \mathcal{C}^{\infty}(M, \Lambda^{1,1}T^*M \otimes \text{End}(T^{(1,0)}M)), \\ R_{\omega}^{TM}(\bar{U}, V) &\in \text{End}(T^{(1,0)}M), \quad \forall U, V \in T^{(1,0)}M, \\ R_{\omega}^{TM}(\bar{U}, V)\xi &= \sum_{j,k=1}^n \langle \mathcal{R}_{j,k}, \bar{U} \wedge V \rangle \xi_k \frac{\partial}{\partial z_j}, \quad \xi = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j}, \quad U, V \in T^{(1,0)}M. \end{aligned}$$

Set

$$(4.72) \quad |R_{\omega}^{TM}|_{\omega}^2 := \sum_{j,k,s,t=1}^n |\langle R_{\omega}^{TM}(\bar{e}_j, e_k)e_s, e_t \rangle_{\omega}|^2,$$

where e_1, \dots, e_n is an orthonormal frame for $T^{(1,0)}M$ with respect to $\langle \cdot, \cdot \rangle_{\omega}$. It is straightforward to see that the definition of $|R_{\omega}^{TM}|_{\omega}^2$ is independent of the choices of orthonormal frames. Thus, $|R_{\omega}^{TM}|_{\omega}^2$ is globally defined. The Ricci curvature with respect to ω is given by

$$(4.73) \quad \text{Ric}_{\omega} := -\sum_{j=1}^n \langle R_{\omega}^{TM}(\cdot, e_j)\cdot, e_j \rangle_{\omega},$$

where e_1, \dots, e_n is an orthonormal frame for $T^{(1,0)}M$ with respect to $\langle \cdot, \cdot \rangle_{\omega}$. That is,

$$\langle \text{Ric}_{\omega}, U \wedge V \rangle = -\sum_{j=1}^n \langle R_{\omega}^{TX}(U, e_j)V, e_j \rangle_{\omega}, \quad U, V \in \mathbb{C}TM.$$

Ric_{ω} is a global $(1, 1)$ form. We can check that

$$\text{Ric}_{\omega} = -\bar{\partial}\partial \log V_{\omega},$$

where V_{ω} is as in (4.68).

Let \mathcal{S}_k , $b(z, w, k)$, $b_j(z, w)$, $j = 0, 1, 2, \dots$, be as in Theorem 4.11. We will calculate $b_1(p, p)$ and $b_2(p, p)$ at a fixed $p \in D$. In a small neighbourhood $D \Subset M(0)$ of the point p there exist local coordinates $(D, z) \cong (D, x)$ centred at p and a local frame s of L , $|s|_{h^L}^2 = e^{-2\phi}$ so that ϕ is a Kähler potential of ω satisfying

(4.74)

$$\phi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + \phi_1(z),$$

$$\phi_1(z) = O(|z|^4), \quad \frac{\partial^{|\alpha|+|\beta|} \phi_1}{\partial z^\alpha \partial \bar{z}^\beta}(0) = 0 \quad \text{for } \alpha, \beta \in \mathbb{N}_0^n, |\alpha| \leq 1 \text{ or } |\beta| \leq 1,$$

and moreover

$$(4.75) \quad \Theta(z) = \sqrt{-1} \sum_{j=1}^n dz_j \wedge d\bar{z}_j + O(|z|), \quad z \rightarrow 0,$$

(this is possible by Ruan [57]). First, we claim that

$$(4.76) \quad \bar{\partial}_s \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})},$$

where $\bar{\partial}_s$ is as in (3.7). We notice that $\square_s^{(0)} \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})}$. Thus, $\square_s^{(1)} \bar{\partial}_s \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})}$. From Theorem 3.13, we know that $\square_s^{(1)}$ has semi-classical parametrices. Thus, $\bar{\partial}_s \mathcal{S}_k \equiv 0 \pmod{O(k^{-\infty})}$ so (4.76) follows. Now, we claim that

$$(4.77) \quad \bar{\partial}_z (i\Psi(z, w) + \phi(z)) \text{ vanishes to infinite order at } z = w.$$

We write $w = (w_1, \dots, w_n) = (y_1, \dots, y_{2n}) = y$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \dots, n$. We assume that there exist $\alpha_0, \beta_0 \in \mathbb{N}_0^{2n}$, $|\alpha_0| + |\beta_0| \geq 1$ and $(z_0, z_0) \in D \times D$, such that

$$(4.78) \quad \partial_x^{\alpha_0} \partial_y^{\beta_0} \left(\bar{\partial}_z (i\Psi(z, w) + \phi(z)) \right) \Big|_{(z_0, z_0)} = C_{\alpha_0, \beta_0} \neq 0,$$

and if $|\alpha| + |\beta| < |\alpha_0| + |\beta_0|$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, then

$$(4.79) \quad \partial_x^\alpha \partial_y^\beta \left(\bar{\partial}_z (i\Psi(z, w) + \phi(z)) \right) \Big|_{(z_0, z_0)} = 0.$$

From (4.78), (4.79) and since $b_0(z_0, z_0) \neq 0$, $\Psi(z_0, z_0) = 0$, we can check that

$$(4.80) \quad \lim_{k \rightarrow \infty} k^{-n-1} \partial_x^{\alpha_0} \partial_y^{\beta_0} \left(\bar{\partial}_s (e^{ik\Psi(z,w)} b(z, w, k)) \right) \Big|_{(z_0, z_0)} = C_{\alpha_0, \beta_0} b_0(z_0, z_0) \neq 0.$$

On the other hand, since $\bar{\partial}_s (e^{ik\Psi(z,w)} b(z, w, k)) \equiv 0 \pmod{O(k^{-\infty})}$, we can check that

$$(4.81) \quad \lim_{k \rightarrow \infty} k^{-n-1} \partial_x^{\alpha_0} \partial_y^{\beta_0} \left(\bar{\partial}_s (e^{ik\Psi(z,w)} b(z, w, k)) \right) \Big|_{(z_0, z_0)} = 0.$$

We get thus a contradiction, hence the claim (4.77) follows. Similarly, we have

$$(4.82) \quad \partial_w (i\Psi(z, w) + \phi(w)) \text{ vanishes to infinite order at } z = w.$$

In particular, we have

$$(4.83)$$

$\bar{\partial}_z (i\Psi(z, 0) + \phi(z))$, $\partial_z (i\Psi(0, z) + \phi(z))$ vanish to infinite order at $z = 0$.

Combining (4.77), (4.82), (4.83) with $\Psi(z, z) = 0$, it is easy to check that for all $\alpha \in \mathbb{N}_0^n$,

$$(4.84) \quad \begin{aligned} i \frac{\partial^{|\alpha|} \Psi(z, 0)}{\partial z^\alpha} \Big|_{z=0} &= -i \frac{\partial^{|\alpha|} \Psi(0, z)}{\partial z^\alpha} \Big|_{z=0} = \frac{\partial^{|\alpha|} \phi}{\partial z^\alpha}(0) = 0, \\ i \frac{\partial^{|\alpha|} \Psi(0, z)}{\partial \bar{z}^\alpha} \Big|_{z=0} &= -i \frac{\partial^{|\alpha|} \Psi(z, 0)}{\partial \bar{z}^\alpha} \Big|_{z=0} = \frac{\partial^{|\alpha|} \phi}{\partial \bar{z}^\alpha}(0) = 0. \end{aligned}$$

Here we used (4.74). From (4.83) and (4.84), we deduce that for every $N \in \mathbb{N}_0$

$$(4.85) \quad \Psi(z, 0) = i\phi(z) + O(|z|^N), \quad \Psi(0, z) = i\phi(z) + O(|z|^N).$$

We claim that

$$(4.86) \quad \bar{\partial}_z b_j(z, w) \text{ and } \partial_w b_j(z, w) \text{ vanish to infinite order at } z = w, \text{ for all } j \in \mathbb{N}_0.$$

In view of (4.77), we see that $\bar{\partial}_z(i\Psi(z, w) + \phi(z))$ vanishes to infinite order at $z = w$. From this observation and (4.76), we conclude that

$$(4.87) \quad e^{ik\Psi(z, w)} \bar{\partial}_z b(z, w, k) = H_k(z, w),$$

where $H_k(z, w) \equiv 0 \pmod{O(k^{-\infty})}$. We assume that there exist $\gamma_0, \delta_0 \in \mathbb{N}_0^{2n}$, $|\gamma_0| + |\delta_0| \geq 1$ and $(z_1, z_1) \in D \times D$, such that

$$\partial_x^{\gamma_0} \partial_y^{\delta_0} (\bar{\partial}_z b_0(z, w)) \Big|_{(z_1, z_1)} = D_{\gamma_0, \delta_0} \neq 0,$$

and

$$\partial_x^\gamma \partial_y^\delta (\bar{\partial}_z b_0(z, w)) \Big|_{(z_1, z_1)} = 0 \quad \text{if } |\gamma| + |\delta| < |\gamma_0| + |\delta_0|, \quad \gamma, \delta \in \mathbb{N}_0^{2n}.$$

From (4.87), we have

$$(4.88) \quad \partial_x^{\gamma_0} \partial_y^{\delta_0} (\bar{\partial}_z b(z, w, k)) \Big|_{(z_1, z_1)} = \partial_x^{\gamma_0} \partial_y^{\delta_0} (e^{-ik\Psi(z, w)} H_k(z, w)) \Big|_{(z_1, z_1)}.$$

Since $\Psi(z_1, z_1) = 0$, we have

$$(4.89) \quad \lim_{k \rightarrow \infty} k^{-n} \partial_x^{\gamma_0} \partial_y^{\delta_0} (e^{-ik\Psi(z, w)} H_k(z, w)) \Big|_{(z_1, z_1)} = 0.$$

On the other hand, we can check that

$$(4.90) \quad \lim_{k \rightarrow \infty} k^{-n} \partial_x^{\gamma_0} \partial_y^{\delta_0} (\bar{\partial}_z b(z, w, k)) \Big|_{(z_1, z_1)} = D_{\gamma_0, \delta_0} \neq 0.$$

From (4.90), (4.89) and (4.88), we get a contradiction. Thus, $\bar{\partial}_z b_0(z, w)$ vanishes to infinite order at $z = w$. Similarly, we can repeat the procedure above and conclude that $\bar{\partial}_z b_j(z, w)$ and $\partial_w b_j(z, w)$ vanish to infinite order at $z = w$, $\forall j \in \mathbb{N}_0$. The claim (4.86) follows.

Now, we are ready to calculate $b_1(0, 0)$ and $b_2(0, 0)$. We note that

$$b_0(z, z) = (2\pi)^{-n} \det \dot{R}^L(z).$$

From this and (4.86), it is easy to see that for all $\alpha \in \mathbb{N}_0^n$,

$$(4.91) \quad \frac{\partial^{|\alpha|} b_0(z, 0)}{\partial z^\alpha} \Big|_{z=0} = (2\pi)^{-n} \frac{\partial^{|\alpha|} (\det \dot{R}^L(z))}{\partial z^\alpha} \Big|_{z=0}, \quad \frac{\partial^{|\alpha|} b_0(z, 0)}{\partial \bar{z}^\alpha} \Big|_{z=0} = 0.$$

Since $\mathcal{S}_k \circ \mathcal{S}_k \equiv \mathcal{S}_k \pmod{O(k^{-\infty})}$, we have

$$(4.92) \quad b(0, 0, k) = \int_D e^{ik(\Psi(0,z) + \Psi(z,0))} b(0, z, k) b(z, 0, k) V_\Theta(z) d\lambda(z) + r_k,$$

where $d\lambda(z) = 2^n dx_1 dx_2 \cdots dx_{2n}$, V_Θ is given by (4.68) and

$$\lim_{k \rightarrow \infty} \frac{r_k}{k^N} = 0, \quad \forall N \geq 0.$$

We notice that since $b(z, w, k)$ is properly supported, we have

$$b(0, z, k) \in C_0^\infty(D), \quad b(z, 0, k) \in C_0^\infty(D).$$

We apply the stationary phase formula of Hörmander [39, Theorem 7.7.5]) to the integral in (4.92) and obtain (for the details see Hsiao [41, Section 4]):

Theorem 4.15. *We have*

$$(4.93) \quad \begin{aligned} b_1(0, 0) &= (2\pi)^n (\det \dot{R}^L(0))^{-1} \left(2b_0(0, 0)b_1(0, 0) \right. \\ &\quad \left. + \frac{1}{2} \Delta_0 (V_\Theta b_0(0, z) b_0(z, 0))(0) - \frac{1}{4} \Delta_0^2 (\phi_1 V_\Theta b_0(0, z) b_0(z, 0))(0) \right) \end{aligned}$$

and

$$(4.94) \quad \begin{aligned} b_2(0, 0) &= (2\pi)^n (\det \dot{R}^L(0))^{-1} \left(2b_0(0, 0)b_2(0, 0) + b_1(0, 0)^2 \right. \\ &\quad + \frac{1}{2} \Delta_0 (V_\Theta (b_0(0, z)b_1(z, 0) + b_1(0, z)b_0(z, 0)))(0) \\ &\quad - \frac{1}{4} \Delta_0^2 (\phi_1 V_\Theta (b_0(0, z)b_1(z, 0) + b_1(0, z)b_0(z, 0)))(0) \\ &\quad + \frac{1}{8} \Delta_0^2 (V_\Theta b_0(0, z) b_0(z, 0))(0) - \frac{1}{24} \Delta_0^3 (\phi_1 V_\Theta b_0(0, z) b_0(z, 0))(0) \\ &\quad \left. + \frac{1}{192} \Delta_0^4 (\phi_1^2 V_\Theta b_0(0, z) b_0(z, 0))(0) \right), \end{aligned}$$

where $\Delta_0 = \sum_{j=1}^n \frac{1}{\lambda_j} \frac{\partial^2}{\partial \bar{z}_j \partial z_j}$, ϕ_1 is as in (4.74) and V_Θ is as in (4.68).

From (4.86), (4.91) and (4.93), it is straightforward to see that (see Section 4.2 in [41], for the details)

$$(4.95) \quad \begin{aligned} b_1(0,0) &= (2\pi)^{-n} \det \dot{R}^L(0) \left(\frac{1}{4\pi} \widehat{r}(0) - \frac{1}{8\pi} r(0) \right) \\ &= \frac{V_\omega(0)}{V_\Theta(0)} \left(\frac{1}{4\pi} (\Delta_\omega \log V_\Theta)(0) - \frac{1}{8\pi} (\Delta_\omega \log V_\omega)(0) \right), \end{aligned}$$

where \widehat{r} and r are as in (4.69) and V_ω is as in (4.68). From this, (1.11) follows.

Similarly, from (1.11) and (4.86), it is easy to see that for all $\alpha \in \mathbb{N}_0^n$,

$$(4.96) \quad \begin{aligned} \left. \frac{\partial^{|\alpha|} b_1(z,0)}{\partial z^\alpha} \right|_{z=0} &= (2\pi)^{-n} \left. \frac{\partial^{|\alpha|} \left(\det \dot{R}^L(z) \left(\frac{1}{4\pi} \widehat{r}(z) - \frac{1}{8\pi} r(z) \right) \right)}{\partial z^\alpha} \right|_{z=0}, \\ \left. \frac{\partial^{|\alpha|} b_1(z,0)}{\partial \bar{z}^\alpha} \right|_{z=0} &= 0. \end{aligned}$$

From (4.86), (4.96) and (4.94), it is straightforward to see that (see Section 4.3 in [41], for the details)

$$(4.97) \quad \begin{aligned} b_2(0,0) &= (2\pi)^{-n} \det \dot{R}^L(0) \left(\frac{1}{128\pi^2} r^2 - \frac{1}{32\pi^2} r\widehat{r} + \frac{1}{32\pi^2} (\widehat{r})^2 - \frac{1}{32\pi^2} \Delta_\omega \widehat{r} \right. \\ &\quad \left. - \frac{1}{8\pi^2} |R_\Theta^{\det}|_\omega^2 + \frac{1}{8\pi^2} \langle \text{Ric}_\omega, R_\Theta^{\det} \rangle_\omega + \frac{1}{96\pi^2} \Delta_\omega r - \frac{1}{24\pi^2} |\text{Ric}_\omega|_\omega^2 \right. \\ &\quad \left. + \frac{1}{96\pi^2} |R_\omega^{TM}|_\omega^2 \right) (0), \end{aligned}$$

where Δ_ω , R_Θ^{\det} , Ric_ω and R_ω^{TM} are as in (4.67), (4.70), (4.73) and (4.71) respectively, and $\langle \cdot, \cdot \rangle_\omega$, $|\cdot|_\omega$ are as as in the discussion after (1.10) and $|R_\omega^{TM}|_\omega^2$ is given by (4.72). From (4.97), (1.12) follows.

5. Asymptotic upper bounds near the degeneracy set

In this section, we will use the heat equation expansion for $\square_k^{(q)}$ of Marinescu [49, Section 1.6] to get an asymptotic upper bound near the degeneracy part of L . The goal of this Section is to prove (1.13).

By the spectral theorem (see Davies [21, Theorem 2.5.1]), there exists a finite measure μ on $\mathbb{S} \times \mathbb{N}$, where \mathbb{S} denotes the spectrum of $\square_k^{(q)}$, and a

unitary operator

$$(5.1) \quad U : L^2_{(0,q)}(M, L^k) \rightarrow L^2(\mathbb{S} \times \mathbb{N}, d\mu)$$

with the following properties. If $h : \mathbb{S} \times \mathbb{N} \rightarrow \mathbb{R}$ is the function $h(s, n) = s$, then the element ξ of $L^2_{(0,q)}(M, L^k)$ lies in $\text{Dom } \square_k^{(q)}$ if and only if $hU(\xi) \in L^2$.

We have $U \square_k^{(q)} U^{-1} \varphi = h\varphi$ for all $\varphi \in U(\text{Dom } \square_k^{(q)})$.

We identify $L^2_{(0,q)}(M, L^k)$ with $L^2(\mathbb{S} \times \mathbb{N}, d\mu)$. Then the heat operator $\exp(-t \square_k^{(q)})$, $t > 0$, is the operator on $L^2(\mathbb{S} \times \mathbb{N}, d\mu)$ given by

$$\exp(-t \square_k^{(q)}) : L^2(\mathbb{S} \times \mathbb{N}, d\mu) \rightarrow L^2(\mathbb{S} \times \mathbb{N}, d\mu), \quad u(s, n) \mapsto e^{-st} u(s, n).$$

Since $\square_k^{(q)}$ is elliptic, the distribution kernel of $\exp(-t \square_k^{(q)})$ is smooth (see [49, Theorem D.1.2]). Let

$$\exp(-t \square_k^{(q)})(x, y) \in \mathcal{C}^\infty(M \times M, L_y^k \otimes \Lambda^q T_y^{*(0,1)} M \boxtimes L_x^k \otimes \Lambda^q T_x^{*(0,1)} M)$$

be the distribution kernel of $\exp(-t \square_k^{(q)})$ with respect to $(\cdot, \cdot)_k$. That is,

$$\begin{aligned} (\exp(-t \square_k^{(q)})u)(x) &= \int_M \exp(-t \square_k^{(q)})(x, y) u(y) dv_M(y), \\ u &\in L^2_{(0,q)}(M, L^k). \end{aligned}$$

Let s be a local section of L over \tilde{X} , where $\tilde{X} \subset M$. Then on $\tilde{X} \times \tilde{X}$ we can write

$$\exp(-t \square_k^{(q)})(x, y) = \exp(-t \square_k^{(q)})_s(x, y) s(x)^k s^*(y)^k,$$

where

$$\exp(-t \square_k^{(q)})_s(x, y) \in \mathcal{C}^\infty(\tilde{X} \times \tilde{X}, \Lambda^q T_y^* M \boxtimes \Lambda^q T_x^* M).$$

For $u \in \Omega_0^{0,q}(\tilde{X}, L^k)$ consider $\tilde{u} \in \Omega_0^{0,q}(\tilde{X})$ with $u = s^k \tilde{u}$. Then for $x \in \tilde{X}$,

$$(5.2) \quad \begin{aligned} (\exp(-t \square_k^{(q)})u)(x) &= s(x)^k \int_M \exp(-t \square_k^{(q)})_s(x, y) \langle u(y), s^*(y)^k \rangle dv_M(y) \\ &= s(x)^k \int_M \exp(-t \square_k^{(q)})_s(x, y) \tilde{u}(y) dv_M(y). \end{aligned}$$

For $x = y$, we can check that the function

$$\exp(-t \square_k^{(q)})_s(x, x) \in \mathcal{C}^\infty(\tilde{X}, \Lambda^q T_x^* M \boxtimes \Lambda^q T_x^* M)$$

is independent of the choices of local section s . We identify the function $\exp(-t \square_k^{(q)})_s(x, x)$ with $\exp(-t \square_k^{(q)})(x, x)$. The trace of $\exp(-t \square_k^{(q)})(x, x)$ is given by

$$\mathrm{Tr} \exp(-t \square_k^{(q)})(x, x) := \sum_{j=1}^d \langle \exp(-t \square_k^{(q)})(x, x) e_{J_j}(x), e_{J_j}(x) \rangle,$$

where $\{e_{J_j}(x)\}_{j=1}^d$ is an orthonormal basis of the space $\Lambda^q T_x^{*(0,1)} M$ with respect to $\langle \cdot, \cdot \rangle$, $\dim \Lambda^q T_x^{*(0,1)} M = d$.

Proposition 5.1. *Fix $t > 0$ and $N_0 \geq 1$. We have for k large,*

$$(5.3) \quad \mathrm{Tr} \exp\left(-\frac{t}{k} \square_k^{(q)}\right)(x, x) \geq (1 - k^{-N_0}) \mathrm{Tr} P_{k, k^{-N_0}}^{(q)}(x, x), \quad \forall x \in M,$$

where $P_{k, k^{-N_0}}^{(q)}(x, x)$ is as in (1.5).

Proof. First, we claim that for all $u \in \Omega_0^{0,q}(M, L^k)$,

$$(5.4) \quad \left(\exp\left(-\frac{t}{k} \square_k^{(q)}\right)u, u\right)_k \geq (1 - k^{-N_0}) (P_{k, k^{-N_0}}^{(q)}u, u)_k.$$

We identify $L^2_{(0,q)}(M, L^k)$ with $L^2(\mathbb{S} \times \mathbb{N}, d\mu)$. Then

$$\exp\left(-\frac{t}{k} \square_k^{(q)}\right) : u(s, n) \in L^2(\mathbb{S} \times \mathbb{N}, d\mu) \mapsto e^{-s \frac{t}{k}} u(s, n)$$

and

$$P_{k, k^{-N_0}}^{(q)} : u(s, n) \in L^2(\mathbb{S} \times \mathbb{N}, d\mu) \mapsto u(s, n) \mathbf{1}_{[0, k^{-N_0}]}(s).$$

For $u(s, n) \in L^2(\mathbb{S} \times \mathbb{N}, d\mu)$, we have

$$(5.5) \quad \begin{aligned} \left(\exp\left(-\frac{t}{k} \square_k^{(q)}\right)u, u\right)_k &= \int_{\mathbb{S} \times \mathbb{N}} e^{-s \frac{t}{k}} |u(s, n)|^2 d\mu \\ &\geq \int_{\mathbb{S} \times \mathbb{N}} e^{-s \frac{t}{k}} |u(s, n)|^2 \mathbf{1}_{[0, k^{-N_0}]}(s) d\mu \\ &\geq \int_{\mathbb{S} \times \mathbb{N}} |u(s, n)|^2 \mathbf{1}_{[0, k^{-N_0}]}(s) d\mu \\ &\quad - \int_{\mathbb{S} \times \mathbb{N}} \left|e^{-s \frac{t}{k}} - 1\right| |u(s, n)|^2 \mathbf{1}_{[0, k^{-N_0}]}(s) d\mu \\ &\geq \left(1 - \sup_{s \in [0, k^{-N_0}]} (1 - e^{-s \frac{t}{k}})\right) (P_{k, k^{-N_0}}^{(q)}u, u)_k. \end{aligned}$$

It is easy to see that fix $t > 0$, we have $\sup_{s \in [0, k^{-N_0}]} (1 - e^{-s \frac{t}{k}}) \leq k^{-N_0}$ if k large. From this observation and (5.5), the claim (5.4) follows.

Now, fix $p \in M$ and let s be a local section of L defined in some open neighbourhood D of p , $|s|_{h^L}^2 = e^{-2\phi}$. Let $e_{J_1}(p), \dots, e_{J_d}(p)$, be an orthonormal basis of $\Lambda^q T_p^{*(0,1)} M$ with respect to $\langle \cdot, \cdot \rangle$. Fix $l \in \{1, \dots, d\}$. For each $j \in \mathbb{N}$, take $\chi_j \in \Omega_0^{0,q}(D, L^k)$ so that for every continuous operator $F : \mathcal{C}^\infty(D, L^k \otimes \Lambda^q T^{*(0,1)} M) \rightarrow \mathcal{C}^\infty(D, L^k \otimes \Lambda^q T^{*(0,1)} M)$ with smooth kernel $F(x, y) \in \mathcal{C}^\infty(M \times M, L_y^k \otimes \Lambda^q T_y^{*(0,1)} M \boxtimes L_x^k \otimes \Lambda^q T_x^{*(0,1)} M)$, we have

$$(F\chi_j, \chi_j)_k \rightarrow \langle F(p, p)e_{J_l}(p), e_{J_l}(p) \rangle, \quad j \rightarrow \infty.$$

Then, we have

$$\begin{aligned} \left(\exp \left(-\frac{t}{k} \square_k^{(q)} \right) \chi_j, \chi_j \right)_k &\rightarrow \left\langle \exp \left(-\frac{t}{k} \square_k^{(q)} \right) (p, p)e_{J_l}(p), e_{J_l}(p) \right\rangle, \quad j \rightarrow \infty, \\ (P_{k, k^{-N_0}}^{(q)} \chi_j, \chi_j)_k &\rightarrow \langle P_{k, k^{-N_0}}^{(q)}(p, p)e_{J_l}(p), e_{J_l}(p) \rangle, \quad j \rightarrow \infty. \end{aligned}$$

Combining this with (5.4), we conclude that

$$\begin{aligned} \left\langle \exp \left(-\frac{t}{k} \square_k^{(q)} \right) (p, p)e_{J_l}(p), e_{J_l}(p) \right\rangle &\geq (1 - k^{-N_0}) \\ &\quad \times \langle P_{k, k^{-N_0}}^{(q)}(p, p)e_{J_l}(p), e_{J_l}(p) \rangle. \end{aligned}$$

Thus,

$$\mathrm{Tr} \exp \left(-\frac{t}{k} \square_k^{(q)} \right) (p, p) \geq (1 - k^{-N_0}) \mathrm{Tr} P_{k, k^{-N_0}}^{(q)}(p, p),$$

so (5.3) follows. \square

Theorem 5.2 [49, Theorem 1.6.1]. *For each $t > 0$ fixed and any $D \Subset M$, $m \in \mathbb{N}$, we have as $k \rightarrow \infty$,*

(5.6)

$$\begin{aligned} &\mathrm{Tr} \exp \left(-\frac{t}{k} \square_k^{(q)} \right) (x, x) \\ &= k^n (2\pi)^{-n} \left(\sum_{j_1 < j_2 < \dots < j_q} \exp \left(-t \sum_{i=1}^q a_{j_i}(x) \right) \right) \prod_{j=1}^n \frac{a_j(x)}{1 - e^{-ta_j(x)}} + o(k^n), \end{aligned}$$

in the \mathcal{C}^m norm on $\mathcal{C}^\infty(D, \Lambda^q T^{*(0,1)} M \boxtimes \Lambda^q T^{*(0,1)} M)$, where $a_1(x), \dots, a_n(x)$ are the eigenvalues of $\hat{R}^L(x)$. Here we use the convention that $\frac{a_j(x)}{1 - e^{-ta_j(x)}} := \frac{1}{t}$, if $a_j(x) = 0$.

From (5.3) and (5.6), we know that

$$(5.7) \quad \begin{aligned} & (1 - k^{-N_0}) \operatorname{Tr} P_{k,k^{-N_0}}^{(q)}(x, x) \\ & \leq k^n (2\pi)^{-n} \left(\sum_{j_1 < j_2 < \dots < j_q} \exp \left(-t \sum_{i=1}^q a_{j_i}(x) \right) \right) \prod_{j=1}^n \frac{a_j(x)}{1 - e^{-ta_j(x)}} + o(k^n), \end{aligned}$$

locally uniformly on M .

Now, let M_{deg} be as in Theorem 1.3. Fix $t > 1$, t large and $x_0 \in M_{\text{deg}}$ and let U be a small neighbourhood of x_0 such that for every point $x \in U$, there is an eigenvalue $a_0(x)$ of $\dot{R}^L(x)$ such that $|ta_0(x)| < 1$. Fix $p \in U$. Set

$$\begin{aligned} \iota(p) = & \{j \in \{1, \dots, n\}; |a_j(p)t| < 1, \\ & \text{where } a_1(p), \dots, a_n(p) \text{ are the eigenvalues of } \dot{R}^L(p)\}. \end{aligned}$$

Fix $1 \leq j_1 < j_2 < \dots < j_q \leq n$. We have

$$(5.8) \quad \begin{aligned} & \exp \left(-t \sum_{i=1}^q a_{j_i}(p) \right) \prod_{j=1}^n \frac{a_j(p)}{1 - e^{-ta_j(p)}} \\ & = \prod_{j_i \in \iota(p)} \frac{e^{-ta_{j_i}(p)} a_{j_i}(p)}{1 - e^{-ta_{j_i}(p)}} \prod_{j_i \notin \iota(p)} \frac{e^{-ta_{j_i}(p)} a_{j_i}(p)}{1 - e^{-ta_{j_i}(p)}} \\ & \quad \times \prod_{j \in \iota(p), j \notin \{j_1, \dots, j_q\}} \frac{a_j(p)}{1 - e^{-ta_j(p)}} \prod_{j \notin \iota(p), j \notin \{j_1, \dots, j_q\}} \frac{a_j(p)}{1 - e^{-ta_j(p)}}. \end{aligned}$$

We observe that there is a constant $C > 0$ such that

$$(5.9) \quad \begin{aligned} & \left| \frac{x}{1 - e^x} \right| \leq C, \quad \left| \frac{xe^x}{1 - e^x} \right| \leq C, \quad \forall x \in \mathbb{R}, \quad |x| \leq 1, \\ & \left| \frac{1}{1 - e^x} \right| \leq C, \quad \left| \frac{e^x}{1 - e^x} \right| \leq C, \quad \forall x \in \mathbb{R}, \quad |x| > 1. \end{aligned}$$

From (5.9) and (5.8), it is straightforward to see that

$$(5.10) \quad \exp \left(-t \sum_{i=1}^q a_{j_i}(p) \right) \prod_{j=1}^n \frac{a_j(p)}{1 - e^{-ta_j(p)}} \leq \prod_{j \in \iota(p)} \frac{C}{t} \prod_{j \notin \iota(p)} C |a_j(p)|,$$

where C is the constant as in (5.9).

The proof of (1.13). Let $\varepsilon > 0$. Let $W \Subset M$ be any open set of x_0 . Take $t > \max\{C, 1\}$ large enough so that

$$(5.11) \quad (2\pi)^{-n} d \frac{C}{t} \left(1 + C \sup \left\{ |a(x)|; a(x): \text{eigenvalue of } \dot{R}^L(x), x \in W \right\} \right)^n < \frac{\varepsilon}{2},$$

where C is the constant as in (5.9) and $d = \dim \Lambda^q T_x^{*(0,1)} M$. Let $U \Subset W$ be a small neighbourhood of x_0 such that for every point $x \in U$, there is an eigenvalue $a_0(x)$ of $\dot{R}^L(x)$ such that $|ta_0(x)| < 1$. From (5.10), (5.7) and (5.11), we see that

$$\text{Tr } P_{k, k-N_0}^{(q)}(x, x) \leq \frac{1}{1 - k^{-N_0}} \frac{\varepsilon}{2} k^n + o(k^n), \quad x \in U,$$

thus (1.13) follows. \square

Theorem 5.2 also holds on the case when the forms take values in $L^k \otimes E$, for a given holomorphic vector bundle E over M . In this case, the right-hand side of (5.6) gets multiplied by $\text{rank}(E)$, see Theorem 1.6.1 in [49]. From this observation, (1.13) remains true with the same proof on the case when the forms take values in $L^k \otimes E$, for a given holomorphic vector bundle E over M .

6. Bergman kernel asymptotic expansion for adjoint semi-positive line bundles

In this section, we prove Theorem 1.7, i.e., the asymptotic expansion of the Bergman kernel of $L^k \otimes K_M$, where L is a semi-positive line bundle over a complete Kähler manifold M and K_M is the canonical line bundle. The existence of the expansion (1.19) follows immediately from Theorem 6.4, while the calculation of the coefficients is given at the end of this section.

We assume that (M, Θ) is a complete Kähler manifold. Let K_M be the canonical line bundle over M . Then, $\Omega^{n,q}(M, L^k) = \Omega^{0,q}(M, L^k \otimes K_M)$. Let $\square_{k, K_M}^{(0)}$ be the Gaffney extension of the Kodaira Laplacian acting on $L^k \otimes K_M$. Then

$$\text{Ker } \square_{k, K_M}^{(0)} = \mathcal{H}^0(M, L^k \otimes K_M) = \left\{ u \in L^2(M, L^k \otimes K_M); \bar{\partial}_k u = 0 \right\}.$$

Set

$$P_{k, K_M}^{(0)} : L^2(M, L^k \otimes K_M) \rightarrow \mathcal{H}^0(M, L^k \otimes K_M)$$

be the orthogonal projection with respect to $(\cdot, \cdot)_k$. The goal of this Section is to prove that the kernel of $P_{k, K_M}^{(0)}$ admits a full asymptotic expansion on the

non-degenerate part of L . We recall the following form of the L^2 -estimates for $\bar{\partial}$ for semi-positive line bundles. Assume that (L, h^L) is a semi-positive Hermitian line bundle over a complex manifold M . Let $g \in \Lambda^{n,1}T^*M \otimes L$. For $x \in M$, we denote by $|g|_{R^L}(x) \in [0, \infty]$ the smallest constant such that $\langle g, g' \rangle^2(x) \leq |g|_{R^L}^2(x) \langle \sqrt{-1}R^L \wedge (\Theta \wedge)^* g', g' \rangle(x)$ for all $g' \in \Lambda^{n,1}T^*M \otimes L$.

Theorem 6.1 [22, Theorem 4.1]. *Let (M, Θ) be a complete Kähler manifold, (L, h^L) be a semi-positive Hermitian line bundle over M . Then for any form $g \in L^2_{(0,1)}(M, L \otimes K_M)$ satisfying $\bar{\partial}g = 0$ and $\int_M |g|_{R^L}^2(x) dv_M(x) < \infty$ there exists $f \in L^2_{(0,0)}(M, L \otimes K_M)$ with $\bar{\partial}f = g$ and*

$$\int_M |f|_{h^L}^2(x) dv_M(x) \leq \int_M |g|_{R^L}^2(x) dv_M(x).$$

Denote by $\gamma(x)$ the smallest eigenvalue of the curvature $\sqrt{-1}R_x^L$ with respect to Θ_x , for $x \in M$; the function $\gamma : M \rightarrow [0, \infty)$ is continuous. Moreover, $|g|_{R^L}^2(x) \leq \gamma^{-1}(x)|g|_{h^L}^2(x)$, for any $x \in M$ and $g \in \Lambda^{n,1}T^*M \otimes L$ (where $\gamma^{-1} := \infty$ if $\gamma = 0$). Therefore we deduce the following.

Theorem 6.2. *Let (M, Θ) be a complete Kähler manifold and (L, h^L) be a smooth semi-positive line bundle over M . Let $D \Subset M(0)$ be a relatively compact open set. There exists a constant $C_D > 0$ such that for any $k > 0$ and any $g \in \Omega_0^{0,1}(D, L^k \otimes K_M)$ satisfying $\bar{\partial}_k g = 0$ there exists $f \in \mathcal{C}^\infty(M, L^k \otimes K_M)$ such that $\bar{\partial}_k f = g$ and*

$$(6.1) \quad \|f\|^2 \leq \frac{C_D}{k} \|g\|^2.$$

We can actually take $C_D = \sup_D \gamma^{-1}$. We need

Lemma 6.3. *Let (M, Θ) be a complete Kähler manifold and (L, h^L) be a smooth semi-positive line bundle over M . Let $D \Subset M(0)$ be a relatively compact open set. Then $\square_{k, K_M}^{(0)}$ has $O(k^{-n_0})$ small spectral on D .*

Proof. Let $u \in \mathcal{C}_0^\infty(D, L^k \otimes K_M)$. We consider $\bar{\partial}_k u \in \Omega_0^{0,1}(D, L^k \otimes K_M)$. From Theorem 6.2, we know that there exists $f \in \mathcal{C}^\infty(M, L^k \otimes K_M)$ such that $\bar{\partial}_k f = \bar{\partial}_k u$ and

$$(6.2) \quad \|f\|^2 \leq \frac{C_D}{k} \|\bar{\partial}_k u\|^2,$$

where $C_D > 0$ is independent of k and u . We note that $(I - P_{k, K_M}^{(0)})u$ has minimal L^2 norm of the set

$$\left\{ f \in \mathcal{C}^\infty(M, L^k \otimes K_M) \cap L^2(M, L^k \otimes K_M); \bar{\partial}_k f = \bar{\partial}_k u \right\}.$$

From this observation and (6.2), we conclude that

$$(6.3) \quad \left\| (I - P_{k, K_M}^{(0)})u \right\|^2 \leq \frac{C_D}{k} \left\| \bar{\partial}_k u \right\|^2.$$

It is easy to check that

$$\left\| \bar{\partial}_k u \right\|^2 \leq \left\| \square_{k, K_M}^{(0)} u \right\|^2 \left\| (I - P_{k, K_M}^{(0)})u \right\|^2.$$

Combining this with (6.3), we get $\left\| (I - P_{k, K_M}^{(0)})u \right\| \leq \frac{C_D}{k} \left\| \square_{k, K_M}^{(0)} u \right\|$. Thus, $\square_{k, K_M}^{(0)}$ has $O(k^{-n_0})$ small spectral on D . The lemma follows. \square

Let s be a local frame of L on an open set $D \Subset M(0)$ and $|s|_{h^L}^2 = e^{-2\phi}$. As in (4.61), we consider the localized Bergman projection

$$(6.4) \quad \begin{aligned} \widehat{P}_{k, s, K_M}^{(0)} : L^2(D, K_M) \cap \mathcal{E}'(D, K_M) &\rightarrow L^2(D, K_M), \\ u &\mapsto e^{-k\phi} s^{-k} P_{k, K_M}^{(0)}(s^k e^{k\phi} u). \end{aligned}$$

From Lemma 6.3 and Theorem 4.14, we get one of the main results of this work:

Theorem 6.4. *Let (M, Θ) be a complete Kähler manifold and (L, h^L) be a smooth semi-positive line bundle over M . Let $D \Subset M(0)$ be a relatively compact open set and s be a local frame of L on D . Then the localized Bergman projection $\widehat{P}_{k, s, K_M}^{(0)}$ satisfies*

$$\widehat{P}_{k, s, K_M}^{(0)} \equiv \mathcal{S}_k \pmod{O(k^{-\infty})}$$

on D , where $\mathcal{S}_k : \mathcal{E}'(D, K_M) \rightarrow \mathcal{C}_0^\infty(D, K_M)$ is a smoothing operator and the distribution kernel $\mathcal{S}_k(z, w) \in \mathcal{C}^\infty(D \times D, K_M \boxtimes K_M)$ of \mathcal{S}_k satisfies

$$\mathcal{S}_k(z, w) \equiv e^{ik\Psi(z, w)} b(z, w, k) \pmod{O(k^{-\infty})},$$

with

$$b(z, w, k) \in S_{\text{loc}}^n(1; D \times D, K_M \boxtimes K_M),$$

$$b(z, w, k) \sim \sum_{j=0}^{\infty} b_j(z, w) k^{n-j} \text{ in } S_{\text{loc}}^n(1; D \times D, K_M \boxtimes K_M),$$

$$b_j(z, w) \in \mathcal{C}^\infty(D \times D, K_M \boxtimes K_M), \quad j = 0, 1, 2, \dots,$$

$$b_0(z, z) = (2\pi)^{-n} \det \dot{R}^L(z) \otimes \text{Id}_{K_M}(z), \quad \text{Id}_{K_M} \text{ is the identity map on } K_M,$$

and $\Psi(z, w)$ is as in Theorems 3.8 and 4.11.

From Theorem 6.4, the existence of the asymptotic expansion (1.19) for $L^k \otimes K_M$ follows immediately.

We prove now the formulas (1.20) for the coefficients. Let s be a local frame of L on an open set $D \Subset M(0)$. We take local coordinates $(D, z) \cong (D, x)$ defined in D . Let \mathcal{S}_k and $\mathcal{S}_k(\cdot, \cdot) \in \mathcal{C}^\infty(D \times D, K_M \boxtimes K_M)$ be as in Theorem 6.4. We may replace \mathcal{S}_k by $\frac{1}{2}(\mathcal{S}_k + \mathcal{S}_k^*)$, where \mathcal{S}_k^* is the formal adjoint of \mathcal{S}_k with respect to (\cdot, \cdot) . Then,

$$(6.5) \quad \mathcal{S}_k^* = \mathcal{S}_k.$$

Let $e(z)$ be a local section of K_M so that $|e(z)|^2 = (V_\Theta(z))^{-1}$, where $V_\Theta(z)$ is given by (4.68). Define the smooth kernels $\mathcal{S}_k(\cdot, \cdot), \widehat{\mathcal{S}}_k(\cdot, \cdot) \in \mathcal{C}^\infty(D \times D)$ by

$$(6.6) \quad \mathcal{S}_k(z, w) = e(z) \widetilde{\mathcal{S}}_k(z, w) e^*(w), \quad \widehat{\mathcal{S}}_k(z, w) = \widetilde{\mathcal{S}}_k(z, w) V_\Theta(w).$$

From Theorem 6.4, we have

$$(6.7) \quad \begin{aligned} \widehat{\mathcal{S}}_k(z, w) &\equiv e^{ik\Psi(z, w)} \widehat{b}(z, w, k) \pmod{O(k^{-\infty})}, \\ \widehat{b}(z, w, k) &\in S_{\text{loc}}^n(1; D \times D), \\ \widehat{b}(z, w, k) &\sim \sum_{j=0}^{\infty} \widehat{b}_j(z, w) k^{n-j} \text{ in } S_{\text{loc}}^n(1; D \times D), \\ \widehat{b}_j(z, w) &\in \mathcal{C}^\infty(D \times D), \quad j = 0, 1, 2, \dots, \\ \widehat{b}_0(z, z) &= (2\pi)^{-n} V_\Theta(z) \det \dot{R}^L(z). \end{aligned}$$

Let $(\cdot, \cdot)_{d\lambda}$ be the inner product on $\mathcal{C}_0^\infty(D)$ given by

$$(u, v)_{d\lambda} = \int u(z) \overline{v(z)} d\lambda(z), \quad u, v \in \mathcal{C}_0^\infty(D),$$

where $d\lambda(z) = 2^n dx_1 dx_2 \cdots dx_{2n}$. Let $\widehat{\mathcal{S}}_k$ be the continuous operator given by

$$\begin{aligned} \widehat{\mathcal{S}}_k &: \mathcal{C}_0^\infty(D) \rightarrow \mathcal{C}_0^\infty(D), \\ u &\mapsto \int \widehat{\mathcal{S}}_k(z, w) u(w) d\lambda(w). \end{aligned}$$

Let $\widehat{\mathcal{S}}_k^{*,d\lambda}$ be the formal adjoint of $\widehat{\mathcal{S}}_k$ with respect to $(\cdot, \cdot)_{d\lambda}$. From (6.5), (6.6) we can check that

$$(6.8) \quad \widehat{\mathcal{S}}_k^{*,d\lambda} = \widehat{\mathcal{S}}_k.$$

Since $\mathcal{S}_k^2 \equiv \mathcal{S}_k \pmod{O(k^{-\infty})}$, we can check that

$$(6.9) \quad (\widehat{\mathcal{S}}_k)^2 \equiv \widehat{\mathcal{S}}_k \pmod{O(k^{-\infty})}.$$

Moreover, it is obviously that

$$(6.10) \quad \bar{\partial}_s \widehat{\mathcal{S}}_k \equiv 0 \pmod{O(k^{-\infty})}.$$

We recall that $\bar{\partial}_s = \bar{\partial} + k(\bar{\partial}\phi)\wedge$.

From (6.8), (6.9) and (6.10), we can repeat the procedure in Section 4.5 and conclude that (see (4.95) and (4.97))

$$(6.11) \quad \begin{aligned} \widehat{b}_1(0,0) &= V_\omega(0) \left(-\frac{1}{8\pi} r(0) \right), \\ \widehat{b}_2(0,0) &= V_\omega(0) \left(\frac{1}{128\pi^2} r^2 + \frac{1}{96\pi^2} \Delta_\omega r - \frac{1}{24\pi^2} |\text{Ric}_\omega|_\omega^2 + \frac{1}{96\pi^2} |R_\omega^{TM}|_\omega^2 \right)(0), \end{aligned}$$

where V_ω , r , Δ_ω , Ric_ω and R_ω^{TM} are as in (4.68), (4.69), (4.67), (4.73) and (4.71) respectively, and $\langle \cdot, \cdot \rangle_\omega$, $|\cdot|_\omega$ are as in the discussion after (1.10) and $|R_\omega^{TM}|_\omega^2$ is given by (4.72). From (6.6) and (6.7), we can check that for $b_{1,K_M}^{(0)}(z)$, $b_{2,K_M}^{(0)}(z)$ in (1.19), we have

$$b_{1,K_M}^{(0)}(0) = \frac{1}{V_\Theta(0)} \widehat{b}_1(0,0) \text{Id}_{K_M}(0), \quad b_{2,K_M}^{(0)}(0) = \frac{1}{V_\Theta(0)} \widehat{b}_2(0,0) \text{Id}_{K_M}(0).$$

Combining this with (6.11) and observing that

$$\frac{1}{V_\Theta(0)} V_\omega(0) = (2\pi)^{-n} \det \dot{R}^L(0),$$

we obtain (1.20).

Remark 6.5. In [49, (4.1.9)], Ma–Marinescu gave a formula for $b_1^{(0)}$ in the presence of a twisting vector bundle E (under the assumption that L is positive everywhere). For $E = K_M$ the formula [49, (4.1.9)] reads:

$$(6.12) \quad b_{1,K_M}^{(0)} = (2\pi)^{-n} (\det \dot{R}^L) \frac{1}{8\pi} \left(r - 2\Delta_\omega (\log(\det \dot{R}^L)) + 4\sqrt{-1}\Lambda_\omega(R^{K_M}) \right) \text{Id}_{K_M},$$

where $\Lambda_\omega(R^{K_M})$ is given by $nR^{K_M} \wedge \omega^{n-1} = \Lambda_\omega(R^{K_M})\omega^n$. Formula (1.20) gives

$$(6.13) \quad b_{1,K_M}^{(0)} = (2\pi)^{-n} \det \dot{R}^L \left(-\frac{1}{8\pi} r \right) \text{Id}_{K_M}.$$

We show that the right-hand sides of (6.12) and (6.13) are equal. By definition, we have

$$(6.14) \quad r = \Delta_\omega \log V_\omega, \quad \det \dot{R}^L = (2\pi)^n \frac{V_\omega}{V_\Theta},$$

where V_ω and V_Θ are given by (4.68). Using (6.14), (6.12) becomes

$$(6.15) \quad b_{1,K_M}^{(0)} = (2\pi)^{-n} (\det \dot{R}^L) \frac{1}{8\pi} \left(-r + 2\Delta_\omega \log V_\Theta + 4\sqrt{-1}\Lambda_\omega(R^{K_M}) \right) \text{Id}_{K_M}.$$

Moreover, it is straightforward to see that

$$(6.16) \quad 4\sqrt{-1}\Lambda_\omega(R^{K_M}) = -2\Delta_\omega \log V_\Theta.$$

Combining (6.16) with (6.15), we conclude that our claim holds true.

7. Singular L^2 -estimates

In Section 9, we need a singular version of L^2 estimates. We assume that (M, Θ) is a compact Hermitian manifold and (L, h^L) is a holomorphic line bundle over M , endowed with a singular Hermitian metric h^L . We solve the $\bar{\partial}$ -equation $\bar{\partial}_k f = g$ for $(0, 1)$ forms with values in L^k with a rough L^2 -estimate, namely $\|f\|^2 \leq C_D k^N \|g\|^2$ with $N > 0$, instead of the estimate $\|f\|^2 \leq \frac{C_D}{k} \|g\|^2$ from (6.1).

For a singular Hermitian metric h^L on L (see, e.g., [49, Definition 2.3.1]) the local weight with respect to a holomorphic frame $s : D \rightarrow L$ is a function

$\phi \in L^1_{\text{loc}}(D)$, $D \Subset M$, defined by

$$|s|_{h^L}^2 = e^{-2\phi} \in [0, \infty].$$

The curvature current R^L is given locally by $R^L := 2\partial\bar{\partial}\phi$ and does not depend on the choice of local frame s , is thus well-defined as a $(1, 1)$ current on M .

We say that $\sqrt{-1}R^L$ is strictly positive if there exists $\varepsilon > 0$ such that $\sqrt{-1}R^L \geq \varepsilon\Theta$, that is, $\sqrt{-1}R^L - \varepsilon\Theta$ is a positive current in the sense of Lelong (see e.g., [49, Definition B.2.11]). If $\sqrt{-1}R^L$ is strictly positive then ϕ is strictly psh on D (in particular ϕ is bounded above on D). The goal of this Section is to prove the following.

Theorem 7.1. *Let (L, h^L) be a singular Hermitian holomorphic line bundle over a compact Hermitian manifold (M, Θ) . We assume that h^L is smooth outside a proper analytic set Σ and*

$$(7.1) \quad \sqrt{-1}R^L \geq \varepsilon\Theta, \quad \varepsilon > 0.$$

Let $D \Subset M \setminus \Sigma$. Then, there exist $k_0 > 0$, $N > 0$ and $C_D > 0$, such that for all $k \geq k_0$, and $g \in \Omega^{0,1}(D, L^k)$ with $\bar{\partial}_k g = 0$, there is $u \in \mathcal{C}^\infty(M, L^k)$ such that $\bar{\partial}_k u = g$ and

$$(7.2) \quad \|u\|_{h^k, \Theta}^2 \leq k^N C_D \|g\|_{h^k, \Theta}^2,$$

where $\|u\|_{h^k, \Theta}^2 := \int_M |u|_{h^k}^2 dv_M$, $dv_M := \frac{\Theta^n}{n!}$, and similarly for $\|g\|_{h^k, \Theta}^2$.

Proof. Let Θ_{ε_0} be the generalized Poincaré metric on $M \setminus \Sigma$ (see [49, p. 276]). Let $\mathcal{T}_{\varepsilon_0} := [(\Theta_{\varepsilon_0} \wedge)^*, \partial\Theta_{\varepsilon_0}]$ be the Hermitian torsion of Θ_{ε_0} . Let $R_{\Theta_{\varepsilon_0}}^{\det}$ denote the curvature of the holomorphic line bundle $\Lambda^n T^{*(1,0)}M$ induced by Θ_{ε_0} . By [49, Lemma 6.2.1] we have

$$(7.3) \quad \begin{aligned} &\Theta_{\varepsilon_0} \text{ is a complete Hermitian metric of finite volume on } M \setminus \Sigma, \\ &\Theta_{\varepsilon_0} \geq c_0\Theta \text{ for some } c_0 > 0, \\ &-C\Theta_{\varepsilon_0} < \sqrt{-1}R_{\Theta_{\varepsilon_0}}^{\det} < C\Theta_{\varepsilon_0}, \quad |\mathcal{T}_{\varepsilon_0}|_{\Theta_{\varepsilon_0}} < C, \end{aligned}$$

where $C > 0$ is a constant and $|\mathcal{T}_{\varepsilon_0}|_{\Theta_{\varepsilon_0}}$ is the norm with respect to Θ_{ε_0} . Moreover, by [49, Section 6.2] there is a Hermitian metric $h^L_{\varepsilon_0}$ of L on $M \setminus \Sigma$

such that $h_{\epsilon_0}^L$ is smooth on $M \setminus \Sigma$ and

$$(7.4) \quad h_{\epsilon_0}^L > h^L, \quad \sqrt{-1}R_{\epsilon_0}^L > c\Theta_{\epsilon_0},$$

where $c > 0$ is a constant and $R_{\epsilon_0}^L$ is the curvature of L induced by $h_{\epsilon_0}^L$.

Let s be a local frame of L and define local weights ϕ_{ϵ_0} and ϕ for $h_{\epsilon_0}^L$ and h^L by $|s|_{h_{\epsilon_0}^L}^2 = e^{-2\phi_{\epsilon_0}}$, $|s|_{h^L}^2 = e^{-2\phi}$. Let \widehat{h}^k be the Hermitian metric on L^k locally given by

$$|s|_{\widehat{h}^k}^2 := \exp(-2(\log k)\phi_{\epsilon_0} - 2(k - \log k)\phi).$$

Since $h_{\epsilon_0}^L > h^L$, we have $\widehat{h}^k > h^k$. Moreover, from (7.1) and (7.4), we can check that

$$(7.5) \quad \sqrt{-1}\widehat{R}^{L^k} > c(\log k)\Theta_{\epsilon_0},$$

where \widehat{R}^{L^k} denotes the curvature of L^k associated to \widehat{h}^k and $c > 0$ is the constant as in (7.4). Let $(\cdot, \cdot)_{\widehat{h}^k, \Theta_{\epsilon_0}}$ denote the L^2 inner product on $\Omega_0^{0,q}(M \setminus \Sigma, L^k)$ with respect to \widehat{h}^k and Θ_{ϵ_0} as (2.3). For $f \in \Omega_0^{0,q}(M \setminus \Sigma, L^k)$, we write $\|f\|_{\widehat{h}^k, \Theta_{\epsilon_0}}^2 := (f, f)_{\widehat{h}^k, \Theta_{\epsilon_0}}$. Let $\widehat{L}_{(0,q)}^2(M \setminus \Sigma, L^k)$ be the completion of $\Omega_0^{0,q}(M \setminus \Sigma, L^k)$ with respect to $\|\cdot\|_{\widehat{h}^k, \Theta_{\epsilon_0}}$. Let

$$\widehat{\square}_k^{(1)} = \bar{\partial}_k \bar{\partial}_k^* + \bar{\partial}_k^* \bar{\partial}_k : \text{Dom } \widehat{\square}_k^{(1)} \subset \widehat{L}_{(0,1)}^2(M \setminus \Sigma, L^k) \rightarrow \widehat{L}_{(0,1)}^2(M \setminus \Sigma, L^k)$$

be the Gaffney extension of the Kodaira Laplacian with respect to \widehat{h}^k and Θ_{ϵ_0} (see (2.8)). Here $\bar{\partial}_k^*$ is the Hilbert space adjoint of $\bar{\partial}_k$ with respect to $(\cdot, \cdot)_{\widehat{h}^k, \Theta_{\epsilon_0}}$. From (7.3) and (7.5), we can repeat the procedure in [49, p. 272–273] and conclude that for k large, we have

$$(7.6) \quad \|g\|_{\widehat{h}^k, \Theta_{\epsilon_0}}^2 \leq \frac{1}{c(\log k)} \left\| \widehat{\square}_k^{(1)} g \right\|_{\widehat{h}^k, \Theta_{\epsilon_0}}^2,$$

for all $g \in \Omega_0^{0,1}(M \setminus \Sigma, L^k)$, where $c > 0$ is a positive constant. From this, we can repeat the method in [49, p. 272–273] and conclude that $\widehat{\square}_k^{(1)}$ has closed range in $\widehat{L}_{(0,1)}^2(M \setminus \Sigma, L^k)$, $\text{Ker } \widehat{\square}_k^{(1)} \cap \widehat{L}_{(0,1)}^2(M \setminus \Sigma, L^k) = \{0\}$ and there is a bounded operator $G_k : \widehat{L}_{(0,1)}^2(M \setminus \Sigma, L^k) \rightarrow \text{Dom } \widehat{\square}_k^{(1)}$ such that $\widehat{\square}_k^{(1)} G_k = I$

on $\widehat{L}_{(0,1)}^2(M \setminus \Sigma, L^k)$, $G_k \widehat{\square}_k^{(1)} = I$ on $\text{Dom } \widehat{\square}_k^{(1)}$ and

$$(7.7) \quad \|G_k g\|_{\widehat{h}^k, \Theta_{\epsilon_0}}^2 \leq \frac{1}{c(\log k)} \|g\|_{\widehat{h}^k, \Theta_{\epsilon_0}}^2$$

for k large, for all $g \in \widehat{L}_{(0,1)}^2(M \setminus \Sigma, L^k)$, where $c > 0$ is independent of g and k , and

$$(7.8) \quad G_k : \Omega^{0,1}(M \setminus \Sigma, L^k) \rightarrow \Omega^{0,1}(M \setminus \Sigma, L^k),$$

$$(7.9) \quad g = \widehat{\square}_k^{(1)} G_k g = \bar{\partial}_k \bar{\partial}_k^* G_k g, \quad \text{if } \bar{\partial}_k g = 0, \quad g \in \widehat{L}_{(0,1)}^2(M \setminus \Sigma, L^k).$$

Now, fix $D \Subset M \setminus \Sigma$ and let $g \in \Omega_0^{0,1}(D, L^k)$ with $\bar{\partial}_k g = 0$ and set

$$u = \bar{\partial}_k^* G_k g \in \Omega^{0,1}(M \setminus \Sigma, L^k) \cap \widehat{L}_{(0,0)}^2(M \setminus \Sigma, L^k).$$

From (7.9) and (7.7), it is not difficult to see that

$$(7.10) \quad \begin{aligned} \bar{\partial}_k u &= g \quad \text{on } M \setminus \Sigma, \\ \|u\|_{\widehat{h}^k, \Theta_{\epsilon_0}}^2 &\leq \frac{1}{c_1 \sqrt{\log k}} \|g\|_{\widehat{h}^k, \Theta_{\epsilon_0}}^2, \end{aligned}$$

where $c_1 > 0$ is a constant independent of g and k . Now, let us compare the norms $\|\cdot\|_{\widehat{h}^k, \Theta_{\epsilon_0}}$ and $\|\cdot\|_{h^k, \Theta}$. Let s be a local section of L on D and $|s|_{h^L}^2 = e^{-2\phi_{\epsilon_0}}$, $|s|_{h^L}^2 = e^{-2\phi}$. Then,

$$|s|_{\widehat{h}^k}^2 = e^{-2k\phi} e^{2\log k(\phi - \phi_{\epsilon_0})} = |s|_{h^k}^2 e^{2\log k(\phi - \phi_{\epsilon_0})}.$$

Thus, on D , we have

$$(7.11) \quad |s|_{\widehat{h}^k}^2 < k^N |s|_{h^k}^2,$$

where $N > \sup_{x \in D} |2\phi(x) - 2\phi_{\epsilon_0}(x)|$. Thus,

$$(7.12) \quad \|g\|_{\widehat{h}^k, \Theta_{\epsilon_0}}^2 < \widetilde{C}_D k^N \|g\|_{h^k, \Theta}^2,$$

where $\widetilde{C}_D > 0$ is a constant independent of g and k . From $\widehat{h}^k > h^k$ and the second property in (7.3), we have $\|u\|_{h^k, \Theta}^2 < \widetilde{c} \|u\|_{\widehat{h}^k, \Theta_{\epsilon_0}}^2$, where $\widetilde{c} > 0$ is a constant independent of k and u . Combining this with (7.12) and (7.10), we

obtain

$$(7.13) \quad \|u\|_{h^k, \Theta}^2 \leq C_D k^N \|g\|_{h^k, \Theta}^2,$$

where $C_D > 0$ is a constant independent of k and g . Note that h^L is bounded away from zero and Σ has Lebesgue measure zero. From this observation and (7.13), we see that u is L^2 integrable with respect to some smooth metric of L over M . Combining this with Skoda's Lemma (see Lemma 7.2 below), we get $\bar{\partial}_k u = g$ on M and $u \in \Omega^{0,1}(M, L^k)$. The theorem follows. \square

We recall the following result of Skoda (for a proof, see Demailly [24, Lemma 7.3, Ch. VIII]).

Lemma 7.2. *Let $u \in \mathcal{D}'(M, L^k)$, $g \in \mathcal{D}'(M, L^k \otimes T^{*(0,1)}M)$. We assume that u and g are L^2 integrable with respect to some smooth metric of L and Θ over M . If $\bar{\partial}_k u = g$ on $M \setminus \Sigma$ in the sense of distributions, then $\bar{\partial}_k u = g$ on M in the sense of distributions.*

8. Bergman kernel asymptotics for semi-positive line bundles

In this Section we prove Theorem 1.10. Let (M, Θ) a compact Hermitian manifold. Assume that $(L, h^L) \rightarrow M$ is a smooth semi-positive line bundle which is positive at some point of M . By Siu's criterion [49, Theorem 2.2.27] (see also Corollary 10.8) we know that L is big and M is Moishezon. By [49, Lemma 2.3.6], L admits a singular Hermitian metric h_{sing}^L , smooth outside a proper analytic set Σ , and with strictly positive curvature current.

Lemma 8.1. *With the assumptions and notations above, let $D \Subset M \setminus \Sigma$ be an open set. Then, there exist $k_0 > 0$, $N > 0$ and $C_D > 0$, such that for all $k \geq k_0$, and $g \in \Omega_0^{0,1}(D, L^k)$ with $\bar{\partial}_k g = 0$, there is $u \in \mathcal{C}^\infty(M, L^k)$ such that $\bar{\partial}_k u = g$ and*

$$\|u\|^2 \leq k^N C_D \|g\|^2.$$

Proof. Let ϕ and $\hat{\phi}$ denote local weights for h^L and h_{sing}^L respectively. Then, $\hat{\phi}$ is smooth on $M \setminus \Sigma$ and bounded above. We may assume that

$$\hat{\phi} \leq \phi.$$

Let \tilde{h}^k be the Hermitian metric on L^k induced by the local weight

$$\tilde{\phi} := (\log k)\hat{\phi} + (k - \log k)\phi.$$

We can check that \tilde{h}^k is a strictly positive singular Hermitian metric, smooth outside a proper analytic set Σ . Let $\|\cdot\|_{\tilde{h}^k}$ and $\|\cdot\|_{h^k}$ denote the corresponding L^2 norms for sections with respect to \tilde{h}^k and h^k respectively. We can repeat the proof of Theorem 7.1 and conclude that for a given $g \in \Omega_0^{0,1}(D, L^k)$ with $\bar{\partial}_k g = 0$, there is $u \in \mathcal{C}^\infty(M, L^k)$ such that $\bar{\partial}_k u = g$ and

$$(8.1) \quad \|u\|_{\tilde{h}^k}^2 \leq \frac{1}{c\sqrt{\log k}} \|g\|_{\tilde{h}^k}^2,$$

where $c > 0$ is independent of k and g . Since $\hat{\phi} \leq \phi$, we have

$$(8.2) \quad \|u\|_{h^k} \leq \|u\|_{\tilde{h}^k}.$$

On the other hand, we have

$$(8.3) \quad \begin{aligned} \|g\|_{h^k}^2 &= \int_D |g|^2 e^{-2(\log k)\hat{\phi} - 2(k - \log k)\phi} dv_M(x) \\ &\leq \left(\sup_{x \in D} e^{2(\log k)(\phi(x) - \hat{\phi}(x))} \right) \int_D |g|^2 e^{-2k\phi} dv_M(x) \\ &\leq k^N \|g\|_{\tilde{h}^k}^2, \end{aligned}$$

where $N = \sup_{x \in D} 2(\phi(x) - \hat{\phi}(x))$. From (8.2) and (8.3), the lemma follows. \square

For a holomorphic line bundle L over a compact Hermitian manifold (M, Θ) we set $\text{Herm}(L) = \{\text{singular Hermitian metrics on } L\}$,

$$\mathcal{M}(L) = \{h^L \in \text{Herm}(L); h^L \text{ is smooth outside a proper analytic set, } \sqrt{-1}R^L > \varepsilon\Theta, \varepsilon > 0\}.$$

By [49, Lemma 2.3.6], $\mathcal{M}(L) \neq \emptyset$ under the hypotheses of Theorem 8.2 below. Set

$$(8.4) \quad M' := \{p \in M; \exists h^L \in \mathcal{M}(L) \text{ with } h^L \text{ smooth near } p\}.$$

From Lemma 8.1, we can repeat the proof of Lemma 6.3 with minor changes and conclude the following.

Theorem 8.2. *Let (M, Θ) be a compact Hermitian manifold. Let $(L, h^L) \rightarrow M$ be a Hermitian holomorphic line bundle with smooth Hermitian metric h^L having semi-positive curvature and with $M(0) \neq \emptyset$. Let $D \Subset M' \cap M(0)$*

be an open set, where M' is given by (8.4). Then, $\square_k^{(0)}$ has $O(k^{-n_0})$ small spectral gap on D .

Let s be a local frame of L on an open set $D \Subset M$ and $|s|_{h^L}^2 = e^{-2\phi}$. We define the *localized Bergman projection* (with respect to s) by

$$(8.5) \quad \begin{aligned} \widehat{P}_{k,s}^{(0)} : L^2(D) \cap \mathcal{E}'(D) &\rightarrow \mathcal{C}_0^\infty(D), \\ u &\rightarrow e^{-k\phi} s^{-k} P_k^{(0)}(s^k e^{k\phi} u). \end{aligned}$$

That is, if $P_k^{(0)}(s^k e^{k\phi} u) = s^k v$ on D , then $\widehat{P}_{k,s}^{(0)} u = e^{-k\phi} v$.

From Theorem 8.2 and Theorem 4.14, we get the following result.

Theorem 8.3. *Let (M, Θ) be a compact Hermitian manifold. Let $(L, h^L) \rightarrow M$ be a Hermitian holomorphic line bundle with smooth Hermitian metric h^L having semi-positive curvature and with $M(0) \neq \emptyset$. Let s be a local frame of L on an open set $D \Subset M' \cap M(0)$. Then the localized Bergman projection $\widehat{P}_{k,s}^{(0)}$ satisfies*

$$\widehat{P}_{k,s}^{(0)} \equiv \mathcal{S}_k \pmod{O(k^{-\infty})}$$

on D , where \mathcal{S}_k is as in Theorem 4.11.

Theorem 8.3 immediately implies Theorem 1.10.

9. Multiplier ideal Bergman kernel asymptotics. Proof of Theorem 1.8

Let us first recall the notion of multiplier ideal sheaf. Let M be a compact complex manifold and $\varphi \in L_{loc}^1(M, \mathbb{R})$. The *Nadel multiplier ideal sheaf* $\mathcal{I}(\varphi) \subset \mathcal{O}_M$ is the ideal subsheaf of germs of holomorphic functions $f \in \mathcal{O}_{M,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in local coordinates near x for all $x \in M$.

Consider now a singular Hermitian metric h^L on a holomorphic line bundle L over M . If h_0^L is a smooth Hermitian metric on L then $h^L = h_0^L e^{-2\varphi}$ for some function $\varphi \in L_{loc}^1(M, \mathbb{R})$. The Nadel multiplier ideal sheaf of h^L is defined by $\mathcal{I}(h^L) = \mathcal{I}(\varphi)$; the definition does not depend on the choice of h_0^L . Put

$$(9.1) \quad \begin{aligned} &\mathcal{C}^\infty(M, L \otimes \mathcal{I}(h^L)) \\ &:= \left\{ S \in \mathcal{C}^\infty(M, L); \int_M |S|_{h^L}^2 dv_M = \int_M |S|_{h_0^L}^2 e^{-2\varphi} dv_M < \infty \right\}, \end{aligned}$$

where $|\cdot|_{h^L}$ and $|\cdot|_{h_0^L}$ denote the pointwise norms for sections induced by h^L and h_0^L respectively. With the help of h^L and the volume form dv_M we can define an L^2 inner product on $\mathcal{C}^\infty(M, L \otimes \mathcal{I}(h^L))$:

$$(9.2) \quad (S, S') = \int_M \langle S, S' \rangle_{h_0^L} e^{-2\varphi} dv_M, \quad S, S' \in \mathcal{C}^\infty(M, L \otimes \mathcal{I}(h^L)).$$

The singular Hermitian metric h^L induces a singular Hermitian metric $h^k = h_0^k e^{-2k\varphi}$ on L^k , $k > 0$. We denote by $(\cdot, \cdot)_k$ the natural inner products on $\mathcal{C}^\infty(M, L^k \otimes \mathcal{I}(h^k))$ defined as in (9.2) and by $L^2(M, L^k)$ the completion of $\mathcal{C}^\infty(M, L^k \otimes \mathcal{I}(h^k))$ with respect to $(\cdot, \cdot)_k$. The space of global sections in the sheaf $\mathcal{O}(L^k) \otimes \mathcal{I}(h^k)$ is given by

$$(9.3) \quad \begin{aligned} & H^0(M, L^k \otimes \mathcal{I}(h^k)) \\ &= \left\{ s \in \mathcal{C}^\infty(M, L^k); \bar{\partial}_k s = 0, \int_M |s|_{h^k}^2 dv_M = \int_M |s|_{h_0^k}^2 e^{-2k\varphi} dv_M < \infty \right\}. \end{aligned}$$

Let

$$(9.4) \quad P_{k, \mathcal{I}}^{(0)} : L^2(M, L^k) \rightarrow H^0(M, L^k \otimes \mathcal{I}(h^k))$$

be the orthogonal projection.

Now, we assume that h^L is a strictly positive singular Hermitian metric on L , smooth outside a proper analytic set Σ of M . Let $L^2(M \setminus \Sigma, L^k)$ be the completion of $\mathcal{C}_0^\infty(M \setminus \Sigma, L^k)$ with respect to $(\cdot, \cdot)_k$. We notice that Σ is closed and has Lebesgue measure zero. From this observation, it is straightforward to see that

$$(9.5) \quad L^2(M \setminus \Sigma, L^k) = L^2(M, L^k).$$

We consider the Gaffney extension $\square_k^{(0)}$ of the Kodaira Laplacian $\bar{\partial}_k^* \bar{\partial}_k$ on $M \setminus \Sigma$ (see (2.8)), where $\bar{\partial}_k^*$ is the formal adjoint of $\bar{\partial}_k$ with respect to $(\cdot, \cdot)_k$ on $M \setminus \Sigma$. It is easy to see that $\text{Ker } \square_k^{(0)} = L^2(M \setminus \Sigma, L^k) \cap \text{Ker } \bar{\partial}_k$. The local weights of h^L are strictly psh, so they are bounded above, hence elements in $L^2(M \setminus \Sigma, L^k)$ are locally square integrable with respect to smooth metrics on M and L . Since holomorphic sections on $M \setminus \Sigma$ which are locally square integrable extend to holomorphic sections on M (see Lemma 7.2), we see

that

$$(9.6) \quad \text{Ker } \square_k^{(0)} = L^2(M, L^k) \cap \text{Ker } \bar{\partial}_k = H^0(M, L^k \otimes \mathcal{I}(h^k)).$$

Let

$$P_k^{(0)} : L^2(M \setminus \Sigma, L^k) \rightarrow \text{Ker } \square_k^{(0)}$$

be the Bergman projection. From (9.5) and (9.6), we see that

$$(9.7) \quad P_{k, \mathcal{I}}^{(0)} = P_k^{(0)} \quad \text{on } L^2(M, L^k) = L^2(M \setminus \Sigma, L^k).$$

From Theorem 7.1, we can repeat the proof of Lemma 6.3 and conclude that

Theorem 9.1. *With the notations and assumptions above. Let $D \Subset M \setminus \Sigma$. Then, $\square_k^{(0)}$ has $O(k^{-n_0})$ small spectral gap on D .*

Let s be a local frame of L on an open set $D \Subset M \setminus \Sigma$ and $|s|_{h^L}^2 = e^{-2\phi}$. Then, ϕ is smooth on D and $\partial\bar{\partial}\phi$ is positive defined at each point of D . Let us denote by

$$(9.8) \quad \widehat{P}_{k, s, \mathcal{I}}^{(0)} : L^2(D) \cap \mathcal{E}'(D) \longrightarrow L^2(D), \quad u \longmapsto e^{-k\phi} s^{-k} P_{k, \mathcal{I}}^{(0)}(s^k e^{k\phi} u),$$

the localized (multiplier ideal) Bergman projection.

From Theorems 9.1, 4.14 and (9.7), we get one of the main results of this work

Theorem 9.2. *Let (L, h^L) be a singular Hermitian holomorphic line bundle with strictly positive curvature current over a compact Hermitian manifold (M, Θ) . We assume that h^L is smooth outside a proper analytic set Σ . Let s be a local frame of L on an open set $D \Subset M \setminus \Sigma$. Then the localized multiplier ideal Bergman projection $\widehat{P}_{k, s, \mathcal{I}}^{(0)}$ (see (9.8)) satisfies*

$$\widehat{P}_{k, s, \mathcal{I}}^{(0)} \equiv \mathcal{S}_k \quad \text{mod } O(k^{-\infty})$$

on D , where \mathcal{S}_k is as in Theorem 4.11.

From Theorem 9.2, we get Theorem 1.8.

10. Further applications

In this section, we collect further applications of the methods developed here. In Section 10.1, we show the existence of manifolds and line bundles whose Kodaira–Laplace operator has no $O(k^{-n_0})$ small spectral gap. In Section 10.2, we show that under an integral condition (due to Bouche) on the first eigenvalue of the curvature, the asymptotic expansion of the Bergman kernel of a semi-positive line bundle holds. In Section 10.3, we apply our results to prove a result of Berman about the Bergman kernel associated to an arbitrary semi-positive Hermitian metric on an ample line bundle. In Section 10.4, we give a local version of the Bergman kernel expansion for q -forms. In Section 10.5, we obtain precise semiclassical estimates for the dimension of the spectral spaces of the Kodaira Laplacian. Using them one obtains immediately the holomorphic Morse inequalities of Demailly. Finally, we prove in Section 10.6, a version of Tian’s theorem about the convergence of the induced Fubini–Study metrics in the case of singular metrics on a big line bundle. This implies the equidistribution of the zeros of sections in the high tensor powers twisted with the Nadel ideal sheaves.

10.1. Existence of “small” eigenvalues of the Kodaira Laplacian

The hypothesis on the existence of a $O(k^{-n_0})$ small spectral gap was of central importance in our approach. It is interesting to know if there is a compact complex manifold M and a holomorphic line bundle L over M such that the associated Kodaira Laplacian does not exhibit such a spectral gap. We will construct a compact manifold and a holomorphic line bundle L over M such that the associated Kodaira Laplacian $\square_k^{(q)}$ has non-vanishing eigenvalues of order $O(k^{-\infty})$.

Theorem 10.1. *Let $0 \leq q \leq n$, $q \in \mathbb{N}_0$. There exists a compact complex manifold M of dimension n and a holomorphic line bundle L over M such that for*

$$\lambda_k := \inf \left\{ \lambda; \lambda : \text{non-zero eigenvalues of } \square_k^{(q)} \right\},$$

we have for every $N > 0$

$$\lim_{k \rightarrow \infty} k^N \lambda_k = 0.$$

Let S be a compact Riemann surface with a smooth Hermitian metric. Let (L_0, h^{L_0}) be a holomorphic line bundle over S . We assume that $\sqrt{-1}R^{L_0}$

is positive. It is not difficult to see that L_0 admits another smooth Hermitian fibre metric \tilde{h}^{L_0} such that the associated curvature form $\sqrt{-1}\tilde{R}^{L_0}$ is positive on $S_+ \subset S$, negative on $S_- \subset S$ and degenerate on $S_0 \subset S$, where $S = S_+ \cup S_- \cup S_0$, S_+, S_- contain non-empty open subsets of S .

Let M_1 be a compact complex manifold of dimension $n - 1$ with a smooth Hermitian metric and let (L_1, h^{L_1}) be a holomorphic line bundle over M_1 . We assume that $\sqrt{-1}R^{L_1}$ is non-degenerate of constant signature (n_-, n_+) , $n_- + n_+ = n - 1$, at each point of M_1 . Put

$$M := M_1 \times S, \quad L := L_1 \otimes L_0.$$

Then, M is a compact complex manifold of dimension n and L is a holomorphic line bundle over M . The Hermitian metrics on M_1 and S induce a Hermitian metric $\langle \cdot, \cdot \rangle$ on M . Consider the metric $h^L = h^{L_0} \otimes h^{L_1}$ on L ; then the associated curvature $\sqrt{-1}R^L$ is non-degenerate of constant signature $(n_-, n_+ + 1)$ at each point of M . Similarly, setting $\tilde{h}^L = \tilde{h}^{L_0} \otimes h^{L_1}$, the associated curvature $\sqrt{-1}\tilde{R}^L$ is non-degenerate of constant signature $(n_-, n_+ + 1)$ on $M_+ \subset M$, non-degenerate of constant signature $(n_- + 1, n_+)$ on $M_- \subset M$ and degenerate on $M_0 \subset M$, where $M = M_- \cup M_+ \cup M_0$, M_-, M_+ contain non-empty open subsets of M . First, we need

Lemma 10.2. *Under the notations above let $q = n_-$. Then*

$$\dim \mathcal{H}^q(M, L^k) = (-1)^q \frac{k^n}{n!} \left(\int_{M_+} \left(\frac{\sqrt{-1}}{2\pi} \tilde{R}^L \right)^n + \int_{M_-} \left(\frac{\sqrt{-1}}{2\pi} \tilde{R}^L \right)^n \right) + o(k^n).$$

Proof. Note that L admits a smooth Hermitian fibre metric such that the induced curvature is non-degenerate of constant signature $(n_-, n_+ + 1)$ at each point of M . From this observation and Andreotti–Grauert vanishing theorem, we know that if k large, then

$$(10.1) \quad \mathcal{H}^j(M, L^k) = 0 \quad \text{if } j \neq n_-.$$

From the Riemann–Roch–Hirzebruch theorem (see e.g., [49, (4.1.10)]), we see that

$$(10.2) \quad \sum_{j=0}^n (-1)^j \dim \mathcal{H}^j(M, L^k) = \frac{k^n}{n!} \int_M c_1(L)^n + O(k^{n-1}),$$

where $c_1(L)$ is the first Chern class. Combining (10.2) with (10.1), we have for k large enough

$$(10.3) \quad \dim \mathcal{H}^q(M, L^k) = (-1)^q \frac{k^n}{n!} \int_M c_1(L)^n + O(k^{n-1}).$$

But $\frac{\sqrt{-1}}{2\pi} \tilde{R}^L$ represents the Chern class so

$$\int_M c_1(L)^n = \int_M \left(\frac{\sqrt{-1}}{2\pi} \tilde{R}^L \right)^n.$$

The lemma follows from (10.3). \square

The Hermitian fibre metric \tilde{h}^L induces a Hermitian fibre metric \tilde{h}^k on the k -th tensor power of L . As before, let $\square_k^{(q)}$ be the Kodaira Laplacian with values in L^k associated to \tilde{h}^k .

Theorem 10.3. *Under the notations above let $q = n_-$. Then, for any $N > 2n$, we have*

$$\lim_{k \rightarrow \infty} k^N \lambda_k = 0.$$

Proof. Fix $N_0 > 2n$. From Corollary 10.7 below and Lemma 10.2, we know that

$$\begin{aligned} \dim \mathcal{E}_{k^{-N_0}}^q(M, L^k) &= (-1)^q \frac{k^n}{n!} \int_{M_+} \left(\frac{\sqrt{-1}}{2\pi} \tilde{R}^L \right)^n + o(k^n) \\ &> (-1)^q \frac{k^n}{n!} \left(\int_{M_+} \left(\frac{\sqrt{-1}}{2\pi} \tilde{R}^L \right)^n + \int_{M_-} \left(\frac{\sqrt{-1}}{2\pi} \tilde{R}^L \right)^n \right) + o(k^n) \\ &> \dim \mathcal{H}^q(M, L^k) + o(k^n). \end{aligned}$$

Thus, for k large, we have

$$\dim \mathcal{E}_{0 < \lambda \leq k^{-N_0}}^q(M, L^k) > 0,$$

where $\mathcal{E}_{0 < \lambda \leq k^{-N_0}}^q(M, L^k)$ denotes the spectral space spanned by the eigenforms of $\square_k^{(q)}$ whose eigenvalues are bounded by k^{-N_0} and > 0 . We notice that since M is compact, $\square_k^{(q)}$ has a discrete spectrum, each eigenvalues occurs with finite multiplicity. Thus, $\lambda_k \leq k^{-N_0}$ for k large. The theorem follows. \square

From Theorem 10.3, we get Theorem 10.1.

10.2. Bouche integral condition

Let (L, h^L) be a semi-positive holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n . Let $0 \leq \lambda_1(x) \leq \lambda_2(x) \leq \dots$ be the eigenvalues of $\dot{R}^L(x)$. We say that (L, h^L) satisfies the Bouche integral condition [8] if

$$(10.4) \quad \int_M \lambda_1^{-6n} < \infty.$$

If (L, h^L) satisfies (10.4) then Bouche [8] proved that

$$\inf \left\{ \lambda \in \text{Spec}(\square_k^{(q)}); \lambda \neq 0 \right\} \geq k^{\frac{10n+1}{12n+1}},$$

for k large. From this and Theorem 1.6, we deduce

Corollary 10.4. *Let (L, h^L) be a semi-positive holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n . If (L, h^L) satisfies (10.4) then*

$$P_k^{(0)}(x) \sim \sum_{j=0}^{\infty} k^{n-j} b_j^{(0)}(x) \quad \text{locally uniformly on } M(0),$$

where $b_j^{(0)}(x) \in \mathcal{C}^\infty(M(0))$, $j = 0, 1, 2, \dots$, are as in (1.8).

10.3. Asymptotics for arbitrary semi-positive metrics on ample line bundles

We consider now the Bergman kernel of a metric with semi-positive curvature on an ample line bundle and recover the following result of Berman [2].

Corollary 10.5. *Let L be an ample line bundle over a compact projective manifold M of dimension n . We endow M with a Hermitian metric Θ and L with a Hermitian metric h^L with semi-positive curvature. Then the Bergman kernel function associated to these metric data admits an asymptotic expansion*

$$P_k^{(0)}(x) \sim \sum_{j=0}^{\infty} k^{n-j} b_j^{(0)}(x) \quad \text{locally uniformly on } M(0),$$

where $b_j^{(0)}(x) \in \mathcal{C}^\infty(M(0))$, $j = 0, 1, 2, \dots$, are as in (1.8).

Proof. By a result due to Donnelly [28] there exist $C > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\inf \left\{ \lambda \in \text{Spec}(\square_k^{(0)}); \lambda \neq 0 \right\} \geq C.$$

In particular, $\square_k^{(0)}$ has $O(k^{-n_0})$ small spectral gap on any open set $D \subset M(0)$. By applying Theorem 1.6 we immediately deduce the result. \square

10.4. Expansion for Bergman kernel on forms

Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n . Given $q \in \mathbb{N}_0$, $0 \leq q \leq n$, \dot{R}^L is said to satisfy condition $Z(q)$ at $p \in M$ if $\dot{R}^L(p)$ has at least $n + 1 - q$ positive eigenvalues or at least $q + 1$ negative eigenvalues. If $\dot{R}^L(p)$ is non-degenerate of constant signature (n_-, n_+) , then $Z(q)$ holds at p if and only if $q \neq n_-$. It is well-known that if $Z(q - 1)$ and $Z(q + 1)$ hold at each point of M , then $\square_k^{(q)}$ has a “large” spectral gap, i.e., there exists a constant $C > 0$ such that for all k we have

$$(10.5) \quad \inf \left\{ \lambda \in \text{Spec}(\square_k^{(q)}); \lambda \neq 0 \right\} \geq Ck.$$

This fact essentially follows from the L^2 method for $\bar{\partial}$ of Hörmander (see Hörmander [38] for the classical case and Sjöstrand [61, Appendix] for the semi-classical case). From this and Theorem 1.6, we deduce the following local version of the results due to Catlin [12], Zelditch [70], Dai et al. [19] (for $q = 0$) and Berman–Sjöstrand [4], Ma–Marinescu [48] (for $q > 0$):

Corollary 10.6. *Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n . Given $q \in \mathbb{N}_0$, $0 \leq q \leq n$. We assume that $Z(q - 1)$ and $Z(q + 1)$ hold at each point of M . If \dot{R}^L is non-degenerate of constant signature (n_-, n_+) on an open set $D \subset M$, where $q = n_-$, then we have*

$$P_k^{(q)}(x) \sim \sum_{j=0}^{\infty} k^{n-j} b_j^{(q)}(x) \quad \text{locally uniformly on } D,$$

where $b_j^{(q)}(x) \in \mathcal{C}^\infty(D, \text{End}(\Lambda^q T^{*(0,1)} M))$, $j = 0, 1, 2, \dots$, are as in (1.8).

Let us illustrate Corollary 10.6 in the case $q = 0$: if at each point the curvature R^L has either only positive eigenvalues or at least two negative

eigenvalues, then the Bergman kernel of the sections of L^k has an asymptotic expansion on $M(0)$ as $k \rightarrow \infty$.

10.5. Holomorphic Morse inequalities

Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n . Since M is compact, $\square_k^{(q)}$ has a discrete spectrum, each eigenvalues occurs with finite multiplicity. From (1.15), (1.16) and the Lebesgue dominated convergence theorem, we deduce the following.

Corollary 10.7. *Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n . Given $q \in \mathbb{N}_0$, $0 \leq q \leq n$. If $N_0 > 2n$, then*

$$\dim \mathcal{E}_{k^{-N_0}}^q(M, L^k) = k^n (2\pi)^{-n} \int_{M(q)} \left| \det \dot{R}^L(x) \right| dv_M(x) + o(k^n).$$

Fix $N_0 \geq 1$. Let $\mathcal{E}_{0 < \lambda \leq k^{-N_0}}^q(M, L^k)$ denote the spectral space spanned by the eigenforms of $\square_k^{(q)}$ whose eigenvalues are bounded by k^{-N_0} and > 0 . Since the operator $\bar{\partial}_k \oplus \bar{\partial}_k^*$ maps $\mathcal{E}_{0 < \lambda \leq k^{-N_0}}^q(M, L^k)$ injectively into $\mathcal{E}_{0 < \lambda \leq k^{-N_0}}^{q+1}(M, L^k) \oplus \mathcal{E}_{0 < \lambda \leq k^{-N_0}}^{q-1}(M, L^k)$. Thus,

$$\dim \mathcal{E}_{0 < \lambda \leq k^{-N_0}}^q(M, L^k) \leq \dim \mathcal{E}_{0 < \lambda \leq k^{-N_0}}^{q+1}(M, L^k) + \dim \mathcal{E}_{0 < \lambda \leq k^{-N_0}}^{q-1}(M, L^k).$$

From this observation and Corollary 10.7, we deduce:

Corollary 10.8. *Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n . Given $q \in \mathbb{N}_0$, $0 \leq q \leq n$. If $N_0 > 2n$, then*

$$\begin{aligned} \dim \mathcal{H}^q(M, L^k) + \dim \mathcal{E}_{0 < \lambda \leq k^{-N_0}}^{q-1}(M, L^k) + \dim \mathcal{E}_{0 < \lambda \leq k^{-N_0}}^{q+1}(M, L^k) \\ \geq k^n (2\pi)^{-n} \int_{M(q)} \left| \det \dot{R}^L(x) \right| dv_M(x) + o(k^n). \end{aligned}$$

In particular, we have

$$(10.6) \quad \begin{aligned} & \dim \mathcal{H}^q(M, L^k) \\ & \geq k^n (2\pi)^{-n} \left(\int_{M^{(q)}} \left| \det \dot{R}^L(x) \right| dv_M(x) - \int_{M^{(q-1)}} \left| \det \dot{R}^L(x) \right| dv_M(x) \right. \\ & \quad \left. - \int_{M^{(q+1)}} \left| \det \dot{R}^L(x) \right| dv_M(x) \right) + o(k^n). \end{aligned}$$

Hence, if $M^{(q-1)} = \emptyset$, $M^{(q+1)} = \emptyset$, then

$$(10.7) \quad \dim \mathcal{H}^q(M, L^k) = k^n (2\pi)^{-n} \left(\int_{M^{(q)}} \left| \det \dot{R}^L(x) \right| dv_M(x) \right) + o(k^n).$$

By Corollary 10.7 and a straightforward application of the linear algebra result from Demailly [23, Lemma 4.2] or [49, Lemma 3.2.12] to the complex $(\mathcal{E}_{k^{-n_0}}^\bullet(M, L^k), \bar{\partial}_k)$, we obtain the following fundamental result due to Demailly's [23, Theorem 0.1]. We refer the reader to [49, Ch. 1–3] for a thorough discussion of the holomorphic Morse inequalities.

Corollary 10.9 (strong holomorphic Morse inequalities). *Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n . Then for any $q \in \{0, 1, \dots, n\}$ we have for $k \rightarrow \infty$*

$$\begin{aligned} & \sum_{j=0}^q (-1)^{q-j} \dim \mathcal{H}^j(M, L^k) \\ & \leq k^n (2\pi)^{-n} \sum_{j=0}^q (-1)^{q-j} \int_{M^{(j)}} \left| \det \dot{R}^L(x) \right| dv_M(x) + o(k^n). \end{aligned}$$

It is well-known that the strong Morse inequalities provide the solution of the Grauert-Riemenschneider conjecture [34] (cf. [23], [49], [58]).

Let us also give an example of a quite general holomorphic Morse inequalities on arbitrary complete Kähler manifolds.

Corollary 10.10. *Let (M, Θ) be a complete Kähler manifold and (L, h^L) be a semi-positive Hermitian holomorphic line bundle on M . Then*

$$(10.8) \quad \liminf_{k \rightarrow \infty} k^{-n} \dim \mathcal{H}^0(M, L^k \otimes K_M) \geq \frac{1}{n!} \int_M \left(\frac{\sqrt{-1}}{2\pi} R^L \right)^n.$$

Proof. Let $\{S_j^k\}_{j=1}^{d_k}$ be an orthonormal basis of $\mathcal{H}^0(M, L^k \otimes K_M)$, $d_k \in \mathbb{N} \cup \{\infty\}$. Then the Bergman kernel function is given by $\text{Tr } P_{k, K_M}(x) = \sum_{j=1}^{d_k} |S_j^k(x)|^2$, $x \in X$, where $|\cdot|$ denotes the pointwise norm in the metric $h^k \otimes h^{K_M}$. By integrating this relation we obtain

$$\dim \mathcal{H}^0(M, L^k \otimes K_M) = d_k = \int_M \text{Tr } P_{k, K_M}(x) dv_M.$$

By (1.19) we know that the sequence $k^{-n} \text{Tr } P_{k, K_M}(x)$ converges pointwise on $M(0)$ to $\text{Tr } b_{0, K_M}^{(0)}$ as $k \rightarrow \infty$. By Fatou's lemma we obtain

$$\begin{aligned} \liminf_{k \rightarrow \infty} k^{-n} \int_M \text{Tr } P_{k, K_M}(x) dv_M &\geq \liminf_{k \rightarrow \infty} \int_{M(0)} k^{-n} \text{Tr } P_{k, K_M}(x) dv_M \\ &\geq \int_{M(0)} \text{Tr } b_{0, K_M}^{(0)}(x) dv_M = \frac{1}{n!} \int_{M(0)} \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n = \frac{1}{n!} \int_M \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n. \end{aligned}$$

Hence (10.8) follows. \square

Let us close with an amusing by-product of Theorem 1.1. Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n . Assume that \dot{R}^L is non-degenerate of constant signature (n_-, n_+) at each point of M . From Theorem 1.1, we see that if $q \neq n_-$, then $P_k^{(q)}(x) = O(k^{-N})$, for every $N \geq 0$. Thus,

$$\dim \mathcal{H}^q(M, L^k) = O(k^{-N}), \quad \forall N \geq 0.$$

Since $\dim \mathcal{H}^q(M, L^k)$ is an integer, we obtain the Andreotti–Grauert coarse vanishing theorem (see [48, Theorem 1.5], [49, Rem. 8.2.6]):

$$(10.9) \quad \dim \mathcal{H}^q(M, L^k) = 0, \quad \text{for } k \text{ large enough.}$$

This proof uses just estimates of the spectral spaces. The original proof of Andreotti–Grauert was based on cohomology finiteness theorems for the disc bundle L^* . Ph. Griffiths gave a proof using the Bochner–Kodaira–Nakano formula. For a proof using Lichnerowicz formula and a comparison of methods, see [48, Theorem 1.5], [48, Rem. 1.6]. Note that the above proof of (10.9) provides a positive answer to a question of Bouche [9] whether one could get vanishing theorems by just using (heat or Bergman) kernel methods.

10.6. Tian's theorem and equidistribution of zeros

Given a positive line bundle L on a compact manifold M one can consider the Kodaira embeddings $\Phi_k : M \rightarrow \mathbb{P}(H^0(M, L^k)^*)$ for large k , where $H^0(M, L^k) = \{u \in \mathcal{C}^\infty(M, L^k); \bar{\partial}_k u = 0\}$. Denote by ω_{FS} the Fubini-Study metric on $\mathbb{P}(H^0(M, L^k)^*)$. Tian [63, Theorem A] proved that $\frac{1}{k}\Phi_k^*(\omega_{FS})$ converges to the curvature $\frac{\sqrt{-1}}{2\pi}R^L$ as $k \rightarrow \infty$ in the \mathcal{C}^2 -topology. This answered a conjecture of Yau [69]. Ruan [57] proved the convergence in the \mathcal{C}^∞ -topology and improved the estimate of the convergence speed. Both papers use the peak section method, based on L^2 -estimates for $\bar{\partial}$. A proof of the convergence in the \mathcal{C}^0 -topology using the heat kernel appeared in Bouche [7]. Catlin [12] and Zelditch [70] deduced the convergence from the asymptotic expansion of the Bergman kernel.

We will consider here a compact Hermitian manifold (M, Θ) and a big line bundle $L \rightarrow M$. Let h^L be a strictly positive singular Hermitian metric on L , smooth outside a proper analytic set Σ of M . We endow $H^0(M, L^k \otimes \mathcal{I}(h^k))$ with the L^2 scalar product (9.2) induced by h^L and $dv_M = \Theta^n/n!$. Consider the Kodaira map

$$(10.10) \quad \begin{aligned} \Phi_k : M \setminus B_k &\rightarrow \mathbb{P}(H^0(M, L^k \otimes \mathcal{I}(h^k))^*), \\ x &\longmapsto \{s \in H^0(M, L^k \otimes \mathcal{I}(h^k)); s(x) = 0\}, \end{aligned}$$

where B_k is the base locus of $H^0(M, L^k \otimes \mathcal{I}(h^k))$. To the Hermitian structure (9.2) corresponds a Fubini-Study metric ω_{FS} on $\mathbb{P}(H^0(M, L^k \otimes \mathcal{I}(h^k))^*)$, defined as the curvature of the hyperplane line bundle (see, e.g., [49, (5.1.3)]). The induced Fubini-Study metric is the metric $\frac{1}{k}\Phi_k^*(\omega_{FS})$ on $M \setminus B_k$.

Theorem 10.11. *Let (M, Θ) be a compact Hermitian manifold and let $L \rightarrow M$ be a big line bundle. Let h^L be a strictly positive singular Hermitian metric on L , smooth outside a proper analytic set Σ of M . Then for any compact set $K \subset M \setminus \Sigma$, there exists k_0 such that for $k \geq k_0$ the base locus B_k of $H^0(M, L^k \otimes \mathcal{I}(h^k))$ is disjoint of K . Moreover, for any $\ell \in \mathbb{N}$, there exists $C_{\ell, K} > 0$ independent of k such that for $k \geq k_0$ the following holds*

$$(10.11) \quad \left| \frac{1}{k}\Phi_k^*(\omega_{FS}) - \frac{\sqrt{-1}}{2\pi}R^L \right|_{\mathcal{C}^\ell(K)} \leq \frac{C_{\ell, K}}{k}.$$

Proof. Let $\{S_j^k\}_{j=1}^{m_k}$ be an orthonormal basis of $H^0(M, L^k \otimes \mathcal{I}(h^k))$. Then the multiplier Bergman kernel function (1.21) is given by

$$P_{k, \mathcal{I}}^{(0)}(x) = \sum_{j=1}^{m_k} |S_j^k(x)|_{h^k}^2, \quad x \in M \setminus \Sigma.$$

Let $K \subset M \setminus \Sigma$ be a compact set. The expansion (1.22) yields $P_{k, \mathcal{I}}^{(0)}(x) = b_0^{(0)}(x)k^n + o(k^n)$, as $k \rightarrow \infty$, uniformly on K . Since $\inf_{x \in K} b_0^{(0)}(x) > 0$, there exists k_0 such that for all $k \geq k_0$ we have $\inf_{x \in K} P_{k, \mathcal{I}}^{(0)}(x) > 0$. Hence, $K \cap B_k = \emptyset$, for all $k \geq k_0$.

For a local holomorphic frames e_L of L over an open set $U \subset M$, we set $S_j^k = f_j^k e_L^{\otimes k}$, where $f_j^k \in \mathcal{O}(U)$. The choice of the basis $\{S_j^k\}_{j=1}^{m_k}$ induces an isometric identification $\mathbb{P}(H^0(M, L^k \otimes \mathcal{I}(h^k))^*) \cong \mathbb{P}^{m_k-1}$ and in terms of this identification Φ_k has the form

$$\Phi_k : M \setminus B_k \rightarrow \mathbb{P}^{m_k-1}, \quad \Phi_k(x) = [f_1^k(x), \dots, f_{m_k}^k(x)],$$

hence

$$\Phi_k^*(\omega_{FS}) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left(\sum_{j=1}^{m_k} |f_j^k(x)|^2 \right) \quad \text{on } U \setminus B_k,$$

thus

$$(10.12) \quad \frac{1}{k} \Phi_k^*(\omega_{FS}) - \frac{\sqrt{-1}}{2\pi} R^L = -\frac{\sqrt{-1}}{2\pi k} \partial \bar{\partial} \log P_{k, \mathcal{I}}^{(0)}(x), \quad \text{on } M \setminus B_k.$$

The expansion (1.22) shows that $\partial \bar{\partial} \log P_{k, \mathcal{I}}^{(0)}(x) = O(1)$, for $k \rightarrow \infty$ in the \mathcal{C}^ℓ -topology, since $\log P_{k, \mathcal{I}}^{(0)}(x) = \log k^n + \log(b_0^{(0)}(x) + O(\frac{1}{k}))$ in the $\mathcal{C}^{\ell+2}$ -topology. Hence (10.11) is a consequence of (10.12). \square

An important application of the convergence of the Fubini–Study currents is the study of the asymptotic distribution of zeros of random holomorphic sections. After the pioneering work of Nonnenmacher–Voros [56], general methods were developed by Shiffman–Zelditch [59] and Dinh–Sibony [26] to describe the asymptotic distribution of zeros of random holomorphic sections of a positive line bundle over a projective manifold endowed with a smooth positively curved metric. The paper [26] gives moreover very good convergence speed and applies to general measures (e.g., equidistribution of complex zeros of homogeneous polynomials with real coefficients). Some important technical tools for higher dimension used in the previous works

were introduced by Fornæss–Sibony [33]. For the non-compact setting and the case of singular Hermitian metrics, see [16–18, 25].

Using the results of the present paper we can further generalize some results from [16].

We define positive $(1, 1)$ currents γ_k on M , called Fubini–Study currents, by

$$(10.13) \quad \gamma_k|_U = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{j=1}^{m_k} |f_j^k(x)|^2 \right).$$

Then

$$(10.14) \quad \frac{1}{k} \gamma_k - \frac{\sqrt{-1}}{2\pi} R^L = -\frac{\sqrt{-1}}{2\pi k} \bar{\partial}\partial \log P_{k, \mathcal{S}}^{(0)}(x), \quad \text{on } M \setminus \Sigma.$$

This shows that the definition (10.13) of γ_k is independent of the choice of holomorphic frame e_L and basis $\{S_j^k\}_{j=1}^{m_k}$.

Let λ_k be the normalized surface measure on the unit sphere \mathcal{S}^k of $H^0(M, L^k \otimes \mathcal{S}(h^k))$, defined in the natural way by using a fixed orthonormal basis. Consider the probability space $\mathcal{S}_\infty = \prod_{p=1}^\infty \mathcal{S}^p$ endowed with the probability measure $\lambda_\infty = \prod_{p=1}^\infty \lambda_p$. Denote by $[S = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{S = 0\}$ determined by a non-trivial section $S \in H^0(M, L^k \otimes \mathcal{S}(h^k))$.

Corollary 10.12. *Let (M, Θ) be a compact Hermitian manifold and let $L \rightarrow M$ be a big line bundle. Then we have in the weak sense of currents on M*

$$\lim_{k \rightarrow \infty} \frac{1}{k} \gamma_k = \frac{\sqrt{-1}}{2\pi} R^L,$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} [\sigma_k = 0] = \frac{\sqrt{-1}}{2\pi} R^L, \quad \text{for } \lambda_\infty - \text{a.e. sequence } \{\sigma_k\}_{k \geq 1} \in \mathcal{S}_\infty.$$

Proof. Let us observe that $H^0(M, L^k \otimes \mathcal{S}(h^k)) = H_{(2)}^0(M \setminus \Sigma, L^k, h^k, dv_M)$ and h^L and dv_M satisfy the conditions (A)–(C) of [16]. Owing to the asymptotic expansion (1.22) on $M \setminus \Sigma$ the conclusion follows from Theorems 1.1 and 4.3 from [16]. \square

Corollary 10.12 generalizes [16, Theorem 6.5], where the result was obtained under the hypothesis that M is Kähler.

The results of this paper allow also to extend Tian’s convergence theorem to the situation considered in Theorem 1.7. Let $\{S_j^k\}_{j=1}^{d_k}$, $d_k \in \mathbb{N} \cup \{\infty\}$, be

an orthonormal basis of $H_{(2)}^0(X, L^k \otimes K_M)$. We define the Fubini–Study currents γ_k on M in analogy to (10.13) as follows. Let U be an open set and let e_L be a local holomorphic frame for L on U . Set

$$(10.15) \quad \gamma_k|_U = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{j=1}^{d_k} |f_j^k(x)|^2 \right).$$

The currents γ_k don't depend on the choice of the local frame e_L and are globally defined $(1, 1)$ currents (see [16, Lemma 3.2 (ii)], [49], [50]). If $d_k < \infty$, then $\gamma_k = \Phi_k^*(\omega_{FS})$, where $\Phi_k : M \setminus B_k \rightarrow \mathbb{P}(H_{(2)}^0(X, L^k \otimes K_M)^*)$, $x \mapsto \{s \in H_{(2)}^0(X, L^k \otimes K_M); s(x) = 0\}$ is the Kodaira map. We have moreover

$$(10.16) \quad \frac{1}{k} \gamma_k - \frac{\sqrt{-1}}{2\pi} R^L = -\frac{\sqrt{-1}}{2\pi k} \partial\bar{\partial} \log P_{k, K_M}(x), \quad \text{on } M.$$

Hence Theorem 1.7 implies immediately the following.

Theorem 10.13. *Let (M, Θ) be a complete Kähler manifold and (L, h^L) be a holomorphic semi-positive line bundle over M , with smooth Hermitian metric h^L . Let $M(0)$ be the set where (L, h^L) be a positive. Then for any compact set $K \subset M(0)$, there exists k_0 such that for $k \geq k_0$ the base locus B_k of $H^0(M, L^k \otimes K_M)$ is disjoint of K . Moreover, for any $\ell \in \mathbb{N}$, there exists $C_{\ell, K} > 0$ independent of k such that for $k \geq k_0$ the following holds*

$$(10.17) \quad \left| \frac{1}{k} \gamma_k - \frac{\sqrt{-1}}{2\pi} R^L \right|_{\mathcal{C}^\ell(K)} \leq \frac{C_{\ell, K}}{k}.$$

Theorem 10.13 can be used as above to prove the analogue of Corollary 10.12. Note that in [18, Theorem 3.1 (ii)] the equidistribution of sections of adjoint bundles was actually obtained in the presence of singular Hermitian metrics.

Acknowledgments

The methods of microlocal analysis used in this work are marked by the influence of Professor Johannes Sjöstrand. The first-named author in particular wishes to express his hearty thanks for discussions on similar subjects and for giving us the idea of the proof of Theorem 10.1. This work was supported by the DFG funded project MA 2469/2-1 and also partially supported by the SFB/TR 12.

References

- [1] R. Berman, *Bergman kernels and local holomorphic Morse inequalities*, Math. Z. **248**(2) (2004), 325–344.
- [2] R. Berman, *Bergman kernels and equilibrium measures for line bundles over projective manifolds for high powers*, Amer. J. Math. **131**(5) (2009), 1485–1524.
- [3] R. Berman, B. Berndtsson and J. Sjöstrand, *A direct approach to Bergman kernel asymptotics for positive line bundles*, Ark. Math. **46**(2) (2008), 197–217.
- [4] R. Berman and J. Sjöstrand, *Asymptotics for Bergman–Hodge kernels for high powers of complex line bundles*, Ann. Fac. des Sci. Toulouse. **XIV**(4) (2007), 719–771.
- [5] J.-M. Bismut, *Demailly’s asymptotic Morse inequalities: a heat equation proof*, J. Funct. Anal. **72**(2) (1987), 263–278.
- [6] M. Bordemann and E. Meinrenken and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and $\mathfrak{gl}(N)$, $N \rightarrow \infty$ limits*, Comm. Math. Phys. **165**(2) (1994), 281–296.
- [7] T. Bouche, *Convergence de la métrique de Fubini–Study d’un fibré linéaire positif*, Ann. Inst. Fourier (Grenoble) **40** (1990), 117–130.
- [8] T. Bouche, *Two vanishing theorems for holomorphic vector bundles of mixed sign*, Math. Z. **218** (1995), 519–526.
- [9] T. Bouche, *Asymptotic results for Hermitian line bundles over complex manifolds: the heat kernel approach*, Higher-dimensional complex varieties (Trento, 1994), 67–81, de Gruyter, Berlin, 1996.
- [10] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegő*, Astérisque **34–35** (1976), 123–164.
- [11] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Ann. Math. Stud. **99** (1981), v+161 pp.
- [12] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables, Katata, Trends in Mathematics 1–23.

- [13] X.-X. Chen, S. Donaldson and S. Sun, *Kähler-Einstein metrics on Fano manifolds, I: approximation of metrics with cone singularities*, preprint available at [arXiv:1211.4566](https://arxiv.org/abs/1211.4566).
- [14] X.-X. Chen, S. Donaldson and S. Sun, *Kähler-Einstein metrics on Fano manifolds, II: limits with cone angle less than 2π* , preprint available at [arXiv:1212.4714](https://arxiv.org/abs/1212.4714).
- [15] X.-X. Chen, S. Donaldson and S. Sun, *Kähler-Einstein metrics on Fano manifolds, III: limits as cone angle approaches 2π and completion of the main proof*, preprint available at [arXiv:1302.0282](https://arxiv.org/abs/1302.0282).
- [16] D. Coman and G. Marinescu, *Equidistribution results for singular metrics on line bundles*, preprint available at [arXiv:1108.5163](https://arxiv.org/abs/1108.5163), 2011.
- [17] D. Coman and G. Marinescu, *Convergence of Fubini-Study currents for orbifold line bundles*, preprint available at [arXiv:1210.5604](https://arxiv.org/abs/1210.5604), to appear in *Internat. J. Math.*
- [18] D. Coman and G. Marinescu, *On the approximation on closed positive currents on compact Kähler manifolds*, preprint available at [arXiv:1302.0292](https://arxiv.org/abs/1302.0292), to appear in *Math. Rep.*
- [19] X. Dai, K. Liu and X. Ma, *On the asymptotic expansion of Bergman kernel*, *C. R. Math. Acad. Sci. Paris* **339**(3) (2004), 193–198.
- [20] X. Dai, K. Liu and X. Ma, *On the asymptotic expansion of Bergman kernel*, *J. Differ. Geom.* **72**(1) (2006), 1–41.
- [21] E.-B. Davies, *Spectral Theory and Differential Operators*, *Cambridge Stud. Adv. Math.* **42**, (1995), 182.
- [22] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au dessus d'une variété kählérienne complète*, *Ann. Sci. Ec. Norm. Sup.* **15** (1982), 457–511.
- [23] J.-P. Demailly, *Champs magnétiques et inégalités de Morse pour la d'' -cohomologie*, *Ann. Inst. Fourier (Grenoble)* **35** (1985), 189–229.
- [24] J.-P. Demailly, *Complex analytic and algebraic geometry*, available at www-fourier.ujf-grenoble.fr/~demailly/books.html.
- [25] T.C. Dinh, G. Marinescu and V. Schmidt, *Asymptotic distribution of zeros of holomorphic sections in the non compact setting*, preprint available at [arXiv:1106.5244](https://arxiv.org/abs/1106.5244), 2011.

- [26] T.C. Dinh and N. Sibony, *Distribution des valeurs de transformations méromorphes et applications*, Comment. Math. Helv. **81** (2006), 221–258.
- [27] S.K. Donaldson, *Scalar curvature and projective embeddings. I*, J. Differ. Geom. **59** (2001), 479–522.
- [28] H. Donnelly, *Spectral theory for tensor products of Hermitian holomorphic line bundles*, Math. Z. **245**(1) (2003), 31–35.
- [29] M.R. Douglas and S. Klevtsov, *Bergman kernel from path integral*, Comm. Math. Phys. **293** (2010), 205–230.
- [30] M. Engliš, *Weighted Bergman kernels and quantization*, Comm. Math. Phys. **227**, (2002), 211–241.
- [31] J. Fine, *Calabi flow and projective embeddings*, with an Appendix by K. Liu and X. Ma, J. Differ. Geom. **84** (2010), 489–523.
- [32] J. Fine, *Quantisation and the Hessian of Mabuchi energy*, Duke Math. J. **161**(14) (2012), 2753–2798.
- [33] J.E. Fornæss and N. Sibony, *Oka’s inequality for currents and applications*, Math. Ann. **301** (1995), 399–419.
- [34] M. Garcia-Fernandez and J. Ross, *Limits of balanced metrics on vector bundles and polarised manifolds*, Proc. London Math. Soc. **106**(5) (2013), 1143–1156.
- [35] H. Grauert and O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970), 263–292.
- [36] A. Grigis and J. Sjöstrand, *Microlocal analysis for differential operators. An introduction*, London Mathematical Society Lecture Note Series, vol. 196, Cambridge University Press, Cambridge, 1994.
- [37] B. Helffer and J. Sjöstrand, *Puits multiples en mécanique semi-classique. IV. tude du complexe de Witten*, Commun. Partial Differ. Equ. **10**(3) (1985), 245–340.
- [38] L. Hörmander, *An introduction to complex analysis in several variables*, North-Holland mathematical library, North-Holland Publishing Co., Amsterdam, **7**, 1990.

- [39] L. Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003.
- [40] C.-Y. Hsiao, *Projections in several complex variables*, Mém. Soc. Math. France, Nouv. Sér. **123** (2010), 131.
- [41] C.-Y. Hsiao, *On the coefficients of the asymptotic expansion of the kernel of Berezin–Toeplitz quantization*, Ann. Glob. Anal. Geom. **42**(2) (2012), 207–245.
- [42] C.-Y. Hsiao, *The second coefficient of the asymptotic expansion of the weighted Bergman kernel for $(0, q)$ forms on \mathbb{C}^n* , preprint available at [arXiv:1208.3818](https://arxiv.org/abs/1208.3818).
- [43] C.-Y. Hsiao and G. Marinescu, *Szegő kernel asymptotics and Morse inequalities on CR manifolds*, Math. Z. **271** (2012), 509–553.
- [44] Z. Lu, *On the lower order terms of the asymptotic expansion of Tian–Yau–Zelditch*, Amer. J. Math. **122**(2) (2000), 235–273.
- [45] W. Lu, *The second coefficient of the asymptotic expansion of the Bergman kernel of the Hodge–Dolbeault operator*, preprint available at [arXiv:1212.6034](https://arxiv.org/abs/1212.6034), to appear in the Journal of Geometric Analysis.
- [46] X. Ma, *Geometric quantization on Kähler and symplectic manifolds*, Proceedings of the international congress of mathematicians (ICM 2010), Hyderabad, India, August 19–27, 2010. Vol II, 785–810.
- [47] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, C. R. Math. Acad. Sci. Paris **339** (2004), 493–498.
- [48] X. Ma and G. Marinescu, *The first coefficients of the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator*, Int. J. Math. **17**(6) (2006), 737–759.
- [49] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Math., vol. 254, Birkhäuser, Basel, 2007, 422 pp.
- [50] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, Adv. Math. **217**(4) (2008), 1756–1815.
- [51] X. Ma and G. Marinescu, *Toeplitz operators on symplectic manifolds*, J. Geom. Anal. **18**(2) (2008), 565–611.
- [52] X. Ma and G. Marinescu, *Berezin–Toeplitz Quantization and its kernel expansion* Travaux mathématiques **19** (2011), 125–166.

- [53] X. Ma and G. Marinescu, *Berezin–Toeplitz quantization on Kähler manifolds*, J. Reine Angew. Math. **662** (2012), 1–56.
- [54] A. Melin and J. Sjöstrand, *Fourier integral operators with complex-valued phase functions*, Springer Lecture Notes in Mathematics **459** (1975), 120–223.
- [55] A. Menikoff and J. Sjöstrand, *On the eigenvalues of a class of hypoelliptic operators*, Math. Ann. **235**(1) (1978), 55–85.
- [56] S. Nonnenmacher and A. Voros, *Chaotic eigenfunctions in phase space*, J. Stat. Phys. **92**(3-4) (1998), 451–518.
- [57] W.-D. Ruan, *Canonical coordinates and Bergmann metrics*, Comm. Anal. Geom. **6**(3) (1998), 589–631.
- [58] Y.T. Siu, *A vanishing theorem for semipositive line bundles over non-Kähler manifolds*, J. Differ. Geom. **20** (1984), 431–452.
- [59] B. Shiffman and S. Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Commun. Math. Phys. **200** (1999), 661–683.
- [60] B. Shiffman and S. Zelditch, *Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds*, J. Reine Angew. Math. **544** (2002), 181–222.
- [61] J. Sjöstrand, *Function spaces associated to global I-Lagrangian manifolds*, World Sci. Publishing (1996), River Edge, NJ, 369–423.
- [62] M.E. Taylor, *Partial differential equations. 1: Basic theory*, Applied Mathematical Sciences, vol. 115, Springer–Verlag, Berlin, 1996.
- [63] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differ. Geom. **32** (1990), 99–130.
- [64] G. Tian, *K-stability and Kähler–Einstein metrics*, preprint available at [arXiv:1211.4669](https://arxiv.org/abs/1211.4669).
- [65] H. Xu, *A closed formula for the asymptotic expansion of the Bergman kernel*, preprint available at [arXiv:1103.3060](https://arxiv.org/abs/1103.3060).
- [66] L. Wang, *Bergman kernel and stability of holomorphic vector bundles with sections*, MIT Ph.D. Dissertation (2003), 85 pages.
- [67] X. Wang, *Canonical metrics on stable vector bundles*, Commun. Anal. Geom. **13** (2005), 253–285.

- [68] E. Witten, *Supersymmetry and Morse theory*, J. Differ. Geom. **17** (1982), 661–692.
- [69] S.T. Yau, *Nonlinear analysis in geometry*, L'Enseignement Math. **33** (1987), 109–158.
- [70] S. Zelditch, *Szegő kernels and a theorem of Tian*, Int. Math. Res. Not. (6) (1998), 317–331.

INSTITUTE OF MATHEMATICS
ACADEMIA SINICA, 6F
ASTRONOMY-MATHEMATICS BUILDING
NO.1, SEC.4, ROOSEVELT ROAD
TAIPEI 10617
TAIWAN
E-mail address: `chsiao@math.sinica.edu.tw`

UNIVERSITÄT ZU KÖLN
MATHEMATISCHES INSTITUT
WEYERTAL 86-90, 50931 KÖLN
GERMANY
E-mail address: `gmarines@math.uni-koeln.de`

RECEIVED SEPTEMBER 23, 2012