

Stable weighted minimal surfaces in manifolds with non-negative Bakry–Emery Ricci tensor

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In this paper, we study stable weighted minimal hypersurfaces in manifolds with non-negative Bakry–Emery Ricci curvature. We will give some geometric and topological applications. In particular, we give some partial classification of complete 3-manifolds with non-negative Bakry–Emery Ricci curvature assuming that f is bounded.

1. Introduction

A smooth metric measure space is a triple $(M, g, e^{-f} dvol)$, where M is a smooth manifold; g is the Riemannian metric on M ; f is a smooth function and $dvol$ is the volume form induced by g . This object has been studied extensively in geometric analysis in recent years, e.g., [18, 22–24, 27, 28]. Perelman [28] introduces a functional that involves an integral of the scalar curvature with respect to a weighted measure. The Ricci flow is thus a gradient flow of such a functional. Metric measure spaces also arise in smooth collapsed Gromov–Hausdorff limits. In the physics literature, f is referred to as the dilation field. On the smooth metric measure space, there is an important curvature quantity called the Bakry–Emery Ricci curvature, which is defined in [5] by

$$\text{Ric}_f = \text{Ric} + \nabla^2 f.$$

One observes that $\text{Ric}_f = \lambda g$ for some constant λ is exactly the gradient Ricci soliton equation, which plays an essential role in the analysis of the singularities of the Ricci flow.

A lower bound for Bakry–Emery curvature is a natural assumption to make and it has significant geometric consequences. More generally, Ric_f has a natural extension to metric measure spaces; see [19, 30, 31]. Recently, in [27], Wei and Wylie proved the weighted volume comparison theorems; Munteanu and Wang established the gradient estimate for positive weighted harmonic functions. It should be noted that a while back, Lichnerowicz [17]

has generalized the classical Cheeger–Gromoll splitting theorem [6] to the metric measure spaces with $\text{Ric}_f \geq 0$ and f is bounded (see [11] for more generalizations).

In Riemannian geometry, minimal surfaces arise naturally in the variation of the area functional. A minimal surface is called stable if the second variation of the area is non-negative for any compactly supported variations. Minimal surfaces have their own beauties, e.g., Bernstein’s theorem. Moreover, they have important applications to the geometry and topology of manifolds. For example, more than 60 years ago, the Synge theorem and the Bonnet–Meyers theorem were proved by the variation of geodesics (one-dimensional minimal surface). More recently, by using minimal surfaces, Schoen and Yau proved the famous positive mass conjecture [34, 35]. Meeks and Yau [25, 26] proved the loop theorem, sphere theorem and Dehn lemma together with the equivariant forms. In [33], Schoen and Yau proved that a complete non-compact 3-manifold with positive Ricci curvature is diffeomorphic to \mathbb{R}^3 . Anderson [1] studied the restriction of the first betti number for manifolds with non-negative Ricci curvature; Liu [16] used the minimal surface theory to classify complete three dimensional manifolds with non-negative Ricci curvature.

In the study of smooth metric measure spaces, it is natural to add a weight e^{-f} on the area functional of the surface. The critical points of the weighted area functional are called weighted minimal surfaces. A weighted minimal surface is called stable if the second variation of the weighted area is non-negative.

Very recently, Cheng et al. [8] studied the stability condition and compactness of f -minimal surfaces. They [9] also gave eigenvalue estimates for certain closed f -minimal surfaces.

In this paper, we will investigate some geometric and topological results for smooth metric measure spaces via analyzing stable weighted minimal surfaces. We shall assume that the Bakry–Emery Ricci curvature is non-negative.

Below is the organization of this paper. In Section 2, we will derive the second variation formula for the weighted area (see also [4, 8] for the derivation). We give an application to compact stable f -minimal surfaces in Section 3. This generalizes some previous works of Heintze and Karcher [15]. An example is given in Section 4 to show that a result of Schoen and Fischer-Colbrie [12] cannot be extended to the case when Bakry–Emery Ricci curvature is non-negative. In Section 5, we give an application of the stability inequality to non-compact case. In Section 6, we study the topology of complete 3-manifolds with non-negative Bakry–Emery Ricci curvature.

2. Second variation formula

Definition. Let $(M^m, g, e^{-f} dv)$ be a complete smooth metric measure space and Σ be a complete submanifold in M . We say Σ is f -minimal in M , if the first variation of the e^{-f} weighted area functional vanishes at Σ . Σ is called stable f -minimal if the second variation of the e^{-f} weighted area functional is non-negative along any compactly supported variational normal vector field.

Proposition 1. Let $(M^m, g, e^{-f} dv)$ be a complete smooth metric measure space and Σ^n be a complete f -minimal submanifold in M . Let $e_i (0 \leq i \leq n)$ be an orthonormal frame in an open set of Σ . Define ∇^T and ∇^\perp to be the connections projected to the tangential and normal spaces on Σ . Then

$$H = \nabla^\perp f,$$

where $H = -\sum_i \nabla_{e_i}^\perp e_i$ is the mean curvature vector. If $\Sigma_t (-\epsilon < t < \epsilon)$ is a smooth family of the submanifolds such that $\Sigma_0 = \Sigma$ and the variational normal vector field ν is compactly supported on Σ_t , then at $t = 0$,

$$\begin{aligned} \frac{d^2 \int_{\Sigma_t} e^{-f}}{dt^2} &= \int_{\Sigma} e^{-f} \left(-\sum_{i=1}^n R_{i\nu\nu i} - \frac{1}{2} \Delta_{\Sigma}(|\nu|^2) + |\nabla_{\Sigma} \nu|^2 - 2|A^\nu|^2 - f_{\nu\nu} \right. \\ &\quad \left. + \frac{1}{2} \langle \nabla^T f, \nabla^T(|\nu|^2) \rangle \right), \end{aligned}$$

where $A^\nu_{ij} = -\langle \nabla_{e_i} e_j, \nu \rangle$.

Proof. For any point $p \in \Sigma_0$, consider a local frame $e_i (1 \leq i \leq n)$ near p such that they are tangential to Σ_t and $[e_i, \nu] = 0$ for all small t . We can also assume that at p , e_i is an orthonormal frame and $\nabla_{e_i}^T e_j = 0$. Let $g_{ij} = \langle e_i, e_j \rangle$ and g^{ij} be the inverse matrix of g_{ij} . We have

$$\frac{d \int_{\Sigma_t} e^{-f}}{dt} = \int_{\Sigma_t} e^{-f} \langle H - \nabla^\perp f, \nu \rangle,$$

where

$$H = -(\nabla_{e_i} e_j)^\perp g^{ij}.$$

Thus, if Σ_0 is e^{-f} minimal,

$$H = \nabla^\perp f.$$

At p , we have

$$\begin{aligned}
 (2.1) \quad \frac{d\langle H, \nu \rangle}{dt} &= -(\langle \nabla_\nu \nabla_{e_i} e_j, \nu \rangle g^{ij} + \langle \nabla_{e_i} e_j, \nabla_\nu \nu \rangle g^{ij} + \langle \nabla_{e_i} e_j, \nu \rangle \nu(g^{ij})) \\
 &= -\left(\sum_{i=1}^n R_{\nu i i \nu} + \langle \nabla_{e_i} \nabla_\nu e_i, \nu \rangle \right. \\
 &\quad \left. - \langle H, \nabla_\nu \nu \rangle - \sum_{i,j=1}^n \langle \nabla_{e_i} e_j, \nu \rangle (\langle \nabla_\nu e_i, e_j \rangle + \langle \nabla_\nu e_j, e_i \rangle) \right) \\
 &= -\left(\sum_{i=1}^n R_{\nu i i \nu} + \frac{1}{2} \Delta_\Sigma(|\nu|^2) - \sum_{i=1}^n |\nabla_{e_i} \nu|^2 \right. \\
 &\quad \left. + 2 \sum_{i,j=1}^n |\langle \nabla_{e_i} e_j, \nu \rangle|^2 - \langle H, \nabla_\nu \nu \rangle \right).
 \end{aligned}$$

$$\begin{aligned}
 (2.2) \quad \frac{d\langle \nabla^\perp f, \nu \rangle}{dt} &= \nu \nu(f) \\
 &= f_{\nu\nu} + \langle \nabla^T f, \nabla_\nu \nu \rangle + \langle \nabla^\perp f, \nabla_\nu \nu \rangle \\
 &= f_{\nu\nu} + \sum_{i=1}^n e_i(f) \langle e_i, \nabla_\nu \nu \rangle + \langle \nabla^\perp f, \nabla_\nu \nu \rangle \\
 &= f_{\nu\nu} - \frac{1}{2} \langle \nabla^T f, \nabla^T(|\nu|^2) \rangle + \langle \nabla^\perp f, \nabla_\nu \nu \rangle.
 \end{aligned}$$

Since Σ_0 is f minimal, by the two equalities above, we have

$$\begin{aligned}
 (2.3) \quad \frac{d^2 \int_{\Sigma_t} e^{-f}}{dt^2} &= \frac{d \int_{\Sigma_t} e^{-f} \langle H - \nabla^\perp f, \nu \rangle}{dt} \\
 &= \int_\Sigma e^{-f} \left(-\sum_{i=1}^n R_{i\nu\nu i} - \frac{1}{2} \Delta_\Sigma(|\nu|^2) + |\nabla_\Sigma \nu|^2 - 2|A^\nu|^2 - f_{\nu\nu} \right. \\
 &\quad \left. + \frac{1}{2} \langle \nabla^T f, \nabla^T(|\nu|^2) \rangle \right).
 \end{aligned}$$

□

Corollary 1. *Let $(M^m, g, e^{-f} dv)$ be a complete oriented Riemannian manifold and Σ_t be a smooth family of oriented hypersurfaces in M . Let N be the unit normal vector field on Σ_t . Suppose the variational vector field*

for Σ_t is given by λN where λ is smooth function with compact support on Σ_t . If Σ_0 is e^{-f} minimal, then the mean curvature of Σ_0 satisfies

$$H = f_n.$$

where f_n is the normal derivative of f . Moreover,

$$\frac{d^2 \int_{\Sigma_t} e^{-f}}{dt^2} \Big|_{t=0} = \int_{\Sigma_0} (|\nabla \lambda|^2 - \lambda^2(\text{Ric}_f(n, n) + |A|^2))e^{-f},$$

where $\text{Ric}_f = \text{Ric} + \nabla^2 f$, A is the second fundamental form. Therefore, the stability inequality is

$$\int_{\Sigma_0} (|\nabla \lambda|^2 - \lambda^2(\text{Ric}_f(n, n) + |A|^2))e^{-f} \geq 0$$

for any compactly supported function λ on Σ_0 .

Proof. Since Σ_0 is weighted minimal, according to Proposition 1,

$$H = \langle \nabla^\perp f, N \rangle = f_n.$$

Let $\nu = \lambda N$. For an orthonormal frame e_i at a point on Σ_0 ,

$$\begin{aligned} |\nabla_\Sigma \nu|^2 &= |\langle \nabla_{e_i}(\lambda N), \nabla_{e_i}(\lambda N) \rangle|^2 \\ (2.4) \qquad &= |\nabla \lambda|^2 + \sum_{i,j} |\langle \nabla_{e_i}(\lambda N), e_j \rangle|^2 \\ &= |\nabla \lambda|^2 + \lambda^2 |A|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} (2.5) \qquad \frac{d^2 \int_{\Sigma_t} e^{-f}}{dt^2} &= \int_{\Sigma_0} e^{-f} \left(- \sum_{i=1}^n R_{i\nu\nu i} - \frac{1}{2} \Delta_\Sigma(|\nu|^2) + |\nabla_\Sigma \nu|^2 - 2|A^\nu|^2 - f_{\nu\nu} \right. \\ &\quad \left. + \frac{1}{2} \langle \nabla^T f, \nabla^T(|\nu|^2) \rangle \right) \\ &= \int_{\Sigma_0} e^{-f} (-\lambda^2 \text{Ric}_f(n, n) - \lambda \Delta \lambda - \lambda^2 |A|^2 + \langle \nabla f, \nabla \lambda \rangle \lambda) \\ &= \int_{\Sigma_0} (|\nabla \lambda|^2 - \lambda^2(\text{Ric}_f(n, n) + |A|^2))e^{-f}. \end{aligned}$$

In the last step, we have used the integration by parts. □

3. An application to the compact case

In [29], Simons observed that there are no closed, stable minimal 2-sided hypersurfaces in a manifold with positive Ricci curvature. Later Heintze and Karcher [15] proved that the exponential map of the normal bundle of a hypersurface $\Sigma \in M$ is area decreasing, if Σ is stable, minimal and M has non-negative Ricci curvature. Anderson extended this result, he also proved that a version of the Cheeger–Gromoll splitting theorem in the compact case; see [2]. More recently, Morgan [20] obtained the upper bound of weighted volume of one side of a hypersurface which generalizes some works in [15]. See also Chapter 18 in [21] for more discussion.

In this section, we shall prove the following:

Theorem 1. *Let $(M^m, g, e^{-f} dv)$ be an oriented complete Riemannian manifold and Σ be a closed oriented stable f -minimal hypersurface in M . If $\text{Ric}_f \geq 0$, then Σ is totally geodesic and $\text{Ric}_f(n, n) = 0$. If Σ is weighted f -area-minimizing in its homology class, then M^m is isometric to a quotient of $\Sigma \times \mathbb{R}$. In this case, if $m = 3$, then topologically Σ is either a sphere or a torus. In the torus case, M^3 is flat.*

Proof. The first conclusion follows if we take $\lambda = 1$ in corollary 1. Let N be the unit normal vector field on Σ . For x close to Σ in M , consider the oriented distance function $d(x) = \text{Sign}(x)\text{dist}(x, \Sigma)$, where $\text{Sign}(x)$ is 1 if x is on one side of Σ ; $\text{Sign}(x) = -1$ if x is on the other side of Σ . Then $d(x)$ is smooth near Σ and let Σ_t be the level set of $d(x)$. Then for t small, Σ_t is a smooth family of hypersurfaces on M and we have

$$\frac{d(H - f_n)}{dt} = -\text{Ric}(n, n) - |A|^2 - f_{nn} = -\text{Ric}_f(n, n) - |A|^2 \leq 0.$$

Note that Σ_0 is totally geodesic and $f_n = H = 0$ at $t = 0$. Therefore

$$H - f_n \leq 0$$

for all t and

$$\frac{d \int_{\Sigma_t} e^{-f}}{dt} = \int_{\Sigma_t} (H - f_n) e^{-f} \leq 0.$$

Since Σ_0 is area-minimizing in its homology class, Σ_t are all totally geodesic. By induction, one can easily show that M is isometric to the quotient of

$\Sigma_0 \times \mathbb{R}$. Therefore

$$f_n = H = 0, \quad f_{nn} = \frac{\partial f_n}{\partial t} = 0, \quad \text{Ric}_{nn} = 0$$

for all t .

Now consider the case when $m = 3$. Let e_1, e_2 be a local orthonormal frame on Σ_0 . Let S be the scalar curvature on M ; $S_f = S + \Delta f$; K_Σ be the Gaussian curvature on Σ . Since Σ_0 is totally geodesic,

$$2K_{\Sigma_0} = 2R_{1221} = S - 2\text{Ric}_{nn} = S_f - f_{11} - f_{22} = S_f - \Delta_{\Sigma_0} f.$$

In the above equality, we have used the fact that $f_{nn} = 0$. Since $S_f \geq 0$, the Gauss-Bonnet theorem says that Σ_0 is either a sphere or a torus. In the torus case, $S_f = 0$ everywhere, thus on Σ , $\text{Ric} + \nabla^2 f = 0$. So Σ is a 2-dimensional steady soliton. Thus the Gaussian curvature on Σ is non-negative. This means that Σ and M are flat. □

4. An example

In [12], Schoen and Fischer–Colbrie proved the following theorem:

Theorem 2 (Schoen and Fischer–Colbrie). *Let M be a complete oriented 3-manifold with non-negative scalar curvature. Let Σ be an oriented complete stable minimal surface in M , then if Σ is compact, then it is conformal to \mathbb{S}^2 or a torus \mathbb{T}^2 ; if Σ is not compact, it is conformally covered by \mathbb{C} .*

In view of Theorem 2, it is natural to ask whether we can weaken the condition in Theorem 1 when $\dim(M) = 3$. We will show that at least locally, even if the Bakry–Emery Ricci curvature is non-negative, the stability of a weighted stable minimal surface Σ does not provide any information on the conformal structure on Σ .

Let M^3 be an oriented manifold with non-negative Bakry–Emery Ricci curvature and Σ be an oriented stable f -minimal surface in M . In this section we will give an explicit example so that Σ is hyperbolic.

Let (Σ, ds_Σ^2) be a complete surface with curvature -1 . Let $M = (-\frac{1}{2}, \frac{1}{2}) \times \Sigma$ and define metric on M by

$$ds^2 = dt^2 + g(t)ds_\Sigma^2.$$

Note that the metric on M is not complete. Let $p \in M$ and consider a product chart $U \ni p$ such that $e_1 = \frac{\partial}{\partial x_1}$, $e_2 = \frac{\partial}{\partial x_2}$ are tangential to Σ_t and $\frac{\partial}{\partial t} = e_3$ on U . We may assume that e_1, e_2, e_3 is an orthogonal frame in U and $ds_\Sigma^2(e_1, e_1) = ds_\Sigma^2(e_2, e_2) = 1$. Then

$$\begin{aligned}\langle \nabla_{e_1} e_3, e_1 \rangle &= \langle \nabla_{e_2} e_3, e_2 \rangle = \frac{1}{2} g'(t), \\ \langle \nabla_{e_1} e_3, e_2 \rangle &= \langle \nabla_{e_2} e_3, e_1 \rangle = 0.\end{aligned}$$

Therefore, $\nabla^{\Sigma_t} A = 0$ for all t . By Gauss equation,

$$K_{\Sigma_t} - \frac{R_{1221}}{g^2} = \frac{A_{11}A_{22}}{g^2} = \frac{\langle \nabla_{e_1} e_3, e_1 \rangle \langle \nabla_{e_2} e_3, e_2 \rangle}{g^2}.$$

Since the Gaussian curvature $K_{\Sigma_t} = -\frac{1}{g}$,

$$R_{1221} = -g - \frac{1}{4} g'^2.$$

It is easy to see that $\nabla_{e_3} e_3 \equiv 0$, thus

$$\begin{aligned}R_{1331} &= \langle \nabla_{e_1} \nabla_{e_3} e_3, e_1 \rangle - \langle \nabla_{e_3} \nabla_{e_1} e_3, e_1 \rangle \\ &= -\langle \nabla_{e_3} \nabla_{e_1} e_3, e_1 \rangle \\ (4.1) \quad &= -(e_3(\langle \nabla_{e_1} e_3, e_1 \rangle) - |\nabla_{e_3} e_1|^2) \\ &= -\frac{1}{2} g'' + \frac{1}{4} \frac{g'^2}{g}.\end{aligned}$$

From the same computation, we see that $R_{1332} = 0$. By Codazzi equation,

$$R_{1223} = (\nabla_{e_1}^{\Sigma_t} A)(e_2, e_2) - (\nabla_{e_2}^{\Sigma_t} A)(e_1, e_2) = 0.$$

Therefore,

$$\begin{aligned}\text{Ric}_{11} &= \frac{R_{1221}}{g} + R_{1331} = -1 - \frac{1}{2} g'' = \text{Ric}_{22}, \\ \text{Ric}_{33} &= -\frac{g''}{g} + \frac{1}{2} \left(\frac{g'}{g}\right)^2, \\ \text{Ric}_{12} &= \text{Ric}_{13} = \text{Ric}_{23} = 0.\end{aligned}$$

Let $f = f(t)$ be a function of M , then

$$\begin{aligned} f_{11} &= -\langle \nabla f, \nabla_{e_1} e_1 \rangle = \frac{g' f'}{2} = f_{22}, \\ f_{12} &= f_{13} = f_{23} = 0, \quad f_{33} = f''. \end{aligned}$$

Therefore,

$$\text{Ric}_f(e_1, e_1) = -1 - \frac{g''}{2} + \frac{f' g'}{2}, \quad \text{Ric}_f(e_3, e_3) = \frac{-2g''g + g'^2 + 2g^2 f''}{2g^2}.$$

If $f = -2t^2$, $g = 1 - 2t^2$, then one gets that

$$\begin{aligned} \text{Ric}_f(e_2, e_2) &= \text{Ric}_f(e_1, e_1) = 1 + 8t^2 \geq 0, \\ \text{Ric}_f(e_3, e_3) &= 4 \left(\frac{1}{(1 - 2t^2)^2} - 1 \right) \geq 0. \end{aligned}$$

Therefore, M has non-negative Bakry–Emery Ricci curvature. Moreover, the second fundamental form and $\text{Ric}_f(e_3, e_3)$ vanish at $t = 0$. According to corollary 1, Σ_0 is a stable f -minimal surface in M . However, Σ is hyperbolic.

5. Applications to the non-compact case

Now consider the case when Σ is non-compact. The following proposition follows from a simple cut-off argument:

Proposition 2. *Let $(M^m, g, e^{-f} dv)$ be an oriented complete Riemannian manifold and Σ be a complete non-compact oriented stable f -minimal hypersurface in M . If $\text{Ric}_f \geq 0$ on Σ and that the weighted volume growth of Σ with respect to its intrinsic distance to a point $p \in \Sigma$ satisfy*

$$V_{\Sigma, f}(B_p(r)) \leq Cr^2$$

for all large r , then Σ is totally geodesic and $\text{Ric}_f(n, n) = 0$.

Proof. Let r be a distance function to $p \in M$. Given any $a > 1$, consider the cut-off function

$$(5.1) \quad \lambda(r) = \begin{cases} 1 & 0 \leq r \leq a \\ \frac{2 \log a - \log r}{\log a} & a < r < a^2 \\ 0 & r \geq a^2. \end{cases}$$

Define $V(r) = \int_{B_{\Sigma}(p,r)} e^{-f}$. Plugging this in the stability inequality in corollary 1, we find that

$$\begin{aligned}
 & \int_{B_{\Sigma}(a)} (\text{Ric}_f(n, n) + |A|^2) |\lambda|^2 e^{-f} \\
 & \leq \int_{B_{\Sigma}(a^2)} |\nabla \lambda|^2 e^{-f} \\
 (5.2) \quad & = \int_a^{a^2} \frac{V'(r)}{r^2 \log^2 a} dr \\
 & = \frac{V(r)}{r^2 \log^2 a} \Big|_{r=a}^{r=a^2} - \int_a^{a^2} V(r) \left(\frac{1}{r^2 \log^2 a} \right)' dr \\
 & \leq \frac{C}{\log^2 a} + C \frac{1}{\log^2 a} \int_a^{a^2} \frac{dr}{r} \\
 & = O\left(\frac{1}{\log a}\right).
 \end{aligned}$$

The proposition follows by taking $a \rightarrow \infty$. □

Now recall the following theorem in [22, 27].

Lemma 1. *Let $(M^m, e^{-f} dv)$ be a smooth metric measure space with $\text{Ric}_f \geq 0$, then along any minimizing geodesic starting from $x \in B_p(R)$ we have*

$$\frac{J_f(x, r_2, \xi)}{J_f(x, r_1, \xi)} \leq e^{4A(R)} \left(\frac{r_2}{r_1} \right)^{m-1}$$

for $0 < r_1 < r_2 < R$. In particular, for $0 < r_1 < r_2 < R$, the weighted area of the geodesic spheres satisfy

$$\frac{A_f(\partial B_x(r_2))}{A_f(\partial B_x(r_1))} \leq e^{4A(R)} \frac{r_2^{m-1}}{r_1^{m-1}}.$$

Here $A(R) = \text{Sup}_{x \in B_x(3R)} |f|(x)$ and $J_f(x, r, \xi) = e^{-f} J(x, r, \xi)$ is the e^{-f} weighted volume in geodesic polar coordinates.

If f is bounded, $\text{Vol}_f(B_x(r))$ has polynomial growth of order at most m .

Proposition 3. *Let $(M^3, e^{-f} dv)$ be a smooth metric measure space with $\text{Ric}_f \geq 0$ and f is bounded. If Σ is a complete weighted area-minimizing hypersurface which is the boundary of least weighted area in M , then Σ is totally geodesic and $\text{Ric}_f(n, n) = 0$.*

Proof. According to Lemma 1, the weighted volume of the geodesic sphere has at most quadratic growth. Since Σ is weighted area minimizing and is a boundary of least weighted area in M , $\text{vol}_f(\Sigma \upharpoonright B_x(r)) \leq A_f(\partial B_x(r)) \leq Cr^2$. Proposition 3 follows from Proposition 2. \square

6. Application to complete 3-manifolds with non-negative Bakry–Emery Ricci curvature

The classification of complete 3-manifolds with non-negative Ricci curvature has been complete by various authors' works. By using the Ricci flow, Hamilton [13, 14] classified all compact 3-manifolds with non-negative Ricci curvature. He proved that the universal cover is either diffeomorphic to \mathbb{S}^3 , $\mathbb{S}^2 \times \mathbb{R}$ or \mathbb{R}^3 . In the latter two cases, the manifold splits.

In [33], Schoen and Yau proved that a complete non-compact 3-manifold with positive Ricci curvature is diffeomorphic to the Euclidean space. Anderson-Rodriguez [3] and Shi [32] classified complete non-compact 3-manifolds with non-negative Ricci curvature by assuming an upper bound of the sectional curvature. Very recently, the author [16] classified all complete non-compact 3-manifolds with non-negative Ricci curvature.

In view of the results above, it is natural to ask what happens to a 3-manifold when the Bakry–Emery Ricci curvature is non-negative. Below is a partial classification when f is bounded.

Theorem 3. *Let $(M^3, g, e^{-f}dv)$ be a complete 3-manifold with bounded f and $\text{Ric}_f \geq 0$.*

- *If M is non-compact, then either M is contractible or through each point in M , there exists a totally geodesic surface with $\text{Ric}_f(n, n) = 0$. If in addition the rank of Ric_f is at least 2 everywhere, then the universal of M splits as a Riemann product as $\Sigma \times \mathbb{R}$. In particular, if the Bakry–Emery Ricci curvature is positive, then M is contractible.*
- *If M is compact, then either it is a quotient of \mathbb{S}^3 or the universal cover splits as a product $\Sigma \times \mathbb{R}$.*

In each splitting case, Σ is conformal to \mathbb{S}^2 or \mathbb{C} and f is constant along the \mathbb{R} factor.

Proof. First, we consider the case when M is non-compact. The argument is similar to [16, 33]. Assume M is simply connected, if $\pi_2(M) \neq 0$, according

to Lemma 2 in [33], M must have at least two ends. From Lichnerowicz’s extension of the Cheeger–Gromoll splitting theorem [17], the universal cover splits. So we assume $\pi_2(M) = 0$. Therefore, the universal cover of M is contractible. If M is not simply connected, Schoen and Yau [33] proved that $\pi_1(M)$ must have no torsion elements. Thus, after replacing M by a suitable covering, we may assume that $\pi_1(M) = \mathbb{Z}$ and that M is orientable.

Recall lemma 2.2 in [1] by Anderson:

Lemma 2 (Anderson). *Let M be a complete Riemannian manifold with finitely generated homology $H_1(M, \mathbb{Z})$. Then any non-zero line $\mathbb{R} \cdot \alpha, \alpha \in H_1(M, \mathbb{Z})$ gives rise to a complete homologically area-minimizing hypersurface Σ_α , which is the boundary of least area in a cover $\mathbb{Z} \rightarrow \bar{M} \rightarrow M$. Moreover, the volume growth of Σ_α satisfies $\text{vol}(\Sigma \upharpoonright B^{\bar{M}}(r)) \leq \text{vol}(\partial B^{\bar{M}}(r))$ and the intersection number $I(\Sigma, \alpha) \neq 0$.*

The proof of the above lemma in [1] can be carried out without any modification to weighted volume case. Taking α to be the generator of $H_1(M, \mathbb{Z})$, we can find a complete oriented boundary Σ of least weighted area in the universal cover \bar{M} . By Proposition 3, Σ is totally geodesic and $\text{Ric}_f(n, n) = 0$. If $\text{Ric}_f > 0$ on M , then this is a contradiction.

Now consider the case when $\text{Ric}_f \geq 0$. We shall use a perturbation argument in [10, 16]. For any point $p \in M$, consider a family of metric $g(t) = e^{2t\lambda}g_0$, where $\lambda = \lambda(x)$ is a function on M . Let (U, g_{ij}, x_i) be a normal coordinate for g_0 at p such that $\frac{\partial}{\partial x_i} = e_i$. We have

$$f_{ij}^t = e_j e_i(f) - (\nabla_{e_j}^t e_i)f,$$

$$\Gamma_{ij}^s(g(t)) = \frac{1}{2}g^{sl}(t) \left(\frac{\partial g_{il}(t)}{\partial x_j} + \frac{g_{jl}(t)}{\partial x_i} - \frac{\partial g_{ij}(t)}{\partial x_l} \right).$$

Then at p ,

$$\Gamma_{ij}^s(g(t)) = t(\lambda_j \delta_{is} + \lambda_i \delta_{js} - \lambda_s \delta_{ij}).$$

Therefore,

$$\begin{aligned} f_{ij}^t - f_{ij} &= -\Gamma_{ij}^s(g(t))f_s \\ (6.1) \qquad \qquad &= t(f_s \lambda_s \delta_{ij} - \lambda_i f_j - \lambda_j f_i) \\ &\geq -3t|\nabla f| |\nabla \lambda|. \end{aligned}$$

Let $m = \dim(M) = 3$. Recall that

$$\text{Ric}^t(v, v) = (\text{Ric}(v, v) - t(m - 2)\lambda_{vv} - t\Delta\lambda + t^2(m - 2)(v(\lambda)^2 - |\nabla\lambda|^2))$$

for $|\nu|_{g_0} = 1$. Let $r(x) = \text{dist}(x, p)$ on M . For a very small $R > 0$, consider the function $\rho = R - r$ for $\frac{R}{2} < r < R$. Then we extend ρ to be a positive smooth function for $0 \leq r < \frac{R}{2}$. Define $\lambda = -\rho^5$.

Now

$$\nabla^2(\rho^5)(v, v) = 20\rho^3v(\rho)^2 + 5\rho^4\nabla^2(\rho)(v, v).$$

For $aR < r < R$, we have

(6.2)

$$\begin{aligned} \text{Ric}^t(v, v) + f_{vv}^t &\geq \text{Ric}^0(v, v) + f_{vv}^0 + 20t\rho^3 + 5t\rho^4(\Delta\rho \\ &\quad + (m - 2)\nabla^2(\rho)(v, v)) - 25(m - 2)t^2\rho^8 - 15t\rho^4|\nabla f|. \end{aligned}$$

Using the fact that the manifold is almost Euclidean near p , for small R , we have

$$|\Delta\rho + (m - 2)\nabla^2\rho(v, v)| \leq \frac{9(2m - 3)}{8(R - \rho)}.$$

Therefore, there exists small $R > 0$ such that for all small t , $\text{Ric}_f^t(v, v) > 0$ in an annulus $B_p(R) \setminus B_p(aR)$ for $a = \frac{7}{8}$. The metric remains the same outside $B_p(R)$. The deformation is C^4 continuous with respect to the metric and C^∞ with respect to t .

Let γ be a closed curve in M which represents the generator of $\pi_1(M)$. We can apply the perturbation finitely many times such that $\text{Ric}_f > 0$ on γ and Ric_f is non-negative on M except a small neighborhood U of p . Then for the perturbed metric g_t , we can apply Lemma 2 to obtain a complete oriented boundary Σ of least weighted area in the universal cover \tilde{M} . Since g_t is uniformly equivalent to g_0 , we can show Σ_t has quadratic weighted volume growth. Let $q \in \Sigma_t$, then for any $r > 0$,

$$\begin{aligned} \text{vol}_{g(t)}(\Sigma_t \upharpoonright B_{g(t)}(q, \tilde{M})(r)) &\leq \text{vol}_{g(t)}(\Sigma_t \upharpoonright B_{g(0)}(q, \tilde{M})(Cr)) \\ (6.3) \qquad \qquad \qquad &\leq \text{vol}_{g(t)}(\partial B_{g(0)}(q, \tilde{M})(Cr)) \\ &\leq C\text{vol}_{g(0)}(\partial B_{g(0)}(q, \tilde{M})(Cr)) \\ &\leq C_1r^2. \end{aligned}$$

If Σ_t does not intersect the preimage of U in \tilde{M} , then on Σ_t , $\text{Ric}_f \geq 0$ and $\text{Ric}_f > 0$ at $\Sigma_t \cap \gamma$. This contradicts Proposition 2.

For each Σ_t , we can find deck transformation l_t on \tilde{M} such that $l_t(\Sigma_t)$ intersects the preimage of U at some fixed compact set in \tilde{M} . Therefore, if we shrink the size of the neighborhood of p and let $t \rightarrow 0$ sufficiently fast,

a subsequence of Σ_t will converge to a weighted area minimizing surface Σ satisfying

$$\text{vol}_{g(0)}(\Sigma \upharpoonright B_{g(0)}(q, \tilde{M})(r)) \leq Cr^2.$$

Thus, by Proposition 2, Σ is totally geodesic and $\text{Ric}_f(n, n) = 0$. Since p is arbitrary, though each point there exists a totally geodesic surface with $\text{Ric}_f(n, n) = 0$.

Now we use the assumption that the rank of Ric_f is at least 2 everywhere. Then through each point $p \in \tilde{M}$, there exists a unique totally geodesic surface. Therefore we have a foliation on \tilde{M} . We can parameterize the surfaces as Σ_t .

Let N be the unit normal vector and λN be the variational vector field of Σ_t . Since the smooth family of surfaces Σ_t never intersect with each other, λ is non-negative. A simple computation shows that the variational vector field of these totally geodesic surfaces satisfies

$$\Delta\lambda + \lambda\text{Ric}(n, n) = 0.$$

Since

$$H = f_n = 0,$$

$$\begin{aligned} 0 &= \frac{df_n}{dt} \\ &= \lambda f_{nn} + \langle \nabla f, \nabla_{\lambda N} N \rangle \\ (6.4) \quad &= \lambda f_{nn} + \sum_{i=1}^2 \langle \nabla f, e_i \rangle \langle e_i, \nabla_{\lambda N} N \rangle \\ &= \lambda f_{nn} - \langle \nabla f, \nabla \lambda \rangle. \end{aligned}$$

In the above computation, e_i is an orthonormal frame on an open set of Σ .

But

$$0 = \text{Ric}_f(n, n) = \text{Ric}(n, n) + f_{nn},$$

thus we have

$$\Delta_f \lambda = \Delta\lambda - \langle \nabla \lambda, \nabla f \rangle = 0$$

on Σ .

The lemma below is close to corollary 1 in [7].

Lemma 3. *For a smooth metric measured space $(M, g, e^{-f} dv)$ with quadratic weighted volume growth, if λ is a positive function which satisfies $\Delta_f \lambda = 0$, then λ is a constant.*

Proof. Let $\lambda = e^h$, then

$$\Delta h + |\nabla h|^2 - \langle \nabla h, \nabla f \rangle = 0.$$

Let φ be a cut-off function, we find

$$\int \varphi^2 \Delta h e^{-f} + \int \varphi^2 |\nabla h|^2 e^{-f} - \int \varphi^2 \langle \nabla h, \nabla f \rangle e^{-f} = 0.$$

By integration by parts,

$$\int \varphi^2 (\Delta h) e^{-f} = - \int h_i 2\varphi \varphi_i e^{-f} + \int h_i \varphi^2 f_i e^{-f}.$$

Therefore,

$$\int \varphi^2 |\nabla h|^2 e^{-f} = 2 \int \varphi_i h_i \varphi e^{-f} \leq 2 \left(\int \varphi^2 |\nabla h|^2 e^{-f} \right)^{\frac{1}{2}} \left(\int |\nabla \varphi|^2 e^{-f} \right)^{\frac{1}{2}}.$$

Thus,

$$\int \varphi^2 |\nabla h|^2 e^{-f} \leq 4 \int |\nabla \varphi|^2 e^{-f}.$$

Now we can use the same cut-off function in Proposition 2 to show that $\nabla h \equiv 0$. Thus λ is a constant. □

Since λ is non-negative, by Lemma 3, λ is constant. After a reparametrization of Σ_t , we may assume $\lambda = 1$. Now for $X \in T\Sigma_t$, $\nabla_X N = 0$, since Σ_t is totally geodesic. Since λ is a constant, we may assume $[X, N] = 0$. $\langle \nabla_N N, X \rangle = -\langle N, \nabla_N X \rangle = -\langle N, \nabla_X N \rangle = 0$. Thus, $\nabla N \equiv 0$. Therefore, M is locally isometric to $\Sigma \times \mathbb{R}$. f is constant along the \mathbb{R} factor, since $f_n = 0$.

Now consider the case when M is compact. If the universal cover is compact, then according to Perelman’s solution to the Poincare conjecture, M is covered by \mathbb{S}^3 . If the universal cover \tilde{M} is non-compact, then according to Theorem 6.6 in [27], \tilde{M} splits as a product $\Sigma \times \mathbb{R}$.

Finally, we show that in the splitting case, Σ is conformal to \mathbb{C} or \mathbb{S}^2 . There are two methods to do this. Note that on Σ ,

$$\text{Ric}_\Sigma + \nabla^2 f \geq 0.$$

Consider the conformal change of the metric $\tilde{g} = e^{-f}g$ on Σ , then the tensor

$$\text{Ric}_\Sigma(\tilde{g}) = \text{Ric}_\Sigma(g) + \frac{1}{2}(\Delta_\Sigma f)g \geq 0.$$

As f is bounded, \tilde{g} is complete. Since Σ is simply connected, Σ is conformal to \mathbb{C} or \mathbb{S}^2 .

The second way is this: by Lemma 1, the weighted volume growth of Σ is at most quadratic. Since f is bounded, the volume growth of Σ is at most quadratic. If Σ is conformal to the Poincaré disk, then there exists a non-trivial bounded harmonic function on Σ . But according to Corollary 1 in [7], the function is a constant. This is a contradiction. \square

Remark 1. *The bounded condition of f cannot be dropped in the above theorem. For example, consider the warped product metric $ds^2 = dt^2 + g(t)ds_{\mathbb{S}^2}^2$ on $M = \mathbb{S}^2 \times \mathbb{R}$. Here $ds_{\mathbb{S}^2}^2$ is the standard metric on \mathbb{S}^2 with curvature 1. Consider an orthogonal frame e_1, e_2, e_3 on M such that $ds_{\mathbb{S}^2}^2(e_1, e_1) = ds_{\mathbb{S}^2}^2(e_2, e_2) = 1$ and $\frac{\partial}{\partial t} = e_3$.*

If we take f as a function of t on M , then by similar computations in Section 4, we see

$$\operatorname{Ric}_f(e_1, e_1) = 1 - \frac{g''}{2} + \frac{f'g'}{2}, \quad \operatorname{Ric}_f(e_3, e_3) = \frac{-2g''g + g'^2 + 2g^2f''}{2g^2}.$$

If $f(t) = t^2$, $g(t) = e^t$, then one can check that $\operatorname{Ric}_f > 0$, however, M is not a Riemann product or a contractible manifold.

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