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Mean curvature flow of higher codimension in hyperbolic spaces

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In this paper, we investigate the convergence for the mean curvature flow of closed submanifolds with arbitrary codimension in space forms. Particularly, we prove that the mean curvature flow deforms a closed submanifold satisfying a pinching condition in a hyperbolic space form to a round point in finite time.

1. Introduction

In this paper, we study the convergence for the mean curvature flow of submanifolds in space forms. Let $F : M^n \to \mathbb{F}^{n+d}(c)$ be a smooth immersion from an n-dimensional closed Riemannian manifold M^n to an $(n + d)$-dimensional complete simply connected space form $\mathbb{F}^{n+d}(c)$ with constant sectional curvature c. Consider a one-parameter family of smooth immersions $F : M \times [0, T) \to \mathbb{F}^{n+d}(c)$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t}F(x,t) = H(x,t), \\ F(x,0) = \ F(x), \end{cases} \tag{1.1}$$

where $H(x, t)$ is the mean curvature vector of $M_t, M_t = F_t(M)$ and $F_t(x) = F(x, t)$. We call $F : M \times [0, T) \to \mathbb{F}^{n+d}(c)$ the mean curvature flow with initial value $F(\cdot)$.

The mean curvature flow was proposed by Mullins [18] to describe the formation of grain boundaries in annealing metals. In [3], Brakke introduced the motion of a submanifold by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. For the classical solution of the mean curvature flow, most works have been done on hypersurfaces. Huisken [11, 12] showed that if the initial hypersurface in a Riemannian manifold is uniformly convex, then the mean curvature flow converges to a round point in finite time. Later, Huisken [13] extended this result to hypersurfaces satisfying a pinching condition in a sphere. Many other beautiful results have been obtained, and there are various approaches to study the mean curvature flow of hypersurfaces (see [6, 7], etc.). For the mean curvature flow of submanifolds in higher codimension, some special cases have been studied by Wang, Smoczyk and others; see [20–26] etc. for example. Recently, Andrews–Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying a pinching condition in the Euclidean space. Using Ricci flow, Gu and Xu [10, 27] proved the same differentiable sphere theorem as in [1] independently. In fact, they [10, 27] obtained a general differentiable sphere theorem for submanifolds in a Riemannian manifold. In [2], Baker proved a convergence result for the mean curvature flow of submanifolds in a sphere. In this paper, we study the mean curvature flow of closed submanifolds in hyperbolic spaces and extend the convergence results in [1, 2] to the mean curvature flow of arbitrary codimension in space forms.

Theorem 1.1. *Let $F : M^n \to \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold in a complete simply connected hyperbolic space form with constant curvature $c < 0$. Assume F satisfies*

$$|A|^2 \leq \begin{cases} \dfrac{4}{3n}|H|^2 + \dfrac{n}{2}c, & n = 2, 3, \\ \dfrac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases} \tag{1.2}$$

Then the mean curvature flow with initial value F converges to a round point in finite time.

As an immediate consequence of Theorem 1.1, we obtain the following differentiable sphere theorem.

Corollary 1.2. *Let $F : M^n \to \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold in a complete simply connected hyperbolic space form with constant curvature $c < 0$. Assume F satisfies*

$$|A|^2 \leq \begin{cases} \dfrac{4}{3n}|H|^2 + \dfrac{n}{2}c, & n = 2, 3, \\ \dfrac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases}$$

Then M is diffeomorphic to the unit n-sphere.

Remark 1.3. This differentiable sphere theorem was also obtained by Gu and Xu [10, 27] provided the submanifold is simply connected. For more topics in sphere theorems and mean curvature flow of submanifolds, we refer the readers to [1, 2, 9, 10, 16, 19, 27–30], etc.

Combining Theorem 1.1 and the convergence results in [1, 2], we obtain the following theorem.

Theorem 1.4. *Let $F : M^n \to \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold in a complete simply connected space form with $|H|^2 + n^2c > 0$. Assume F satisfies*

$$(1.3) \qquad |A|^2 \leq \begin{cases} \dfrac{4}{3n}|H|^2 + \dfrac{1}{12}[7n - 4 + \mathrm{sgn}(c)(n-4)]c, & n = 2, 3, \\ \dfrac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases}$$

Then either $F_t(M)$ converges to a round point in finite time, or $c > 0$ and $F_t(M)$ converges to a total geodesic sphere in $\mathbb{F}^{n+d}(c)$ as $t \to \infty$.

Remark 1.5. For $c > 0$, $|H|^2 + n^2c > 0$ is automatically satisfied. For $c = 0$, $|H|^2 + n^2c > 0$ is equivalent to that the mean curvature is nowhere vanishing. For $c < 0$, $|H|^2 + n^2c > 0$ is implied by condition (1.3).

Remark 1.6. For $c > 0$, the maximal existence time of the mean curvature flow may be finite or infinite. For $c \leq 0$, the mean curvature flow with a closed initial submanifold always has finite maximal existence time.

Most recently, Liu et al. [17] obtained a general convergence theorem for the mean curvature flow of higher codimension in Riemannian manifolds.

2. Basic equations

Let $F : M \times [0, T) \to \mathbb{F}^{n+d}(c)$ be a smooth mean curvature flow with initial closed immersion $F_0 : M \to \mathbb{F}^{n+d}(c)$. Denote by $g(t)$ and $d\mu_t$ the induced metric and the volume form on M. Let A and H be the second fundamental form and the mean curvature vector of M in $\mathbb{F}^{n+d}(c)$, respectively. We shall make use of the following convention on the range of indices.

$$1 \leq i, j, k, \ldots \leq n, \;\; 1 \leq A, B, C, \ldots \leq n + d \;\; \textit{and} \;\; n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n+d.$$

As in [1, 2], we consider the evolution on the spatial tangent bundle. Choose a local orthonormal frame $\{e_i\}$ for the spatial tangent bundle and a local orthonormal frame $\{\nu_\alpha\}$ for the normal bundle. Let $\{\omega_i\}$ be the dual frame of $\{e_i\}$. Then A and H can be written as

$$A = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j, \quad H = \sum_{\alpha} H_\alpha \nu_\alpha,$$

where $h_{ij} = \sum_\alpha h_{ij\alpha}\nu_\alpha$ and $H_\alpha = \sum_i h_{ii\alpha}$. We have the following evolution equations.

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2R_1 + 4c|H|^2 - 2nc|A|^2, \tag{2.1}$$

$$\frac{\partial}{\partial t}|H|^2 = \Delta|H|^2 - 2|\nabla H|^2 + 2R_2 + 2nc|H|^2, \tag{2.2}$$

where

$$R_1 = \sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij\alpha}h_{ij\beta}\Big)^2 + |R^\perp|^2, \tag{2.3}$$

$$|R^\perp|^2 = \sum_{i,j,\alpha,\beta}\Big(\sum_p \Big(h_{ip\alpha}h_{jp\beta} - h_{jp\alpha}h_{ip\beta}\Big)\Big)^2, \tag{2.4}$$

$$R_2 = \sum_{i,j}\Big(\sum_\alpha H_\alpha h_{ij\alpha}\Big)^2. \tag{2.5}$$

The contracted form of Simons' identity for tracefree second fundamental form $\mathring{A} = \sum_{i,j} \mathring{h}_{ij}\omega_i \otimes \omega_j := A - \frac{1}{n} g \otimes H$ is

$$\frac{1}{2}\Delta|\mathring{A}|^2 = \sum_{i,j} \mathring{h}_{ij}\nabla_i\nabla_j H + |\nabla \mathring{A}|^2 + Z + nc|\mathring{A}|^2. \tag{2.6}$$

Here

$$Z = -R_1 + \sum_{i,j,p,\alpha,\beta} H_\alpha h_{ip\alpha}h_{ij\beta}h_{pj\beta}. \tag{2.7}$$

We also have the following inequality.

$$|\nabla A|^2 \geq \frac{3}{n+2}|\nabla H|^2. \tag{2.8}$$

3. Preserved curvature pinching condition

Now we prove that the pinching condition (1.2) is preserved under the mean curvature flow with arbitrary codimension in $\mathbb{F}^{n+d}(c)$.

Lemma 3.1. *For $c < 0$ and $n \geq 2$, if the initial immersion satisfies (1.2), then this condition is preserved along the mean curvature flow.*

Proof. We consider $Q = |A|^2 - \alpha|H|^2 - \beta c$, where the constants

$$\alpha \leq \begin{cases} \dfrac{4}{3n}, & n = 2, 3, \\ \dfrac{1}{n-1}, & n \geq 4, \end{cases} \quad \text{and } \beta \geq \begin{cases} \dfrac{n}{2}, & n = 2, 3, \\ 2, & n \geq 4. \end{cases}$$

By (2.1) and (2.2) we have

$$\begin{aligned} \frac{\partial}{\partial t}Q &= \Delta Q - 2(|\nabla A|^2 - \alpha|\nabla H|^2) \\ &\quad + 2R_1 - 2\alpha R_2 - 2nc|\mathring{A}|^2 - 2n\left(\alpha - \frac{1}{n}\right)c|H|^2. \end{aligned} \tag{3.1}$$

It is sufficient to show that if $Q = 0$ at a point $x \in M$, then

$$2R_1 - 2\alpha R_2 - 2nc|\mathring{A}|^2 - 2n\left(\alpha - \frac{1}{n}\right)c|H|^2 < 0$$

holds at x. We also have $H \neq 0$ at x. Choose $\{\nu_\alpha\}$ such that $\nu_{n+1} = \frac{H}{|H|}$. Let $A_H = \sum_{i,j} h_{ij,n+1}\omega_i \otimes \omega_j$. Set $\mathring{A}_H = A_H - \frac{|H|}{n}g$ and $|\mathring{A}_I|^2 = |\mathring{A}|^2 - |\mathring{A}_H|^2$.

We replace $|H|^2$ with $\frac{|\mathring{A}|^2 - \beta c}{\alpha - \frac{1}{n}}$. Then

(3.2)

$$\begin{aligned} &2R_1 - 2\alpha R_2 - 2nc|\mathring{A}|^2 - 2n\left(\alpha - \frac{1}{n}\right)c|H|^2 \\ &\quad \leq 2|\mathring{A}_H|^2 - 2\left(\alpha - \frac{1}{n}\right)|\mathring{A}_H|^2|H|^2 + \frac{2}{n}|\mathring{A}_H|^2|H|^2 - \frac{2}{n}\left(\alpha - \frac{1}{n}\right)|H|^4 \\ &\qquad + 8|\mathring{A}_H|^2|\mathring{A}_I|^2 + 3|\mathring{A}_I|^2 - 2nc(|\mathring{A}_H|^2 + |\mathring{A}_I|^2) - 2n\left(\alpha - \frac{1}{n}\right)c|H|^2 \\ &\quad = \left(6 - \frac{2}{n(\alpha - \frac{1}{n})}\right)|\mathring{A}_H|^2|\mathring{A}_I|^2 + \left(3 - \frac{2}{n(\alpha - \frac{1}{n})}\right)|\mathring{A}_I|^4 \\ &\qquad + \left(2\beta - 4n + \frac{2\beta}{n(\alpha - \frac{1}{n})}\right)c|\mathring{A}_H|^2 + 4\left(\frac{\beta}{n(\alpha - \frac{1}{n})} - n\right)c|\mathring{A}_I|^2 \\ &\qquad - 2\beta\left(\frac{\beta}{n(\alpha - \frac{1}{n})} - n\right)c^2. \end{aligned}$$

By the definitions of α and β, we know that the right-hand side of (3.2) is negative for $n \geq 2$. This completes the proof of the lemma. □

For $\theta \geq 0$, set

$$\alpha_\theta = \begin{cases} \dfrac{4}{3n+n\theta}, & n=2,3, \\ \dfrac{1}{n-1+\theta}, & n \geq 4, \end{cases} \quad \text{and } \beta_\theta = \begin{cases} \dfrac{n}{2}(1+\theta), & n=2,3, \\ 2(1+\theta), & n \geq 4. \end{cases}$$

If the initial immersion satisfies $|A|^2 < \alpha_0|H^2| + \beta_0 c$, then there exists an $\epsilon > 0$ such that $|A|^2 \leq \alpha_\epsilon |H|^2 + \beta_\epsilon c$ holds on M_0. From the proof of Lemma 3.1, this inequality also holds for $t > 0$. If $|A|^2 \leq \alpha_0|H^2| + \beta_0 c$ and the equality holds somewhere on M_0, then by the maximum principle, we see that either the equality holds everywhere on M_0, or the strict inequality holds everywhere for $t > 0$. For the first case, we have $\nabla A = 0$ and $\mathring{A}_I = 0$ on M_0. By [8], M_0 lies in an $(n+1)$-dimensional totally geodesic submanifold of $\mathbb{F}^{n+d}(c)$. Since $\nabla A = 0$, from Theorem 4 of [15], M_0 is either isometric to an Euclidean space, or isometric to a product $\mathbb{F}^k(c_1) \times \mathbb{F}^{n-k}(c_2)$ for some $c_1 > 0$, $c_2 < 0$ and $k = 0, \ldots, n$. Since M_0 is closed, we see that M_0 is a totally umbilical sphere. Then $|A|^2 \leq \alpha_\epsilon|H|^2 + \beta_\epsilon c$ holds on M_0 for some $\epsilon > 0$. For the second case, we see that after a short time, we also have $|A|^2 \leq \alpha_\epsilon|H|^2 + \beta_\epsilon c$ for some $\epsilon > 0$. Hence, we may assume that $|A|^2 \leq \alpha_\epsilon|H|^2 + \beta_\epsilon c$ for some $\epsilon \in (0,1)$ and $t \geq t_0 > 0$.

4. Pinching of $\mathring{A}$ along the mean curvature flow

Assume that $c < 0$. We prove a pinching estimate for the tracefree second fundamental form, which guarantees that M_t becomes spherical along the mean curvature flow.

Theorem 4.1. *There are positive constants C_0 and σ_0 independent of t such that*

$$|\mathring{A}|^2 \leq C_0|H|^{2-\sigma_0} \tag{4.1}$$

holds along the mean curvature flow.

Proof. We consider the function $f_\sigma = \frac{|\mathring{A}|^2}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}$, where $\sigma \in (0,1)$ and

$$a = \begin{cases} \dfrac{1}{3n+n\epsilon}, & n=2,3, \\ \dfrac{1}{n(n-1+\epsilon)}, & n \geq 4. \end{cases}$$

Note that

$$
\begin{aligned}
& a|H|^2+\beta_\epsilon c-\Big(\Big(\alpha_\epsilon-\frac{1}{n}\Big)|H|^2+\beta_\epsilon c\Big) \\
\geq\ & \frac{\epsilon}{3n+n\epsilon}|H|^2 \\
\geq\ & \frac{\epsilon}{3n+n\epsilon}\cdot\beta(-c) \\
>\ & 0
\end{aligned} \tag{4.2}
$$

for $n=2,3$, and

$$
\begin{aligned}
& a|H|^2+\beta_\epsilon c-\Big(\Big(\alpha_\epsilon-\frac{1}{n}\Big)|H|^2+\beta_\epsilon c\Big) \\
\geq\ & \frac{\epsilon}{n(n-1+\epsilon)}|H|^2 \\
\geq\ & \frac{\epsilon}{n(n-1+\epsilon)}\cdot\beta(-c) \\
>\ & 0
\end{aligned} \tag{4.3}
$$

for $n\geq 4$. So f_σ is well defined. From (4.2) and (4.3) we also have

$$
a|H|^2+\beta_\epsilon c\geq b|H|^2, \tag{4.4}
$$

where

$$
b=\begin{cases}\dfrac{\epsilon}{3n+n\epsilon}, & n=2,3,\\[2mm] \dfrac{\epsilon}{n(n-1+\epsilon)}, & n\geq 4.\end{cases}
$$

By a similar computation as in [1], we have

(4.5)

$$
\begin{aligned}
\frac{\partial}{\partial t}f_\sigma=\ & \Delta f_\sigma+\frac{2a(1-\sigma)}{a|H|^2+\beta_\epsilon c}\langle\nabla|H|^2,\nabla f_\sigma\rangle \\
& -\frac{2}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}\Big(|\nabla A|^2-\frac{1}{n}|\nabla H|^2-\frac{a|\mathring{A}|^2}{a|H|^2+\beta_\epsilon c}|\nabla H|^2\Big) \\
& -\frac{4a^2\sigma(1-\sigma)}{(a|H|^2+\beta_\epsilon c)^2}f_\sigma|H|^2\cdot\Big|\nabla|H|\Big|^2-\frac{2a\sigma f_\sigma}{a|H|^2+\beta_\epsilon c}|\nabla H|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}\bigg(R_1-\frac{1}{n}R_2-\frac{aR_2|\mathring{A}|^2}{a|H|^2+\beta_\epsilon c}-nc|\mathring{A}|^2\\
&-\frac{an(1-\sigma)c|\mathring{A}|^2|H|^2}{a|H|^2+\beta_\epsilon c}\bigg)\\
&+\frac{2a\sigma R_2 f_\sigma}{a|H|^2+\beta_\epsilon c}.
\end{aligned}
$$

By (2.8), we have

$$
\begin{aligned}
&|\nabla A|^2-\frac{1}{n}|\nabla H|^2-\frac{a|\mathring{A}|^2}{a|H|^2+\beta_\epsilon c}|\nabla H|^2\\
&\quad\geq\left(\frac{3}{n+2}-\frac{1}{n}-\frac{a\Big((\alpha_\epsilon-\frac{1}{n})H^2+\beta_\epsilon c\Big)}{a|H|^2+\beta_\epsilon c}\right)|\nabla H|^2\\
&\quad\geq\left(\frac{3}{n+2}-\frac{1}{n}-a\right)|\nabla H|^2\\
&\quad:=\epsilon_\nabla|\nabla H|^2.
\end{aligned}
\tag{4.6}
$$

Here ϵ_∇ is a positive constant for $n\geq 2$.

We also have the following estimate.

$$
\begin{aligned}
&R_1-\frac{1}{n}R_2-\frac{aR_2|\mathring{A}|^2}{a|H|^2+\beta_\epsilon c}\\
&\quad\leq R_1-\frac{1}{n}R_2-\frac{R_2|\mathring{A}|^2}{|H|^2}\\
&\quad= R_1-\frac{R_2|A|^2}{|H|^2}\\
&\quad\leq|\mathring{A}_H|^4-2\left(\frac{|A|^2}{|H|^2}-\frac{2}{n}\right)|\mathring{A}_H|^2|H|^2-\frac{2}{n}\left(\frac{|A|^2}{|H|^2}-\frac{1}{n}\right)|H|^4\\
&\quad-4|\mathring{A}_H|^2|\mathring{A}_I|^2-\frac{3}{2}|\mathring{A}_I|^4\\
&\quad\leq 0.
\end{aligned}
\tag{4.7}
$$

In (4.7) we have used the pinching condition $|A|^2\leq\alpha_\epsilon|H|^2+\beta_\epsilon c<\alpha_\epsilon|H|^2$ for $\epsilon\in(0,1)$.

By (4.4), we have

$$
\begin{aligned}
&-nc|\mathring{A}|^2 - \frac{an(1-\sigma)c|\mathring{A}|^2|H|^2}{a|H|^2+\beta_\epsilon c} \\
&\leq -nc|\mathring{A}|^2 - \frac{an(1-\sigma)c|\mathring{A}|^2(a|H|^2+\beta_\epsilon c)}{b(a|H|^2+\beta_\epsilon c)} \\
&\leq -nc|\mathring{A}|^2 - \frac{anc}{b}|\mathring{A}|^2 \\
&:= \bar{b}|\mathring{A}|^2.
\end{aligned} \tag{4.8}
$$

For the last term of right-hand side of (4.5), we have by (4.4)

$$
\frac{2a\sigma R_2 f_\sigma}{a|H|^2+\beta_\epsilon c} \leq \frac{2a\sigma|H|^2|A|^2 f_\sigma}{a|H|^2+\beta_\epsilon c} \leq \frac{2a\sigma}{b}|A|^2 f_\sigma := \tilde{b}\sigma|A|^2 f_\sigma. \tag{4.9}
$$

Combining (4.5), (4.6), (4.7), (4.8) and (4.9), we have

$$
\begin{aligned}
\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma &+ \frac{2a(1-\sigma)}{a|H|^2+\beta_\epsilon c}\langle \nabla|H|^2, \nabla f_\sigma\rangle - \frac{2\epsilon_\nabla}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}|\nabla H|^2 \\
&+ 2\bar{b} f_\sigma + \tilde{b}\sigma|A|^2 f_\sigma.
\end{aligned} \tag{4.10}
$$

To deal with the last term of the right-hand side of (4.10), we need the following estimate.

Proposition 4.2. *There exists a positive constant ε independent of t such that*

$$
Z + nc|\mathring{A}|^2 \geq \varepsilon|\mathring{A}|^2(a|H|^2+\beta_\epsilon c) \tag{4.11}
$$

holds for $t \geq t_0$.

Proof. By the argument in the proof of Lemma 5.4 in [2], we only need to show

$$
-\frac{\beta_\epsilon}{\alpha_\epsilon - \frac{1}{n}}\left(\frac{1}{n} - \frac{n-2}{2n(n-1)}\right) + n \leq 0.
$$

This is true by our choices of α_ϵ and β_ϵ. □

Proposition 4.3. *For any $\eta > 0$, $p \geq 2$ and $t \geq t_0$, we have*

$$\begin{aligned}\int_{M_t} f_\sigma^p(a|H|^2 + \beta_\epsilon c)d\mu_t \leq \frac{2p\eta + 5}{b\varepsilon} \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\ + \frac{2(p-1)}{b\eta\varepsilon} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t.\end{aligned} \tag{4.12}$$

Proof. Using a similar argument as in [1, 2], we have the following inequality.

$$\begin{aligned}&2\int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} Z d\mu_t + 2nc \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\mathring{A}|^2 d\mu_t \\ &\leq 2(p-1) \int_{M_t} \frac{f_\sigma^{p-2}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla f_\sigma||\mathring{A}||\nabla H| d\mu_t \\ &\quad + \frac{2(n-1)}{n} \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\ &\quad + 4 \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{2-\sigma}} |H||\mathring{A}||\nabla H|^2 d\mu_t \\ &\quad + 4(1-\sigma)(p-2) \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)} |H||\nabla H||\nabla f_\sigma| d\mu_t \\ &\quad + 4 \int_{M_t} \frac{f_\sigma^{p}}{(a|H|^2 + \beta_\epsilon c)^2} |H|^2 |\nabla H|^2 d\mu_t.\end{aligned} \tag{4.13}$$

Since $|\mathring{A}|^2 \leq f_\sigma(a|H|^2 + \beta_\epsilon c)^{1-\sigma}$ and $f_\sigma \leq (a|H|^2 + \beta_\epsilon c)^\sigma$, by choosing $\sigma \in (0,1)$ we have the following estimates.

$$\begin{aligned}&2(p-1) \int_{M_t} \frac{f_\sigma^{p-2}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla f_\sigma||\mathring{A}||\nabla H| d\mu_t \\ &\leq \frac{1-\sigma}{\eta} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t + (p-1)\eta \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t,\end{aligned} \tag{4.14}$$

$$\begin{aligned}4\int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{2-\sigma}} |H||\mathring{A}||\nabla H|^2 d\mu_t \\ \leq \frac{4}{b} \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t,\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&4(p-2)\int_{M_t}\frac{f_\sigma^{p-1}}{a|H|^2+\beta_\epsilon c}|H||\nabla H||\nabla f_\sigma|d\mu_t\\
&\quad\le(p-2)\int_{M_t}\frac{1}{a|H|^2+\beta_\epsilon c}\left(\frac{2}{\eta}f_\sigma^{p-2}|H|^2|\nabla f_\sigma|^2+2\eta f_\sigma^p|\nabla H|^2\right)d\mu_t\\
&\quad\le\frac{2(p-2)}{b\eta}\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2d\mu_t\\
&\qquad+2(p-2)\eta\int_{M_t}\frac{f_\sigma^{p-1}}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}|\nabla H|^2d\mu_t,
\end{aligned}\tag{4.16}$$

$$4\int_{M_t}\frac{f_\sigma^p}{(a|H|^2+\beta_\epsilon c)^2}|H|^2|\nabla H|^2d\mu_t\le\frac{4}{b}\int_{M_t}\frac{f_\sigma^{p-1}}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}|\nabla H|^2d\mu_t.\tag{4.17}$$

In (4.15), (4.16) and (4.17), we have used (4.4).

By (4.11), we have

$$\begin{aligned}
&2\int_{M_t}\frac{f_\sigma^{p-1}}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}(Z+2nc|\mathring{A}|^2)d\mu_t\\
&\quad\ge2\varepsilon\int_{M_t}f_\sigma^p(a|H|^2+\beta_\epsilon c)d\mu_t.
\end{aligned}\tag{4.18}$$

Combining (4.13) to (4.18), we obtain

$$\begin{aligned}
2\varepsilon\int_{M_t}f_\sigma^p(a|H|^2+\beta_\epsilon c)d\mu_t\le{}&\frac{3p\eta+10}{b}\int_{M_t}\frac{f_\sigma^{p-1}}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}|\nabla H|^2d\mu_t\\
&+\frac{3(p-1)}{b\eta}\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2d\mu_t.
\end{aligned}\tag{4.19}$$

Dividing through by 2ε completes the proof. □

Now we show that the L^p-norm of f_σ is bounded for sufficiently high p.

Lemma 4.4. *For any $p\ge\max\{2,\frac{8}{b\epsilon_\nabla}+1\}$ and $\sigma\le\min\left\{\frac{b^2\varepsilon\epsilon_\nabla}{10\tilde{b}\alpha_\epsilon},\frac{b^2\varepsilon\sqrt{\epsilon_\nabla}}{4\tilde{b}\alpha_\epsilon\sqrt{p}},\frac{1}{2}\right\}$, there exists a constant C independent of t such that for all $t\in[0,T_{\max})$, where $T_{\max}<\infty$, we have*

$$\left(\int_{M_t}f_\sigma^pd\mu_t\right)^{\frac{1}{p}}\le C.\tag{4.20}$$

Proof. For $t \geq t_0$, from (4.10), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p d\mu_t &\leq \int_{M_t} p f_\sigma^{p-1}\frac{\partial}{\partial t} f_\sigma d\mu_t \\
&\leq -p(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu_t \\
&\quad + 4(1-\sigma)p\int_{M_t}\frac{f_\sigma^{p-1}}{a|H|^2+\beta_\epsilon c}|H||\nabla|H|||\nabla f_\sigma| d\mu_t \\
&\quad - 2p\epsilon_\nabla \int_{M_t}\frac{f_\sigma^{p-1}}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}|\nabla H|^2 d\mu_t \\
&\quad + 2\bar{b}p\int_{M_t} f_\sigma^p d\mu_t + \tilde{b}\sigma p\int_{M_t}|A|^2 f_\sigma^p d\mu_t.
\end{aligned} \tag{4.21}
$$

As in (4.16), we have

$$
\begin{aligned}
&4(1-\sigma)p\int_{M_t}\frac{f_\sigma^{p-1}}{a|H|^2+\beta_\epsilon c}|H||\nabla|H|||\nabla f_\sigma| d\mu_t \\
&\quad\leq \frac{2p}{b\mu}\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu_t + 2p\mu\int_{M_t}\frac{f_\sigma^{p-1}}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}|\nabla H|^2 d\mu_t.
\end{aligned} \tag{4.22}
$$

Substituting (4.22) into (4.21), letting $\mu = \frac{4}{b(p-1)}$ and $p \geq \max\{2, \frac{8}{b\epsilon_\nabla}+1\}$, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p d\mu_t \leq& -\frac{p(p-1)}{2}\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu_t \\
&- p\epsilon_\nabla\int_{M_t}\frac{f_\sigma^{p-1}}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}|\nabla H|^2 d\mu_t \\
&+ 2\bar{b}p\int_{M_t} f_\sigma^p d\mu_t + \frac{\tilde{b}\sigma\alpha_\epsilon p}{b}\int_{M_t} f_\sigma^p(a|H|^2+\beta_\epsilon c)d\mu_t.
\end{aligned}
$$

This together with (4.12) implies

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p d\mu_t \leq& -p(p-1)\left(\frac{1}{2}-\frac{2\tilde{b}\sigma\alpha_\epsilon}{b^2\eta\varepsilon}\right)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu_t \\
&-\left(p\epsilon_\nabla - \frac{(2p\eta+5)\tilde{b}\sigma\alpha_\epsilon p}{b^2\varepsilon}\right)\int_{M_t}\frac{f_\sigma^{p-1}}{(a|H|^2+\beta_\epsilon c)^{1-\sigma}}|\nabla H|^2 d\mu_t \\
&+ 2\bar{b}p\int_{M_t} f_\sigma^p d\mu_t.
\end{aligned} \tag{4.23}
$$

Now we pick $\eta = \frac{4\tilde{b}\alpha_\epsilon\sigma}{b^2\varepsilon}$ and let $\sigma \le \min\left\{\frac{b^2\varepsilon\epsilon_\nabla}{10\tilde{b}\alpha_\epsilon}, \frac{b^2\varepsilon\sqrt{\epsilon_\nabla}}{4\tilde{b}\alpha_\epsilon\sqrt{p}}, \frac{1}{2}\right\}$. Then (4.23) reduces to

$$\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p d\mu_t \le 2\bar{b}p\int_{M_t} f_\sigma^p d\mu_t.$$

This implies

$$\int_{M_t} f_\sigma^p d\mu_t \le \mathrm{e}^{2\bar{b}pt}\int_{M_{t_0}} f_\sigma^p d\mu_t. \tag{4.24}$$

If $t \in [0, t_0]$, by the smoothness of the mean curvature flow we see that $\int_{M_t} f_\sigma^p d\mu_t$ is bounded. For $t \ge t_0$, we need to show that $T_{\max}$ is finite.

Proposition 4.5. *The maximal existence time $T_{\max}$ of the mean curvature flow is finite.*

Proof. For a fixed point $y \in \mathbb{F}^{n+d}(c)$, let r be the distance function on $\mathbb{F}^{n+d}(c)$ from y. Denote also by r the composition $r \circ F_t$. We may assume that $r > 0$ on M_t for $t \in [0, T_{\max})$. In fact, if $d = 1$, we may choose y such that it is outside of a geodesic ball in $\mathbb{F}^{n+1}(c)$ that encloses M_0. By the maximum principle we see that y does not lie in any M_t. If $d > 1$, then the Haussdorff dimension of $F(M \times [0, T_{\max}))$ is not bigger than $n+1$. So, we can also pick a point y such that it does not lie in any M_t. In both cases, we have $r > 0$ on each M_t.

From [4], we know that

$$\Delta r = \langle H, \partial_r\rangle + \mathrm{co_c}(r)(n - |\partial_r^T|^2). \tag{4.25}$$

Here $\mathrm{co_c}(r) = \frac{\sqrt{-c}\cosh(\sqrt{-c}r)}{\sinh(\sqrt{-c}r)} = \frac{\sqrt{-c}(e^{\sqrt{-c}r}+e^{-\sqrt{-c}r})}{e^{\sqrt{-c}r}-e^{-\sqrt{-c}r}}$, ∂_r is the gradient of r in $\mathbb{F}^{n+d}(c)$, and ∂_r^T is the tangential part of ∂_r to M_t. Clearly we have $\mathrm{co_c}(r) \ge \sqrt{-c}$ and $|\partial_r^T|^2 \le 1$.

On the other hand, since F_t satisfies (1.1), we have

$$\frac{\partial}{\partial t} r = \langle H, \partial_r\rangle. \tag{4.26}$$

Combining (4.25) and (4.26), we obtain

$$\frac{\partial}{\partial t} r = \Delta r - \mathrm{co}_c(r)(n - |\partial_r^T|^2). \tag{4.27}$$

Suppose that $r(x,0) < R$ for all $x \in M$. By the maximum principle we see that

$$r(x,t) < R - (n-1)\sqrt{-c}t \tag{4.28}$$

for all $x \in M$ and $t \in [0, T_{\max})$.

Then $T_{\max} < \frac{R}{(n-1)\sqrt{-c}}$, i.e., the maximal existence time of the mean curvature flow is finite. □

By Proposition 4.5, we complete the proof of Lemma 4.4. □

Now we can proceed as in [11] or [14] via a Stampacchia iteration procedure to complete the proof of Theorem 4.1. □

5. A gradient estimate for the mean curvature

We establish a gradient estimate for the mean curvature flow, which will be used to compare the mean curvatures at different points of the submanifold. We also assume that $c < 0$.

Theorem 5.1. *For every $\eta > 0$, there exists a constant C_η independent of t such that for all $t \in [0, T_{\max})$, there holds*

$$|\nabla H|^2 \le \eta |H|^4 + C_\eta. \tag{5.1}$$

Proof. By a direct computation, we have

$$\frac{\partial}{\partial t}|H|^4 \ge \Delta |H|^4 - 12|H|^2|\nabla H|^2 + \frac{4}{n}|H|^6 + 4nc|H|^4, \tag{5.2}$$

$$\frac{\partial}{\partial t}|\nabla H|^2 \le \Delta |\nabla H|^2 + C_1|H|^2|\nabla A|^2 + C_2|\nabla A|^2, \tag{5.3}$$

for constants C_1 and C_2 independent of t.

We also have the following estimate for sufficiently large positive constants N_1 and N_2 independent of t.

(5.4)
$$\begin{aligned}\frac{\partial}{\partial t}\Big((N_1 + N_2|H|^2)|\mathring{A}|^2\Big) \le \Delta\Big((N_1 + N_2|H|^2)|\mathring{A}|^2\Big) & \\ - \frac{4(n-1)}{3n}(N_2 - 1)|H|^2|\nabla A|^2 & \\ - \frac{4(n-1)}{3n}(N_1 - C_3(N_2))|\nabla A|^2 & \\ + C_4(N_1, N_2)|\mathring{A}|^2(|H|^4 + 1) - 2ncN_1|\mathring{A}|^2. &\end{aligned}$$

In (5.4), $C_3(N_2)$ is a constant depending only on N_2, and $C_4(N_1,\ N_2)$ is a constant depending on N_1 and N_2. Consider the function $f = |\nabla H|^2 + (N_1 + N_2|H|^2)|\mathring{A}|^2 - \eta|H|^4$. From (5.2), (5.3) and (5.4), we have

(5.5)

$$\begin{aligned}\frac{\partial}{\partial t} f \leq \Delta f - \frac{4(n-1)}{3n}(N_2 - 1)|H|^2|\nabla A|^2 - \frac{4(n-1)}{3n}(N_1 - C_5(N_2))|\nabla A|^2 \\ + C_6(N_1, N_2)|\mathring{A}|^2(|H|^4 + 1) + 12\eta|H|^2|\nabla H|^2 - \frac{4\eta}{n}|H|^6 - 4nc\eta|H|^4,\end{aligned}$$

where $C_5(N_2)$ is a constant depending only on N_2, and $C_6(N_1,\ N_2)$ is a constant depending on N_1 and N_2.

In (5.5), we have consumed $C_1|H|^2|\nabla A|^2 + C_2|\nabla A|^2$ by firstly choosing sufficiently large N_2 and secondly choosing sufficiently large N_1. Notice that $|\nabla H|^2 \leq n|\nabla A|^2$. We can choose larger N_2 and N_1 depending on η to consume $12\eta|H|^2|\nabla H|^2$ and make the second, third terms of the right-hand side of (5.5) negative. Since $|\mathring{A}|^2 \leq C_0|H|^{2-\sigma_0}$ for $t \geq t_0$ and $|\mathring{A}|^2$ is uniformly bounded for $t \in [0, t_0]$, using Young's inequality, we get

$$C_4(N_1, N_2)|\mathring{A}|^2(|H|^4 + 1) - 4nc\eta|H|^4 \leq \frac{4\eta}{n}|H|^6 + C_\eta.$$

Here C_η is a constant depending on η and other quantities but independent of t. Then we obtain

$$\frac{\partial}{\partial t} f \leq \Delta f + C_\eta.$$

Note that $T_{\max}$ is finite. Then the theorem follows from the maximum principle and the definition of f. □

6. Convergence of MCF in a hyperbolic space

Theorem 6.1. *Let $F : M^n \to \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold, where $n \geq 2$ and $c < 0$. Assume F satisfies*

$$|A|^2 \leq \begin{cases} \dfrac{4}{3n}|H|^2 + \frac{n}{2}c, & n = 2, 3, \\ \dfrac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases} \tag{6.1}$$

Then $F_t(M)$ converges to a round point in finite time.

Proof. By the curvature estimate in [5], we see that

$$K_{\min}(x) \geq \frac{1}{2}\left(\frac{1}{n-1} - \alpha_\epsilon\right)|H|^2(x) + \frac{1}{2}(2-\beta_\epsilon)c.$$

By our choices of α_ϵ and β_ϵ, and the preserved pinching condition, we see that there exists a positive constant ε_0 independent of t such that

$$K_{\min}(x) \geq \varepsilon_0 |H|^2. \tag{6.2}$$

Since $T_{\max}$ is finite, $\max_{M_t} |A|^2 \to \infty$ as $t \to T_{\max}$. By a similar argument as in [1, 11, 12] we have $\frac{\max_{M_t}|H|}{\min_{M_t}|H|} \to 1$ as $t \to T_{\max}$, and M_t converges to a single point o as $t \to T_{\max}$. If we take a rescaling around o (since $\mathbb{F}^{n+d}(c)$ can be considered as a linear space that isomorphic to $\mathbb{R}^{n+d}$) such that the total area of the expanded submanifold is fixed, then the rescaled immersions converge to a closed and totally umbilical immersion as $t \to T_{\max}$. □

When $p = 1$ and $n = 3$, we have the following proposition.

Proposition 6.2. *Let $F : M^3 \to \mathbb{F}^4(c)$ be a smooth closed hypersurface in a complete simply connected hyperbolic space form with constant curvature $c < 0$. Assume F satisfies*

$$|A|^2 \leq \frac{1}{2}|H|^2 + 2c.$$

Then the mean curvature flow with initial value F converges to a round point in finite time.

Proof. Since $d = 1$, we have $\mathring{A}_I = 0$. If we take $\alpha \leq \frac{1}{n-1}$ and $\beta \geq 2$ for all $n \geq 2$, the left-hand side of (3.2) is negative. Hence, $|A|^2 \leq \frac{1}{n-1}|H|^2 + 2c$ is preserved along the mean curvature flow for all $n \geq 2$. When $n = 3$, if we set $a = \frac{1}{3(2+\epsilon)}$, then $\epsilon_\nabla = \frac{3}{4} - \frac{1}{3} - a > 0$, and Theorem 4.1 also holds. Then Theorem 5.1 follows. By a similar argument as in the proof of Theorem 6.1, we show that the mean curvature flow converges to a round point in finite time. □

Remark 6.3. When $d=1$ and $n=3$, the pinching condition $|A|^2 \leq \frac{1}{2}|H|^2 + 2c$ is better than condition (6.1). In fact, we have $\frac{1}{2}|H|^2 + 2c = \frac{4}{9}|H|^2 + \frac{3}{2}c + \frac{1}{18}|H|^2 + \frac{1}{2}c > \frac{4}{9}|H^2| + \frac{3}{2}c$. Here, we have used the fact that $|H|^2 + 9c > 0$, which is implied by $|A|^2 \leq \frac{1}{2}|H|^2 + 2c$.

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