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Adiabatic limit and connections in Finsler geometry

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In this paper, we identify the Bott connection on the natural foliation of the projective sphere bundle of a Finsler manifold to the Chern connection of this manifold. As a consequence, the symmetrization of the Bott connection turns out to be the Cartan connection of the Finsler manifold. Following Liu–Zhang [9], the Cartan connection can also be obtained through an adiabatic limit process. Furthermore, a Chern–Simons type form is defined and its conformal properties are discussed.

Introduction

In Finsler geometry, the Chern connection and the Cartan connection are two basic connections which have remarkable properties. Let (M, F) be a Finsler manifold. Let $\pi : SM \to M$ be the projective sphere bundle of M. Then the Finsler structure F on M defines naturally an Euclidean structure on the pull-back vector bundle $\pi^*TM \to SM$ and a Sasaki-type Riemannian metric on SM. The Chern and Cartan connections are connections on π^*TM defined from different geometric reasons.

On the other hand, the Finsler structure F gives rise to a natural splitting of $T(SM)$. One part is the vertical tangent bundle $V(SM)$ formed by the tangent vectors of the (vertical) projective spheres, which is an integrable subbundle of $T(SM)$. Another part is the horizontal tangent bundle $H(SM)$, which is defined as the orthogonal complement of $V(SM)$ in $T(SM)$ with respect to the Sasaki-type Riemannian metric on SM. It is well-known that $H(SM)$ with its restriction metric is isometric to π^*TM.

In this paper, we consider SM as a foliation foliated by projective spheres. So the well-known Bott connection in foliation theory is now a connection on $H(SM)$. We will prove that the Bott connection is the Chern connection under the identification of $H(SM)$ and π^*TM. As a consequence, the symmetrization of the Bott connection turns out to be the Cartan connection. These also partially answer a question of M. Abate and G. Patrizio (cf. [1, p.28]). Following Liu–Zhang [9], the relations between the Chern connection, the Cartan connection and the Levi-Civita connection associated to the Sasaki-type Riemannian metric are also established through an adiabatic limit process.

We then consider a special Chern-Simons transgressed term of the Chern and Cartan connections. In the case of dimension 2, the explicit formula of this term is given. Inspired by this formula, we define a Chern-Simons type form of (M, F), which is a non-Riemannian geometric invariant of the Finsler manifold. Some conformal properties of this form are also discussed.

This paper is organized as follows. In Section 1, we give a review of some basic facts in Finsler geometry. In Section 2, we study the relations between the Bott connection, the Chern connection and the Cartan connection for a Finsler manifold. In Section 3, we define a Chern–Simons type form of a Finsler manifold and discuss its conformal properties.

1. Finsler manifolds and related structures

In this section we give a review of some basic facts in Finsler geometry which will be used in this paper.

Let M be an n dimensional smooth manifold and $\pi : TM \to M$ the tangent bundle of M. Let $(U; x = (x^1, x^2, \ldots, x^n))$ be a local coordinate system on an open subset U of M. Then by the standard procedure one gets a local coordinate system $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)$ on $\pi^{-1}(U)$. Set $TM_0 = TM \setminus 0$, where 0 denotes the zero section of TM. Then (x, y) with $y \neq 0$ is a local coordinate system on TM_0.

Definition 1. A Finsler structure on M is a smooth function $F : TM_0 \to \mathbb{R}^+$, which satisfies the following conditions:

(i) $F(x, \lambda y) = \lambda F(x, y)$, $\forall (x, y) \in TM_0$, and $\lambda \in \mathbb{R}^+$;

(ii) The $n \times n$ Hessian matrix

$$(g_{ij}) = \left(\frac{1}{2} [F^2]_{y^i y^j} \right)$$

is positive definite at every point of TM_0. A manifold M with a Finsler structure F is called a Finsler manifold, and denoted by (M, F).

In this paper, lower case Latin indices will run from 1 to n and lower case Greek indices will run from 1 to $n - 1$. We also adopt the summation convention of Einstein.

Let (M, F) be an n-dimensional Finsler manifold. Set

$$G^i = \frac{1}{4} g^{ij} \left([F^2]_{y^j x^k} y^k - [F^2]_{x^j} \right), \tag{1.1}$$

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \frac{\partial G^j}{\partial y^i} \frac{\partial}{\partial y^j}, \qquad \frac{\delta}{\delta y^i} = F \frac{\partial}{\partial y^i}, \tag{1.2}$$

where $(g^{ij}) = (g_{ij})^{-1}$. Clearly, the vectors

$$\left\{ \frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \ldots, \frac{\delta}{\delta x^n}, \frac{\delta}{\delta y^1}, \frac{\delta}{\delta y^2}, \ldots, \frac{\delta}{\delta y^n} \right\} \tag{1.3}$$

form a local tangent frame of TM_0. For another local coordinate system $(U; \tilde{x})$ on M, a routine computation shows that

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}, \qquad \frac{\delta}{\delta y^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{y}^j}. \tag{1.4}$$

Now by (1.4), one gets a well-defined linear map $J : T(TM_0) \to T(TM_0)$

$$J\left(\frac{\delta}{\delta x^i} \right) = \frac{\delta}{\delta y^i}, \quad J\left(\frac{\delta}{\delta y^i} \right) = -\frac{\delta}{\delta x^i}, \tag{1.5}$$

which is in fact an almost complex structure on TM_0. Let

$$\{\delta x^1, \delta x^2, \ldots, \delta x^n, \delta y^1, \delta y^2, \ldots, \delta y^n\} \tag{1.6}$$

be the dual frame of (1.3). One has

$$\delta x^i = dx^i, \quad \delta y^i = \frac{1}{F} \left(dy^i + \frac{\partial G^i}{\partial y^j} dx^j \right), \tag{1.7}$$

and

$$J^*(\delta x^i) = -\delta y^i, \quad J^*(\delta y^i) = \delta x^i, \tag{1.8}$$

where J^* denotes the dual map of J.

Let $\pi : SM = TM_0/\mathbb{R}^+ \to M$ denote the projective sphere bundle. Now the fundamental tensor $g = g_{ij} dx^i \otimes dx^j$ defines an Euclidean metric on the pull back bundle π^*TM over SM. Note that π^*TM admits a distinguished global section $l : SM \to \pi^*TM$, which is defined by

$$l(x,[y]) = \left(x,[y],\frac{y}{F(x,y)}\right). \tag{1.9}$$

For any local orthonormal frame $\{e_1,\ldots,e_n\}$ of (π^*TM,g) with $e_n = l$, let $\{\omega^1,\ldots,\omega^n\}$ be the dual frame. Clearly, ω^i's can also be viewed naturally as (local) one forms on SM as well as on TM_0. Here ω^n, the so-called Hilbert form, is a globally defined one form and $\omega^n = F_{y^i}\delta x^i$. Set

$$\omega^{n+i} = J^*(\omega^i), \quad i = 1,2,\ldots,n. \tag{1.10}$$

The one forms $\omega^1,\omega^2,\ldots,\omega^{2n-1}$ and $\omega^{2n} = -F_{y^i}\delta y^i$ give rise to a local coframe of TM_0. Moreover, one verifies easily that the forms $\omega^{n+\alpha}$, $\alpha = 1,2,\ldots,n-1$, are actually the one forms on SM (cf. [8, p.269]) and the set

$$\theta = \left\{\omega^1,\ldots,\omega^n,\omega^{n+1},\ldots,\omega^{2n-1}\right\} \tag{1.11}$$

forms a local coframe of SM. By using the local coframe (1.11), the tensor

$$g^{T(SM)} = \sum_{i=1}^{n}\omega^i\otimes\omega^i + \sum_{\alpha=1}^{n-1}\omega^{n+\alpha}\otimes\omega^{n+\alpha} \tag{1.12}$$

gives a well-defined Riemannian metric on SM, which is called the Sasaki-type Riemannian metric on SM. Moreover, the fundamental tensor g can be written as

$$g = \sum_{i,j=1}^{n} g_{ij}dx^i\otimes dx^j = \sum_{i,j=1}^{n} g_{ij}\delta x^i\otimes\delta x^j = \sum_{i=1}^{n}\omega^i\otimes\omega^i \quad \text{on} \quad SM. \tag{1.13}$$

As mentioned in Introduction of this paper, the vertical and horizontal subbundles $V(SM)$ and $H(SM)$ of $T(SM)$ are orthogonal to each other with respect to $g^{T(SM)}$. Let $\{\mathbf{e}_1,\ldots,\mathbf{e}_n,\mathbf{e}_{n+1},\ldots,\mathbf{e}_{2n-1}\}$ denote the dual frame of θ. Note that

$$\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_{n-1},\mathbf{e}_n\} \tag{1.14}$$

is a local orthonormal frame of $H(SM)$.

Remark 1. (π^*TM,g) can be identified with $H(SM)$ with the restricting metric of $g^{T(SM)}$ as Euclidean bundles. In fact, this identification is given by identifying $\frac{\partial}{\partial x^i}$ with $\frac{\delta}{\delta x^i}$ and so e_i with $\mathbf{e}_i$. In particular, the distinguished section $l = e_n = \frac{y^i}{F}\frac{\partial}{\partial x^i}$ in (1.9) turns out to be the Reeb vector field $\mathbf{G} = \mathbf{e}_n = \frac{y^i}{F}\frac{\delta}{\delta x^i}$ of (M, F) on SM.

Write that

$$\omega^j = v_i^j \delta x^i, \quad \text{and} \quad \text{so} \quad \omega^{n+\alpha} = J^*(v_i^\alpha \delta x^i) = -v_i^\alpha \delta y^i. \tag{1.15}$$

Then one has

$$\mathbf{e}_i = u_i^j \frac{\delta}{\delta x^j} \quad \text{and} \quad \mathbf{e}_{n+\alpha} = -u_\alpha^j \frac{\delta}{\delta y^j}, \tag{1.16}$$

where $(u_j^i) = (v_i^j)^{-1}$. Here also note that $v_i^n = F_{y^i}$ and $u_n^i = \frac{y^i}{F}$. By Definition 1, one gets easily that

$$\sum_{\alpha=1}^{n-1} v_i^\alpha v_j^\alpha = F F_{y^i y^j}, \quad F F_{y^i} g^{ij} = y^j. \tag{1.17}$$

The following lemma gives an explicit expression of the exterior derivative of the Hilbert form ω^n with respect to the local coframe (1.11). This formula is usually obtained as one of the structure equations of the Chern connection in Finsler geometry.

Lemma 1. *The exterior derivative of Hilbert form is given by*

$$d\omega^n = \sum_{\alpha=1}^{n-1} \omega^\alpha \wedge \omega^{n+\alpha}. \tag{1.18}$$

Proof. Note that

$$\begin{aligned} d\omega^n &= d(F_{y^i}\delta x^i) = F_{y^i x^j}\delta x^j \wedge \delta x^i + F_{y^i y^j} dy^j \wedge \delta x^i \\ &= F F_{y^i y^j}\delta y^j \wedge \delta x^i + \left(F_{y^i x^j} - \frac{\partial G^k}{\partial y^j} F_{y^i y^k}\right)\delta x^j \wedge \delta x^i \\ &= F F_{y^i y^j}\delta y^j \wedge \delta x^i + \left(\frac{\delta F}{\delta x^j}\right)_{y^i} \delta x^j \wedge \delta x^i + \frac{\partial^2 G^k}{\partial y^j \partial y^i} F_{y^k}\delta x^j \wedge \delta x^i. \end{aligned}$$

By (1.15) and (1.17), the term $F F_{y^i y^j}\delta y^j \wedge \delta x^i = \sum_{\alpha=1}^{n-1} \omega^\alpha \wedge \omega^{n+\alpha}$. Clearly, $\frac{\partial^2 G^k}{\partial y^j \partial y^i} F_{y^k}\delta x^j \wedge \delta x^i = 0$. Now the lemma follows from the following result (cf. [4, p.36]),

$$\frac{\delta F}{\delta x^j} = F_{x^j} - \frac{\partial G^k}{\partial y^j} F_{y^k} = 0. \tag{1.19}$$

□

The following lemma is actually obtained by Mo in [11, p.317]. We will give it a direct proof without using any concepts of connections.

Lemma 2 (cf. [11, p.317])**.** *The Lie derivative of the fundamental tensor g along the Reeb vector field* **G** *(cf.* Remark 1.*) is given by*

$$\mathcal{L}_{\mathbf{G}} g = -\sum_{\alpha=1}^{n-1} \left(\omega^\alpha \otimes \omega^{n+\alpha} + \omega^{n+\alpha} \otimes \omega^\alpha\right). \tag{1.20}$$

Proof. Firstly one has

$$\begin{aligned}
\mathbf{G}(g_{ij}) &= \frac{y^k}{F} \frac{\delta}{\delta x^k} \left(\frac{1}{2}[F^2]_{y^i y^j}\right) = \frac{1}{2} \frac{y^k}{F} \left(\frac{\partial}{\partial x^k}[F^2]_{y^i y^j} - \frac{\partial G^l}{\partial y^k} \frac{\partial}{\partial y^l}[F^2]_{y^i y^j}\right) \\
&= \frac{1}{2} \frac{y^k}{F} \left(\frac{\delta [F]^2}{\delta x^k}\right)_{y^i y^j} + \frac{1}{F} \left(g_{lj} \frac{\partial G^l}{\partial y^i} + g_{li} \frac{\partial G^l}{\partial y^j}\right) \\
&= \frac{1}{F} \left(g_{lj} \frac{\partial G^l}{\partial y^i} + g_{li} \frac{\partial G^l}{\partial y^j}\right).
\end{aligned}$$

Then by (1.13) and Cartan homotopy formula (cf. [13, p.30]), one has

$$\begin{aligned}
\mathcal{L}_{\mathbf{G}} g &= \mathcal{L}_{\mathbf{G}}(g_{ij} dx^i \otimes dx^j) \\
&= \mathbf{G}(g_{ij}) dx^i \otimes dx^j + g_{ij} \mathcal{L}_{\mathbf{G}}(dx^i) \otimes dx^j + g_{ij} dx^i \otimes \mathcal{L}_{\mathbf{G}}(dx^j) \\
&= \frac{1}{F} \left(g_{lj} \frac{\partial G^l}{\partial y^i} + g_{li} \frac{\partial G^l}{\partial y^j}\right) dx^i \otimes dx^j + g_{ij} \frac{dy^i}{F} \otimes dx^j \\
&\quad - g_{ij} \frac{y^i}{F} d\log F \otimes dx^j + g_{ij} dx^i \otimes \frac{dy^j}{F} - g_{ij} dx^i \otimes \frac{y^j}{F} d\log F \\
&= g_{ij} \delta y^i \otimes dx^j + g_{ij} dx^i \otimes \delta y^j - d\log F \otimes F_{y^j} dx^j - F_{y^i} dx^i \otimes d\log F \\
&= -\sum_{i=1}^{n} \omega^i \otimes \omega^{n+i} - \sum_{i=1}^{n} \omega^{n+i} \otimes \omega^i - d\log F \otimes \omega^n - \omega^n \otimes d\log F \\
&= -\sum_{\alpha=1}^{n-1} \left(\omega^\alpha \otimes \omega^{n+\alpha} + \omega^{n+\alpha} \otimes \omega^\alpha\right).
\end{aligned}$$

The last equation comes from that $\omega^{2n} = -d\log F$, a direct corollary of (1.19). $\square$

Remark 2. We denote the Hilbert form as $\omega = \omega^n$. By Lemma 1, one has that $\omega \wedge (d\omega)^{n-1} \neq 0$. So ω is a contact form of SM.

2. The relations of some connections related to a Finsler manifold

In this section, we will use the same notations as in Section 1. Note that there exists a natural foliation structure on the Riemannian manifold $(SM, g^{T(SM)})$, which is foliated by the vertical bundle $V(SM)$. Following Liu–Zhang [9] and Zhang [13, Section 1.7], set

$$\mathcal{F} = V(SM), \quad \mathcal{F}^\perp = H(SM). \tag{2.1}$$

Let $\nabla^{T(SM)}$ be the Levi–Civita connection on $T(SM)$ associated to the Sasaki-type Riemannian metric $g^{T(SM)}$ on SM. Let p, $p^\perp$ denote the orthogonal projections from $T(SM)$ to $\mathcal{F}$, $\mathcal{F}^\perp$ respectively. Set

$$\nabla^{\mathcal{F}} = p\nabla^{T(SM)}p, \quad \nabla^{\mathcal{F}^\perp} = p^\perp\nabla^{T(SM)}p^\perp. \tag{2.2}$$

Let $g^{\mathcal{F}}$, $g^{\mathcal{F}^\perp}$ be the restriction of $g^{T(SM)}$ on $\mathcal{F}$, $\mathcal{F}^\perp$ respectively. Then $\nabla^{\mathcal{F}}$, $\nabla^{\mathcal{F}^\perp}$ are metric-preserving connections of $\mathcal{F}$, $\mathcal{F}^\perp$ respectively.

Now the Bott connection $\widetilde{\nabla}^{\mathcal{F}^\perp}$ on $\mathcal{F}^\perp$ is determined by the following definition

Definition 2 (cf. [9], [13, Section 1.7]). For any $X \in \Gamma(T(SM))$, $U \in \Gamma(\mathcal{F}^\perp)$,

(i) If $X \in \Gamma(\mathcal{F})$, set $\widetilde{\nabla}^{\mathcal{F}^\perp}_X U = p^\perp[X, U]$;

(ii) If $X \in \Gamma(\mathcal{F}^\perp)$, set $\widetilde{\nabla}^{\mathcal{F}^\perp}_X U = \nabla^{\mathcal{F}^\perp}_X U$.

In general, the Bott connection $\widetilde{\nabla}^{\mathcal{F}^\perp}$ is not a metric-preserving connection of $g^{\mathcal{F}^\perp}$. One defines the dual connection $\widetilde{\nabla}^{\mathcal{F}^\perp,*}$ of the Bott connection as follows,

$$d\langle U, V\rangle = \langle \widetilde{\nabla}^{\mathcal{F}^\perp} U, V\rangle + \langle U, \widetilde{\nabla}^{\mathcal{F}^\perp,*} V\rangle,$$

where $U, V \in \Gamma(\mathcal{F}^\perp)$.

Following Bismut–Zhang [5, p.62] and Liu–Zhang [9], set

$$2H = \widetilde{\nabla}^{\mathcal{F}^\perp,*} - \widetilde{\nabla}^{\mathcal{F}^\perp} \quad \text{and} \quad \widehat{\nabla}^{\mathcal{F}^\perp} = \widetilde{\nabla}^{\mathcal{F}^\perp} + H. \tag{2.3}$$

The connection $\widehat{\nabla}^{\mathcal{F}^\perp}$ is the symmetrization of the Bott connection with respect to the metric $g^{\mathcal{F}^\perp}$ on $\mathcal{F}^\perp$ and so a metric-preserving connection on $\mathcal{F}^\perp$.

Some basic properties of the $\Omega^1(SM)$-valued endomorphism H are also established in [5, p.62] and [9].

Lemma 3 ([5, p.62], [9])**.** *For any $U, V \in \Gamma(\mathcal{F}^\perp)$, one has that*

(1) $\langle HU, V\rangle = \langle U, HV\rangle$,

(2) $H(U) = 0$,

(3) $H = \frac{1}{2}(g^{\mathcal{F}^\perp})^{-1}\widetilde{\nabla}^{\mathcal{F}^\perp} g^{\mathcal{F}^\perp}$.

Write $H = (H_{ij})$ under the local frame (1.14). As a corollary of Lemma 3, one has that $H_{ij} = H_{ji}$ and $H_{ij} = H_{ij\gamma}\omega^{n+\gamma}$ for some functions $H_{ij\gamma}$.

Lemma 4. *Set $A_{ijk} = \frac{1}{4}F[F^2]_{y^i y^j y^k}$. With respect to (1.16), one has*

$$H_{ij\gamma} = -A_{pqk} u_i^p u_j^q u_\gamma^k. \tag{2.4}$$

Moreover, $H_{ij\gamma} = 0$ if $i = n$ or $j = n$.

Proof. For any $X \in \Gamma(\mathcal{F})$ and $U, V \in \Gamma(\mathcal{F}^\perp)$, one gets easily that

$$\langle 2H(X)U, V\rangle = (\mathcal{L}_X g^{\mathcal{F}^\perp})(U, V).$$

So by (1.13), (1.16) and (2.4),

$$\begin{aligned} H_{ij\gamma} &= \langle H(\mathbf{e}_{n+\gamma})\mathbf{e}_i, \mathbf{e}_j\rangle = \frac{1}{2}(\mathcal{L}_{\mathbf{e}_{n+\gamma}} g^{\mathcal{F}^\perp})(\mathbf{e}_i, \mathbf{e}_j) = \frac{1}{2}(\mathcal{L}_{\mathbf{e}_{n+\gamma}} g)(\mathbf{e}_i, \mathbf{e}_j) \\ &= \frac{1}{2}(\mathbf{e}_{n+\gamma} g_{pq}) dx^p \otimes dx^q(\mathbf{e}_i, \mathbf{e}_j) = -\frac{1}{2} F u_\gamma^k \frac{\partial g_{pq}}{\partial y^k} dx^p \otimes dx^q(\mathbf{e}_i, \mathbf{e}_j) \\ &= -\frac{1}{4} F[F^2]_{y^p y^q y^k} u_i^p u_j^q u_\gamma^k = -A_{pqk} u_i^p u_j^q u_\gamma^k. \end{aligned}$$

By the Euler lemma, it is clear that $H_{ij\gamma} = 0$ if $i = n$ or $j = n$. □

Remark 3. Traditionally, the Cartan tensor is defined as $\mathbf{A} = A_{ijk} dx^i \otimes dx^j \otimes dx^k$, and the Cartan form is that $\mathbf{I} = g^{ij} A_{ijk} dx^k := A_k dx^k$ (cf. [10, p.11–12]). From this reason, we call H the Cartan endomorphism, and call the one form $\eta = \mathrm{tr}[H] \in \Omega^1(SM)$ the Cartan-type form for a Finsler manifold (M, F).

Let $\boldsymbol{\omega} = (\omega^i_j)$ be the connection matrix of the Bott connection with respect to the orthonormal frame (1.14), i.e.,

$$\widetilde{\nabla}^{\mathcal{F}^\perp} \mathbf{e}_i = \omega_i^j \mathbf{e}_j. \tag{2.5}$$

Theorem 1. *Under the identification of π^*TM with $H(SM)$ as in Remark 1, the Chern connection is the Bott connection. As a consequence, the Cartan connection is the symmetrization of the Bott connection.*

Proof. Under the identification of π^*TM with $H(SM)$, it is well known that the Chern connection matrix $\boldsymbol{\omega} = (\omega^i_j)$ is defined by the following structure equations (cf. [3, p.140–146], [4, p.38], [7, p.97–104], [8, p.274–282], [10, p.23–33]),

$$\begin{cases} d\vartheta = \vartheta \wedge \boldsymbol{\omega}, \\ \boldsymbol{\omega} + \boldsymbol{\omega}^t = -2H, \end{cases} \tag{2.6}$$

where $\vartheta = (\omega^1, \dots, \omega^n)$.

So to prove the theorem, we only need to show that the solution of (2.6) is unique and the Bott connection forms in (2.5) satisfy (2.6).

To prove the uniqueness, let $\widetilde{\boldsymbol{\omega}} = (\widetilde{\omega}^i_j)$ be another solution of (2.6). One has

$$\omega^j \wedge (\widetilde{\omega}^i_j - \omega^i_j) = 0.$$

It deduces that

$$\widetilde{\omega}^i_j - \omega^i_j = a^i_{jk}\omega^k, \quad \text{with} \quad a^i_{jk} = a^i_{kj}.$$

From the second equation of (2.6), one has that

$$0 = (\omega^i_j + \omega^j_i) - (\widetilde{\omega}^i_j + \widetilde{\omega}^j_i) = (a^i_{jk} + a^j_{ik})\omega^k,$$

and so $a^i_{jk} + a^j_{ik} = 0$. Thus

$$(a^i_{jk} + a^j_{ik}) + (a^k_{ij} + a^i_{kj}) - (a^j_{ki} + a^k_{ji}) = 2a^i_{jk} = 0.$$

So, we conclude that $\widetilde{\omega}^i_j - \omega^i_j = 0$.

Now we will show that the Bott connection forms in (2.5) satisfy (2.6). For any $X, Y \in \Gamma(T(SM)$,

$$\begin{aligned}&(d\omega^i - \omega^j \wedge \omega_j^i)(X,Y)\\ &\quad = X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i([X,Y]) - \left(\omega^j(X)\omega_j^i(Y) - \omega^j(Y)\omega_j^i(X)\right).\end{aligned}$$

Now for any $X, Y \in \Gamma(\mathcal{F})$, and $U, V \in \Gamma(\mathcal{F}^\perp)$, one has

$$\begin{aligned}&(d\omega^i - \omega^j \wedge \omega_j^i)(X,Y) = -\omega^i([X,Y]) = 0,\\ &(d\omega^i - \omega^j \wedge \omega_j^i)(X,U) = X(\omega^i(U)) + \omega^j(U)\omega_j^i(X) - \omega^i([X,U])\\ &\quad = \omega^i\left(\widetilde{\nabla}_X^{\mathcal{F}^\perp} U - [X,U]\right) = 0,\end{aligned}$$

and

$$\begin{aligned}&(d\omega^i - \omega^j \wedge \omega_j^i)(U,V)\\ &\quad = \left(U(\omega^i(V)) + \omega^j(V)\omega_j^i(U)\right) - \left(V(\omega^i(U)) + \omega^j(U)\omega_j^i(V)\right) - \omega^i([U,V])\\ &\quad = \omega^i\left(\widetilde{\nabla}_U^{\mathcal{F}^\perp} V - \widetilde{\nabla}_V^{\mathcal{F}^\perp} U - [U,V]\right) = \omega^i\left(\nabla_U^{T(SM)} V - \nabla_V^{T(SM)} U - [U,V]\right)\\ &\quad = 0.\end{aligned}$$

Hence, the Bott connection matrix $\boldsymbol{\omega}$ satisfies the first equation of (2.6).

The second equation of (2.6) comes directly from the definition of H. So we have proved that the Chern connection is the Bott connection.

Moreover, under the orthonormal frame (1.14), the symmetrization $\widehat{\nabla}^{\mathcal{F}^\perp}$ of the Bott connection has the connection matrix

$$\widehat{\boldsymbol{\omega}} = \boldsymbol{\omega} + H. \tag{2.7}$$

In [4, p.39], an expression of the Cartan connection is given in the local coordinate system on SM. One can check easily that these two expressions are differ from a gauge transformation of the connection. So $\widehat{\nabla}^{\mathcal{F}^\perp}$ turns out to be the Cartan connection in Finsler geometry. □

Remark 4. Note that in this case, the Chern connection and the Cartan connection are globally characterized without using local coordinate construction. These partially answer the question of M. Abate and G. Patrizio to find global characterizations of connections in Finsler geometry (cf. [1, p. 28]).

Lemma 5. *The connection forms of the Bott connection in (2.5) satisfy*

$$\omega_\alpha^n = -\omega_n^\alpha = \omega^{n+\alpha}, \quad \text{and} \quad \omega_n^n = 0.$$

Proof. The formula $\omega_n^n = 0$ comes directly from Lemma 4. By Lemma 1, the connection forms ω_α^n can be written as

$$\omega_\alpha^n = \omega^{n+\alpha} + c_{\alpha\beta}\omega^\beta \quad \text{with} \quad c_{\alpha\beta} = c_{\beta\alpha}. \tag{2.8}$$

The second equation of (2.6) and Lemma 4 imply that

$$\omega_n^\alpha = -\omega_\alpha^n = -\omega^{n+\alpha} - c_{\alpha\beta}\omega^\beta. \tag{2.9}$$

By (2.9) and the first equation of (2.6), one has that

$$\begin{aligned} d\omega^\alpha &= \omega^\beta \wedge \omega_\beta^\alpha + \omega^n \wedge \omega_n^\alpha \\ &= \omega^\beta \wedge (\omega_\beta^\alpha + c_{\alpha\beta}\omega^n) + \omega^n \wedge (-\omega^{n+\alpha}). \end{aligned}$$

Set $\widetilde{\omega}_\beta^\alpha = \omega_\beta^\alpha + c_{\alpha\beta}\omega^n$, $\widetilde{\omega}_\alpha^n = -\widetilde{\omega}_n^\alpha = \omega^{n+\alpha}$ and $\widetilde{\omega}_n^n = 0$. Clearly, $\widetilde{\boldsymbol{\omega}} = (\widetilde{\omega}_j^i)$ satisfies the first equation of (2.6). Moreover,

$$\widetilde{\omega}_\beta^\alpha + \widetilde{\omega}_\beta^\alpha = 2c_{\alpha\beta}\omega^n - 2H_{\alpha\beta\gamma}\omega^{n+\gamma}.$$

Note that by Cartan homotopy formula, one has

$$\begin{aligned} \mathcal{L}_{\mathbf{e}_n} g &= \mathcal{L}_{\mathbf{e}_n} \sum_{i=1}^n \omega^i \otimes \omega^i = \sum_{i=1}^n \big((\mathcal{L}_{\mathbf{e}_n}\omega^i) \otimes \omega^i + \omega^i \otimes (\mathcal{L}_{\mathbf{e}_n}\omega^i) \big) \\ &= \sum_{i=1}^n \big((i_{\mathbf{e}_n} d\omega^i) \otimes \omega^i + \omega^i \otimes (i_{\mathbf{e}_n} d\omega^i) \big) \\ &= \sum_{\alpha=1}^{n-1} ((i_{\mathbf{e}_n} d\omega^\alpha) \otimes \omega^\alpha + \omega^\alpha \otimes (i_{\mathbf{e}_n} d\omega^\alpha)) \\ &= \sum_{\alpha,\beta=1}^{n-1} \Big(i_{\mathbf{e}_n}(\omega^\beta \wedge \widetilde{\omega}_\beta^\alpha) \otimes \omega^\alpha + \omega^\alpha \otimes i_{\mathbf{e}_n}(\omega^\beta \wedge \widetilde{\omega}_\beta^\alpha) \\ &\quad + i_{\mathbf{e}_n}(\omega^n \wedge \widetilde{\omega}_n^\alpha) \otimes \omega^\alpha + \omega^\alpha \otimes i_{\mathbf{e}_n}(\omega^n \wedge \widetilde{\omega}_n^\alpha)) \\ &= - \sum_{\alpha,\beta=1}^{n-1} (\widetilde{\omega}_\beta^\alpha(\mathbf{e}_n) + \widetilde{\omega}_\alpha^\beta(\mathbf{e}_n))\omega^\alpha \otimes \omega^\beta - \sum_{\alpha=1}^{n-1}(\omega^\alpha \otimes \omega^{n+\alpha} + \omega^{n+\alpha} \otimes \omega^\alpha). \end{aligned}$$

Comparing with Lemma 2, we conclude that

$$2c_{\alpha\beta} = (\widetilde{\omega}_\alpha^\beta + \widetilde{\omega}_\beta^\alpha)(\mathbf{e}_n) = 0.$$

Now by (2.8), the corollary is proved. □

Now we consider the rescaled metrics on SM with $\epsilon > 0$,

$$g^{T(SM),\epsilon} = \frac{1}{\epsilon^2}\sum_{i=1}^{n}\omega^i \otimes \omega^i + \sum_{\alpha=1}^{n-1}\omega^{n+\alpha}\otimes\omega^{n+\alpha}. \tag{2.10}$$

Let $\nabla^{T(SM),\epsilon}$ be the Levi-Civita connection of $g^{T(SM),\epsilon}$ and $\nabla^{\mathcal{F}^\perp,\epsilon} = p^\perp \nabla^{T(SM),\epsilon} p^\perp$.

Following Liu–Zhang [9] and Zhang [13, Section 1.7], the Cartan connection $\widehat{\nabla}^{\mathcal{F}^\perp}$ now can also be obtained through the adiabatic limit technique, i.e.,

Proposition 1. *Let $\nabla^{\mathcal{F}^\perp,\epsilon} = p^\perp \nabla^{T(SM),\epsilon} p^\perp$, then*

$$\lim_{\epsilon\to 0}\nabla^{\mathcal{F}^\perp,\epsilon} = \widehat{\nabla}^{\mathcal{F}^\perp}.$$

Furthermore, by using the technique of the adiabatic limit, we can prove the following property of the Cartan endomorphism H.

Proposition 2. *Let (M, F) be a Finsler manifold. For any $\sigma \in C^\infty(M)$, let $\bar{g}^{T(SM)} = e^{2\sigma} g^{T(SM)}$ and $\bar{H}$ be the associated Cartan endomorphism, then*

$$\bar{H} = H.$$

Proof. Let $\overline{\widetilde{\nabla}^{\mathcal{F}^\perp}}$ and $\overline{\widehat{\nabla}^{\mathcal{F}^\perp}}$ be the Bott connection and its symmetrization corresponding to $\bar{g}^{T(SM)} = e^{2\sigma} g^{T(SM)}$, respectively. Then the corresponding Cartan endomorphism $\bar{H}$ is

$$\bar{H} = \overline{\widehat{\nabla}^{\mathcal{F}^\perp}} - \overline{\widetilde{\nabla}^{\mathcal{F}^\perp}}.$$

Consider the rescaled coformal metrics

$$\bar{g}^{T(SM),\epsilon} = \frac{1}{\epsilon^2} e^{2\sigma} g^{\mathcal{F}^\perp} \oplus e^{2\sigma} g^{\mathcal{F}}$$

and the projection connections $\overline{\nabla^{\mathcal{F}^\perp,\epsilon}}$ on $\mathcal{F}^{\perp,\epsilon}$. It is clear that $\bar{H}(U) = H(U) = 0$ for any $U \in \Gamma(\mathcal{F}^\perp)$. For any $X \in \Gamma(\mathcal{F})$, $U, V \in \Gamma(\mathcal{F}^\perp)$, we have

$$\begin{aligned}
\langle \bar{H}(X)U, V\rangle &= \langle \overline{\widehat{\nabla}^{\mathcal{F}^\perp}_X} U, V\rangle - \langle [X,U], V\rangle \\
&= \lim_{\epsilon\to 0} \langle \overline{\nabla^{\mathcal{F}^\perp,\epsilon}_X} U, V\rangle - \langle [X,U], V\rangle \\
&= \lim_{\epsilon\to 0} \frac{1}{2} e^{-2\sigma}\epsilon^2 \{X\langle U,V\rangle_{\sigma,\epsilon} + U\langle X,V\rangle_{\sigma,\epsilon} - V\langle X,U\rangle_{\sigma,\epsilon} \\
&\quad + \langle [X,U],V\rangle_{\sigma,\epsilon} - \langle [X,V],U\rangle_{\sigma,\epsilon} - \langle [U,V],X\rangle_{\sigma,\epsilon}\} - \langle [X,U],V\rangle \\
&= \lim_{\epsilon\to 0} \frac{1}{2} \{X\langle U,V\rangle + 2X(\sigma)\langle U,V\rangle \\
&\quad + \langle [X,U],V\rangle - \langle [X,V],U\rangle - \epsilon^2\langle [U,V],X\rangle\} - \langle [X,U],V\rangle \\
&= \frac{1}{2}\{X\langle U,V\rangle - \langle [X,U],V\rangle - \langle [X,V],U\rangle\} \\
&= \langle H(X)U, V\rangle.
\end{aligned}$$

□

3. Geometric classes of Finsler manifolds

Let (M, F) be an oriented and closed Finsler manifold of dimension n. As in the previous section, let $\widetilde{\nabla}^{\mathcal{F}^\perp}$ and $\widehat{\nabla}^{\mathcal{F}^\perp}$ denote the Chern connection and the Cartan connection on $\mathcal{F}^\perp = H(SM)$, respectively.

Let $\nabla^{\mathcal{F}^\perp}_t$, $t \in [0,1]$, be a family of connections on $\mathcal{F}^\perp$ defined by

$$\nabla^{\mathcal{F}^\perp}_t = (1-t)\widetilde{\nabla}^{\mathcal{F}^\perp} + t\widehat{\nabla}^{\mathcal{F}^\perp} = \widetilde{\nabla}^{\mathcal{F}^\perp} + tH.$$

Let $R^{\mathcal{F}^\perp}_t = (\nabla^{\mathcal{F}^\perp}_t)^2$ be the curvature of $\nabla^{\mathcal{F}^\perp}_t$. The term

$$-n\int_0^1 \operatorname{tr}\left[H(R^{\mathcal{F}^\perp}_t)^{n-1}\right] dt \tag{3.1}$$

appears naturally in the transgression formula associated to $\operatorname{tr}\left[(R^{\mathcal{F}^\perp}_t)^n\right]$ (cf. [13, p.16]).

With respect to (1.14), the curvature two forms of $R^{\mathcal{F}^\perp}_0$ are $\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k$. By the first equation of (2.6) (also Lemma 1.14 in [13]), one can write Ω^i_j as

$$\Omega^i_j = \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l + P^i_{jk\gamma}\omega^k \wedge \omega^{n+\gamma}, \tag{3.2}$$

where R^i_{jkl} and $P^i_{jk\gamma}$ are some functions on SM.

In the following we will compute the term (3.1) for a Finsler surface.

Theorem 2. *Let (M,F) be an oriented and closed Finsler surface. The term (3.1) is given by*

$$-2\int_0^1 \operatorname{tr}\left[HR_t^{\mathcal{F}^\perp}\right]dt = \eta\wedge d\eta, \tag{3.3}$$

where $\eta = \operatorname{tr}[H] = H_{111}\omega^3$ is the Cartan-type form of (M,F) (cf. Remark 3).

Proof. Firstly one has that

$$\begin{aligned}\int_0^1 \operatorname{tr}\left[HR_t^{\mathcal{F}^\perp}\right]dt &= \int_0^1 \operatorname{tr}\left[HR_0^{\mathcal{F}^\perp} + tH\left[\nabla_0^{\mathcal{F}^\perp}, H\right] + t^2H^3\right]dt \\ &= \operatorname{tr}\left[HR_0^{\mathcal{F}^\perp} + \frac{1}{2}H\left[\nabla_0^{\mathcal{F}^\perp}, H\right] + \frac{1}{3}H^3\right].\end{aligned}$$

In the case of $\dim M = 2$, by Corollary 1 and (3.2), one has

$$d\omega^3 = -R^1_{212}\omega^1\wedge\omega^2 - P^1_{211}\omega^1\wedge\omega^3.$$

With respect to the local frame (1.14), one gets

$$H = \begin{bmatrix} H_{111}\omega^3 & 0 \\ 0 & 0\end{bmatrix},\quad R_0^{\mathcal{F}^\perp} = \begin{bmatrix}\Omega^1_1 & * \\ * & *\end{bmatrix} = \begin{bmatrix} H_{111}R^1_{212}\omega^1\wedge\omega^2 + \cdots & * \\ * & *\end{bmatrix}.$$

Thus

$$\begin{aligned} HR_0^{\mathcal{F}^\perp} &= \begin{bmatrix}(H_{111})^2R^1_{212}\omega^1\wedge\omega^2\wedge\omega^3 & * \\ 0 & 0\end{bmatrix}, \\ H[\nabla_0^{\mathcal{F}^\perp}, H] &= \begin{bmatrix}-(H_{111})^2R^1_{212}\omega^1\wedge\omega^2\wedge\omega^3 & 0 \\ 0 & 0\end{bmatrix},\end{aligned}$$

and

$$\int_0^1 \operatorname{tr}\left[-HR_t^{\mathcal{F}^\perp}\right]dt = \frac{1}{2}(H_{111})^2R^1_{212}\omega^1\wedge\omega^2\wedge\omega^3.$$

On the other hand,

$$\eta\wedge d\eta = H_{111}\omega^3\wedge d(H_{111}\omega^3) = -(H_{111})^2R^1_{212}\omega^1\wedge\omega^2\wedge\omega^3.$$

So Theorem 2 follows. □

Remark 5. In [12], Szabó proved that any two dimensional Berwald manifold is either locally Minkowskian or Riemannian. So the term (3.1) is identically zero for any two dimensional Berwald manifold. On the other hand, in [6], Bryant constructed a family of two dimensional non-Riemannian Finsler manifolds with $R^1_{212} = 1$. From Theorem 2, the cohomology class associated to the term (3.1) of these Finsler manifolds are not zero.

Motivated by Theorem 2 and Remark 5, we make the following definition.

Definition 3. For a closed and oriented Finsler manifold (M, F) of dimension n, the top form $\eta \wedge (d\eta)^{n-1}$ on SM is called the Chern-Simons type form of (M, F). The corresponding class

$$[\eta \wedge (d\eta)^{n-1}] \in H^{2n-1}_{\mathrm{dR}}(SM)$$

is called the Chern–Simons type secondary class of (M, F).

When $(d\eta)^k = 0$ for some $k \geq 1$, one gets a closed form $\eta \wedge (d\eta)^{k-1}$ and so a class $[\eta \wedge (d\eta)^{k-1}] \in H^{2k-1}_{\mathrm{dR}}(SM)$. It would be interesting to explore the properties of the Finsler manifolds with $(d\eta)^k = 0$ and $[\eta \wedge (d\eta)^{k-1}] \neq 0$.

Note that the form $\eta \wedge (d\eta)^{n-1}$ is unchanged about the conformal metrics in Proposition 2. In the following proposition, a condition on conformal Finsler metrics is given which leaves η unchanged.

Proposition 3. *Let (M, F) be a Finsler manifold. Let $\bar{F} = e^\sigma F$ be a conformal deformation of F, where $\sigma \in \pi^* C^\infty(M)$. Let η and $\bar{\eta}$ be the Cartan-type forms of (M, F) and $(M, \bar{F})$, respectively. Then $\bar{\eta} = \eta$ if and only if σ satisfies*

$$\mathbf{G}(\sigma)\mathbf{I} + \mathbf{A}(\mathbf{I}^*, d\sigma^*) = 0 \quad \text{and} \quad \langle \mathbf{I}^*, d\sigma^* \rangle = 0, \tag{3.4}$$

where $\mathbf{G} = \frac{y^i}{F}\frac{\delta}{\delta x^i}$ is the Reeb vector field on SM; $\mathbf{A}$ is the Cartan tensor and $\mathbf{I}$ is the usual Cartan form (cf. Remark 3); $\mathbf{I}^*$, $d\sigma^*$ are the dual vector fields of $\mathbf{I}$, $d\sigma$ with respect to the metric $g^{T(SM)}$, respectively.

Proof. By (1.1), one has $\overline{G}^i = G^i + \sigma_{x^k} y^k y^i - \frac{1}{2} F^2 \sigma_{x^k} g^{ki}$. Furthermore,

$$\frac{\partial \overline{G}^i}{\partial y^j} = \frac{\partial G^i}{\partial y^j} + \sigma_{x^j} y^i + \sigma_{x^k} y^k \delta^i_j - F F_{y^j} \sigma_{x^k} g^{ki} + F A_{pqj} g^{ip} g^{qk} \sigma_{x^k},$$

and

$$\begin{aligned}\overline{\delta y^i} &= \frac{1}{\bar{F}}\left(dy^i + \frac{\partial \overline{G}^i}{\partial y^j}dx^j\right)\\ &= e^{-\sigma}\frac{1}{F}\Bigg[dy^i + \left(\frac{\partial G^i}{\partial y^j} + \sigma_{x^j}y^j + \sigma_{x^k}y^k\delta^i_j - FF_{y^j}\sigma_{x^k}g^{ki}\right.\\ &\quad\left. +\ FA_{pqj}g^{ip}g^{qk}\sigma_{x^k}\right)dx^j\Bigg]\\ &= e^{-\sigma}\delta y^i + e^{-\sigma}\frac{1}{F}\Big(\sigma_{x^j}y^i + \sigma_{x^k}y^k\delta^i_j - FF_{y^j}\sigma_{x^k}g^{ki}\\ &\quad + FA_{pqj}g^{ip}g^{qk}\sigma_{x^k}\Big)\delta x^j.\end{aligned}$$

Corresponding to $\bar{F}$, one has that $\overline{\omega^i} = e^{\sigma}\omega^i$ and $\overline{\omega^{n+\gamma}} = -e^{\sigma}v^i_j\overline{\delta y^j}$. Now,

$$\begin{aligned}-\overline{\omega^{n+\gamma}} &= v^\gamma_j\delta y^j + \frac{1}{F}v^\gamma_j(\sigma_{x^k}y^j + \sigma_{x^l}y^l\delta^j_k - FF_{y^k}\sigma_{x^l}g^{lj} + FA_{pqk}g^{jp}g^{ql}\sigma_{x^l})\delta x^k\\ &= -\omega^{n+\gamma} + \frac{1}{F}\sigma_{x^l}y^l v^\gamma_k\delta x^k - g^{lj}v^\gamma_j\sigma_{x^l}F_{y^k}\delta x^k + v^\gamma_j A_{pqk}g^{jp}g^{ql}\sigma_{x^l}\delta x^k\\ &= -\omega^{n+\gamma} + \frac{1}{F}\sigma_{x^l}y^l\omega^\gamma + v^\gamma_j A_{pqk}g^{jp}g^{ql}\sigma_{x^l}\delta x^k - g^{lj}v^\gamma_j\sigma_{x^l}\omega^n.\end{aligned}$$

On the other hand, one sees easily from (2.4) that functions $H_{\alpha\beta\gamma}$ are unchanged under the above conformal deformations. Finally, we obtain

$$\begin{aligned}\bar{\eta} &= \bar{H}_{ii\gamma}\overline{\omega^{n+\gamma}}\\ &= H_{ii\gamma}\omega^{n+\gamma} - \frac{1}{F}\sigma_{x^l}y^l H_{ii\gamma}\omega^\gamma - H_{ii\gamma}v^\gamma_j A_{pqk}g^{jp}g^{ql}\sigma_{x^l}\delta x^k + H_{ii\gamma}v^\gamma_j g^{lj}\sigma_{x^l}\omega^n\\ &= \eta + \frac{y^l}{F}\sigma_{x^l}\mathbf{I} + A_j g^{jp}A_{pqk}g^{ql}\sigma_{x^l}\delta x^k - A_j g^{jl}\sigma_{x^l}\omega^n\\ &= \eta + \mathbf{G}(\sigma)\mathbf{I} + \mathbf{A}(\mathbf{I}^*, d\sigma^*) - \langle\mathbf{I}^*, d\sigma^*\rangle\omega^n.\end{aligned}$$

□

By the above proposition, the Chern–Simons type form $\eta\wedge(d\eta)^{n-1}$ is a conformal invariant when the conformal factor σ satisfies (3.4).

It should be noted that the second equation in (3.4) also appears as the conformal invariance condition of the so called S-curvature (cf. [2, p.231]).

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