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Characterization of isolated complete intersection singularities with $\mathbb{C}^*$-action of dimension $n \geq 2$ by means of geometric genus and irregularity

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Dedicated to Professor Michael Artin on the occasion of his 79th birthday

It is well known that geometric genus p_g and irregularity q are two important invariants for isolated singularities. In this paper, we give a formula relating p_g and q for isolated singularities with $\mathbb{C}^*$-action in any dimension. We also give a simple characterization of the quasi-homogeneous isolated complete intersection singularities using p_g and q. As a corollary, we prove that q is an invariant of topological type for two-dimensional weighted homogeneous hypersurface singularities.

1. Introduction

Let $(V, 0)$ be a Stein germ of an analytic space with an isolated singularity at 0. $(V, 0)$ is a singularity with a (good) $\mathbb{C}^*$-action if the complete local ring of V at 0 is the completion of a (positively) graded ring. $(V, 0)$ is a quasi-homogeneous singularity if there exists an analytic isomorphism type of $(V, 0)$ which is defined by weighted homogeneous polynomials.

Let $f \in \mathbb{C}\{z_0, z_1, \ldots, z_n\}$ be a holomorphic function germ with an isolated singularity at the origin. It is well known that

$$\mu = \dim\mathbb{C}\{z_0, z_1, \ldots, z_n\}/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$$

and

$$\tau = \dim\mathbb{C}\{z_0, z_1, \ldots, z_n\}/(f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n)$$

are two very important invariants for hypersurface singularities. Clearly, $\mu \geq \tau$, and the equality holds if and only if f is quasi-homogeneous singularity by a well-known theorem of Saito [13]. Both μ and τ can also be defined for n-dimensional isolated complete intersection singularity (ICIS) with $n \geq 1$ in the following manner:

$$\mu = \mathrm{rk} H_n(F)$$

and

$$\tau = \dim T^1_{V,0},$$

where F is the Milnor fiber of a Milnor fibration of $(V, 0)$ (see [7]), and τ is the dimension of the base space of a semi-universal deformation of $(V, 0)$. From the defining equations of $(V, 0)$, one can give formulas for μ and τ as dimensions of certain finite length modules, but it is no longer clear what the relation between these invariants is. This problem was first considered by Greuel [2], who conjectured $\mu \geq \tau$, and proved the inequality in two cases: $n = 1$ or the link of V is a rational homotopy sphere. Greuel also proved that (in every dimension) $\mu = \tau$ if $(V, 0)$ is quasi-homogeneous. Looijenga [6] proved that for ICIS of dimension $n = 2$, $\mu \geq \tau + b$ where $b =$ number of loops in the resolution dual graph of $(V, 0)$. Then Looijenga and Steenbrink [7] generalized this result for all $n \geq 2$. In [23], Wahl proved that for two-dimensional ICIS $(V, 0)$, $\mu \geq \tau + b$ and $\mu = \tau + b$ if and only if $(V, 0)$ is quasi-homogeneous (for $b = 0$) or $(V, 0)$ is cusp ($b = 1$). More recently, Vosegaard [20] generalized this result for general n. If $(V, 0)$ is an ICIS of any dimension, he proved that $(V, 0)$ is quasi-homogeneous if and only if $\mu = \tau$.

Let $(V, 0)$ be a normal surface singularity. Wagreich [22] first defined an invariant geometric genus p_g for the singularity $(V, 0)$. It turns out that this is an important invariant for the theory of normal surface singularities. In [29], Yau introduced another invariant called irregularity q of the singularity $(V, 0)$. This invariant is interesting for the following reason. It is a long-term conjecture that normal surface singularities are not rigid, i.e., $\dim T^1_{V,0} \geq 1$ where $T^1_{V,0}$ is the set of isomorphism classes of first order infinitesimal deformations of V. In the case of Gorenstein surface singularities this irregularity actually gives a lower bound for $\dim T^1_{V,0}$. In fact, both geometric genus and irregularity can also be defined for general n-dimensional isolated singularities. Let $(V, 0)$ be a normal isolated singularity of dimension $n (\geq 2)$. Let $\pi : \tilde{V} \to V$ be a resolution of the singularity of V with exceptional set $E = \pi^{-1}(0)_{\mathrm{red}}$. Then $p_g := \dim R^{n-1}\pi_* \mathscr{O}_{\tilde{V}}$, and $q := \dim H^0(\Omega^{n-1}_{\tilde{V}-E})/H^0(\Omega^{n-1}_{\tilde{V}})$.

In [29], Yau gave a formula for the irregularity in case $(V, 0)$ is a hypersurface singularity or a two-dimensional singularity with $\mathbb{C}^*$-action. Moreover, for an n-dimensional singularity with $\mathbb{C}^*$-action, Yau gave a lower estimate for irregularity in terms of geometric genus. He proved

$$q \geq p_g - h^{n-1}(\mathscr{O}_E).$$

In this paper, one of our main results is to prove that the above inequality is actually an equality.

Main Theorem A. *Let $(V, 0)$ be a normal isolated singularity of dimension $n(\geq 2)$ with $\mathbb{C}^*$-action. Let $\pi : \tilde{V} \to V$ be a good resolution of the singularity $(V, 0)$ with $E = f^{-1}(V)_{\mathrm{red}}$. Then $q = p_g - h^{n-1}(\mathscr{O}_E)$.*

A natural question is how to use p_g and q to characterize quasi-homogeneous singularities. We prove that the converse of the above theorem is also correct for non-Du Bois ICISs.

Remark 1.1. If $(V, 0)$ is a two-dimensional isolated Gorenstein singularity, then $(V, 0)$ is a Du Bois singularity if and only if $(V, 0)$ is either rational, simple elliptic or cusp (see [3]).

Main Theorem B. *Let $(V, 0)$ be a normal ICIS of dimension $n(\geq 2)$, and $\pi : \tilde{V} \to V$ be a good resolution of the singularity $(V, 0)$ with $E = f^{-1}(0)_{\mathrm{red}}$. If $q = p_g - h^{n-1}(\mathscr{O}_E)$, then either $(V, 0)$ has a $\mathbb{C}^*$-action or $(V, 0)$ is a Du Bois singularity.*

Theorem B is a generalization of Wahl's well-known theorem in two-dimensional case (Theorem 1.9 [23]). Let $(V, 0)$ be a normal surface singularity, $\pi : \tilde{V} \to V$ a good resolution, and $E \subset \tilde{V}$ the (reduced) exceptional fibre E is a union of smooth curves E_i, $i = 1, \ldots, k$. Let g_i be genus of E_i, $g = \sum_{i=1}^{k} g_i$. Also define $b =$ first Betti number of the dual graph of $E(=$ number of loops). Then $h^1(\mathscr{O}_E) = g + b$, $\dim H^1(E, \mathbb{C}) = 2g + b$. Denote the geometric genus by $p_g = h^1(\mathscr{O}_{\tilde{V}})$ and irregularity by $q = \dim H^0(\Omega^1_{\tilde{V}-E})/H^0(\Omega^1_{\tilde{V}})$.

Steenbrink introduced three other invariants α, β, γ that are non-negative integers (see (1.8) [23]).

Theorem 1.1 (Steenbrink). *Let $E \subset \tilde{V} \to V$ be a good resolution of a normal surface singularity, with p_g, g, b and $\alpha, \beta, \gamma \geq 0$ as above. Then the irregularity q is given by*

$$q = p_g - g - b - \alpha - \beta - \gamma.$$

Theorem 1.2 (Wahl). *Let* $(V,0)$ *be a two-dimensional Gorenstein surface singularity. Then* $\alpha=\beta=\gamma=0$ *iff either* $(V,0)$ *is quasi-homogeneous* (*so* $b=0$), *or* $(V,0)$ *is a cusp* (*so* $b=1$).

Our Theorem B is a generalization of Theorem 1.2 to ICIS in arbitrary dimension.

In Section 2, we recall the basic propositions of geometric genus and irregularity. Theorem A and Theorem B are proved in Sections 3 and 4, respectively. In Section 5, we give a formula of irregularity in terms of weights for weighted homogeneous polynomial in three variables. In particular, the irregularity is an invariant of topological type in this case.

2. Geometric genus and irregularity

We first recall the concept of the geometric genus of normal isolated singularities; for more details the reader is referred to Laufer [4] and Yau [26, 27, 29].

Let $(V,0)$ be a normal isolated singularity of dimension $n(\geq 2)$. Let $\pi:\tilde{V}\to V$ be a resolution of the singularity of V with exceptional set $E=\pi^{-1}(0)_{\mathrm{red}}$. We define $s^{(i)}, 1\leq i\leq n$, of singularity $(V,0)$ to be $\dim\Gamma(\tilde{V}-E,\Omega^i_{\tilde{V}})/\Gamma(\tilde{V},\Omega^i_{\tilde{V}})$ where $\Omega^i_{\tilde{V}}$ is the sheaf of germs of holomorphic i-forms on $\tilde{V}$.

Let $\bar{\Omega}^i_V$ be the 0th direct image sheaf $\pi_*\Omega^i_{\tilde{V}}$ of $\Omega^i_{\tilde{V}}$. By Grauert's direct image theorem ([1], P.207), $\bar{\Omega}^i_V$ is a coherent sheaf. Let $\iota: V-\{0\}\to V$ be the inclusion map. Then the 0-th direct image sheaf $\bar{\bar{\Omega}}^i_V:=\iota_*\Omega^i_{V-\{0\}}$ is coherent [15]. Hence, the quotient sheaf $\bar{\bar{\Omega}}^i_V/\bar{\Omega}^i_V$ is coherent and supported on 0. $s^{(i)}$ is exactly $\dim\bar{\bar{\Omega}}^i_V/\bar{\Omega}^i_V$. Therefore the invariants $s^{(i)}, 1\leq i\leq n$, are indeed invariants of isolated singularities. Wagreich defined geometric genus p_g of the singularity to be $\dim R^{n-1}\pi_*\mathscr{O}_{\tilde{V}}$. It is proved in [4, 5, 26] that $p_g=s^{(n)}$. Irregularity q of singularity $(V,0)$ is defined to be $s^{(n-1)}$.

The geometric genus of the singularity $(V,0)$ can also be calculated using square integrable forms.

Definition 2.1. $\omega\in\Gamma(V-\{0\},\mathscr{O}(K))$ is called square integrable if

$$\int_{W-\{0\}}\omega\wedge\bar{\omega}<\infty$$

where W is any sufficiently small relatively compact neighborhood of 0 in V.

Let $L^2(V-\{0\},\Omega^n)$ be the subspace of $\Gamma(V-\{0\},\mathscr{O}(K)$ consisting of holomorphic n-form on $V-\{0\}$, which are square integrable near the origin. Then $\Gamma(V-\{0\},\mathscr{O}(K))/L^2(V-\{0\},\Omega^n)$ is a finite-dimensional vector space. It can be shown that this integer $\dim\Gamma(V-\{0\},\mathscr{O}(K))/L^2(V-\{0\},\Omega^n)$ is independent of the choice of the Stein neighborhoods [5].

Theorem 2.1 (Laufer [5],Yau [26, 27]). *The geometric genus of a normal isolated singularity* $(V,0)$ *is*

$$p_g = dim\Gamma(V-\{0\},\mathscr{O}(K))/L^2(V-\{0\},\Omega^n).$$

3. Proof of Theorem A

Let E be a reduced divisor on a smooth manifold $\tilde{V}$, and let $\Omega^j_{\tilde{V}}(E)$ stand for the sheaf of differential j-forms on $\tilde{V}$ with at most simple poles along E.

Definition 3.1. A differential j-form with logarithmic poles along E on an open subset $U \subset \tilde{V}$ is a meromorphic j-form ω regular on $\tilde{V} \setminus E$ and such that both ω and $d\omega$ have at most simple poles along E.

Differential j-forms with logarithmic poles along E form a sheaf denoted by $\Omega^j_{\tilde{V}}(\log E)$. For any open subset $U \subset \tilde{V}$ we have

$$\Gamma(U,\Omega^j_{\tilde{V}}(\log E)) = \{\omega \in \Gamma(U,\Omega^j_{\tilde{V}}(E)) : \, d\omega \in \Gamma(U,\Omega^{j+1}_{\tilde{V}}(E))\}.$$

A normal crossing divisor E in $\tilde{V}$, is a reduced divisor which is locally defined by an equation of the form $f = x_1 \dots x_p$, where $x_1,\dots,x_n$ are local coordinates for $\tilde{V}$, $p \leq n$.

If E is a normal crossing divisor then $\Omega^j_{\tilde{V}}(\log E)$ is locally-free sheaf. In this case a form $\omega \in \Omega^j_{\tilde{V}}(\log E)$ can be written locally in the following way:

$$\omega = \sum_{1\leq k_1<\cdots<k_j\leq n} a_{k_1\dots k_j}\delta_{k_1} \wedge \cdots \wedge \delta_{k_j},$$

where

$$\delta_i = \begin{cases} \dfrac{dx_i}{x_i}, & \text{if } i \leq p, \\ dx_i, & \text{if } i > p. \end{cases}$$

In particular, we have

$$\Omega^j_{\tilde{V}}(\log E) = \wedge^j \Omega^1_{\tilde{V}}(\log E).$$

Definition 3.2. Let $\pi : \tilde{V} \to V$ be a resolution of $(V, 0)$ is called a good resolution if $\tilde{V}$ is non-singular, $E = \pi^{-1}(0)_{\text{red}}$ is a divisor with normal crossings, and π is a surjective and proper holomorphic map that restricts to an isomorphism $\pi : \tilde{V} \setminus E \to V \setminus \{0\}$.

By Hironaka's big theorem, there exists a good resolution for any singularity $(V, 0)$.

Theorem 3.1 (Steenbrink [17]). *Let $(V, 0)$ be a normal isolated singularity of dimension n, where V is a contractible Stein space. Let $\pi : \tilde{V} \to V$ be a good resolution of the singularity $(V, 0)$ with $E = \pi^{-1}(0)_{\text{red}}$. Then*

$$H^q(\tilde{V}, \mathscr{I}_E(\Omega^p_{\tilde{V}}(\log E))) = 0, \ \textit{for } p + q > n,$$

where $\mathscr{I}_E$ is the ideal sheaf of the divisor E.

Proof. See Theorem 2b [17]. □

Lemma 3.1 (Straten–Steenbrink [19]). *Let $(V, 0)$ be a normal isolated singularity of dimension n, where V is a contractible Stein space. Let $U = V - \{0\}$. Let $\pi : \tilde{V} \to V$ be a resolution of the singularity $(V, 0)$ with $E = f^{-1}(0)_{\text{red}}$. Then the map $d : H^0(\Omega^{n-1}_U)/H^0(\Omega^{n-1}_{\tilde{V}}) \to H^0(\Omega^n_U)/H^0(\Omega^n_{\tilde{V}}$ $(\log E))$ induced by differentiation is injective.*

Proof. See Corollary 1.4 [19]. □

Proposition 3.1. *Let $(V, 0)$ be a normal isolated singularity of dimension n where V is a contractible Stein space. Let $U = V - \{0\}$, and $\pi : \tilde{V} \to V$ be a good resolution of the singularity $(V, 0)$ with $E = f^{-1}(0)_{\text{red}}$. Then $q \leq p_g - h^{n-1}(\mathscr{O}_E)$.*

Proof. Since $p_g = \dim H^0(U, \Omega^n_U)/H^0(\tilde{V}, \Omega^n_{\tilde{V}})$, we have

$$\begin{aligned}
&\dim H^0(\Omega^n_U)/H^0(\Omega^n_{\tilde{V}}(\log E)) \\
&\quad = \dim H^0(\Omega^n_U)/H^0(\Omega^n_{\tilde{V}}) - \dim H^0(\Omega^n_{\tilde{V}}(\log E)/H^0(\Omega^n_{\tilde{V}}) \\
&\quad = p_g - \dim H^0(\Omega^n_{\tilde{V}}(\log E)/H^0(\Omega^n_{\tilde{V}}).
\end{aligned}$$

The exact sequence of sheaves

$$0 \to \mathscr{I}_E \otimes \Omega^n_{\tilde{V}}(\log E) \to \Omega^n_{\tilde{V}}(\log E) \to \Omega^n_{\tilde{V}}(\log E) \otimes \mathscr{O}_E \to 0$$

gives the long exact sequence

$$0 \to H^0(\mathscr{I}_E \otimes \Omega^n_{\tilde{V}}(\log E)) \to H^0(\Omega^n_{\tilde{V}}(\log E)) \to H^0(\Omega^n_{\tilde{V}}(\log E) \otimes \mathscr{O}_E)$$
$$\to H^1(\mathscr{I}_E \otimes \Omega^n_{\tilde{V}}(\log E)) \to \cdots.$$

By Theorem 3.1, $H^1(\mathscr{I}_E \otimes \Omega^n_{\tilde{V}}(\log E)) = 0$. Since $h^0(\mathscr{I}_E \otimes \Omega^n_{\tilde{V}}(\log E)) = h^0(\Omega^n_{\tilde{V}})$ and the dualing sheaf ω_E satisfies

$$\omega_E \cong \Omega^n_{\tilde{V}}(\log E) \otimes \mathscr{O}_E$$

([16] page 523), we have $H^0(\Omega^n_{\tilde{V}}(\log E) \otimes \mathscr{O}_E) \cong H^{n-1}(\mathscr{O}_E)$ by Serre duality. Thus we have $\dim H^0(\Omega^n_{\tilde{V}}(\log E))/H^0(\Omega^n_{\tilde{V}}) = h^{n-1}(\mathscr{O}_E)$. By Lemma 3.1, we get $q \leq p_g - h^{n-1}(\mathscr{O}_E)$. □

Theorem 3.2 (Yau [28]). *Let $(V,0)$ be a $n(\geq 2)$-dimensional normal isolated singularity with $\mathbb{C}^*$-action. Let $\pi : \tilde{V} \to V$ be a good resolution of the singularity $(V,0)$ with $E = \pi^{-1}(0)_{\text{red}}$. Then $q \geq p_g - h^{n-1}(\mathscr{O}_E)$.*

Proof. See [28]. □

Proof of Theorem A. It is a immediately corollary of Proposition 3.1 and Theorem 3.2. □

4. Proof of Theorem B

We first recall the definition of Du Bois singularity.

Definition 4.1. A normal isolated singularity $(V,0)$ is called a Du Bois singularity if the canonical map $(R^i\pi_*\mathscr{O}_{\tilde{V}})_0 \to H^i(E, \mathscr{O}_E)$ is an isomorphism for each $i > 0$, where $\pi : \tilde{V} \to V$ is a good resolution and $E = \pi^{-l}(0)_{\text{red}}$.

If $(V,0)$ is an isolated complete intersection, then we have more simple equivalent definition as follows.

Definition 4.2. Let $(V,0)$ be an ICIS of dimension $n \geq 2$. We can assume that V is contractible. Put $U = V \setminus 0$. Let $\pi : \tilde{V} \to V$ be a good resolution of $(V,0)$, and $E = \pi^{-1}(0)_{\text{red}}$. The ICIS $(V,0)$ is said to be

(1) rational if $R^{n-1}\pi_*(\mathscr{O}_{\tilde{V}}) = 0$;

(2) Du Bois if $R^{n-1}\pi_*(\mathscr{O}_{\tilde{V}}(-E)) = 0$;

(3) purely elliptic if it is Du Bois and not rational.

Remark 4.1. These definitions 4.2 do not apply to more general situations in the given formulation. For instance, the definition 4.1 amounts to saying that an arbitrary normal singularity is Du Bois if the natural map π^* : $\mathbf{m}_{V,0} \to \pi_*(\mathscr{O}_{\tilde{V}}(-E))$ is an isomorphism and $R^j\pi_*(\mathscr{O}_{\tilde{V}}(-E))_0 = 0$ for $j > 0$. In the setup above for ICIS, this is equivalent to (2) (see 4.2 [20]).

Looijenga and Steenbrink [7] gave the formula

$$\mu - \tau = \sum_{j=0}^{n-2} h^{0,j} + a_1 + a_2 + a_3 \tag{4.1}$$

expressing the difference between the Milnor number and the Tjurina number of $(V, 0)$ as a sum of non-negative integers.

In the formula, $h^{i,j}$ denotes the (i, j)-th Hodge number of the mixed Hodge structure on $(n-1)$-cohomology of the link of $(V, 0)$.

The term a_1 is the dimension of the vector space

$$A_1 = \operatorname{Coker}\left(H^0(\Omega_U^{n-1}) \xrightarrow{d} \frac{H^0(\Omega_U^n)}{H^0(\Omega_{\tilde{V}}^n(\log E))} \right).$$

We will not define a_2 and a_3 explicitly here. According to Lemma 2.7 [21] we have the following equality:

$$a_2 + a_3 = e_2^{n,0} + b_2^{(1)},$$

where $e_2^{n,0}$ and $b_2^{(1)}$ are the dimensions of the vector spaces

$$E_2^{n,0} = \operatorname{Coker}(H^0(\Omega_{\tilde{V}}^{n-1}(\log E)(-E)) \xrightarrow{d} H^0(\Omega_{\tilde{V}}^n)), \text{ and}$$
$$B_2^{(1)} = \operatorname{Coker}(H^0(\Omega_{\tilde{V}}^{n-1}(\log E)) \to H^0(\Omega_{\tilde{V}}^{n-1}(\log E) \otimes \mathscr{O}_E))$$

respectively.

Set $\chi = \sum_{j=0}^{n-2} h^{0,j} + b_2^{(1)}$. By equation (4.1), we have $\mu - \tau = e_2^{n,0} + \chi + a_1$.

The following theorem that is crucial ingredient in our proof is due to Vosegaard.

Theorem 4.1 (Vosegaard [20]). *Let $(V, 0)$ be an ICIS of dimension $n \geq 2$. Then $(V, 0)$ is quasi-homogeneous if and only if*

(1) $e_2^{n,0} = 0$, *in case $(V, 0)$ is rational;*

(2) $\chi = 0$ *in case* $(V, 0)$ *is purely elliptic;*

(3) $a_1 = 0$, *in case* $(V, 0)$ *is non-Du Bois.*

In particular, $(V, 0)$ *is quasi-homogeneous if and only if* $\mu = \tau$.

Proof of Theorem B. Let $\pi : \tilde{V} \to V$ be a good resolution of $(V, 0)$, and $E = \pi^{-1}(0)_{\mathrm{red}}$. We claim that $a_1 = p_g - h^{n-1}(\mathscr{O}_E) - q$. Since $p_g = \dim H^0(U, \Omega^n_U)/H^0(\tilde{V}, \Omega^n_{\tilde{V}})$ and by Lemma 3.1, the map

$$d : H^0(\Omega^{n-1}_U)/H^0(\Omega^{n-1}_{\tilde{V}}) \to H^0(\Omega^n_U)/H^0(\Omega^n_{\tilde{V}}(\log E))$$

induced by differentiation is injective. Hence,

$$\begin{aligned} a_1 &= \dim\mathrm{Coker}(H^0(\Omega^{n-1}_U) \xrightarrow{d} \frac{H^0(\Omega^n_U)}{H^0(\Omega^n_{\tilde{V}}(\log E))}) \\ &= \dim\mathrm{Coker}(\frac{H^0(\Omega^{n-1}_U)}{H^0(\Omega^{n-1}_{\tilde{V}})} \xrightarrow{d} \frac{H^0(\Omega^n_U)}{H^0(\Omega^n_{\tilde{V}}(\log E))}) \\ &= \dim H^0(\Omega^n_U)/H^0(\Omega^n_{\tilde{V}}(\log E)) - q. \end{aligned}$$

By proposition 3.1 $\dim H^0(\Omega^n_U)/H^0(\Omega^n_{\tilde{V}}(\log E)) = p_g - h^{n-1}(\mathscr{O}_E)$. It follows that $a_1 = p_g - h^{n-1}(\mathscr{O}_E) - q$. By assumption we have $a_1 = 0$ and if $(V, 0)$ is non-Du Bois, then $(V, 0)$ is quasi-homogeneous by Theorem 4.1 (3). □

5. Applications

In this section, we shall give two ways of applications of Theorem A.

5.1. Singularities with $q = 0$

Proposition 5.1. *Let* $(V, 0)$ *be an* n*-dimensional normal isolated Du Bois singularity with a* $\mathbb{C}^*$*-action. Let* $\pi : \tilde{V} \to V$ *be a good resolution of the singularity* $(V, 0)$ *with* $E = \pi^{-1}(0)_{\mathrm{red}}$. *Then* $q = 0$

Proof. Since a singularity is Du Bois, then by definition 4.1 the canonical map $R^i\pi_*\mathscr{O}_{\tilde{V}} \to H^i(E, \mathscr{O}_E)$ is an isomorphism for each i, in particular $p_g = \dim(R^{n-1}\pi_*\mathscr{O}_{\tilde{V}})_0 = h^{n-1}(\mathscr{O}_E)$. By Theorem A, we have $q = 0$. □

The converse of the above proposition is also correct.

We recall the following lemma.

Lemma 5.1 (Steenbrink [16]). *Let $(V, 0)$ be a normal isolated singularity of dimension n, where V is a contractible Stein space, with $\mathbb{C}^*$-action. Let $\pi : \tilde{V} \to V$ be a good resolution of the singularity $(V, 0)$ with $E = \pi^{-1}(0)_{\text{red}}$. Then for all $i \geq 0$ the natural map*

$$H^i(\tilde{V}, \mathscr{O}_{\tilde{V}}) \to H^i(E, \mathscr{O}_E)$$

is surjective.

Proposition 5.2. *Let $(V, 0)$ be a normal isolated Cohen–Macaulay singularity of dimension $n \geq 2$, with $\mathbb{C}^*$-action. Let $\pi : \tilde{V} \to V$ be a good resolution of the singularity $(V, 0)$ with $E = \pi^{-1}(0)_{\text{red}}$. Then $(V, 0)$ is a Du Bois singularity if and only if $q = 0$.*

Proof. Let $\bar{h}^{p,q} := \dim H^q(\tilde{V}, \Omega^p_{\tilde{V}}(\log E)(-E))$. Since $(V, 0)$ is Cohen-Macaulay, i.e., has depth n, then by Proposition 1 [18] the only possible non-zero $\bar{h}^{0,q}$ is $\bar{h}^{0,n-1}$. Therefore, $(V, 0)$ is Du Bois singularity if and only if $\bar{h}^{0,n-1} = 0$ (see [18]), i.e., $H^{n-1}(\tilde{V}, \mathscr{O}_{\tilde{V}}(-E)) = 0$. We have the short exact sequence

$$0 \to \mathscr{O}_{\tilde{V}}(-E) \to \mathscr{O}_{\tilde{V}} \to \mathscr{O}_E \to 0,$$

which give a long exact sequence

$$\begin{aligned} \cdots \to H^{n-2}(\tilde{V}, \mathscr{O}_{\tilde{V}}) \to H^{n-2}(E, \mathscr{O}_E) \to H^{n-1}(\tilde{V}, \mathscr{O}_{\tilde{V}}(-E)) \\ \to H^{n-1}(\tilde{V}, \mathscr{O}_{\tilde{V}}) \to H^{n-1}(E, \mathscr{O}_E) \to H^n(\tilde{V}, \mathscr{O}_{\tilde{V}}(-E)) \to \cdots . \end{aligned}$$

Since by Lemma 5.1, $H^i(\tilde{V}, \mathscr{O}_{\tilde{V}}) \to H^i(E, \mathscr{O}_E)$ for $i \geq 0$ are surjective, in particular, $H^{n-2}(\tilde{V}, \mathscr{O}_{\tilde{V}}) \to H^{n-2}(E, \mathscr{O}_E)$ is surjective. Hence, we get a long exact sequence

$$\begin{aligned} 0 \to H^{n-1}(\tilde{V}, \mathscr{O}_{\tilde{V}}(-E)) \to H^{n-1}(\tilde{V}, \mathscr{O}_{\tilde{V}}) \to H^{n-1}(E, \mathscr{O}_E) \\ \to H^n(\tilde{V}, \mathscr{O}_{\tilde{V}}(-E)) \to \cdots \end{aligned}$$

By Siu's theorem [14], we have $H^n(\tilde{V}, \mathscr{O}_{\tilde{V}}(-E)) = 0$, thus $H^{n-1}(\tilde{V}, \mathscr{O}_{\tilde{V}} (-E)) = 0$ if and only if $h^{n-1}(\tilde{V}, \mathscr{O}_{\tilde{V}}) = h^{n-1}(E, \mathscr{O}_E)$, i.e., $p_g = h^{n-1}(E, \mathscr{O}_E)$ which is equivalent to $q = 0$ by Theorem A. □

5.2. Topological invariants of singularities

In this section, we will give a formula of q in terms of weights for isolated weighted homogeneous hypersurface singularities of dimension 2. As a byproduct we prove q is a invariant of topological type for those singularities.

A polynomial $f(z_0, z_1, \ldots, z_n)$ is a weighted homogenous of type $(w_0, w_1, \ldots, w_n)$, where $(w_0, w_1, \ldots, w_n)$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $\frac{i_0}{w_0} + \frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} = 1$. $(w_0, w_1, \ldots, w_n)$ are called weights of f. In [30] the first author and Xu proved the following theorem.

Theorem 5.1 [30]. *Let $(V, 0)$ be an isolated quasi-homogeneous surface singularity defined by a weighted homogenous polynomial in $\mathbb{C}^3$ with weights (w_0, w_1, w_2). Then the topological type of $(V, 0)$ determines and is determined by its weights (w_0, w_1, w_2).*

Let $f(z_0, z_1, z_2)$ be weighted homogeneous with weights $w_i = \frac{u_i}{v_i}, i = 0, 1, 2$ in reduced form. For integers a_1, a_2, a_3, let (a_1, a_2, a_3) denote their greatest common divisor. Define

$$c = (u_0, u_1, u_2);\ c_0 = (u_1, u_2)/c;\ c_1 = (u_0, u_2)/c;\ c_2 = (u_0, u_2)/c.$$

Then for some positive integers $\gamma_0, \gamma_1, \gamma_2$, we have

$$u_0 = cc_1c_2\gamma_0;\ u_1 = cc_0c_2\gamma_1;\ u_2 = cc_0c_1\gamma_2.$$

Note that c_0, c_1, c_2 are pairwise relatively prime, $\gamma_0, \gamma_1, \gamma_2$ likewise and $(c_i, \gamma_i) = 1$, for $i = 0, 1, 2$. Thus we have

$$d = < w_0, w_1, w_2 > = cc_0c_1c_2\gamma_0\gamma_1\gamma_2$$

and

$$q_0 = v_0c_0\gamma_1\gamma_2;\ q_1 = v_1c_1\gamma_0\gamma_2;\ q_2 = v_2c_2\gamma_0\gamma_1.$$

The link $K_f = f^{-1}(0) \cap S^5$, where S^5 is a sphere with center at origin, is a Seifert fibered 3-manifold. Orlik and Wagreich [11] have calculated the Seifert invariants of K_f,

$$\{-b; g; n_1(\alpha_1, \beta_1), n_2(\alpha_2, \beta_2), n_3(\alpha_3, \beta_3), n_4(\alpha_4, \beta_4)\},$$

where g is the genus of the central curve in the minimal good resolution of the singularity $(V, 0)$ which is defined by f and b is the self intersection number of the central curve. Since here we are only interested in g , hence we do not give the definitions of n_i, α_i and β_i, which can be found in [11]. Explicitly, g is given by the following formula.

Proposition 5.3 [11]. *With the notation above*

$$g = \frac{c^2c_0c_1c_2 - c(c_0v_0 + c_1v_1 + c_2v_2) + v_0v_1 + v_1v_2 + v_2v_0 - v_0v_1v_2}{2v_0v_1v_2},$$

where $c, c_i, v_i, 0 \le i \le 2$ depend only on w_0, w_1, w_2.

In fact p_g can also be calculated with respect to weights for non-degenerate weighted homogeneous hypersurface singularities. Let $f(x_0, \dots, x_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose that f has an isolated critical point at the origin. f can be developed in a convergent Taylor series $f(x_0, \dots, x_n) = \sum a_\lambda x^\lambda$, where $x^\lambda = x_0^{\lambda_0} \cdots x_n^{\lambda_n}$. Recall that the Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbb{R}^+)^{n+1}\}$ for λ such that $a_\lambda \neq 0$. Finally, let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with cone point at 0. To any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(x) = \sum_{\lambda \in \Delta} a_\lambda x^\lambda$. We say that f is non-degenerate if f_Δ has no critical point in $(\mathbb{C}^*)^{n+1}$ for any $\Delta \in \Gamma(f)$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$. We say that a point p of the integral lattice $\mathbb{Z}^{n+1}$ in $\mathbb{R}^{n+1}$ is positive if all the coordinates of p are positive. The following beautiful theorem is due to Merle–Teissier.

Theorem 5.2 (Merle–Teissier [8]). *Let $(V, 0)$ be an isolated hypersurface singularity defined by a non-degenerate holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Then the geometric genus $p_g = \sharp\{p \in \mathbb{Z}^{n+1} \cap \Gamma_-(f) \ : p$ is positive$\}$.*

A polynomial $f(x_0, x_1, \dots, x_n)$ is a non-degenerate weighted homogeneous singularity of type $(w_0, w_1, \dots, w_n)$. As a consequence of the theorem of Merle–Teissier, we know that in case of isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in a tetrahedron, i.e., we have

$$p_g = \sharp\{(x_0, x_1, \dots, x_n) \in \mathbb{Z}_+^n : \ \frac{x_0}{\omega_0} + \frac{x_1}{\omega_1} + \cdots + \frac{x_n}{\omega_n} \le 1\},$$

where $\mathbb{Z}_+$ is the set of positive integers.

Theorem 5.3. *With the notation as above, let $(V, 0)$ be an isolated singularity defined by a weighted homogeneous polynomial $f(x_0, x_1, x_2)$ of type*

(w_0, w_1, w_2). *Then*

$$q = \sharp\{(x_0, x_1, x_2) \in \mathbb{Z}^3_+ : \frac{x_0}{\omega_0} + \frac{x_1}{\omega_1} + \frac{x_2}{\omega_2} \leq 1\}$$
$$- \frac{c^2 c_0 c_1 c_2 - c(c_0 v_0 + c_1 v_1 + c_2 v_2) + v_0 v_1 + v_1 v_2 + v_2 v_0 - v_0 v_1 v_2}{2 v_0 v_1 v_2}.$$

Proof. By Theorem A, $q = p_g - h^1(\mathscr{O}_E)$. It is well known that the dual resolution graph of a weighted homogeneous singularity is a star-shaped dual graph, and all components of the star-shaped dual graph are rational except for the central curve. Thus, $h^1(\mathscr{O}_E) = g$, where g is the genus of the curve correspond to the central curve of the dual graph of resolution. By Proposition 5.3, we get

$$h^1(\mathscr{O}_E) = \frac{c^2 c_0 c_1 c_2 - c(c_0 v_0 + c_1 v_1 + c_2 v_2) + v_0 v_1 + v_1 v_2 + v_2 v_0 - v_0 v_1 v_2}{2 v_0 v_1 v_2}.$$

By Theorem 5.2 we have

$$p_g = \sharp\{(x_0, x_1, x_2) \in \mathbb{Z}^3_+ : \frac{x_0}{\omega_0} + \frac{x_1}{\omega_1} + \frac{x_2}{\omega_2} \leq 1\}.$$

Therefore,

$$q = \sharp\{(x_0, x_1, x_2) \in \mathbb{Z}^3_+ : \frac{x_0}{\omega_0} + \frac{x_1}{\omega_1} + \frac{x_2}{\omega_2} \leq 1\}$$
$$- \frac{c^2 c_0 c_1 c_2 - c(c_0 v_0 + c_1 v_1 + c_2 v_2) + v_0 v_1 + v_1 v_2 + v_2 v_0 - v_0 v_1 v_2}{2 v_0 v_1 v_2}.$$ □

Corollary 5.1. *Let $(V, 0)$ be an isolated singularity defined by a weighted homogeneous polynomial $f(x_0, x_1, x_2)$ of type (w_0, w_1, w_2). Then the irregularity q is a topological invariant.*

Proof. By Theorem 5.3 the formula for q only depends on the weights of the weighted homogeneous polynomial and by Theorem 5.1 the weights are topological invariant of weighted homogeneous surface singularities. We then obtain that the irregularity q of weighted homogeneous isolated surface singularity f is also a topological invariant. This means that if f, f' are two weighted homogeneous isolated surface singularities and $(\mathbb{C}^3, V(f), 0)$ is homeomorphic to $(\mathbb{C}^3, V(f'), 0)$, then $q = q'$ where q and q' are irregularities of f and f' respectively. □

Recall that p_g can be bounded by Milnor number for weighted homogenous singularities.

Theorem 5.4 (Yau–Zhang [25]). *Let $(V,0)$ be an isolated singularity defined by a weighted homogeneous polynomial $f(z_0,z_1,\ldots,z_n)$. Then $(n+1)!p_g \leq \mu$ and equality holds if and only if $\mu = 0$.*

In general, it is difficult to compute q. However, there is an upper bound for q using the weights for weighted homogenous surface singularities.

Corollary 5.2. *Let $(V,0)$ be an isolated singularity defined by a weighted homogeneous polynomial $f(x_0,x_1,x_2)$ of type (w_0,w_1,w_2). Then*

$$q \leq \frac{1}{6}\prod_{i=0}^{2}(w_i-1) - \frac{\begin{array}{c} c^2c_0c_1c_2 - c(c_0v_0+c_1v_1+c_2v_2) \\ +v_0v_1+v_1v_2+v_2v_0-v_0v_1v_2 \end{array}}{2v_0v_1v_2}.$$

Proof. Milnor and Orlik (see [9]) proved that $\mu = \prod_{i=0}^{2}(w_i-1)$. Then the above inequality follows from Theorems 5.4 and 5.2. □

Example 5.1. Consider $f = z_0^{105} + z_1^9 + z_1z_2^{14}$ with weights $(105, 9, 63/4)$. Thus $c = 3, c_0 = 3, c_1 = 7, c_2 = 1, v_0 = v_1 = 1, v_2 = 4$, and by Proposition (5.3) we get $g = 19$. Since $\mu = (105-1)(9-1)(63/4-1) = 12272$; therefore by Corollary5.2 we get $q \leq 2026$.

Example 5.2. Consider $f = z_0^7 + z_1^3 + z_2^2$ with weights $(7,3,2)$. Obviously f is non-degenerate. By Proposition 5.3, we have $g = 0$, and it is easy to see only one positive point $(1,1,1)$ lying in the Newton polyhedron of f. Thus, $p_g = 1$. Since the dual graph of the resolution of singularity defined by f is star-shaped, then there is no loop. By Theorem A, we get $q = 1$. We have $q = p_g$ for this example. In fact, the singularity is very special, and we know that its dual resolution graph is

Note that a surface singularity has a rational homology sphere (QHS) link if and only if the dual minimal resolution (not necessarily good) graph is tree and each irreducible component is rational. It is easy to check that the singularity in Example 2 has a QHS link. In fact, it has an integral homology sphere link (i.e., QHS and $\det(E_i \cdot E_j) = \pm 1$). For those special surface singularities with QHS links, we have the following theorem.

Theorem 5.5. *Let $(V,0)$ be an isolated weighted homogeneous surface singularity with a QHS link, then $q = p_g$ is a topological invariant.*

Proof. Since $(V,0)$ has a QHS link, the dual graph of the resolution is a tree and has rational components, therefore $h^1(\mathscr{O}_E) = 0$, hence by Theorem A we have $q = p_g$. Observe that weighted homogenous singularities with QHS are splice quotient singularities (see [10]), and it is known that p_g can be calculated from the dual graph of the resolution for splice quotient singularities [12]. Therefore both p_g and q are topological invariants. □

The following beautiful result is due to Wahl.

Proposition 5.4 (Wahl [23]). *Let $(V, 0)$ be defined by $f(z_0, z_1, z_2)$ that is a weighted homogeneous singularity with weights (w_0, w_1, w_2). Let $w_i = \frac{u_i}{v_i}, i = 0, 1, 2$ in reduced form. Let $d = \langle u_0, u_1, u_2 \rangle$ (i.e., least common multiple of u_0, u_1, u_2). Define $q_i = d/w_i$ and $l = d - (q_0 + q_1 + q_2)$. Then $l < 0$ iff $(V, 0)$ is an rational double point, $l = 0$ iff $(V, 0)$ is simple elliptic. The Milnor algebra*

$$J = \mathbb{C}[z_0, z_1, z_2] / \left\langle \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2} \right\rangle$$

is graded with $J = \oplus_0^N J_i$, where $N = 2l + d$. Moreover,

$$p_g = dim \oplus_{i \leq l} J_i,$$
$$g = dim J_l.$$

Proof. See Lemma 4.3 in [23]. □

Remark 5.1. Proposition 5.4 is true not only for hypersurface surface singularities, but for Gorenstein surface singularities with $\mathbb{C}^*$-action (see [24]).

We get the following useful corollary.

Corollary 5.3. *With the same assumption as in* Proposition 5.4. *We have*

$$q = dim \oplus_{i < l} J_i.$$

Proof. It is immediately corollary by Theorem A and Proposition 5.4. □

An immediately application of above Corollary 5.3 is that we can reprove the following Theorem 5.6 of Yau (i.e., Theorem A in [28]) easily.

Theorem 5.6 (Yau [28]). *Let $(V, 0)$ be a Gorenstein surface singularity with a $\mathbb{C}^*$-action. Then $q = 0$ if and only if $(V, 0)$ is either a rational double point or a simple elliptic singularity.*

Proof. *Case 1.* If $(V, 0)$ is a hypersurface singularity, then by Corollary 5.3, $q = 0$ if and only if $l \leq 0$. It follows from Proposition 5.4 that $(V, 0)$ is either a rational double point or a simple elliptic singularity.

Case 2. If $(V, 0)$ is not hypersurface singularities, by Remark 5.1, the result can be derived using the same arguments given in Case 1. □

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