

Reduction of β -integrable 2-Segre structures

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We show that locally every β -integrable $(2, n)$ -Segre structure can be reduced to a torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure. This is done by observing that such reductions correspond to sections with holomorphic image of a certain ‘twistor bundle’. For the homogeneous $(2, n)$ -Segre structure on the oriented 2-plane Grassmannian, the reductions are shown to be in one-to-one correspondence with the smooth quadrics $Q \subset \mathbb{C}\mathbb{P}^{n+1}$ without real points.

1. Introduction

We study the problem of reducing the G -structure associated to a certain type of Segre structure to a torsion-free substructure.

Segre — or closely related structures and their counterparts in the category of complex manifolds were studied under various names, including tensor product structure [13], generalized conformal structure [11], complex paraconformal structure [3], (almost) Grassmann structure [1, 9, 14, 16], Segre structure [6, 12], and in [4] as an example of a class of structures called almost symmetric hermitian manifolds.

Here, by an (m, n) -Segre structure on a manifold M we mean a smoothly varying family of cones $\mathcal{S}_p \subset T_p M$ in the tangent spaces of M , each linearly isomorphic to the Segre cone of linear maps $\mathbb{R}^m \rightarrow \mathbb{R}^n$ of rank one. The tangent planes to M which are contained in some Segre cone \mathcal{S}_p come in two types, called α - and β -planes. An immersed submanifold $\Sigma \subset M$ whose tangent planes are all β -planes and which is maximal in the sense of inclusion is called a β -surface. A Segre structure is called β -integrable, if every β -plane is tangent to a unique β -surface.

In [12] Grossman showed that the space of paths of a certain class of geodesically simple path geometries, which he calls torsion-free, inherits a Segre structure. Bryant observed in [8] that the space of oriented geodesics Λ of a geodesically simple Finsler structure of constant flag curvature (CFC) 1 inherits a Kähler structure and a torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure

satisfying a certain positivity condition. Conversely, he shows that every torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure satisfying the positivity condition (and an integrability condition for $n = 2$) arises via a (generalized) CFC 1 Finsler structure.

The main result of the article is that locally every β -integrable $(2, n)$ -Segre structure \mathcal{S} can be reduced to a torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure. It follows with Bryant's result, that locally every β -integrable $(2, n)$ -Segre structure admitting a $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -reduction satisfying the positivity condition of [8] arises via a (generalized) CFC 1 Finsler structure.

Note that an $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -reduction of a β -integrable $(2, n)$ -Segre structure \mathcal{S} equips the underlying manifold with an integrable almost complex structure, which preserves \mathcal{S} and for which the β -surfaces are totally real.

This paper is organized as follows. In Section 2, we review the construction of a 'twistor bundle' $\rho : X_{\mathcal{S}} \rightarrow M$ over a manifold M which is equipped with a β -integrable $(2, n)$ -Segre structure and show in Section 3 that ρ -sections having holomorphic image are in one-to-one correspondence with reductions of \mathcal{S} to torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structures on M . It follows that locally every β -integrable $(2, n)$ -Segre structure can be reduced to a torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ structure. In Section 4 we show that for the homogeneous $(2, n)$ -Segre structure on the oriented 2-plane Grassmannian $M = G_2^+(\mathbb{R}^{n+2})$, the reductions are in one-to-one correspondence with the smooth quadrics $Q \subset \mathbb{C}\mathbb{P}^{n+1}$ without real points.

Remark 1.1. Before this work begun Robert Bryant informed the author about his private notes regarding the generality of positive CFC Finsler structures on the n -sphere. He shows that a positive CFC Finsler structure on the n -sphere all of whose geodesics are closed and of the same length gives rise to a D^2 -bundle $\rho : X \rightarrow \Lambda$, fibering over the space of oriented geodesics Λ , whose total space is a complex manifold. This bundle is isomorphic to $\rho_0 : \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1} \rightarrow G_2^+(\mathbb{R}^{n+2})$ in the case of a rectilinear Finsler structure. In addition, the Finsler structure induces a ρ -section having holomorphic image (isomorphic to a quadric in the rectilinear case) and conversely every such section satisfying a certain convexity condition gives rise to a Finsler structure on S^n sharing the same geodesics. Using Kodaira deformation theory this allows Bryant to determine the generality of such Finsler structures sharing the same unparametrized geodesics. Although being related, the results in this paper were arrived at independently.

2. 2-Segre structures

2.1. Definitions and examples

Let $m, n \geq 2$ be integers. A vector $v \in \mathbb{R}^m \otimes \mathbb{R}^n$ is called *decomposable* or *simple* if there exists $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ such that $v = x \otimes y$. Let $X = (X_\alpha)$ be linear coordinates on \mathbb{R}^m and $Y = (Y^k)$ on \mathbb{R}^n . Writing $Z_\alpha^k = X_\alpha \otimes Y^k$, the set of simple vectors in $\mathbb{R}^m \otimes \mathbb{R}^n$ is the zero locus of the homogeneous quadratic equations

$$(2.1) \quad Z_\alpha^i Z_\beta^k - Z_\beta^i Z_\alpha^k = 0$$

and thus is a cone. A subset \mathcal{S} in a real vector space V is called an (m, n) -Segre cone, if there exists an isomorphism $V \simeq \mathbb{R}^m \otimes \mathbb{R}^n$ which yields a bijection between \mathcal{S} and the cone of simple vectors in $\mathbb{R}^m \otimes \mathbb{R}^n$.

Let $\mathcal{S} \subset V$ be a Segre cone. Clearly, the isomorphism $V \simeq \mathbb{R}^m \otimes \mathbb{R}^n$ is unique up to composition with an element of the group $G(m, n)$, the subgroup of $\text{GL}(\mathbb{R}^m \otimes \mathbb{R}^n)$ consisting of maps preserving the cone of simple vectors. Let $H(m, n) = \text{GL}(m, \mathbb{R}) \otimes \text{GL}(n, \mathbb{R})$ and $\mathbb{Z}_2 \subset G(n, n)$ be the subgroup generated by the involution $x \otimes y \mapsto y \otimes x$ for $x, y \in \mathbb{R}^n$. Then we have an isomorphism of Lie groups¹

$$(2.2) \quad G(m, n) \simeq \begin{cases} H(m, n) & n \neq m, \\ H(n, n) \rtimes \mathbb{Z}_2 & n = m. \end{cases}$$

Definition 2.1. An (m, n) -Segre structure \mathcal{S} on a smooth mn -manifold M is a choice of an (m, n) -Segre cone $\mathcal{S}_p \subset T_p M$ in each tangent space of M which varies smoothly from point to point.

An isomorphism $f : T_p M \rightarrow \mathbb{R}^m \otimes \mathbb{R}^n$ will be called a *Segre coframe* at p if it maps \mathcal{S}_p to the cone of simple vectors in $\mathbb{R}^m \otimes \mathbb{R}^n$.

The set of Segre coframes at p will be denoted by $(F_{\mathcal{S}})_p$ and is the fiber of a principal right $G(m, n)$ -bundle $\pi : F_{\mathcal{S}} \rightarrow M$, with right action given by $R_g(f) = g^{-1} \circ f$ for $g \in G(m, n)$. The tautological 1-form ζ is defined by requiring that $\zeta_f = f \circ \pi'_f : T_f F_{\mathcal{S}} \rightarrow \mathbb{R}^m \otimes \mathbb{R}^n$ for $f \in F_{\mathcal{S}}$. It satisfies $R_g^* \zeta = g^{-1} \circ \zeta$ for $g \in G(m, n)$.

A linear subspace $\Pi \subset \mathcal{S}_p$ is called *simple*. The simple linear subspaces which are of the form $\Pi \simeq \mathbb{R}^m \otimes y$ for some $y \in \mathbb{R}^n$ are called α -planes and the simple linear subspaces which are of the form $\Pi \simeq x \otimes \mathbb{R}^n$ for some $x \in$

¹For a proof, see the appendix.

\mathbb{R}^m are called β -planes.² Note that α - and β -planes are not well defined for $n = m$ unless the Segre structure has been reduced to an $H(n, n)$ -structure. An immersed connected manifold $\Sigma \rightarrow M$ for which $T_p\Sigma$ is a β -plane for every point $p \in \Sigma$ is called a *proto β -surface*. If, in addition, $\Sigma \rightarrow M$ is maximal in the sense of inclusion, then Σ is called a *β -surface*. A Segre structure \mathcal{S} is called *β -integrable* if every β -plane Π is tangent to a unique β -surface $\Sigma \rightarrow M$. The notion of a (proto) α -surface and α -integrability are defined analogously. The necessary and sufficient conditions for a Segre structure of type (m, n) to be α - or β -integrable were given in [2, 16] (see also [3] for the complex case).

Example 2.1. Recall that a pseudo-Riemannian metric g on a smooth 4-manifold M with signature $(+, +, -, -)$ is said to have split-signature. Locally g may be written as

$$(2.3) \quad g = \varepsilon_1^1 \odot \varepsilon_2^2 - \varepsilon_2^1 \odot \varepsilon_1^2$$

for some linearly independent 1-forms ε_j^i . A vector $v \in TM$ is called *null* if $g(v, v) = 0$. It follows with (2.3) that the g -null vectors give rise to a (2,2)-Segre structure on M and conversely it can be shown that every (2,2)-Segre structure on a 4-manifold M gives rise to a unique conformal structure of split-signature on M .

Closely related to Segre structures is the notion of an almost Grassmann structure.

Definition 2.2. A smooth mn -manifold M is said to carry an *almost Grassmann structure* if there exist smooth vector bundles $E_m \rightarrow M$ and $E_n \rightarrow M$ of rank m, n respectively together with an isomorphism $TM \simeq E_m \otimes E_n$.

Remark 2.1. Clearly, an almost Grassmann structure on M induces a unique Segre structure on M , but the existence of a Segre structure \mathcal{S} on M is in general not sufficient for the existence of an almost Grassmann structure inducing \mathcal{S} . The two definitions are however equivalent when $m + n$ is odd (see the appendix).

Example 2.2. The prototypical example of a manifold carrying an almost Grassmann structure is the Grassmannian of m -planes in \mathbb{R}^{m+n} (see for

²The reader is warned that often the opposite convention regarding α - and β -planes is used. The convention used here is chosen to be consistent with [15].

instance [16] for details). We will construct the associated Segre structure in Section 4 for the case $m = 2$.

2.2. The structure equations of a 2-Segre structure

We will henceforth restrict our attention to $H(2, n)$ -structures $\pi : F_S \rightarrow M$ and simply speak of 2-Segre structures, thus implicitly assuming that $(2, 2)$ -Segre structures have been reduced to $H(2, 2)$ -structures. We think of $H(2, n)$ as a subgroup of $GL(2n, \mathbb{R})$ via the Kronecker product and consequently of $\pi : F_S \rightarrow M$ as a reduction with structure group $H(2, n)$ of the full coframe bundle $F \rightarrow M$ whose fiber at $p \in M$ consists of all isomorphisms $T_p M \rightarrow \mathbb{R}^{2n}$.

A linear connection θ on $F \rightarrow M$ is said to be *adapted* to the 2-Segre structure $\pi : F_S \rightarrow M$, if θ pulls-back to F_S to become a principal $H(2, n)$ -connection. A 2-Segre structure is called *torsion-free*, if it admits an adapted connection θ with vanishing torsion $\tau = d\zeta + \theta \wedge \zeta$.

Write $H = H(2, n)$ and $\mathfrak{h} \subset \mathfrak{gl}(2, \mathbb{R}) \otimes \mathfrak{gl}(n, \mathbb{R})$ for the Lie algebra of H . For computational purposes it is convenient to introduce the matrices

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and write e_k^i for the $(n \times n)$ -matrix whose entry is 1 at the position (k, i) and 0 otherwise. Using this notation an \mathfrak{h} -basis is given by

$$a \otimes I_n, \quad b_1 \otimes I_n, \quad b_2 \otimes I_n, \quad I_2 \otimes e_k^i,$$

and a principal H -connection θ on F_S may be written as

$$(2.4) \quad \theta = \chi \otimes I_n + I_2 \otimes \phi$$

with $\chi = \omega a + 2\xi_1 b_1 + 2\xi_2 b_2$ and $\phi = \phi_k^i e_i^k$ for some linearly independent 1-forms $\omega, \xi_1, \xi_2, \phi_k^i$ on F_S . Let $\xi = \xi_1 + i\xi_2$. Straightforward computations show that we may linearly identify \mathbb{R}^{2n} with \mathbb{C}^n in such a way that we can write the first structure equation in complex form:

Proposition 2.1. *The connection form (ω, ξ, ϕ) of an F_S -adapted connection θ satisfies*

$$(2.5) \quad d\zeta = -\left(i(\omega - \xi)I_n + \phi\right) \wedge \zeta - i\xi I_n \wedge \bar{\zeta} + \tau.$$

Proof. Omitted. □

Here, the forms ζ and τ are thought to be \mathbb{C}^n -valued and $\bar{\zeta}$ denotes the \mathbb{C}^n -valued 1-form on F_S which is obtained by complex conjugation of the entries of ζ .

We have the curvature forms

$$\begin{aligned}
 (2.6) \quad \Omega &= d\omega + \omega \wedge i(\xi - \bar{\xi}), \\
 \Xi &= d\xi + \xi \wedge i(\bar{\xi} - 2\omega), \\
 \Phi &= d\phi + \phi \wedge \phi - \omega \wedge (\xi + \bar{\xi}) I_n.
 \end{aligned}$$

Differentiating the structure equation (2.5) gives the *Bianchi-identity*

$$(2.7) \quad d\tau = (i(\Omega - \Xi) I_n + \Phi) \wedge \zeta + i\Xi I_n \wedge \bar{\zeta}.$$

2.3. A quasiholomorphic fiber bundle

Let $P = S^1 \cdot \text{GL}(n, \mathbb{R}) \subset H(2, n)$ be the closed subgroup consisting of elements of the form

$$e^{i\varphi} \cdot b = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \otimes b,$$

for some real number φ and $b \in \text{GL}(n, \mathbb{R})$. Equip the quotient $X_S = F_S/P$ with its canonical smooth structure so that the quotient projection $\nu : F_S \rightarrow X_S$ is a smooth surjective submersion. Note that the 1-forms η^k are ν -semibasic since they are π -semibasic. Moreover, since $\theta = (\omega, \xi, \phi)$ is a principal H -connection, it follows with (2.4) that ξ is ν -semibasic as well. Therefore the forms η^k together with ξ_1 and ξ_2 span the ν -semibasic 1-forms on F_S .

Lemma 2.1. *Let $\pi : F_S \rightarrow M^{2n}$ be a 2-Segre structure and $\theta = (\omega, \xi, \phi)$ an adapted connection. Then there exists a unique almost complex structure \mathfrak{J} on X_S such that a complex-valued 1-form on X_S is a (1,0)-form for \mathfrak{J} if and only if its ν -pullback is a linear combination of $\{\zeta^1, \dots, \zeta^n, \xi\}$ with coefficients in $C^\infty(F_S, \mathbb{C})$.*

Proof. Denote by $\frac{\partial}{\partial \eta^l}, \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \omega}, \frac{\partial}{\partial \phi_k^i}$, the vector fields dual to the coframing $(\eta^l, \xi_1, \xi_2, \omega, \phi_k^i)$. Define the map $\tilde{\mathfrak{J}} : TF_S \rightarrow TX_S$ by

$$\tilde{\mathfrak{J}}(v) = \nu' \left(-\eta^{2k}(v) \frac{\partial}{\partial \eta^{2k-1}} + \eta^{2k-1}(v) \frac{\partial}{\partial \eta^{2k}} - \xi_2(v) \frac{\partial}{\partial \xi_1} + \xi_1(v) \frac{\partial}{\partial \xi_2} \right).$$

The 1-forms η^l, ξ_1, ξ_2 are ν -semibasic and thus we have $\tilde{\mathfrak{J}}(v + w) = \tilde{\mathfrak{J}}(v)$ for every $v \in TF_S$ and $w \in TF_S$ tangent to the ν -fibers. Since θ is a principal H -connection, the equivariance $(R_h)^*\theta = h^{-1}\theta h$ for $h \in H$ together with a short computation gives

$$(2.8) \quad (R_{e^{i\varphi} \cdot b})^* \xi = e^{-2i\varphi} \xi,$$

for $e^{i\varphi} \cdot b \in S^1 \cdot \text{GL}(n, \mathbb{R})$. Moreover we have

$$(2.9) \quad (R_h)^* \eta = h^{-1} \eta$$

for every $h \in H$. Identifying $\text{GL}(n, \mathbb{C})$ with the subgroup of $\text{GL}(2n, \mathbb{R})$ commuting with $a \otimes I_n$ and using the fact that $S^1 \cdot \text{GL}(n, \mathbb{R}) \subset \text{GL}(n, \mathbb{C})$ together with (2.8, 2.9) implies $\tilde{\mathfrak{J}} \circ (R_{e^{i\alpha} \cdot b})' = \tilde{\mathfrak{J}}$. In other words there exists an almost complex structure $\tilde{\mathfrak{J}} : TX_S \rightarrow TX_S$ such that $\tilde{\mathfrak{J}} = \tilde{\mathfrak{J}} \circ \nu'$. Clearly, $\tilde{\mathfrak{J}}$ has the desired properties and these properties uniquely characterize $\tilde{\mathfrak{J}}$. \square

It is natural to ask when two F_S -adapted connections induce the same almost complex structure. We have:

Lemma 2.2. *The F_S -adapted connections $\theta = (\phi, \omega, \xi)$ and $\theta' = (\phi', \omega', \xi')$ induce the same almost complex structure on X_S if and only if $\xi - \xi' = \lambda_k \zeta^k$ for some smooth functions $\lambda_k : F_S \rightarrow \mathbb{C}$. In particular any two F_S -adapted connections with the same torsion induce the same almost complex structure.*

Proof. Let $\tilde{\mathfrak{J}}_\theta, \tilde{\mathfrak{J}}_{\theta'}$ denote the almost complex structures with respect to the connections θ, θ' and suppose $\xi' = \xi + \lambda_k \zeta^k$ for some smooth functions $\lambda_k : F_S \rightarrow \mathbb{C}$. Let α be a (1,0)-form for $\tilde{\mathfrak{J}}_\theta$. Then we may write

$$\nu^* \alpha = a_k \zeta^k + a \xi = a_k \zeta^k + a (\xi' - \lambda_k \zeta^k) = (a_k - \lambda_k) \zeta^k + a \xi'$$

for some smooth functions $a, a_k : F_S \rightarrow \mathbb{C}$, thus showing that α is a (1,0)-form for $\tilde{\mathfrak{J}}_{\theta'}$. Conversely suppose $\tilde{\mathfrak{J}}_\theta = \tilde{\mathfrak{J}}_{\theta'}$. Note that $\xi - \xi'$ is π -semibasic and may thus be written as

$$\xi - \xi' = \lambda_k \zeta^k + \lambda'_k \bar{\zeta}^k$$

for some smooth functions $\lambda_k, \lambda'_k : F_S \rightarrow \mathbb{C}$. Let α be a (1,0)-form for $\tilde{\mathfrak{J}}_\theta$. Then we may write

$$\nu^* \alpha = a_k \zeta^k + a \xi = a'_k \zeta^k + a' \xi' = a'_k \zeta^k + a' (\xi - \lambda_k \zeta^k - \lambda'_k \bar{\zeta}^k)$$

for some smooth functions $a, a', a_k, a'_k : F_S \rightarrow \mathbb{C}$. Thus, it follows

$$(a - a')\xi + (a_k - a'_k + a'\lambda_k)\zeta^k + a'\lambda'_k\bar{\zeta}^k = 0$$

which can hold for an arbitrary (1,0)-form α if and only if $\lambda'_k = 0$. Finally it is easy to check that if (ω, ξ, ϕ) and (ω', ξ', ϕ') are two F_S -adapted connections with the same torsion, then there exist n smooth complex-valued functions a_k on F_S such that

$$(2.10) \quad \begin{aligned} \omega' - \omega &= \operatorname{Re}(a_k)\operatorname{Im}(\zeta^k), \\ \xi' - \xi &= \frac{1}{2i}\bar{a}_k\zeta^k, \\ (\phi')^i_k - \phi^i_k &= \operatorname{Re}(a_k\zeta^i) + \delta^i_k \operatorname{Re}(a_l)\operatorname{Re}(\zeta^l). \end{aligned}$$

□

Denote by $\mathcal{A}_S^{1,0}$ and $\mathcal{A}_S^{0,1}$ the complex-valued π -semibasic 1-forms on F_S which can be written as $a_k\zeta^k$ and $a_k\bar{\zeta}^k$ respectively. Here a_k are smooth complex-valued functions on F_S . Furthermore set

$$\mathcal{A}_S^{p,q} = \Lambda^p(\mathcal{A}_S^{1,0}) \otimes \Lambda^q(\mathcal{A}_S^{0,1}),$$

so that the complex-valued π -semibasic k -forms \mathcal{A}_S^k on F_S decompose as

$$\mathcal{A}_S^k = \bigoplus_{p+q=k} \mathcal{A}_S^{p,q}.$$

Proposition 2.2. *The almost complex structure \mathfrak{J} is integrable if and only if Ξ and the torsion components τ^i lie in $\mathcal{A}_S^{2,0} \oplus \mathcal{A}_S^{1,1}$. In particular for $n \geq 3$ every torsion-free F_S -adapted connection gives rise to an integrable \mathfrak{J} .*

Remark 2.2. The integrability conditions for the almost complex structure \mathfrak{J} can also be obtained by applying [18, Theorem 4]. We will instead use Proposition 2.1.

Proof of Proposition 2.2. Using the characterization of \mathfrak{J} provided in Lemma 2.1, the first statement is an immediate consequence of the structure equation (2.5), the definition of the curvature form Ξ in (2.6), and the Newlander–Nirenberg theorem. In order to prove the second statement we need to show that for $n \geq 3$ the condition $\tau = 0$ implies $\Xi \in \mathcal{A}_S^{2,0} \oplus \mathcal{A}_S^{1,1}$.

Since the curvature form Ξ is a π -semibasic 2-form we may write

$$(2.11) \quad \Xi = x_{kl}\zeta^k \wedge \zeta^l + \tilde{x}_{kl}\bar{\zeta}^k \wedge \zeta^l + \hat{x}_{kl}\bar{\zeta}^k \wedge \bar{\zeta}^l$$

for some smooth complex-valued functions $x_{kl}, \tilde{x}_{kl}, \hat{x}_{kl}$ on F_S . Writing out the Bianchi-identity (2.7) in components for $\tau = 0$ gives

$$0 = (i(\Omega - \Xi)\delta_k^i + \Phi_k^i) \wedge \zeta^k + i\Xi \wedge \bar{\zeta}^i,$$

replacing Ξ with the expansion (2.11) we get

$$0 = \dots + i\hat{x}_{kl}\bar{\zeta}^k \wedge \bar{\zeta}^l \wedge \bar{\zeta}^i$$

where the unwritten summands do not contain forms in $\mathcal{A}_S^{0,3}$. If $n \geq 3$ there is for every choice of indices k, l an index $i \neq k, i \neq l$ so that the Bianchi-identity can hold if and only if $\hat{x}_{kl} = 0$, which is equivalent to Ξ lying in $\mathcal{A}_S^{2,0} \oplus \mathcal{A}_S^{1,1}$. □

Remark 2.3. Recall that $H(2, 2)$ -structures $\pi : F_S \rightarrow M$ correspond to oriented conformal structures of split-signature and thus are always torsion-free. In fact, the logical value of the curvature condition $\Xi \in \mathcal{A}_S^{2,0} \oplus \mathcal{A}_S^{1,1}$ does not depend on the choice of a particular adapted torsion-free connection, but only on F_S . We leave it to the reader to check that this curvature condition corresponds to self-duality³ of the associated oriented conformal 4-manifold of split-signature.

In fact, it can be shown that for $n = 2$ the almost complex structure \mathfrak{J} is integrable if and only if θ is torsion-free and the associated split-signature conformal structure is self-dual. For $n \geq 3$, the almost complex structure \mathfrak{J} is integrable if and only if θ is torsion-free.

Suppose \mathfrak{J} is integrable, so that the total space of the bundle $\rho : X_S \rightarrow M$ is a complex $(n + 1)$ -manifold. By construction the ρ -fibers are smoothly embedded submanifolds of X_S diffeomorphic to $GL(2, \mathbb{R})/GL(1, \mathbb{C})$. We will argue next, that (X_S, \mathfrak{J}) is a quasiholomorphic fiber bundle with fiber $\mathbb{C}P^1 \setminus \mathbb{R}P^1$.

³As in the case of $(4, 0)$ -signature, a split-signature metric is called self-dual if its Weyl curvature tensor, considered as a bundle-valued 2-form, is its own Hodge-star.

Definition 2.3. Let $\pi : B \rightarrow M$ be a fiber bundle with fiber F and \mathfrak{J} an almost complex structure on B . Then (B, \mathfrak{J}) is called *quasiholomorphic* if

- (i) the almost complex structure \mathfrak{J} is integrable,
- (ii) there exists a complex structure I on F ,
- (iii) each π -fiber $B_p = \pi^{-1}(p)$ admits a complex structure with respect to which it is biholomorphic to (F, I) and with respect to which the inclusion $B_p \hookrightarrow B$ is a holomorphic embedding.

Pulling back ξ with a local section s of $\nu : F_S \rightarrow X_S$ gives a complex-valued 1-form which pulls back to the ρ -fibers to be non-vanishing and which depends on s only up to complex multiples. It follows that the fibers of $\rho : X_S \rightarrow M$ are holomorphically embedded Riemann surfaces with respect to the complex structure induced by ξ . Using the Maurer–Cartan form of $\mathrm{GL}(2, \mathbb{R})$, it is easy to see that fibers are biholomorphic to $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. In [2, 16] it was shown that for $n \geq 3$ a 2-Segre structure is β -integrable if and only if it is torsion-free and for $n = 2$ if and only if it is self-dual. Summarizing we have:

Theorem 2.1. *Let $\pi : F_S \rightarrow M$ be a β -integrable 2-Segre structure. Then there exists a canonical almost complex structure \mathfrak{J} on X_S , so that (X_S, \mathfrak{J}) is a quasiholomorphic fiber bundle with fiber $\mathbb{CP}^1 \setminus \mathbb{RP}^1$.*

Proof. We pick an F_S -adapted connection without torsion and let \mathfrak{J} be the associated almost complex structure on X_S whose existence is guaranteed by Lemma 2.1. Then by Proposition 2.2 and the above remarks, the almost complex structure \mathfrak{J} is integrable and (X_S, \mathfrak{J}) is a quasiholomorphic fiber bundle with fiber $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. Finally, by Lemma 2.2, any other F_S -adapted torsion-free connection gives rise to the same almost complex structure \mathfrak{J} . \square

3. Reductions of β -integrable 2-Segre structures

We will henceforth consider the β -integrable case and assume $\rho : X_S \rightarrow M$ to be equipped with its canonical integrable almost complex structure \mathfrak{J} with respect to which (X_S, \mathfrak{J}) is a quasiholomorphic fiber bundle. By construction the sections of the bundle $\rho : X_S \rightarrow M$ correspond to reductions of the principal H -bundle $\pi : F_S \rightarrow M$ with structure group $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$. Note that an $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -reduction of a β -integrable $(2, n)$ -Segre structure $\pi : F_S \rightarrow M$ equips M with an almost complex structure preserving

the Segre cones \mathcal{S}_p and for which the β -planes are totally real. In this section we will show that the torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -reductions of $F_{\mathcal{S}}$ are in one-to-one correspondence with the sections $\sigma : M \rightarrow X_{\mathcal{S}}$ having holomorphic image $\sigma(M) \subset X_{\mathcal{S}}$. This will be done using exterior differential systems (EDS). The notation and terminology for EDS are chosen to be consistent with [5].

A basis for the Lie algebra of $S^1 \cdot \text{GL}(n, \mathbb{R})$ is given by

$$a \otimes I_n, \quad I_2 \otimes e_k^i.$$

Suppose $R \rightarrow M$ is a torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure with adapted connection θ . Write

$$\theta = a\alpha \otimes I_n + I_2 \otimes \beta,$$

for some 1-form α and some $\mathfrak{gl}(n, \mathbb{R})$ -valued 1-form β on R . Let ζ denote the pullback of the canonical \mathbb{C}^n -valued 1-form to R , then ζ satisfies

$$(3.1) \quad d\zeta = -(\text{i}\alpha I_n + \beta) \wedge \zeta,$$

as was already observed in [8].

We will need the following Lemma whose proof is straightforward and thus omitted:

Lemma 3.1. *Let (X, J) be a complex $(n + 1)$ -manifold, $(\mu^1, \dots, \mu^n, \kappa) \in \mathcal{A}^1(X, \mathbb{C})$ a basis for the $(1, 0)$ -forms of J and $f : \Sigma \rightarrow X$ a $2n$ -submanifold with*

$$(3.2) \quad f^*(\text{i}\mu^1 \wedge \bar{\mu}^1 \wedge \dots \wedge \text{i}\mu^n \wedge \bar{\mu}^n) \neq 0.$$

Then (f, Σ) is a complex submanifold if and only if

$$f^*(\kappa \wedge \mu^1 \wedge \dots \wedge \mu^n) = 0.$$

Moreover through every point $p \in X$ passes such a complex submanifold.

On $F_{\mathcal{S}}$ define the EDS

$$\mathcal{I} = \langle \xi \wedge \zeta^1 \wedge \dots \wedge \zeta^n \rangle$$

together with the independence condition

$$Z = \text{i}\zeta^1 \wedge \bar{\zeta}^1 \wedge \dots \wedge \text{i}\zeta^n \wedge \bar{\zeta}^n.$$

The EDS (\mathcal{I}, Z) is of interest because of the following:

Lemma 3.2. *Let $\sigma : M \rightarrow X_{\mathcal{S}}$ be an $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -reduction of $F_{\mathcal{S}}$ and $\tilde{\sigma} : U \rightarrow F_{\mathcal{S}}$ a local coframing covering σ . Then $\tilde{\sigma}$ is an integral manifold of (\mathcal{I}, Z) if and only if $\sigma|_U : U \rightarrow X_{\mathcal{S}}$ is a complex submanifold.*

Proof. Let $s : \rho^{-1}(U) \rightarrow F_{\mathcal{S}}$ be a local section of the bundle $\nu : F_{\mathcal{S}} \rightarrow X_{\mathcal{S}}$ and let $\mu^i = s^* \zeta^i$ for $i = 1, \dots, n$ and $\kappa = s^* \xi$ be a local basis for the $(1,0)$ -forms on $\rho^{-1}(U)$. Then

$$\begin{aligned} \nu^* \mu^i &= (s \circ \nu)^* \zeta^i = (R_t)^* \zeta^i, \\ \nu^* \kappa &= (s \circ \nu)^* \xi = (R_t)^* \xi, \end{aligned}$$

for some smooth function $t : \pi^{-1}(U) \rightarrow S^1 \cdot \mathrm{GL}(n, \mathbb{R})$. Recall that for $e^{i\varphi} \cdot b \in S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ we have

$$\begin{aligned} (R_{e^{i\varphi} \cdot b})^* \xi &= e^{-2i\varphi} \xi, \\ (R_{e^{i\varphi} \cdot b})^* \zeta &= (e^{-i\varphi} \cdot b^{-1}) \zeta. \end{aligned}$$

This yields

$$\nu^* (i\mu^1 \wedge \bar{\mu}^1 \wedge \dots \wedge i\mu^n \wedge \bar{\mu}^n) = (\det b)^{-2} Z \neq 0$$

for some smooth map $b : \pi^{-1}(U) \rightarrow \mathrm{GL}(n, \mathbb{R})$ and

$$\nu^* \kappa = e^{-2i\varphi} \xi$$

for some smooth function $\varphi : \pi^{-1}(U) \rightarrow \mathbb{R}$. Hence we get

$$(\sigma|_U)^* (i\mu^1 \wedge \bar{\mu}^1 \wedge \dots \wedge i\mu^n \wedge \bar{\mu}^n) = ((\det b)^{-2} \circ \tilde{\sigma}) \tilde{\sigma}^* Z$$

which vanishes nowhere since $\tilde{\sigma}$ is a π -section. Therefore according to Lemma 3.1, $\sigma|_U : U \rightarrow X_{\mathcal{S}}$ is a complex submanifold if and only if

$$(\sigma|_U)^* (\kappa \wedge \mu^1 \wedge \dots \wedge \mu^n) = \left(\left(\frac{e^{-(n+2)i\varphi}}{\det b} \right) \circ \tilde{\sigma} \right) \tilde{\sigma}^* (\xi \wedge \zeta^1 \wedge \dots \wedge \zeta^n) = 0.$$

□

Recall that $S^1 \cdot \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{C})$ and we can thus look for reductions $\sigma : M \rightarrow X_{\mathcal{S}}$ whose associated almost complex structure \mathfrak{J}_{σ} is integrable.

Proposition 3.1. *Let $\sigma : M \rightarrow X_S$ be an $S^1 \cdot \text{GL}(n, \mathbb{R})$ -reduction of $\pi : F_S \rightarrow M$. Then the following two statements are equivalent:*

- (i) *The almost complex structure \mathfrak{J}_σ is integrable.*
- (ii) *Any local coframing $\tilde{\sigma} : U \rightarrow F_S$ covering σ is an integral manifold of (\mathcal{I}, Z) .*

Proof. Since $\tilde{\sigma}$ is a π -section we have $\tilde{\sigma}^*Z \neq 0$. Write $\chi^i = \tilde{\sigma}^*\zeta^i$. The local coframing $\tilde{\sigma}$ is adapted to the $S^1 \cdot \text{GL}(n, \mathbb{R})$ -reduction σ and thus the forms χ^i are a local basis of the $(1,0)$ -forms of \mathfrak{J}_σ . By Newlander–Nirenberg \mathfrak{J}_σ is integrable if and only if there exist complex-valued 1-forms π_k^i such that

$$d\chi^i = \pi_k^i \wedge \chi^k.$$

Using the structure equation (2.5) we get

$$(3.3) \quad d\chi^i = \tilde{\sigma}^*d\zeta^i = \tilde{\pi}_k^i \wedge \chi^k - i\tilde{\sigma}^*\xi \wedge \bar{\chi}^i$$

for some complex-valued 1-forms $\tilde{\pi}_k^i$. Write

$$\tilde{\sigma}^*\xi = x_k\chi^k + y_k\bar{\chi}^k$$

for some smooth complex-valued functions x_k, y_k on U . Then (3.3) implies that \mathfrak{J}_σ is integrable on U if and only if the functions y_k all vanish. This condition is equivalent to $\tilde{\sigma}$ being an integral manifold of (\mathcal{I}, Z) . \square

We are now ready to prove:

Theorem 3.1. *Let $\pi : F_S \rightarrow M$ be a β -integrable 2-Segre structure. Then an $S^1 \cdot \text{GL}(n, \mathbb{R})$ -reduction $R \subset F_S$ is torsion-free if and only if $\nu(R) \subset X_S$ is a complex submanifold.*

Proof. Let $\nu(R) = \sigma(M)$ for some ρ -section $\sigma : M \rightarrow X_S$ which has holomorphic image, then by Lemma 3.2 and Proposition 3.1, the almost complex structure \mathfrak{J}_σ is integrable. This is equivalent to ξ satisfying $\xi = x_k\zeta^k$ for some smooth complex-valued functions x_k on R . Pulling back the structure Equation (2.5) to $R \subset F_S$ gives

$$(3.4) \quad d\zeta = -\left(i\left(\omega - x_k\zeta^k\right)I_n + \phi\right) \wedge \zeta - ix_k\zeta^k I_n \wedge \bar{\zeta}.$$

Define

$$\begin{aligned} \alpha &= \omega - \operatorname{Im}(x_k) \operatorname{Im}(\zeta^k), \\ \beta_l^j &= \phi_l^j - \operatorname{Re}(i\tilde{x}_l \zeta^j) - \delta_l^j \operatorname{Im}(x_k) \operatorname{Re}(\zeta^k), \end{aligned}$$

then the 1-form $\theta = a\alpha \otimes I_n + I_2 \otimes \beta$ is a linear connection on R which satisfies

$$(3.5) \quad d\zeta = -(iaI_n + \beta) \wedge \zeta,$$

thus R is torsion-free. Conversely suppose the reduction $\sigma : M \rightarrow X_S$ is torsion-free, so that on $R = (\nu^{-1} \circ \sigma)(M)$ there exists a linear connection $\theta = a\alpha \otimes I_n + I_2 \otimes \beta$ satisfying (3.5). Pulling back (ω, ξ, ϕ) to R gives

$$(3.6) \quad \begin{aligned} \omega &= \alpha + a_k (\zeta^k + \bar{\zeta}^k) + i\tilde{a}_k (\zeta^k - \bar{\zeta}^k) \\ \xi &= x_k \zeta^k + \tilde{x}_k \bar{\zeta}^k \\ \phi_k^i &= \beta_k^i + f_{kl}^i (\zeta^k + \bar{\zeta}^k) + i\tilde{f}_{kl}^i (\zeta^k - \bar{\zeta}^k) \end{aligned}$$

for some smooth complex-valued functions $a_k, \tilde{a}_k, x_k, \tilde{x}_k, f_{kl}^i, \tilde{f}_{kl}^i$ on R . Subtracting (3.4) from (3.5) and using (3.6) gives in components

$$(3.7) \quad 0 = \dots + i (x_k \zeta^k + \tilde{x}_k \bar{\zeta}^k) \wedge \bar{\zeta}^i$$

where the unwritten summands are not of the form $\bar{\zeta}^k \wedge \bar{\zeta}^i$. It follows that (3.7) can hold for every $i = 1, \dots, n$ if and only if $\tilde{x}_k = 0$. \square

Corollary 3.1. *Locally every β -integrable 2-Segre structure $\pi : F_S \rightarrow M$ can be reduced to a torsion-free $S^1 \cdot \operatorname{GL}(n, \mathbb{R})$ -structure.*

Proof. For a given point $p \in M$, choose $q \in X_S$ with $\rho(q) = p$ and a coordinate neighborhood U_p . Let $\mu^i, i = 1, \dots, n$ and κ be a basis for the $(1,0)$ -forms on $\rho^{-1}(U_p)$ as constructed in Lemma 3.2. Using Lemma 3.1 there exists a complex $2n$ -submanifold $f : \Sigma \rightarrow \rho^{-1}(U_p)$ passing through q for which

$$f^* (i\mu^1 \wedge \bar{\mu}^1 \wedge \dots \wedge i\mu^n \wedge \bar{\mu}^n) \neq 0.$$

Since the $\pi : F_S \rightarrow M$ pullback of a volume form on M is a nowhere vanishing multiple of $Z = i\mu^1 \wedge \bar{\mu}^1 \wedge \dots \wedge i\mu^n \wedge \bar{\mu}^n$, the ρ pullback of a volume form on U_p is a nowhere vanishing multiple of Z and hence $\rho \circ f : \Sigma \rightarrow U_p$ is a local diffeomorphism. Composing f with the locally available inverse of this local diffeomorphism one gets a local ρ -section, which is defined

in a neighborhood of p and which is a complex submanifold. Applying Theorem 3.1 it follows that $\pi : F_S \rightarrow M$ locally has an underlying torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure. \square

4. The flat case

In this section we apply the obtained results to the Grassmannian of oriented 2-planes in \mathbb{R}^{n+2} which carries a 2-oriented torsion-free 2-Segre structure together with an adapted connection of vanishing curvature.

Here a 2-Segre structure $\pi : F_S \rightarrow M$ is called 2-oriented if the structure group $H(2, n)$ has been reduced to $H^+(2, n) = \text{GL}^+(2, \mathbb{R}) \otimes \text{GL}(n, \mathbb{R})$.

Using Theorem 2.1 we also get: If $F_S \rightarrow M$ is a β -integrable 2-oriented 2-Segre structure, then $\rho : X_S = F_S / (S^1 \cdot \text{GL}(n, \mathbb{R})) \rightarrow M$ together with its canonical almost complex structure \mathfrak{J} is a quasiholomorphic fiber bundle with fiber $\text{GL}^+(2, \mathbb{R}) / \text{GL}(1, \mathbb{C}) \simeq D^2$, the open unit disk in \mathbb{C} .

4.1. The Grassmannian of oriented 2-planes

The projective linear group $\text{PL}(n + 2, \mathbb{R}) = \text{GL}(n + 2, \mathbb{R}) / Z$ acts transitively from the left on the Grassmannian $G_2^+(\mathbb{R}^{n+2})$ of oriented 2-planes in \mathbb{R}^{n+2} and the stabilizer subgroup of any element $\Pi \in G_2^+(\mathbb{R}^{n+2})$ may be identified with the subgroup S consisting of elements of the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a \in \text{GL}^+(2, \mathbb{R}), c \in \text{GL}(n, \mathbb{R})$ and $b \in M_{\mathbb{R}}(2, n)$ is a real $(2 \times n)$ -matrix. Let $\mu : \text{PL}(n + 2, \mathbb{R}) \rightarrow G_2^+(\mathbb{R}^{n+2}) \simeq \text{PL}(n + 2, \mathbb{R}) / S$ be the quotient projection and write

$$\tilde{\theta} = \begin{pmatrix} \alpha & \beta \\ \eta & \gamma \end{pmatrix}$$

for the Maurer–Cartan form of $\text{PL}(n + 2, \mathbb{R})$. The real matrix-valued 1-forms $\alpha, \beta, \gamma, \eta$ have sizes according to the block decomposition of the Lie group S and satisfy $\text{Tr}(\alpha) + \text{Tr}(\gamma) = 0$. Let H_k^i denote the vector fields dual to the forms η with respect to the coframing θ . Let $N \subset \text{PL}(n + 2, \mathbb{R})$ be the closed normal subgroup given by

$$N = \left\{ \begin{bmatrix} \text{I}_2 & b \\ 0 & \text{I}_n \end{bmatrix} \mid b \in M_{\mathbb{R}}(2, n) \right\},$$

whose elements will be denoted by $[b]$. The quotient Lie group S/N is isomorphic to $H^+(2, n)$ and thus $\text{PL}(n + 2, \mathbb{R})/N$ is the total space of a right principal $H^+(2, n)$ -bundle over $G_2^+(\mathbb{R}^{n+2})$. Consider the smooth map

$$\varphi_k^i : \text{PL}(n + 2, \mathbb{R}) \rightarrow TG_2^+(\mathbb{R}^{n+2}), \quad p \mapsto \mu'_p(H_k^i(p)).$$

The Maurer–Cartan equation $d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta} = 0$ implies that the form η is basic for the quotient projection $\text{PL}(n + 2, \mathbb{R}) \rightarrow \text{PL}(n + 2, \mathbb{R})/N$. Therefore the maps φ_k^i are invariant under the right action of N and thus descend to smooth maps $\text{PL}(n + 2, \mathbb{R})/N \rightarrow TG_2^+(\mathbb{R}^{n+2})$. The images $\varphi_k^i(p)$ for a given point $p \in \text{PL}(n + 2, \mathbb{R})$ are linearly independent and thus induce a map φ into the coframe bundle of $G_2^+(\mathbb{R}^{n+2})$. The maps φ_k^i can be arranged so that the induced map φ from $\text{PL}(n + 2, \mathbb{R})/N$ into the coframe bundle of $G_2^+(\mathbb{R}^{n+2})$ pulls back the components of the canonical \mathbb{C}^n -valued 1-form to $i(\eta_1^k + i\eta_2^k)$. It follows again with the Maurer–Cartan equation that φ embeds $\text{PL}(n + 2, \mathbb{R})/N$ as a smooth right principal $H^+(2, n)$ -subbundle of the coframe bundle of $G_2^+(\mathbb{R}^{n+2})$. This subbundle will be denoted by $\pi_0 : F_0 \rightarrow G_2^+(\mathbb{R}^{n+2})$ and the projection $\text{PL}(n + 2, \mathbb{R}) \rightarrow F_0$ by v . Write

$$\begin{aligned} \tilde{\omega} &= \alpha_1^2, \\ 2\tilde{\xi} &= (\alpha_2^1 + \alpha_1^2) + i(\alpha_2^2 - \alpha_1^1), \\ \tilde{\phi} &= \gamma - I_n \alpha_2^2, \\ \zeta^i &= i(\eta_1^i + i\eta_2^i). \end{aligned}$$

Then straightforward computations show that the forms $(\tilde{\omega}, \tilde{\xi}, \tilde{\phi})$ transform under the right action of $H^+(2, n)$ as the connection forms of an \mathcal{S}_0 -adapted connection do. Moreover, we have

$$d\zeta = - \left(i \left(\tilde{\omega} - \tilde{\xi} \right) I_n + \tilde{\phi} \right) \wedge \zeta - i\tilde{\xi} I_n \wedge \bar{\zeta}.$$

This implies that there exists an adapted torsion-free connection (ω, ξ, ϕ) on F_0 such that

$$(4.1) \quad v^*(\omega, \xi, \phi) = (\tilde{\omega}, \tilde{\xi}, \tilde{\phi}), \text{ mod } \eta,$$

i.e., (4.1) holds up to linear combinations of the elements of η . Furthermore the Maurer–Cartan equation implies that the curvature forms of this connection all vanish. Summarizing we have proved:

Proposition 4.1. *The Grassmannian of oriented 2-planes in \mathbb{R}^{n+2} admits a 2-oriented 2-Segre structure \mathcal{S}_0 together with an adapted, torsion-free, flat connection (ω, ξ, ϕ) such that $v^*(\omega, \xi, \phi) = (\tilde{\omega}, \tilde{\xi}, \tilde{\phi})$, mod η , holds.*

Let $\rho : X_0 \rightarrow G_2^+(\mathbb{R}^{n+2})$ be the D^2 -bundle associated to the 2-oriented torsion-free 2-Segre structure $\pi_0 : F_0 \rightarrow G_2^+(\mathbb{R}^{n+2})$ and \mathfrak{J}_0 its canonical almost complex structure which makes (X_0, \mathfrak{J}_0) into a quasiholomorphic fiber bundle with fiber D^2 . Its total space X_0 can be identified with the quotient $\text{PL}(n + 2, \mathbb{R})/\tilde{P}$ where \tilde{P} is the closed Lie subgroup

$$\tilde{P} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a \in \text{GL}(1, \mathbb{C}), b \in M_{\mathbb{R}}(2, n), c \in \text{GL}(n, \mathbb{R}) \right\}.$$

Write an element $[g] \in \text{PL}(n + 2, \mathbb{R})$ as $[g_1, \dots, g_{n+2}]$ where the elements g_k are column-vectors well defined up to a common non-zero factor. Consider the smooth map

$$\lambda : \text{PL}(n + 2, \mathbb{R}) \rightarrow \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}, \quad [g_1, g_2, \dots, g_{n+2}] \mapsto [g_1 + ig_2].$$

Clearly λ is a surjective submersion whose fibers are the \tilde{P} -orbits and thus induces a diffeomorphism $\varphi : X_0 \rightarrow \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$. Therefore $\rho_0 = \rho \circ \varphi^{-1} : \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1} \rightarrow G_2^+(\mathbb{R}^{n+2})$ is a bundle with fiber D^2 . Explicitly ρ_0 is given by $[z] \mapsto \mathbb{R}\{\text{Re}(z), \text{Im}(z)\}$ and the 2-plane $\mathbb{R}\{\text{Re}(z), \text{Im}(z)\}$ is oriented by declaring $\text{Re}(z), \text{Im}(z)$ to be a positively oriented basis.

Proposition 4.2. *There exists a biholomorphic fiber bundle isomorphism $\varphi : (X_0, \mathfrak{J}_0) \rightarrow \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$ covering the identity on $G_2^+(\mathbb{R}^{n+2})$.*

Proof. Using Lemma 2.1 and Proposition 4.1 it sufficient to show that λ pulls-back the $(1,0)$ -forms of $\mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$ to linear combinations of the forms $\zeta^1, \dots, \zeta^n, \tilde{\xi}$. This is a computation that causes no difficulties and so we omit it. □

4.2. Smooth quadrics without real points

If V is a real vector space, $V_{\mathbb{C}} = V \otimes \mathbb{C}$ will denote its complexification and $\mathbb{P}(V_{\mathbb{C}}) = (V_{\mathbb{C}} \setminus \{0\})/\mathbb{C}^*$ its complex projectivization. An element $[z] \in \mathbb{P}(V_{\mathbb{C}})$ for which z is a simple vector is called a *real point*.

The aim of this subsection is to show that the smooth quadrics $Q \subset \mathbb{C}\mathbb{P}^{n+1} = \mathbb{P}(\mathbb{R}_{\mathbb{C}}^{n+2})$ without real points are in one-to-one correspondence with

the sections of the bundle $\rho_0 : \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1} \rightarrow G_2^+(\mathbb{R}^{n+2})$ having holomorphic image. This is done by reducing the problem to the case $n = 1$ which was shown to be true in [17, Corollary 2] (see also [7, Theorem 9]).

Let $\Pi \subset \mathbb{R}^{n+2}$ be a 3-dimensional linear subspace. Choosing an isomorphism $\mathbb{R}^3 \simeq \Pi$ induces an embedding of the 2-sphere $S^2 \simeq G_2^+(\mathbb{R}^3) \hookrightarrow G_2^+(\mathbb{R}^{n+2})$. Clearly the image of this embedding and its induced smooth structure do not depend on the chosen isomorphism and thus Π determines a smoothly embedded 2-sphere in $G_2^+(\mathbb{R}^{n+2})$ which will be denoted by S_Π . Moreover the isomorphism $\mathbb{R}^3 \simeq \Pi$ induces a holomorphic embedding $\mathbb{C}\mathbb{P}^2 \simeq \mathbb{P}(\Pi_{\mathbb{C}}) \hookrightarrow \mathbb{C}\mathbb{P}^{n+1}$ and thus an embedding $\mathbb{C}\mathbb{P}^2 \setminus \mathbb{R}\mathbb{P}^2 \hookrightarrow \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$. Again the image of this embedding and its induced complex structure do not depend on the chosen isomorphism and thus Π determines a holomorphically embedded submanifold $Y_\Pi \subset \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$. Restricting the base point projection $\rho_0 : \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1} \rightarrow G_2^+(\mathbb{R}^{n+2})$ to Y_Π gives a D^2 -bundle $\rho_\Pi : Y_\Pi \rightarrow S_\Pi$ which is isomorphic to the bundle $\rho_0^2 : \mathbb{C}\mathbb{P}^2 \setminus \mathbb{R}\mathbb{P}^2 \rightarrow G_2^+(\mathbb{R}^3)$, $[z] \mapsto \mathbb{R}\{\text{Re}(z), \text{Im}(z)\}$.

Recall that for a smooth algebraic hypersurface $X \subset \mathbb{P}(V_{\mathbb{C}})$, the Gauss map $\mathcal{G}_X : X \rightarrow G_{n-1}(V_{\mathbb{C}})$ sends a point $x \in X$ to the tangent hyperplane of X at x . The dual variety X^* is now defined to be the image of X under the Gauss map. Usually \mathcal{G}_X is assumed to take values in $\mathbb{P}(V_{\mathbb{C}}^*) = \mathbb{P}((V_{\mathbb{C}})^*) \simeq G_{n-1}(V_{\mathbb{C}})$. Note that if $Q \subset \mathbb{P}(V_{\mathbb{C}})$ a smooth quadric without real points. Then the dual of Q is a smooth quadric without real points in $\mathbb{P}(V_{\mathbb{C}}^*)$. It follows that the intersection of a smooth quadric without real points with a real k -plane Π of dimension at least 2 gives again a smooth quadric without real points in $\mathbb{P}(\Pi_{\mathbb{C}})$.

Theorem 4.1. *The sections of the disk bundle $\rho_0 : \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1} \rightarrow G_2^+(\mathbb{R}^{n+2})$ having holomorphic image are in one-to-one correspondence with the smooth quadrics $Q \subset \mathbb{C}\mathbb{P}^{n+1}$ without real points.*

Proof. Let $\sigma : G_2^+(\mathbb{R}^{n+2}) \rightarrow \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$ be a ρ_0 -section with holomorphic image $Q = \text{im } \sigma$. Let $\Pi \subset \mathbb{R}^{n+2}$ be a 3-dimensional linear subspace and $\iota_\Pi : S_\Pi \rightarrow G_2^+(\mathbb{R}^{n+2})$, $\tilde{\iota}_\Pi : Y_\Pi \rightarrow \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$ the corresponding embedded submanifolds. Then the map $\sigma \circ \iota_\Pi : S_\Pi \rightarrow \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$ is smooth and takes values in Y_Π . Consequently the induced map $\sigma_\Pi : S_\Pi \rightarrow Y_\Pi$ is an injective immersion and thus, since S_Π is compact, a smooth embedding. Set $Q_\Pi = Q \cap Y_\Pi = \sigma_\Pi(S_\Pi)$, then $Q_\Pi \subset Y_\Pi$ is a smoothly embedded submanifold. Now Chow's theorem implies that Q is a smooth algebraic hypersurface. Suppose $P : \mathbb{C}^{n+2} \rightarrow \mathbb{C}$ is a homogeneous polynomial defining Q and let $P_\Pi : \mathbb{C}^3 \rightarrow \mathbb{C}$ denote the homogeneous polynomial obtained

by pulling back P to $\Pi_{\mathbb{C}} \simeq \mathbb{C}^3$. The map P_{Π} is a homogeneous polynomial of the same degree as P which has no real points, since P has no real points. Under the identification $Y_{\Pi} \simeq \mathbb{C}\mathbb{P}^2 \setminus \mathbb{R}\mathbb{P}^2$, Q_{Π} becomes a smoothly embedded submanifold of $\mathbb{C}\mathbb{P}^2 \setminus \mathbb{R}\mathbb{P}^2$ defined by the zero locus of the homogeneous polynomial P_{Π} . Since Q_{Π} is diffeomorphic to the 2-sphere, the genus of Q_{Π} is 0 and thus by the degree-genus formula for smooth plane algebraic curves $g = (d - 1)(d - 2)/2$, the degree of P_{Π} must be 1 or 2. However since Q_{Π} has no real points the degree of P_{Π} and thus the degree of P must be 2.

Conversely let $Q \subset \mathbb{C}\mathbb{P}^{n+1}$ be a smooth quadric without real points. Let $\{\Pi^{\iota}\}_{\iota \in I}$ be a family of 3-dimensional linear subspaces of \mathbb{R}^{n+2} so that the submanifolds $S_{\Pi^{\iota}}$ cover $G_2^+(\mathbb{R}^{n+2})$. Let $Q_{\Pi^{\iota}}$ denote the intersection of Q with $\mathbb{P}(\Pi_{\mathbb{C}}^{\iota})$ which is a smooth quadric without real points. According to [17, Corollary 2] each such quadric is the image of a unique section $\sigma_{\iota} : S_{\Pi^{\iota}} \rightarrow X_{\Pi^{\iota}}$. Now for any two $\Pi^{\iota_1}, \Pi^{\iota_2}$ the spheres $S_{\Pi^{\iota_1}}$ and $S_{\Pi^{\iota_2}}$ are either disjoint or intersect in exactly two points. Since for a given $Q_{\Pi^{\iota}}$ the section σ_{ι} is unique, it follows that $Q_{\Pi^{\iota}}$ intersects each $\rho_{\Pi^{\iota}}$ -fiber in exactly one point. This implies that the sections σ_{ι_1} and σ_{ι_2} agree on intersection points and thus the family $\{\sigma_{\iota}\}_{\iota \in I}$ gives rise to a unique global section $\sigma : G_2^+(\mathbb{R}^{n+2}) \rightarrow \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$ with image Q . □

Corollary 4.1. *The torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -reductions $R \subset F_0$ are in one-to-one correspondence with the smooth quadrics $Q \subset \mathbb{C}\mathbb{P}^{n+1}$ without real points.*

Proof. This follows immediately from Theorems 3.1 and 4.1. □

Remark 4.1. For $n = 2$, the case of conformal 4-manifolds of split-signature, Corollary 4.1 can also be deduced by applying results from [15]. One could also look for $S^1 \cdot \text{GL}(2, \mathbb{R})$ -reductions whose associated almost complex structure is not only integrable, but for which the corresponding conformal structure $[g]$ also contains a Kähler-metric. This, and the related problem in (4,0)-signature has been studied in [10] (see also [15, Theorem D]). Moreover for $n = 2$, Theorem 3.1 has an analogue in (4,0)-signature due to Salamon [19].

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Appendix A

A.1. The structure group

We provide a proof for the existence of the isomorphism claimed in (2.2).

Lemma A.1. *Let $g \in G(m, n)$ and $v \in V$. Then precisely one of the two statements holds:*

(i) *There exists $v_0 \in V$ and $b_v \in \text{Isom}(W, W)$ such that for all $w \in W$*

$$g(v \otimes w) = v_0 \otimes b_v(w).$$

(ii) *There exists $w_0 \in W$ and $a_v \in \text{Isom}(W, V)$ such that for all $w \in W$.*

$$g(v \otimes w) = a_v(w) \otimes w_0.$$

Moreover if (i) (or (ii)) is true for some $v \in V$, then for all $v \in V$.

Proof. For $v = 0$ the statement is obvious so let us assume $v \neq 0$. Let $w_1, w_2 \in W$ be linearly independent, write $g(v \otimes w_1) = v_1 \otimes u_1$ and $g(v \otimes w_2) = v_2 \otimes u_2$ for some vectors $v_1, v_2 \in V$ and $u_1, u_2 \in W$ all non-zero. Then $g \in G(m, n)$ implies that one of the two following cases occurs:

(I) $v_1 \wedge v_2 = 0$ and $u_1 \wedge u_2 \neq 0$,

(II) $v_1 \wedge v_2 \neq 0$ and $u_1 \wedge u_2 = 0$.

Assume (I) holds and fix $v_0 \neq 0$ with $v_0 \wedge v_1 = 0$. It follows again with $g \in G(m, n)$, that for every $w \in W$, there exists a unique element $b_v(w) \in W$ such that $g(v \otimes w) = v_0 \otimes b_v(w)$. The assignment $w \mapsto b_v(w)$ is invertible and linear, thus (i) follows. Assuming (II) holds we conclude similarly that (ii) follows.

Suppose case (i) occurs for some $v \in V$ and case (ii) for some $v' \in V$. Let b_v and $a_{v'}$ be the associated isomorphisms. Since $g \in G(m, n)$ either b_v or $a_{v'}$ must have rank 1, thus contradicting the fact that both maps are isomorphisms. \square

We can now show:

Proposition A.1. *We have an isomorphism*

$$G(m, n) \simeq \begin{cases} H(m, n) & n \neq m, \\ H(n, n) \rtimes \mathbb{Z}_2 & n = m. \end{cases}$$

Proof. Let $g \in G(m, n)$. Assume case (i) of Lemma A.1 holds for some and hence all $v \in V$. Let $\hat{v}, \tilde{v} \in V$, then by Lemma A.1 there exists $\hat{v}_0, \tilde{v}_0 \in V$ and $b_{\hat{v}}, b_{\tilde{v}} \in \text{Isom}(W, W)$ such that for all $w \in W$ we have

$$g(\hat{v} \otimes w) = \hat{v}_0 \otimes b_{\hat{v}}(w), \quad \varphi(\tilde{v} \otimes w) = \tilde{v}_0 \otimes b_{\tilde{v}}(w).$$

On the other hand for some $\tilde{w} \in W$ there must exist $\tilde{w}_0 \in W$ and $a_{\tilde{w}} \in \text{Isom}(V, V)$ such that for all $v \in V$

$$g(v \otimes \tilde{w}) = a_{\tilde{w}}(v) \otimes \tilde{w}_0.$$

We thus get

$$g(\hat{v} \otimes \tilde{w}) = a_{\tilde{w}}(\hat{v}) \otimes \tilde{w}_0 = \hat{v}_0 \otimes b_{\hat{v}}(\tilde{w})$$

and

$$g(\tilde{v} \otimes \tilde{w}) = a_{\tilde{w}}(\tilde{v}) \otimes \tilde{w}_0 = \tilde{v}_0 \otimes b_{\tilde{v}}(\tilde{w}).$$

Since this holds for any $\tilde{w} \in W$, the map $b_{\tilde{v}}$ must be a (non-zero) constant multiple of the map $b_{\hat{v}}$. It follows that there exists $a \in \text{Isom}(V, V)$ and $b \in \text{Isom}(W, W)$ such that $g(v \otimes w) = a(v) \otimes b(w)$ for all $v \in V$ and $w \in W$. If the case (ii) of Lemma A.1 holds, we can conclude similarly that there exists $a \in \text{Isom}(W, V)$ and $b \in \text{Isom}(V, W)$ and such that $g(v \otimes w) = a(w) \otimes b(v)$ for all $v \in V$ and $w \in W$. From this the claim follows easily. \square

A.2. Segre and almost Grassmann structures

Finally, we show that for $n + m$ odd, an (m, n) -Segre structure $\pi : F_S \rightarrow M$ is the same as an almost Grassmann structure. Let $S(m, n)$ denote the subgroup of $\text{GL}(m, \mathbb{R}) \times \text{GL}(n, \mathbb{R})$ consisting of pairs (a_m, a_n) satisfying $\det a_m \det a_n = 1$. Clearly for $n + m$ odd, the map

$$\rho : S(m, n) \rightarrow \text{GL}(m, \mathbb{R}) \otimes \text{GL}(n, \mathbb{R}), \quad (a_m, a_n) \mapsto a_m \otimes a_n$$

is a Lie group isomorphism. For $k = m, n$ let $\chi_k : S(m, n) \rightarrow \text{Aut}(\mathbb{R}^k)$ be the representation defined by

$$\chi_k((a_m, a_n))(v) = a_k v.$$

for $v \in \mathbb{R}^k$. The reader will recall that any (real or complex) representation $\chi : H(m, n) \simeq S(m, n) \rightarrow \text{Aut}(V)$ defines a vector bundle $(F_S)_\chi = F_S \times_\chi V$ over M . Let $E_k \rightarrow M$ denote the rank k vector bundle obtained via the representation χ_k . By construction, the vector bundle associated to the representation $\chi_m \otimes \chi_n$ is the tangent bundle of M and we thus obtain an isomorphism

$$TM \simeq E_m \otimes E_n$$

inducing the Segre structure $\pi : F_S \rightarrow M$.

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