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Modified mean curvature flow of star-shaped hypersurfaces in hyperbolic space

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We define a new modified mean curvature flow (MMCF) in hyperbolic space $\mathbb{H}^{n+1}$, which interestingly turns out to be the natural negative L^2-gradient flow of the energy functional introduced by De Silva and Spruck in [5]. We show the existence, uniqueness and convergence of the MMCF of complete embedded star-shaped hypersurfaces with prescribed asymptotic boundary at infinity. The proof of our main theorems follows closely Guan and Spruck's work [9], and may be thought of as a parabolic analog.

1. Introduction

Let $\mathbf{F}(\mathbf{z}, t) : \mathbb{S}^n_+ \times [0, \infty) \to \mathbb{H}^{n+1}$ be a one parameter family of complete embedded star-shaped hypersurfaces which are radial graphs in $\mathbb{H}^{n+1}$ over $\mathbb{S}^n_+$, the upper hemisphere of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, where the half-space model of $\mathbb{H}^{n+1}$ is used. We say the images $\Sigma_t = \mathbf{F}(\mathbf{z}, t)$ move by modified mean curvature flow (MMCF) if

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}\mathbf{F}(\mathbf{z}, t)^{\perp} = (H - \sigma)\nu_{\mathrm{H}}, & (\mathbf{z}, t) \in \mathbb{S}^n_+ \times [0, \infty), \\ \mathbf{F}(\mathbf{z}, 0) = \Sigma_0, & \mathbf{z} \in \mathbb{S}^n_+, \end{cases}$$

where H denotes the hyperbolic mean curvature of Σ_t, $\sigma \in (-1, 1)$ is a constant, and ν_{H} denotes the outward unit normal of Σ_t with respect to the hyperbolic metric. By the half-space model of $\mathbb{H}^{n+1}$, we mean

$$\mathbb{H}^{n+1} = \{(x', x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

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equipped with the hyperbolic metric

$$ds_{\mathrm{H}}^2 = \frac{1}{x_{n+1}^2}\, ds_{\mathrm{E}}^2,$$

where ds_{E}^2 denotes the standard Euclidean metric on $\mathbb{R}^{n+1}$. One identifies the hyperplane $\{x_{n+1} = 0\} = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ as the infinity of $\mathbb{H}^{n+1}$, denoted by $\partial_\infty \mathbb{H}^{n+1}$.

In this paper, we consider the questions of the existence, uniqueness and convergence of the MMCF of complete embedded star-shaped hypersurfaces (as radial graphs) in the hyperbolic space $\mathbb{H}^{n+1}$ with prescribed asymptotic boundary at infinity, under some natural geometric conditions on the initial hypersurfaces. Namely, we consider the following Dirichlet problem of the MMCF:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{F}(\mathbf{z},t)^{\perp} = (H-\sigma)\nu_{\mathrm{H}}, & (\mathbf{z},t) \in \mathbb{S}^n_+ \times [0,\infty), \\ \mathbf{F}(\mathbf{z},0) = \Sigma_0, & \mathbf{z} \in \mathbb{S}^n_+, \\ \mathbf{F}(\mathbf{z},t) = \Gamma(\mathbf{z}), & (\mathbf{z},t) \in \partial\mathbb{S}^n_+ \times [0,\infty), \end{cases} \tag{1.2}$$

where $\sigma \in (-1,1)$ and $\Gamma = \partial\Sigma_0$ is the boundary of a star-shaped C^{1+1} domain in $\{x_{n+1} = 0\}$ (the case of Γ being only continuous will also be discussed). As an application, we shall also show that we can use the MMCF to deform a complete regular hypersurface to one with constant hyperbolic mean curvature σ in hyperbolic space $\mathbb{H}^{n+1}$.

Mean curvature flow (MCF) was first studied by Brakke [3] in the context of geometric measure theory. Later, smooth compact surfaces evolved by MCF in Euclidean space were investigated by Huisken in [13, 17], and on arbitrary ambient manifolds in [14]. The study of the evolution of complete graphs by MCF in $\mathbb{R}^{n+1}$ was also studied in [6], the result being improved in [8]. See also [16] for the nonparametric MCF with Dirichlet boundary condition. In [24], Unterberger considered the MCF in hyperbolic space, namely, the case of $\sigma = 0$ in Equation (1.1). And he obtained that if the initial surface Σ_0 has bounded hyperbolic height over $\mathbb{S}^n_+$ then under the MCF, Σ_t converges in C^∞ to $\mathbb{S}^n_+$ which has constant mean curvature 0. We shall remark that a similar MMCF (which is called the volume preserving MCF) was studied by Huisken in [15] for closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$, where the constant σ in (1.1) was replaced by the average of the mean curvature of Σ_t, see also [4] for this volume preserving MCF in the hyperbolic space. With the average of the mean curvature of Σ_t in the place of the constant σ, a priori one cannot predict what the flow will converge to (if it converges), while we see directly that if the MMCF (1.1) converges then it converges to a hypersurface with constant mean curvature σ. Namely, we can actually prescribe the constant mean curvature $\sigma \in (-1, 1)$ for the limiting hypersurface through the flow. This is the important feature and novelty of our version of MMCF defined in this work, which is also special for the hyperbolic setting. Finally, we shall remark that it would be very interesting to see what the corresponding MMCF is in the Euclidean setting.

The problem of finding smooth complete hypersurfaces of constant mean curvature in hyperbolic space with prescribed asymptotic boundary at infinity (also known as Asymptotic Plateau Problem) has also been studied over the years, see [1, 2, 18, 23] for the approach using geometry measure theory. The first elliptic partial differential equation (PDE) approach to this problem was due to Lin [19], and later on it was used by Nelli and Spruck [22] and Guan and Spruck [9]. In particular, in [9] Guan and Spruck proved the existence and uniqueness of smooth complete hypersurfaces of constant mean curvature $\sigma \in (-1, 1)$ in hyperbolic space with prescribed asymptotic boundary at infinity. In [5], among other, De Silva and Spruck recovered this result using the method of calculus of variations and representation techniques. We remark that our paper can be thought of as a flow version of their variational method, see Section 2. For the existence of hypersurfaces of constant (general) curvature in hyperbolic space $\mathbb{H}^{n+1}$ which have prescribed asymptotic boundary at infinity, see [10, 11].

Due to the degeneracy of the MMCF (1.2) for radial graphs at infinity (see Equation (2.10) below), we will begin with considering the approximate problem. For fixed $\epsilon > 0$ sufficiently small, let Γ_ϵ be the vertical translation of Γ to the plane $\{x_{n+1} = \epsilon\}$ and let Ω_ϵ be the subdomain of $\mathbb{S}^n_+$ such that Γ_ϵ is the radial graph over $\partial\Omega_\epsilon$ (see figure 1). We consider the following Dirichlet problem of the approximate modified mean curvature flow (AMMCF):

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{F}(\mathbf{z}, t)^\perp = (H - \sigma)\nu_{\mathrm{H}}, & (\mathbf{z}, t) \in \Omega_\epsilon \times [0, \infty), \\ \mathbf{F}(\mathbf{z}, 0) = \Sigma_0^\epsilon, & \mathbf{z} \in \Omega_\epsilon, \\ \mathbf{F}(\mathbf{z}, t) = \Gamma_\epsilon(\mathbf{z}), & (\mathbf{z}, t) \in \partial\Omega_\epsilon \times [0, \infty), \end{cases} \tag{1.3}$$

where $\Sigma_0^\epsilon = \mathbf{F}(\Omega_\epsilon, 0)$, $\partial\Sigma_0^\epsilon = \Gamma_\epsilon$ and $\sigma \in (-1, 1)$.

For any $\epsilon \geq 0$ sufficiently small and any point $P \in \partial\Sigma_0^\epsilon = \Gamma_\epsilon$ (denoting $\Sigma_0^0 = \Sigma_0$ and $\Gamma_0 = \Gamma$), the uniform star-shapedness and regularity of Γ_ϵ imply there exist balls $B_{R_1}(a, P)$ and $B_{R_2}(b, P)$ with radii $R_1 > 0$ and

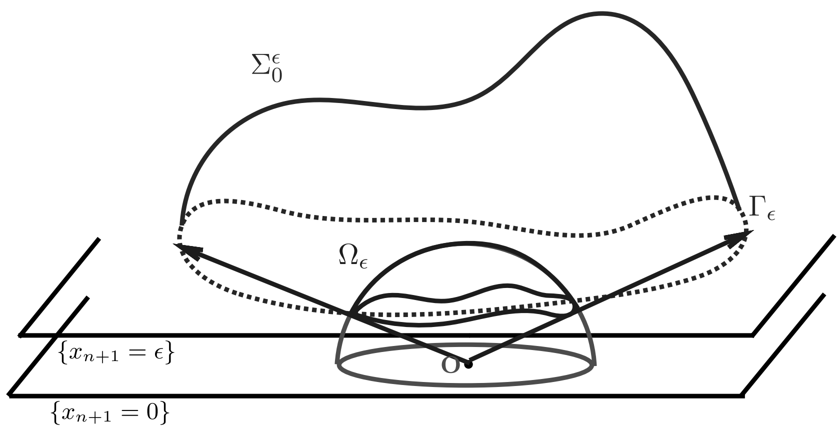

Figure 1: Approximate initial hypersurface.

$R_2 > 0$ and centered at $a = (a', -\sigma R_1)$ and $b = (b', \sigma R_2)$, respectively (see also "equidistant spheres" in Section 3.2 below), such that $\{x_{n+1} = \epsilon\} \cap B_{R_1}(a, P)$ is internally tangent to Γ_ϵ at P and $\{x_{n+1} = \epsilon\} \cap B_{R_2}(b, P)$ is externally tangent to Γ_ϵ at P. Note that in a small neighborhood $B_\delta(P)$ around P for some $\delta > 0$, both $\partial B_{R_1}(a, P) \cap B_\delta(P)$ and $\partial B_{R_2}(b, P) \cap B_\delta(P)$ can be locally represented as radial graphs. To state our main results appropriately, we say that the approximate initial hypersurfaces Σ_0^ϵ's satisfy the uniform interior (resp. exterior) local ball condition if for all $\epsilon \geq 0$ sufficiently small and all $P \in \Gamma_\epsilon$, $\Sigma_0^\epsilon \cap B_\delta(P) \cap B_{R_1}(a, P) = \{P\}$ (resp. $\Sigma_0^\epsilon \cap B_\delta(P) \cap B_{R_2}(b, P) = \{P\}$, see figure 2), and the local radial graph $\partial B_{R_1}(a, P) \cap B_\delta(P)$ (resp. $\partial B_{R_2}(b, P) \cap B_\delta(P)$) has a uniform Lipschitz bound depending only on the star-shapedness of Γ. If Σ_0^ϵ's satisfy both of the uniform interior and exterior local ball conditions, then we say Σ_0^ϵ's satisfy the uniform local ball condition.[1]

The main results in this paper are the following.

Main Theorem 1.1. *Let Γ be the boundary of a star-shaped C^{1+1} domain in $\{x_{n+1} = 0\} = \partial_\infty \mathbb{H}^{n+1}$ and Γ_ϵ be its vertical lift to $\{x_{n+1} = \epsilon\}$ for $\epsilon > 0$ sufficiently small. Let $\Sigma_0 = \lim_{\epsilon \to 0} \Sigma_0^\epsilon$ be the limiting hypersurface of radial*

[1]Such initial hypersurfaces naturally exist and this can be seen explicitly since the balls $B_{R_1}(a, P)$ and $B_{R_2}(b, P)$ can be constructed with uniform radii (see Equation (8.5)) and the tangent plane to them at P can be computed explicitly as well (see Equation (6.2)).

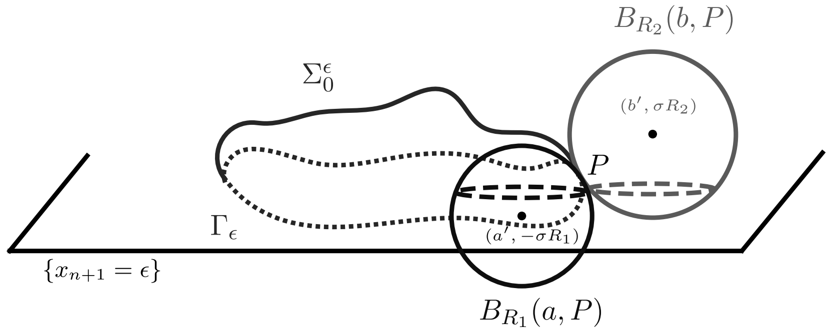

Figure 2: Uniform interior and exterior local ball conditions.

graphs $\Sigma_0^\epsilon \in C^{1+1}(\overline{\Omega_\epsilon})$ *with* $\partial\Sigma_0^\epsilon = \Gamma_\epsilon$. *Suppose* Σ_0^ϵ*'s have a uniform Lipschitz bound and satisfy the uniform local ball condition. Then*

(i) *there exists a unique solution* $\mathbf{F}(\mathbf{z},t) \in C^\infty(\mathbb{S}^n_+ \times (0,\infty)) \cap C^{1+1,\frac{1}{2}+\frac{1}{2}}(\overline{\mathbb{S}^n_+} \times (0,\infty)) \cap C^0(\overline{\mathbb{S}^n_+} \times [0,\infty))$ *to the MMCF* (1.2)*;*

(ii) *there exist* $t_i \nearrow \infty$ *such that* $\Sigma_{t_i} = F(\mathbb{S}^n_+, t_i)$ *converges to a unique stationary smooth complete hypersurface* $\Sigma_\infty \in C^\infty(\mathbb{S}^n_+) \cap C^{1+1}(\overline{\mathbb{S}^n_+})$ *(as a radial graph over* $\mathbb{S}^n_+$*) which has constant hyperbolic mean curvature* σ *and* $\partial\Sigma_\infty = \Gamma$ *asymptotically. Also, each* Σ_t *is a complete radial graph over* $\mathbb{S}^n_+$*;*

(iii) *if additionally* Σ_0^ϵ *has mean curvature* $H^\epsilon \geq \sigma$ *for all* $\epsilon > 0$ *sufficiently small, then* Σ_t *converges uniformly to* Σ_∞ *for all* t*.*

In fact, if Σ_0^ϵ has hyperbolic mean curvature $H^\epsilon \geq \sigma$ for all $\epsilon > 0$ sufficiently small, then the uniform interior local ball condition on Σ_0^ϵ's can be relaxed.

Main Theorem 1.2. *Let* Γ *and* Γ_ϵ *be as in Theorem 1.1 and* $\Sigma_0 = \lim_{\epsilon\to 0} \Sigma_0^\epsilon$ *be the limiting hypersurface of radial graphs* $\Sigma_0^\epsilon \in C^2(\Omega_\epsilon) \cap C^{1+1}(\overline{\Omega_\epsilon})$ *with* $\partial\Sigma_0^\epsilon = \Gamma_\epsilon$. *Suppose* Σ_0^ϵ *has mean curvature* $H^\epsilon \geq \sigma$ *for all* $\epsilon > 0$ *sufficiently small and* Σ_0^ϵ*'s have a uniform Lipschitz bound and satisfy the uniform exterior local ball condition. Then there exists a unique solution* $\mathbf{F}(\mathbf{z},t) \in C^\infty(\mathbb{S}^n_+ \times (0,\infty)) \cap C^{0+1,0+\frac{1}{2}}(\overline{\mathbb{S}^n_+} \times (0,\infty)) \cap C^0(\overline{\mathbb{S}^n_+} \times [0,\infty))$ *to the MMCF* (1.2)*. Moreover,* $\Sigma_t = F(\mathbb{S}^n_+, t)$ *converges uniformly for all* t *to a unique stationary smooth complete hypersurface* $\Sigma_\infty \in C^\infty(\mathbb{S}^n_+) \cap C^{1+1}(\overline{\mathbb{S}^n_+})$ *(as a radial graph over* $\mathbb{S}^n_+$*) which has constant hyperbolic mean curvature* σ *and* $\partial\Sigma_\infty = \Gamma$ *asymptotically. Also, each* Σ_t *is a complete radial graph over* $\mathbb{S}^n_+$*.*

Remark 1.3. We expect that the same results would hold for general star-shaped initial hypersurfaces.

In Section 8, we will give an example of "good" initial hypersurfaces of Theorem 1.2. We point out that many of the techniques and estimates used in the proofs of Theorems 1.1 and 1.2 come from the work of Guan and Spruck [9], and our results could be thought of as the parabolic analog of the results in [9]. Given this fact, we shall also remark that a proof via flow method to the following existence theorem due to Guan and Spruck can be obtained.

Theorem 1.4 [9]. *Suppose* Γ *is the boundary of a star-shaped* C^{1+1} *domain in* $\{x_{n+1} = 0\}$ *and let* $|\sigma| < 1$*. Then there exists a unique smooth complete hypersurface* Σ *of constant hyperbolic mean curvature* σ *in* $\mathbb{H}^{n+1}$ *with asymptotic boundary* Γ*. Moreover,* Σ *may be represented as a radial graph over* $\mathbb{S}^n_+$ *of a function in* $C^\infty(\mathbb{S}^n_+) \cap C^{1+1}(\overline{\mathbb{S}^n_+})$*.*

With the aid of an a priori interior gradient estimate (see Section 9) and via an approximation argument, the regularity of the boundary data Γ in Theorems 1.1 and 1.2 could be further relaxed to be only continuous and a similar result still holds (see Theorem 9.2 below). And again, we note that a parabolic version of proof to the following result due to Guan and Spruck [9] and De Silva and Spruck [5] can be obtained.

Theorem 1.5 [5, 9]. *Suppose* Γ *is the boundary of a continuous star-shaped domain in* $\{x_{n+1} = 0\}$ *and let* $|\sigma| < 1$*. Then there exists a unique smooth complete hypersurface* Σ *of constant hyperbolic mean curvature* σ *in* $\mathbb{H}^{n+1}$ *with asymptotic boundary* Γ*. Moreover,* Σ *may be represented as a radial graph over* $\mathbb{S}^n_+$ *of a function in* $C^\infty(\mathbb{S}^n_+) \cap C^0(\overline{\mathbb{S}^n_+})$*.*

The paper is organized as follows. In Section 2, we set up the problems, namely, the Dirichlet problems for the MMCF and AMMCF for radial graphs in hyperbolic space. In Section 3, we state the short-time existence result for the AMMCF and discuss the equidistant spheres in $\mathbb{H}^{n+1}$ which will serve as good barriers in many situations. We will prove Theorem 1.1 in Sections 4–7. In Section 4, we prove a global gradient estimate for the solution to the AMMCF and therefore the long-time existence of the AMMCF. In Section 5, we prove the uniform gradient estimate (independent of ϵ) for the solutions to the AMMCF's, which leads to the long-time existence of the MMCF. This estimate is the main technical result of the paper. In Section 6, we show the boundary regularity of the MMCF and the uniform

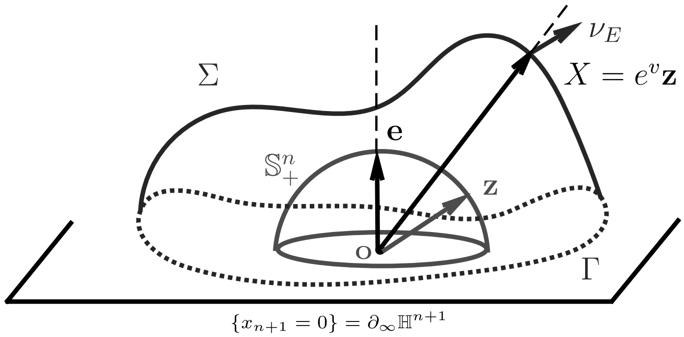

Figure 3: Σ as a radial graph.

convergence of the MMCF in the case of $H^\epsilon \geq \sigma$ initially in Section 7 . In Section 8, we will prove Theorem 1.2 and give an example of "good" initial hypersurfaces in Theorem 1.2. In Section 9, we prove a version of a priori interior gradient estimate and therefore the existence result of the MMCF with only continuous boundary data.

2. MMCF and AMMCF for radial graphs in hyperbolic space

Let $\Omega \subseteq \mathbb{S}^n_+$, and suppose that Σ is a radial graph over Ω with position vector X in $\mathbb{R}^{n+1}$. Then we can write

$$X = \mathrm{e}^{v(\mathbf{z})}\mathbf{z}, \quad \mathbf{z} \in \Omega,$$

for a function v defined over Ω. We call such function v the radial height of Σ. One observes that Σ remains a radial graph as long as

$$X \cdot \nu_{\mathrm{E}} > 0, \tag{2.1}$$

where ν_{E} is the Euclidean outward unit normal vector of Σ (see figure 3).

2.1. Gradient flow

As in [5], one can define the energy functional $\mathcal{I}(\Sigma)$ associated to Σ:

$$\begin{aligned}\mathcal{I}(\Sigma) &= \mathcal{I}_\Omega(v) = A_\Omega(v) + n\sigma V_\Omega(v)\\ &= \int_\Omega \sqrt{1+|\nabla v|^2}\,y^{-n}\,d\mathbf{z} + n\sigma \int_\Omega v(\mathbf{z})y^{-(n+1)}\,d\mathbf{z},\end{aligned} \tag{2.2}$$

where $y = \mathbf{z}_{n+1}$ and ∇ denotes the covariant derivative on the standard unit sphere. Note that in this energy functional $\mathcal{I}(\Sigma)$, the term A_Ω corresponds to the area of Σ (under the hyperbolic metric) and the term V_Ω corresponds to the radial volume of the cone region between Σ and the origin (up to a constant), see [5] for more details.

Then for a smooth solution $\Sigma_t = \mathbf{F}(\mathbf{z}, t)$ to the MMCF (1.1), which can be represented as a complete radial graph over $\Omega = \mathbb{S}^n_+$, namely,

$$\mathbf{F}(\mathbf{z}, t) = \mathbf{X}(\mathbf{z}, t) = \mathrm{e}^{v(\mathbf{z},t)}\mathbf{z}, \quad (\mathbf{z}, t) \in \mathbb{S}^n_+ \times (0, \infty),$$

we have

$$\begin{aligned} \frac{d}{dt}\mathcal{I}(\Sigma_t) &= -n \int_\Omega (H-\sigma)^2\sqrt{1+|\nabla v|^2}\,y^{-n}\,d\mathbf{z} \\ &= -n \int_{\Sigma_t} \langle \partial\mathbf{F}/\partial t, (H-\sigma)\nu_{\mathrm{H}}\rangle_{\mathrm{H}}\, dA \\ &= -n \int_{\Sigma_t} (H-\sigma)^2 dA \le 0, \end{aligned} \tag{2.3}$$

where in the first equality we used integration by parts, Equation (2.10) (see below) and the fact that (see Equation (1.2) of [5])

$$\mathrm{div}_{\mathbf{z}}\left(\frac{y^{-n}\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = nHy^{-(n+1)} \quad \text{in } \Omega,$$

and the second equality is just the first variation formula for $\mathcal{I}$.

From this point of view, one sees that the MMCF is the natural negative L^2-gradient flow of the energy functional $\mathcal{I}(\Sigma)$. We have:

Lemma 2.1. *Let $\mathbf{F}(\mathbf{z}, t) = \mathrm{e}^{v(\mathbf{z},t)}\mathbf{z}$ be a smooth radial graph solution to the AMMCF* (1.3) *in $\Omega \times [0, T]$. Then for all $t \in [0, T)$ we have*

$$I(\Sigma^\epsilon_t) + n\int_0^t \int_\Omega (H-\sigma)^2\, dA\, dt = I(\Sigma^\epsilon_0). \tag{2.4}$$

Remark 2.2. We point out that Equation (2.3) is a natural analog of the well-known formula for the classic MCF:

$$\frac{d}{dt}\,\mathrm{Area}(\Sigma_t) = -\int_{\Sigma_t} H^2\, dA \le 0.$$

2.2. The hyperbolic mean curvature

We will begin with fixing some notations, and collecting some relevant facts about the hyperbolic space $\mathbb{H}^{n+1}$, which can be easily found in [9]. Where necessary, expressions in the Euclidean and hyperbolic spaces will be denoted by the subscript or superscript E and H, respectively. Let ∇ denote the covariant derivative on the standard unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ and

$$y = \mathbf{e} \cdot \mathbf{z}, \quad \text{for } \mathbf{z} \in \mathbb{S}^n \subset \mathbb{R}^{n+1},$$

where, throughout this paper, $\mathbf{e}$ is the unit vector in the positive x_{n+1} direction in $\mathbb{R}^{n+1}$, and '$\cdot$' denotes the Euclidean inner product in $\mathbb{R}^{n+1}$. Let $\tau_1, \ldots, \tau_n$ be a local frame of smooth vector fields on the upper hemisphere $\mathbb{S}^n_+$. We denote by $\gamma_{ij} = \tau_i \cdot \tau_j$ the standard metric of $\mathbb{S}^n_+$ and γ^{ij} its inverse. For a function v on $\mathbb{S}^n_+$, we denote $v_i = \nabla_i v = \nabla_{\tau_i} v$, $v_{ij} = \nabla_j \nabla_i v$, etc.

Suppose that locally Σ is a radial graph over $\Omega \subseteq \mathbb{S}^n_+$. Then the Euclidean outward unit normal vector and mean curvature of Σ are respectively

$$\nu_{\mathrm{E}} = \frac{\mathbf{z} - \nabla v}{w}$$

and

$$H_{\mathrm{E}} = \frac{a^{ij} v_{ij} - n}{n \mathrm{e}^v w},$$

where

$$a^{ij} = \gamma^{ij} - \frac{\gamma^{ik} v_k v_j}{w^2}, \quad 1 \le i, j \le n \quad \text{and} \quad w = (1 + |\nabla v|^2)^{1/2}.$$

Note that

$$X \cdot \nu_{\mathrm{E}} = \frac{\mathrm{e}^v}{w}, \tag{2.5}$$

and therefore as long as w is bounded from above Σ remains a radial graph by (2.1).

We also have the hyperbolic outward unit normal vector

$$\nu_{\mathrm{H}} = u \nu_{\mathrm{E}},$$

where

$$u = \mathbf{e} \cdot \mathbf{X} = \mathbf{e} \cdot \mathrm{e}^v \mathbf{z} = y \mathrm{e}^v$$

is called the height function. Moreover, using the relation between the hyperbolic and Euclidean principle curvatures

$$\kappa_i^H = \mathbf{e}\cdot\nu_{\mathrm{E}} + u\kappa_i^{\mathrm{E}}, \quad i = 1,\ldots,n,$$

we have (see Equation (2.1) of [9], cf. Equation (1.8) of [10])

$$H = \mathbf{e}\cdot\nu_{\mathrm{E}} + uH_{\mathrm{E}}, \tag{2.6}$$

which gives the hyperbolic mean curvature of Σ:

$$H = y\mathrm{e}^{v}H_{\mathrm{E}} + \frac{y - \mathbf{e}\cdot\nabla v}{w} = \frac{ya^{ij}v_{ij}}{nw} - \frac{\mathbf{e}\cdot\nabla v}{w}, \tag{2.7}$$

and therefore

$$a^{ij}v_{ij} = \frac{n}{y}(Hw + \mathbf{e}\cdot\nabla v). \tag{2.8}$$

2.3. Degenerate parabolic equation

The first equation of the MMCF (1.2) implies

$$\left\langle \frac{\partial}{\partial t}\mathbf{F}, \nu_{\mathrm{H}} \right\rangle_{\mathrm{H}} = \left\langle \frac{\partial}{\partial t}(\mathrm{e}^{v}\mathbf{z}), \nu_{\mathrm{H}} \right\rangle_{\mathrm{H}} = \frac{\mathrm{e}^{v}}{uw}\frac{\partial v}{\partial t} = \frac{1}{yw}\frac{\partial v}{\partial t} = H - \sigma. \tag{2.9}$$

Therefore by Equation (2.7) we have

$$\frac{\partial v}{\partial t} = yw(H-\sigma) = y^2\frac{a^{ij}v_{ij}}{n} - y\mathbf{e}\cdot\nabla v - \sigma yw. \tag{2.10}$$

Suppose Γ is the radial graph of a function e^{ϕ} over $\partial\mathbb{S}^n_+$, i.e., Γ can be represented by

$$X = \mathrm{e}^{\phi(\mathbf{z})}\mathbf{z}, \quad \mathbf{z}\in\partial\mathbb{S}^n_+.$$

Then one observes that the Dirichlet problem for the MMCF (1.2) is equivalent to the following (degenerate parabolic) Dirichlet problem (i.e., the MMCF for radial graphs):

$$
\begin{cases}
\dfrac{\partial v(\mathbf{z},t)}{\partial t} = y^2 \frac{a^{ij}v_{ij}}{n} - y\mathbf{e}\cdot\nabla v - \sigma y w, & (\mathbf{z},t) \in \mathbb{S}^n_+ \times (0,\infty),\\
v(\mathbf{z},0) = v_0(\mathbf{z}), & \mathbf{z} \in \mathbb{S}^n_+,\\
v(\mathbf{z},t) = \phi(\mathbf{z}), & (\mathbf{z},t) \in \partial\mathbb{S}^n_+ \times [0,\infty),
\end{cases}
\tag{2.11}
$$

where we represent Σ_0 as the radial graph of the function e^{v_0} over $\mathbb{S}^n_+$ and $v_0\big|_{\partial\mathbb{S}^n_+} = \phi$.

2.4. Approximate problem

Due to the degeneracy of Equation (2.11) at infinity (i.e., $y = 0$), we consider the corresponding approximate problem for a fixed $\epsilon > 0$ sufficiently small. Namely, equivalently to (1.3), we solve the following (non-degenerate parabolic) Dirichlet problem (i.e., the AMMCF for radial graphs):

$$
\begin{cases}
\dfrac{\partial v(\mathbf{z},t)}{\partial t} = y^2 \dfrac{a^{ij}v_{ij}}{n} - y\mathbf{e}\cdot\nabla v - \sigma y w, & (\mathbf{z},t) \in \Omega_\epsilon \times (0,\infty),\\
v(\mathbf{z},0) = v_0^\epsilon(\mathbf{z}), & \mathbf{z} \in \Omega_\epsilon,\\
v(\mathbf{z},t) = \phi^\epsilon(\mathbf{z}), & (\mathbf{z},t) \in \partial\Omega_\epsilon \times [0,\infty),
\end{cases}
\tag{2.12}
$$

where we represent Σ_0^ϵ as the radial graph of the function $\mathrm{e}^{v_0^\epsilon}$ over Ω_ϵ and $v_0^\epsilon\big|_{\partial\Omega_\epsilon} = \phi^\epsilon$, and ϕ^ϵ is a function defined on $\partial\Omega_\epsilon \subset \mathbb{S}^n_+$ such that Γ_ϵ can be represented as a radial graph of $\mathrm{e}^{\phi^\epsilon}$ over $\partial\Omega_\epsilon$, i.e.,

$$
X = \mathrm{e}^{\phi^\epsilon(\mathbf{z})}\mathbf{z}, \quad \mathbf{z} \in \partial\Omega_\epsilon.
\tag{2.13}
$$

We denote the regular solution to (2.12) by v^ϵ.

3. The short-time existence and equidistant spheres

3.1. Short-time existence

In the rest of the paper, we will focus on the case of $\sigma \in [0,1)$ and the case of $\sigma \in (-1,0)$ can be dealt with in the same way after using the hyperbolic reflection over $\mathbb{S}^n_+$. The standard parabolic PDE theory with Schauder estimates guarantees the short-time existence of a regular solution (up to the parabolic boundary) to the AMMCF (2.12) with a C^∞ initial hypersurface and compatible boundary data (i.e., $H = \sigma$ on $\partial\Sigma_0^\epsilon$). For a C^∞ initial hypersurface with incompatible boundary data, a solution exists at least for short time and becomes regular immediately after $t = 0$ (cf. [12]). This is the statement of the next lemma.

Lemma 3.1. *There exists* $T_\epsilon^\star > 0$ *such that the AMMCF* (2.12) *with initial data* $v_0^\epsilon \in C^\infty(\overline{\Omega_\epsilon})$ *has a solution* $v^\epsilon \in C^\infty(\overline{\Omega_\epsilon} \times [0, T_\epsilon^\star))$ *except on the corner* $\partial\Omega_\epsilon \times \{t = 0\}$.

For less regular (e.g., C^{1+1}) initial and boundary data, the short-time existence lemma will remain true (see e.g., [20, Theorem 8.2] and [21, Theorem 4.2, P. 559]).

Lemma 3.2. *There exists* $T_\epsilon^\star > 0$ *such that the AMMCF* (2.12) *with initial data* $v_0^\epsilon \in C^{1+1}(\overline{\Omega_\epsilon})$ *has a solution* $v^\epsilon \in C^\infty(\Omega_\epsilon \times (0, T_\epsilon^\star)) \cap C^0(\overline{\Omega_\epsilon} \times [0, T_\epsilon^\star))$.

Moreover, as we shall see, the passage to the limit of $\{v^\epsilon\}$ as $\epsilon \to 0$ to get the long-time existence of the MMCF (2.11) will be based on a series of estimates uniform in ϵ.

3.2. Equidistant spheres

In the following, let T_ϵ (possibly ∞) be the maximal time up to which the AMMCF (1.3) for radial graphs or equivalently the solution to (2.12) exists, and let $V_\epsilon = \cup_{0 \le t \le T_\epsilon} \Sigma_t^\epsilon$ denote the flow region in $\mathbb{H}^{n+1}$, where $\Sigma_t^\epsilon = \mathbf{F}(\Omega_\epsilon, t)$ is the hypersurface moving by the AMMCF (1.3) at time t.

Our estimates in the proof of the main theorems are all based on the following fact, which was also extensively used in [9]. Let $B_1 = B_R(a)$ be a ball of radius R centered at $a = (a', -\sigma R) \in \mathbb{R}^{n+1}$ where $a' \in \mathbb{R}^n$ and $\sigma \in (-1, 1)$. Then $S_1 = \partial B_1 \cap \mathbb{H}^{n+1}$ has constant hyperbolic mean curvature σ with respect to its outward normal. Similarly, let $B_2 = B_R(b)$ be a ball of radius R centered at $b = (b', \sigma R) \in \mathbb{R}^{n+1}$, then $S_2 = \partial B_2 \cap \mathbb{H}^{n+1}$ has constant hyperbolic mean curvature σ with respect to its inward normal. These so-called equidistant spheres will serve as good barriers in many situations (see Lemma 3.4 below). Let $D \subset \{x_{n+1} = 0\}$ be the domain enclosed by Γ and $D_\epsilon \subset \{x_{n+1} = \epsilon\}$ be the domain enclosed by Γ_ϵ.

The following lemma is an immediate consequence of the fact mentioned above, see [9].

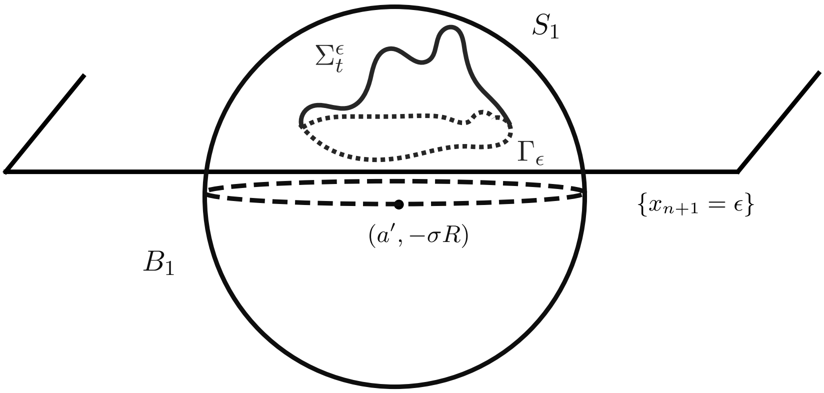

Figure 4: Bounded by equidistant spheres.

Lemma 3.3 ([9], Lemma 3.1). *Let B_1 and B_2 be balls in $\mathbb{R}^{n+1}$ of radius R centered at $a = (a', -\sigma R)$ and $b = (b', \sigma R)$, respectively. Suppose Σ has constant hyperbolic mean curvature σ. Then*

(i) *If $\partial\Sigma \subset B_1$, then $\Sigma \subset B_1$;*

(ii) *If $B_1 \cap \{x_{n+1} = \epsilon\} \subset D_\epsilon$, then $B_1 \cap \Sigma = \emptyset$;*

(iii) *If $B_2 \cap D_\epsilon = \emptyset$, then $B_2 \cap \Sigma = \emptyset$.*

As a parabolic analog of Guan and Spruck's Lemma 3.3, we have the following a priori bound for the flow.

Lemma 3.4. *Let B_1 and B_2 be balls in $\mathbb{R}^{n+1}$ of radius R centered at $a = (a', -\sigma R)$ and $b = (b', \sigma R)$, respectively.*

(i) *If $\Sigma_0^\epsilon \subset B_1$, then $V_\epsilon \subset B_1$ (see figure 4);*

(ii) *If $B_1 \cap \{x_{n+1} = \epsilon\} \subset D_\epsilon$ and $B_1 \cap \Sigma_0^\epsilon = \emptyset$, then $B_1 \cap V_\epsilon = \emptyset$;*

(iii) *If $B_2 \cap D_\epsilon = \emptyset$ and $B_2 \cap \Sigma_0^\epsilon = \emptyset$, then $B_2 \cap V_\epsilon = \emptyset$.*

Proof. The proof is virtually the same as the proof of Lemma 3.3 in [9] and we include it for the convenience of the reader. This lemma follows from the maximum principle by performing homothetic dilations (hyperbolic isometries) from $(a', 0)$ and $(b', 0)$, respectively. For (i), we expand B_1 continuously until it contains Σ_0^ϵ; for (ii) and (iii) we shrink B_1 and B_2 until they are respectively inside and outside Σ_0^ϵ. We note that Σ_t^ϵ satisfies Equation (2.10) as a radial graph and its mean curvature is calculated with respect to its outward normal direction. Also S_1, S_2 have constant mean curvature σ with respect to the outward and inward normal respectively, and locally as radial graphs they both satisfy Equation (2.10) (statically) too. Then from the maximum principle we see that Σ_t^ϵ cannot touch B_1 or B_2 when we reverse this process. □

4. Global gradient bounds and long time existence of the AMMCF

Before we begin our proof, we collect some important formulas that were first derived in [9]. From now on, we assume the local vector fields $\tau_1, \ldots, \tau_n$ to be orthonormal on $\mathbb{S}^n_+$ so that $\gamma_{ij} = \delta_{ij}$ and thus $a^{ij} = \delta_{ij} - \frac{v_i v_j}{w^2}$. The covariant derivatives of y are

$$\begin{aligned} y_i &= \nabla_i y = (\mathbf{e} \cdot \mathbf{z})_i = \mathbf{e} \cdot \tau_i, \\ y_{ij} &= \nabla_i \nabla_j y = \mathbf{e} \cdot \nabla_i \nabla_j \mathbf{z} = \mathbf{e} \cdot \nabla_i \tau_j = -y\delta_{ij}. \end{aligned} \tag{4.1}$$

Therefore

$$\begin{gathered} \mathbf{e} \cdot \nabla y = \sum (\mathbf{e} \cdot \tau_i)^2 = 1 - y^2, \\ \nabla v \cdot \nabla y = \mathbf{e} \cdot \nabla v \quad \text{and} \quad \nabla w \cdot \nabla y = \mathbf{e} \cdot \nabla w. \end{gathered}$$

Note that we also have the identities

$$a^{ij} v_i = \frac{v_j}{w^2}, \quad a^{ij} v_i v_j = 1 - \frac{1}{w^2}, \quad \sum a^{ii} = n - 1 + \frac{1}{w^2}.$$

Moreover,

$$w_i = \frac{v_k v_{ki}}{w}, \quad w_{ij} = \frac{v_k v_{kij}}{w} + \frac{1}{w} a^{kl} v_{ki} v_{lj} \quad \text{and} \quad (\nabla_k a^{ij}) v_{ij} = -\frac{2}{w} a^{ij} w_i v_{kj}. \tag{4.2}$$

Straight forward calculations also show that

$$\begin{aligned} (\mathbf{e} \cdot \nabla v)_i &= (\mathbf{e} \cdot \tau_k v_k)_i = \mathbf{e} \cdot \tau_k v_{ki} - y v_i = y_k v_{ki} - y v_i, \\ (\mathbf{e} \cdot \nabla v)_{ij} &= \mathbf{e} \cdot \tau_k v_{kij} - 2y v_{ij} - \mathbf{e} \cdot \tau_j v_i = y_k v_{kij} - 2y v_{ij} - y_j v_i \end{aligned}$$

and

$$\nabla v \cdot \nabla (\mathbf{e} \cdot \nabla v) = v_i (\mathbf{e} \cdot \tau_k v_{ki} - y v_i) = w \mathbf{e} \cdot \nabla w - y(w^2 - 1). \tag{4.3}$$

We also have the formula for commuting the covariant derivatives

$$v_{ijk} = v_{kij} + v_j\delta_{ik} - v_k\delta_{ij}. \tag{4.4}$$

Now we are ready to state our first main technical lemma.

Lemma 4.1. *Let $v \in C^{3,\frac{3}{2}}(\Omega \times (0,T))$ be a function satisfying Equation* (2.10) *for some $T > 0$ and $\Omega \subseteq \mathbb{S}^n_+$. Then*

$$\left(\frac{\partial}{\partial t} - L\right) w \leq -\sigma(\mathbf{e}\cdot\nabla v) + \frac{y^2(w^2-1)}{nw} - H^2 w \leq 2w, \quad \text{in } \Omega \times (0,T), \tag{4.5}$$

where L is the linear elliptic operator

$$L \equiv \frac{y^2}{n}\left(a^{ij}\nabla_{ij} - \frac{2}{w}a^{ij}w_i\nabla_j - \frac{n}{wy}(\sigma\nabla v + w\mathbf{e})\cdot\nabla\right).$$

Remark 4.2. The main part of the linear elliptic operator L was already used by Guan and Spruck in [9].

Proof. (of Lemma 4.1) By Equation (2.10) we have

$$\begin{aligned}\frac{\partial}{\partial t}w &= \frac{1}{w}\nabla v\cdot\nabla(v_t) = \frac{\nabla v}{w}\cdot\nabla(yw(H-\sigma))\\ &= \frac{\nabla v}{w}\cdot(\nabla y w(H-\sigma) + y\nabla w(H-\sigma) + yw\nabla H)\\ &= \mathbf{e}\cdot\nabla v(H-\sigma) + \frac{y(H-\sigma)}{w}\nabla v\cdot\nabla w + y\nabla v\cdot\nabla H\end{aligned}$$

Differentiating both sides of Equation (2.8) with respect to τ_k gives (using also Equation (4.2))

$$\begin{aligned}(\nabla_k a^{ij})v_{ij} + a^{ij}v_{ijk} &= a^{ij}v_{ijk} - \frac{2}{w}a^{ij}w_i v_{kj}\\ &= \frac{n}{y}(H_k w + Hw_k + (\mathbf{e}\cdot\nabla v)_k) - \frac{n}{y^2}(Hw + \mathbf{e}\cdot\nabla v)y_k.\end{aligned}$$

Therefore

$$\begin{aligned}a^{ij}v_{kij} = {}&\frac{n}{y}(H_k w + Hw_k + (\mathbf{e}\cdot\nabla v)_k) - \frac{n}{y^2}(Hw + \mathbf{e}\cdot\nabla v)y_k + \frac{2}{w}a^{ij}w_i v_{kj}\\ &- \frac{v_k}{w^2} + \left(n - 1 + \frac{1}{w^2}\right)v_k\end{aligned} \tag{4.6}$$

and

$$a^{ij}v_kv_{ijk} - \frac{2}{w}a^{ij}w_iv_kv_{kj} = \frac{n}{y}\nabla v\cdot(\nabla Hw + H\nabla w + \nabla(\mathbf{e}\cdot\nabla v)) \\ - \frac{n\mathbf{e}\cdot\nabla v}{y^2}(Hw + \mathbf{e}\cdot\nabla v).$$

Note that we also have

$$\begin{aligned} a^{ij}w_{ij} &= a^{ij}\left(\frac{v_kv_{kij}}{w} + \frac{1}{w}a^{kl}v_{ki}v_{lj}\right) \\ &= \frac{1}{w}(v_ka^{ij}(v_{ijk} - v_j\delta_{ik} + v_k\delta_{ij})) + \frac{1}{w}a^{ij}a^{kl}v_{ki}v_{lj}. \end{aligned}$$

Now by the definition of the operator L, we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - L\right)w &= \mathbf{e}\cdot\nabla v(H-\sigma) + \frac{y(H-\sigma)}{w}\nabla v\cdot\nabla w + y\nabla v\cdot\nabla H \\ &\quad - \frac{y^2}{n}\left(a^{ij}w_{ij} - \frac{2}{w}a^{ij}w_iw_j - \frac{n}{wy}(\sigma\nabla v + w\mathbf{e})\cdot\nabla w\right) \\ &= \mathbf{e}\cdot\nabla v(H-\sigma) + \frac{y(H-\sigma)}{w}\nabla v\cdot\nabla w + y\nabla v\cdot\nabla H \\ &\quad - \frac{y^2}{n}\left[\frac{n}{wy}\nabla v\cdot\big(\nabla Hw + H\nabla w + \nabla(\mathbf{e}\cdot\nabla v)\big)\right. \\ &\quad \left. - \frac{n\mathbf{e}\cdot\nabla v}{wy^2}(Hw + \mathbf{e}\cdot\nabla v)\right] + \frac{y^2a^{ij}v_iv_j}{nw} - \frac{y^2(w^2-1)}{nw} \\ &\quad \times\left(n - 1 + \frac{1}{w^2}\right) - \frac{y^2}{nw}a^{ij}a^{kl}v_{ki}v_{lj} - \frac{2y^2}{w^2n}a^{ij}w_iv_kv_{kj} \\ &\quad + \frac{2y^2}{wn}a^{ij}w_iw_j + \frac{y}{w}(\sigma\nabla v + w\mathbf{e})\cdot\nabla w \\ &= \mathbf{e}\cdot\nabla v(2H-\sigma) - \frac{y}{w}\left(\nabla v\cdot\nabla(\mathbf{e}\cdot\nabla v) - w\mathbf{e}\cdot\nabla w\right) \\ &\quad + \frac{(\mathbf{e}\cdot\nabla v)^2}{w} + \frac{y^2}{nw}\left(1 - \frac{1}{w^2}\right) - y^2\left(w - \frac{1}{w}\right) \\ &\quad \times\left(1 - \frac{1}{n} + \frac{1}{nw^2}\right) - \frac{y^2}{nw}a^{ij}a^{kl}v_{ki}v_{lj} \end{aligned}$$

$$
\begin{aligned}
&\le \mathbf{e}\cdot\nabla v(2H-\sigma) - \frac{y}{w}(-y(w^2-1)) + \frac{(\mathbf{e}\cdot\nabla v)^2}{w} \\
&\quad + \frac{y^2}{nw}\left(1-\frac{1}{w^2}\right) - y^2\left(w-\frac{1}{w}\right)\left(1-\frac{1}{n}+\frac{1}{nw^2}\right) \\
&\quad - \frac{1}{w}(Hw+\mathbf{e}\cdot\nabla v)^2 \\
&= -\sigma(\mathbf{e}\cdot\nabla v) + \frac{y^2}{n}\left(w-\frac{1}{w}\right) - H^2 w.
\end{aligned}
$$

Here we have used Equations (4.3), (2.8) and (by Cauchy–Schwarz inequality)

$$a^{ij}a^{kl}v_{ki}v_{lj} \ge \frac{1}{n}(a^{ij}v_{ij})^2 = \frac{n}{y^2}(Hw+\mathbf{e}\cdot\nabla v)^2.$$

Hence we conclude that

$$\left(\frac{\partial}{\partial t} - L\right) w \le 2w.$$

□

For any $\epsilon \ge 0$ sufficiently small and at any point $\mathbf{z}_0 \in \partial\Omega_\epsilon$ corresponding to $P_0 = \mathrm{e}^{\phi^\epsilon(\mathbf{z}_0)}\mathbf{z}_0 \in \Gamma_\epsilon$, let $B_1^\epsilon = B_{R_1}^\epsilon(a', -\sigma R_1)$ and $B_2^\epsilon = B_{R_2}^\epsilon(b', \sigma R_2)$ be the (Euclidean) balls with radii $R_1 > 0$ and $R_2 > 0$, respectively, such that B_1^ϵ and B_2^ϵ are tangent at P_0, and $B_1^\epsilon \cap \{x_{n+1} = \epsilon\}$ is internally tangent to Γ_ϵ at P_0, and $B_2^\epsilon \cap \{x_{n+1} = \epsilon\}$ is externally tangent to Γ_ϵ at P_0. Recall that $S_1^\epsilon = \partial B_1^\epsilon \cap \mathbb{H}^{n+1}$ has constant (hyperbolic) mean curvature σ with respect to its outward normal while $S_2^\epsilon = \partial B_2^\epsilon \cap \mathbb{H}^{n+1}$ has constant mean curvature σ with respect to its inward normal. Moreover, we can represent S_1^ϵ and S_2^ϵ near P_0 as local radial graphs $X_i = \mathrm{e}^{\varphi_i^\epsilon}\mathbf{z}, i = 1, 2$ for $\mathbf{z} \in \overline{\Omega_\epsilon} \cap B_{\epsilon_0}(\mathbf{z}_0)$ where ϵ_0 depends only on the radii of B_i^ϵ's and the uniformly star-shapedness of Γ. Then the uniform local ball condition implies

$$\varphi_1^\epsilon(\mathbf{z}) \le v_0^\epsilon \le \varphi_2^\epsilon(\mathbf{z}), \quad \mathbf{z} \in \overline{\Omega_\epsilon} \cap B_{\epsilon_0}(\mathbf{z}_0). \tag{4.7}$$

From this point of view, one sees that S_1^ϵ and S_2^ϵ serve as good local barriers of Σ_0^ϵ around P_0 and moreover we have $|\nabla v_0^\epsilon|(P_0) \le C$, where C is independent of ϵ and $P_0 \in \Gamma_\epsilon$. Also note that S_1^ϵ and S_2^ϵ have constant hyperbolic mean curvature σ (w.r.t. respective normals) and they are static under the MMCF (2.10) as local radial graphs. Therefore by the maximum principle, they also serve as good local barriers of Σ_t^ϵ around (P_0, t) for all $t \in [0, T_\epsilon)$

and we have

$$|\nabla v^{\epsilon}|(P_0, t) \leq C \tag{4.8}$$

for all $t \in [0, T_{\epsilon})$, where C is independent of ϵ and P_0 by the uniform local ball condition.

Lemma 4.3. *Locally S_1^{ϵ} is interior to V_{ϵ} and S_2^{ϵ} is exterior to V_{ϵ}.*

Proof. This follows from the maximum principle. □

Let $P\Omega_{\epsilon}(T_{\epsilon}^{\star}) = \Omega_{\epsilon} \times \{0\} \cup \partial\Omega_{\epsilon} \times [0, T_{\epsilon}^{\star})$ be the parabolic boundary of $\overline{\Omega_{\epsilon}} \times [0, T_{\epsilon}^{\star})$. Then Lemma 4.1, Equation (4.8) and the Lipschitz bound on the initial radial graph Σ_0^{ϵ} immediately yield (see e.g., [20, Theorem 9.5])

$$w^{\epsilon}(\mathbf{z}, t) \leq \mathrm{e}^{3T_{\epsilon}^{\star}} \max_{(\mathbf{z},t)\in P\Omega_{\epsilon}(T_{\epsilon}^{\star})} w^{\epsilon}(\mathbf{z}, t) \leq C(\epsilon), \quad (\mathbf{z}, t) \in \overline{\Omega_{\epsilon}} \times [0, T_{\epsilon}^{\star}). \tag{4.9}$$

With this gradient estimate (and therefore the Hölder gradient estimate, see e.g., [20, Theorem 12.10]), for any fixed $\epsilon > 0$ sufficiently small, the AMMCF with the approximate initial hypersurface satisfying the conditions in Theorem 1.1 exists uniquely by the parabolic comparison principle and $v^{\epsilon} \in C^{\infty}(\Omega_{\epsilon} \times (0, \infty)) \cap C^{0+1,0+\frac{1}{2}}(\overline{\Omega_{\epsilon}} \times (0, \infty)) \cap C^{0}(\overline{\Omega_{\epsilon}} \times [0, \infty))$ by Schauder estimates. Therefore we have proved

Theorem 4.4. *Let Γ, Γ_{ϵ} and Σ_0^{ϵ}'s be as in Theorem 1.1. Then there exists a unique solution $\mathbf{F}(\mathbf{z}, t) \in C^{\infty}(\Omega_{\epsilon} \times (0, \infty)) \cap C^{0+1,0+\frac{1}{2}}(\overline{\Omega_{\epsilon}} \times (0, \infty)) \cap C^{0}(\overline{\Omega_{\epsilon}} \times [0, \infty))$ to the AMMCF* (1.3).

5. Sharp gradient estimates

Since the earlier gradient estimate is too crude to prove the uniform convergence of the AMMCF's to the MMCF as $\epsilon \to 0$, we need a uniform sharp gradient estimate. To do this, we will need the next main technical result.

Theorem 5.1. *Let $v \in C^{3,\frac{3}{2}}(\Omega \times (0, T))$ be a function satisfying Equation* (2.10) *for some $T > 0$ and $\Omega \subseteq \mathbb{S}_{+}^{n}$. Then*

$$\left(\frac{\partial}{\partial t} - L\right)\left(\mathrm{e}^{v}(w + \sigma(y + \mathbf{e} \cdot \nabla v))\right) \leq 0, \quad \text{in } \Omega \times (0, T), \tag{5.1}$$

where L is the linear elliptic operator from Lemma 4.1.

Proof. From the proof of Lemma 4.1 we know that

$$\left(\frac{\partial}{\partial t} - L\right) w \leq -\sigma(\mathbf{e}\cdot\nabla v) + \frac{y^2}{n}\left(w - \frac{1}{w}\right) - H^2 w. \tag{5.2}$$

We also have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - L\right) y &= -L(y) \\ &= -\frac{y^2}{n}\left(a^{ij}y_{ij} - \frac{2}{w}a^{ij}w_i y_j - \frac{n}{wy}(\sigma\nabla v + w\mathbf{e})\cdot\nabla y\right) \\ &= -\frac{y^2}{n}\left(-y\sum a^{ii} - \frac{2}{w}a^{ij}w_i y_j - \frac{n}{wy}(\sigma\nabla v + w\mathbf{e})\cdot\nabla y\right) \\ &= -\frac{y^2}{n}\left(-\frac{2}{w}a^{ij}w_i y_j - \frac{n}{wy}(\sigma\mathbf{e}\cdot\nabla v + w) + y - \frac{y}{w^2}\right) \\ &= \frac{2y^2}{nw}a^{ij}w_i y_j + \frac{y}{w}(\sigma\mathbf{e}\cdot\nabla v + w) - \frac{y^3}{n} + \frac{y^3}{nw^2},\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - L\right)(\mathbf{e}\cdot\nabla v) = \mathbf{e}\cdot\nabla v_t - L(\mathbf{e}\cdot\nabla v) \\ &= \mathbf{e}\cdot\nabla(yw(H-\sigma)) - \frac{y^2}{n}\big[a^{ij}(\mathbf{e}\cdot\nabla v)_{ij} - \frac{2}{w}\,a^{ij}w_i(\mathbf{e}\cdot\nabla v)_j \\ &\quad - \frac{n}{wy}\,(\sigma\nabla v + w\mathbf{e})\cdot\nabla(\mathbf{e}\cdot\nabla v)\big] \\ &= \mathbf{e}\cdot(\nabla y w(H-\sigma) + y\nabla w(H-\sigma) + yw\nabla H) \\ &\quad - \frac{y^2}{n}\big[a^{ij}(y_k v_{kij} - 2yv_{ij} - y_j v_i) - \frac{2}{w}a^{ij}w_i(y_k v_{kj} - yv_j) \\ &\quad - \frac{n\sigma}{wy}\nabla v\cdot\nabla(\mathbf{e}\cdot\nabla v) - \frac{n}{y}\,\mathbf{e}\cdot\nabla(\mathbf{e}\cdot\nabla v)\big] \\ &= (1-y^2)w(H-\sigma) + (\nabla w\cdot\nabla y)y(H-\sigma) + yw\mathbf{e}\cdot\nabla H \\ &\quad - \frac{y^2}{n}\left[y_k\left(\frac{n}{y}(H_k w + Hw_k + (\mathbf{e}\cdot\nabla v)_k) - \frac{n}{y^2}(Hw + \mathbf{e}\cdot\nabla v)y_k\right.\right. \\ &\quad \left. + \frac{2}{w}a^{ij}w_i v_{kj} - \frac{v_k}{w^2} + \left(n - 1 + \frac{1}{w^2}\right)v_k\right) - \frac{\nabla v\cdot\nabla y}{w^2} - 2n(Hw + \mathbf{e}\cdot\nabla v) \\ &\quad \left. -\frac{2}{w}a^{ij}w_i y_k v_{kj} + \frac{2y}{w}a^{ij}w_i v_j - \frac{n\sigma}{wy}\nabla v\cdot\nabla(\mathbf{e}\cdot\nabla v) - \frac{n}{y}\,\mathbf{e}\cdot\nabla(\mathbf{e}\cdot\nabla v)\right]\end{aligned}$$

$$= 2wH - \sigma w(1-y^2) - \sigma y \nabla w \cdot \nabla y + \left(1 + \frac{y^2}{n} + \frac{y^2}{nw^2}\right) \mathbf{e} \cdot \nabla v$$
$$- \frac{2y^3}{nw^3} \nabla v \cdot \nabla w + \frac{y\sigma}{w} \nabla v \cdot \nabla(\mathbf{e} \cdot \nabla v),$$

where we used Equations (2.8), (4.1) to (4.3) and (4.6). Moreover,

(5.4)
$$\begin{aligned}\left(\frac{\partial}{\partial t} - L\right) v &= yw(H-\sigma) - \frac{y^2}{n}\left(a^{ij}v_{ij} - \frac{2}{w}a^{ij}w_i v_j - \frac{n}{wy}(\sigma \nabla v + w\mathbf{e}) \cdot \nabla v\right) \\ &= yw(H-\sigma) - \frac{y^2}{n}\left(\frac{n}{y}Hw - \frac{2}{w^3}\nabla v \cdot \nabla w - \frac{n\sigma w}{y} + \frac{n\sigma}{wy}\right) \\ &= yw(H-\sigma) - yHw + \frac{2y^2}{nw^3}\nabla v \cdot \nabla w + y\sigma w - \frac{y\sigma}{w} \\ &= \frac{2y^2}{nw^3}\nabla v \cdot \nabla w - \frac{y\sigma}{w}.\end{aligned}$$

Next, we note that for a function η defined on $\Omega \times (0,T)$,

(5.5)
$$\mathrm{e}^{-v}\left(\frac{\partial}{\partial t} - L\right)(\mathrm{e}^{v}\eta) = \eta(v_t - Lv) + (\eta_t - L\eta) - \frac{y^2}{n}a^{ij}v_i v_j \eta - \frac{2y^2}{n}a^{ij}v_i \eta_j.$$

In particular,

(5.6)
$$\begin{aligned}\mathrm{e}^{-v}\left(\frac{\partial}{\partial t} - L\right)(\mathrm{e}^{v}w) \le{}& w\left(\frac{2y^2}{nw^3}\nabla v \cdot \nabla w - \frac{y\sigma}{w}\right) \\ &+ \left[-\sigma(\mathbf{e} \cdot \nabla v) + \frac{y^2}{n}\left(w - \frac{1}{w}\right) - H^2 w\right] \\ &- \frac{y^2}{n}a^{ij}v_i v_j w - \frac{2y^2}{n}a^{ij}v_i w_j \\ ={}& \frac{2y^2}{nw^2}\nabla v \cdot \nabla w - y\sigma - \sigma(\mathbf{e} \cdot \nabla v) + \frac{y^2}{n}\left(w - \frac{1}{w}\right) \\ &- H^2 w - \frac{y^2}{n}\left(w - \frac{1}{w}\right) - \frac{2y^2}{nw^2}\nabla v \cdot \nabla w \\ ={}& -y\sigma - \sigma(\mathbf{e} \cdot \nabla v) - H^2 w,\end{aligned}$$

and

$$
\begin{aligned}
&\mathrm{e}^{-v}\left(\frac{\partial}{\partial t}-L\right)(\mathrm{e}^{v}y)\\
&=y\left(\frac{2y^2}{nw^3}\nabla v\cdot\nabla w-\frac{y\sigma}{w}\right)+\frac{2y^2}{nw}a^{ij}w_iy_j\\
&\quad+\frac{y}{w}(\sigma\mathbf{e}\cdot\nabla v+w)-\frac{y^3}{n}+\frac{y^3}{nw^2}-\frac{y^3}{n}a^{ij}v_iv_j-\frac{2y^2}{n}a^{ij}v_iy_j\\
&=\frac{2y^3}{nw^3}\nabla v\cdot\nabla w-\frac{y^2\sigma}{w}+\frac{2y^2}{nw}\nabla y\cdot\nabla w-\frac{2y^2}{nw^3}(\nabla v\cdot\nabla w)(\nabla y\cdot\nabla v)\\
&\quad+\frac{\sigma y}{w}(\mathbf{e}\cdot\nabla v)+y-\frac{2y^3}{n}\left(1-\frac{1}{w^2}\right)-\frac{2y^2}{nw^2}\nabla v\cdot\nabla y,
\end{aligned}
$$

and also

$$
\begin{aligned}
&\mathrm{e}^{-v}\left(\frac{\partial}{\partial t}-L\right)(\mathrm{e}^{v}(\mathbf{e}\cdot\nabla v))\\
&=(\mathbf{e}\cdot\nabla v)\left(\frac{2y^2}{nw^3}\nabla v\cdot\nabla w-\frac{y\sigma}{w}\right)+2wH-\sigma w(1-y^2)-\sigma y\nabla w\cdot\nabla y\\
&\quad+(\mathbf{e}\cdot\nabla v)\left(1+\frac{y^2}{n}+\frac{y^2}{nw^2}\right)-\frac{2y^3}{nw^3}\nabla v\cdot\nabla w+\frac{y\sigma}{w}\nabla v\cdot\nabla(\mathbf{e}\cdot\nabla v)\\
&\quad-\frac{y^2}{n}(\mathbf{e}\cdot\nabla v)\left(1-\frac{1}{w^2}\right)-\frac{2y^2}{n}\frac{\nabla v\cdot\nabla(\mathbf{e}\cdot\nabla v)}{w^2}\\
&=\frac{2y^2}{nw^3}(\nabla v\cdot\nabla w)(\mathbf{e}\cdot\nabla v)-\frac{y\sigma}{w}(\mathbf{e}\cdot\nabla v)+2wH-\sigma w(1-y^2)\\
&\quad-\sigma y\nabla w\cdot\nabla y+(\mathbf{e}\cdot\nabla v)\left(1+\frac{2y^2}{nw^2}\right)-\frac{2y^3}{nw^3}\nabla v\cdot\nabla w+\left(\frac{y\sigma}{w}-\frac{2y^2}{nw^2}\right)\\
&\quad\times(w\mathbf{e}\cdot\nabla w-y(w^2-1)).
\end{aligned}
$$

Therefore, combining the above two Equations gives

$$
\begin{aligned}
&\mathrm{e}^{-v}\left(\frac{\partial}{\partial t}-L\right)(\mathrm{e}^{v}(y+(\mathbf{e}\cdot\nabla v)))\\
&=-\frac{y^2\sigma}{w}+\left(\frac{2y^2}{nw^2}-\frac{\sigma y}{w}\right)y(w^2-1)+y-\frac{2y^3}{n}\left(1-\frac{1}{w^2}\right)\\
&\quad+2wH-\sigma w(1-y^2)+\mathbf{e}\cdot\nabla v\\
&=y+2wH-\sigma w+\mathbf{e}\cdot\nabla v.
\end{aligned}
\tag{5.7}
$$

Finally, combining Equations (5.6) and (5.7) implies

$$\left(\frac{\partial}{\partial t} - L\right)(\mathrm{e}^v(w + \sigma(y + \mathbf{e}\cdot\nabla v))) \le -\mathrm{e}^v(H-\sigma)^2 w \le 0.$$

□

The uniform local ball condition (see Equation (4.8)) and Theorem 5.1, together with the maximum principle allow us to conclude:

Corollary 5.2. *Let v^ϵ be the regular solution to the AMMCF* (2.12) *with initial hypersurface Σ_0^ϵ as in Theorem 1.1. Then we have*

$$|\nabla v^\epsilon(\mathbf{z},t)| \le C, \quad \text{for all } (\mathbf{z},t) \in \overline{\Omega_\epsilon} \times [0,\infty), \tag{5.8}$$

where C is a constant independent of ϵ.

With the aid of Corollary 5.2 and the Arzelà–Ascoli theorem, letting $\epsilon \to 0$, we can extract a subsequence of the regular solutions $\{\Sigma_t^{\epsilon_i}\}$ to the AMMCF (1.3), converging uniformly to $\Sigma_t \in C^\infty(\mathbb{S}_+^n \times (0,\infty)) \cap C^{0+1,0+\frac{1}{2}}(\overline{\mathbb{S}_+^n} \times (0,\infty)) \cap C^0(\overline{\mathbb{S}_+^n} \times [0,\infty))$ which solves the MMCF (1.2) with initial hypersurface $\Sigma_0 = \lim_{\epsilon_i\to 0}\Sigma_0^{\epsilon_i}$.

6. The boundary regularity

In this section, we show the boundary regularity of the MMCF (1.2) in Theorem 1.1. The proof follows closely the idea in Section 4.3 of [9], cf. [22]. Under the uniform local ball condition, let $P_0 \in \Gamma$ and set $\epsilon = 0$ in Equation (4.7). Let us denote $S_i = S_i^0$ and $\varphi_i = \varphi_i^0, i = 1,2$. Then for some $\epsilon_2 > 0$ we have

$$\varphi_1(\mathbf{z}) \le v(\mathbf{z},t) \le \varphi_2(\mathbf{z}), \quad (\mathbf{z},t) \in (\mathbb{S}_+^n \cap B_{\epsilon_2}(\mathbf{z}_0)) \times [0,\infty). \tag{6.1}$$

Note that the tangent plane T to S_1 and S_2 at P_0 is a radial graph $T = \mathrm{e}^\eta \mathbf{z}$ in $\mathbb{S}_+^n \cap \{\mathbf{z}\cdot\nu_0 > 0\}$ with

$$\eta(\mathbf{z}) = \log\frac{P_0\cdot\mathbf{e}_1}{\lambda y + \mathbf{z}\cdot\mathbf{e}_1}, \tag{6.2}$$

where $\lambda = \frac{\sigma}{\sqrt{1-\sigma^2}}$, $\mathbf{e}_1$ is the exterior unit normal to Γ at P_0 and $\nu_0 = \sigma\mathbf{e} + \sqrt{1-\sigma^2}\mathbf{e}_1$ is the unit normal vector to S_1 and S_2 at P_0. We also have

$$\varphi_1(\mathbf{z}) \le \eta(\mathbf{z}) \le \varphi_2(\mathbf{z}), \quad \mathbf{z} \in \mathbb{S}_+^n \cap B_{\epsilon_2}(\mathbf{z}_0). \tag{6.3}$$

We will need the following more precise estimate on v.

Lemma 6.1. *$v(\mathbf{z},t) = \eta(\mathbf{z}) + O(|\mathbf{z}-\mathbf{z}_0|^2)$ in $(\mathbb{S}^n_+ \cap B_{\epsilon_2}(\mathbf{z}_0)) \times [0,\infty)$.*

Proof. This follows immediately from Equation (6.1) and the estimates $|\varphi_i - \eta|(\mathbf{z}) = O(|\mathbf{z}-\mathbf{z}_0|^2), i = 1,2$ from [9, Lemma 4.5]. □

Now let $p \in \mathbb{S}^n_+$ and δ be the geodesic distance of p to $\partial\mathbb{S}^n_+$ with $\delta < \epsilon_2$. Let $q \in \partial\mathbb{S}^n_+$ be the closest point to p. Introduce normal coordinates $x = (x_1, \ldots, x_n)$ in $T_q\mathbb{S}^n_+$ with $x(p) = (0,\ldots,0,\delta)$. We observe that Equation (2.10) may be written as

$$\frac{\partial v}{\partial t} - \frac{y^2 w}{n}\nabla_i\left(\frac{\nabla^i v}{w}\right) + y\nabla y \cdot \nabla v + \sigma y w = 0,$$

or in local coordinates (cf. Equation (4.33) of [9]):

$$\frac{\partial v}{\partial t} - \frac{y^2 w}{n\sqrt{\gamma}}\frac{\partial}{\partial x_i}\left(\frac{\sqrt{\gamma}\gamma^{ij}}{w}\frac{\partial v}{\partial x_j}\right) + y\gamma^{kl}\frac{\partial y}{\partial x_k}\frac{\partial v}{\partial x_l} + \sigma y w = 0, \tag{6.4}$$

where $\gamma = \det(\gamma_{ij})$ and $w^2 = 1 + \gamma^{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}$. One sees easily that both v and η satisfy Equation (6.4) (note that the hyperplane T has constant hyperbolic mean curvature σ as well and locally as radial graph T is static under the MMCF).

Set $\tilde{v}(x,t) = \frac{1}{\delta}v(\delta x, t)$ and $\tilde{\eta}(x) = \frac{1}{\delta}\eta(\delta x)$. Then (6.4) transforms to

$$\frac{\partial \tilde{v}}{\partial t} - \frac{\tilde{y}^2 \tilde{w}}{n\sqrt{\tilde{\gamma}}}\frac{\partial}{\partial x_i}\left(\frac{\sqrt{\tilde{\gamma}}\tilde{\gamma}^{ij}}{\tilde{w}}\frac{\partial \tilde{v}}{\partial x_j}\right) + \tilde{y}\tilde{\gamma}^{kl}\frac{\partial \tilde{y}}{\partial x_k}\frac{\partial \tilde{v}}{\partial x_l} + \sigma \tilde{y}\tilde{w} = 0, \tag{6.5}$$

where $\tilde{y}(x) = \frac{1}{\delta}y(\delta x)$, $\tilde{\gamma}_{ij}(x,t) = \gamma_{ij}(\delta x, t)$, $\tilde{\gamma} = \det(\tilde{\gamma}_{ij})$ and $\tilde{w}^2 = 1 + \tilde{\gamma}^{ij}\frac{\partial \tilde{v}}{\partial x_i}\frac{\partial \tilde{v}}{x_j}$.

Under this transformation we can move point p to the "interior" point $\tilde{p} = (0,\ldots,0,1)$. For any $T > 0$ and in $B_T = B_{\frac{1}{2}}(\tilde{p}) \times (0,T)$, one observes that $\tilde{y} = O(1)$. Also since $\sup|\nabla\tilde{v}| = \sup|\nabla v| \leq C$ and by [20, Theorem 12.10], $\tilde{v}$ is uniformly $C^{1+\alpha,\frac{1+\alpha}{2}}$. Moreover, since $\tilde{\eta}$ satisfies the same Equation (6.4), $\tilde{v} - \tilde{\eta}$ satisfies a linear uniformly parabolic equation $\overline{L}(\tilde{v}-\tilde{\eta}) = 0$ with uniformly Hölder continuous coefficients. Then by the standard parabolic Schauder-type estimates and Lemma 6.1 we obtain

$$\sup_{B_T}\left(|\nabla(\tilde{v}-\tilde{\eta})| + |\nabla^2(\tilde{v}-\tilde{\eta})|\right) \leq C_1 \sup_{B_T}|\tilde{v}-\tilde{\eta}| \leq C\delta.$$

Returning to the original variable we obtain

$$|\nabla v| + |\nabla^2 v| \leq C, \quad \text{where } C \text{ is independent of } \delta. \tag{6.6}$$

Now by Equation (2.3) and Lemma 2.1, the energy functional $\mathcal{I}$ is non-increasing as time t increases and the MMCF subconverges to a smooth complete hypersurface $\Sigma_\infty \in C^\infty(\mathbb{S}^n_+) \cap C^{1+1}(\overline{\mathbb{S}^n_+})$ with constant hyperbolic mean curvature σ and $\partial\Sigma_\infty = \Gamma \subset \partial_\infty \mathbb{H}^{n+1}$. Thus we have proved

Theorem 6.2. *Let $v \in C^\infty(\mathbb{S}^n_+ \times (0,\infty)) \cap C^{0+1,0+\frac{1}{2}}(\overline{\mathbb{S}^n_+} \times (0,\infty)) \cap C^0(\overline{\mathbb{S}^n_+} \times [0,\infty))$ be a solution to the MMCF* (2.11) *and $\phi \in C^{1+1}(\partial\overline{\mathbb{S}^n_+})$. Then $v \in C^\infty(\mathbb{S}^n_+ \times (0,\infty)) \cap C^{1+1,\frac{1}{2}+\frac{1}{2}}(\overline{\mathbb{S}^n_+} \times (0,\infty)) \cap C^0(\overline{\mathbb{S}^n_+} \times [0,\infty))$. Moreover, there exist $t_i \nearrow \infty$ such that $\Sigma_{t_i} = F(\mathbb{S}^n_+, t_i)$ converges to a unique stationary smooth complete hypersurface $\Sigma_\infty \in C^\infty(\mathbb{S}^n_+) \cap C^{1+1}(\overline{\mathbb{S}^n_+})$ (as a radial graph over $\mathbb{S}^n_+$) which has constant hyperbolic mean curvature σ and $\partial\Sigma_\infty = \Gamma$ asymptotically.*

So now all that is left to prove of Theorem 1.1 is the uniform convergence of the MMCF in the case that Σ_0^ϵ has mean curvature $H^\epsilon \geq \sigma$ for all $\epsilon > 0$ sufficiently small.

7. Uniform convergence

In this section, we will show the uniform convergence of the regular solution to the MMCF (1.2) as $t \to \infty$ in the case of $H^\epsilon \geq \sigma$ initially for all $\epsilon > 0$ sufficiently small. To do this, we first show that for any fixed $\epsilon > 0$ sufficiently small and for any $\mathbf{z}_0 \in \Omega_\epsilon$, $v^\epsilon(\mathbf{z}_0, t)$ is non-decreasing along the flow in this case, where v^ϵ is the regular solution to the AMMCF (2.12) for radial graphs. This is an immediate corollary of the following lemma.

Lemma 7.1. *Let $v \in C^{3,\frac{3}{2}}(\Omega \times (0,T))$ be a function satisfying Equation* (2.10) *for some $T > 0$ and $\Omega \subseteq \mathbb{S}^n_+$. Then*

$$\left(\frac{\partial}{\partial t} - \widetilde{L}\right)(yw(H-\sigma)) = 0, \quad \text{in } \Omega \times (0,T), \tag{7.1}$$

where $\widetilde{L}$ is the linear elliptic operator

$$\widetilde{L} \equiv \frac{y^2}{n} a^{ij} \nabla_{ij} + \left[\frac{2y^2}{nw^3}(\nabla w \cdot \nabla v)\nabla v - \frac{2y^2 \nabla w}{nw} - \frac{\sigma y}{w}\nabla v - y\mathbf{e}\right] \cdot \nabla.$$

Proof. Let $g = H - \sigma$ and $h = ywg$, we have

$$\frac{\partial v}{\partial t} = yw(H-\sigma) = ywg = h, \tag{7.2}$$

$$\frac{\partial w}{\partial t} = \frac{1}{w}\nabla v \cdot \nabla(ywg) = \frac{1}{w}\nabla v \cdot \nabla h, \tag{7.3}$$

$$\frac{\partial a^{ij}}{\partial t} = \frac{2v_i v_j \nabla v \cdot \nabla h}{w^4} - \frac{h_i v_j + h_j v_i}{w^2}, \tag{7.4}$$

and

$$\frac{\partial H}{\partial t} = \frac{y}{nw}(a_t^{ij} v_{ij} + a^{ij}(v_t)_{ij}) - \frac{y a^{ij} v_{ij} w_t}{nw^2} - \frac{(\mathbf{e}\cdot\nabla v)_t}{w} + \frac{(\mathbf{e}\cdot\nabla v)w_t}{w^2}. \tag{7.5}$$

Therefore by Equations (7.3) to (7.5) and (2.8), we have

$$\begin{aligned}
\frac{\partial h}{\partial t} &= yw_t g + ywg_t \\
&= yw_t g + yw\left[\frac{y a_t^{ij} v_{ij} + y a^{ij} h_{ij}}{nw} - \frac{y a^{ij} v_{ij} w_t}{nw^2} - \frac{(\mathbf{e}\cdot\nabla v)_t}{w} + \frac{(\mathbf{e}\cdot\nabla v)w_t}{w^2}\right] \\
&= yHw_t - \sigma y w_t + \frac{y^2 v_{ij}}{n}\left(\frac{2v_i v_j \nabla v\cdot\nabla h}{w^4} - \frac{h_i v_j + h_j v_i}{w^2}\right) + \frac{y^2}{n} a^{ij} h_{ij} \\
&\quad - \frac{y(Hw + \mathbf{e}\cdot\nabla v)}{w} w_t - y(\mathbf{e}\cdot\nabla v)_t + \frac{y}{w}(\mathbf{e}\cdot\nabla v)w_t \\
&= yHw_t - \frac{\sigma y}{w}\nabla v\cdot\nabla h + \frac{2y^2}{nw^3}(\nabla w\cdot\nabla v)(\nabla v\cdot\nabla h) - \frac{2y^2 \nabla w\cdot\nabla h}{nw} \\
&\quad + \frac{y^2}{n} a^{ij} h_{ij} - yHw_t - y(\mathbf{e}\cdot\nabla v)_t \\
&= \frac{y^2}{n} a^{ij} h_{ij} + \frac{2y^2}{nw^3}(\nabla w\cdot\nabla v)(\nabla v\cdot\nabla h) - \frac{2y^2\nabla w\cdot\nabla h}{nw} \\
&\quad - \frac{\sigma y}{w}\nabla v\cdot\nabla h - y(\mathbf{e}\cdot\nabla h).
\end{aligned}$$

This completes the proof of the lemma using the definition of the operator $\widetilde{L}$. □

Corollary 7.2. *Suppose Σ_0^ϵ has mean curvature $H^\epsilon \ge \sigma$. Then $\frac{\partial v^\epsilon}{\partial t} = yw^\epsilon (H^\epsilon - \sigma) \ge 0$ for all $(\mathbf{z}, t) \in \overline{\Omega_\epsilon} \times [0, \infty)$.*

Proof. Since for any $\epsilon > 0$ sufficiently small, $v^\epsilon(\mathbf{z}, t) \equiv \phi^\epsilon(\mathbf{z})$ for all $(\mathbf{z}, t) \in \partial\Omega_\epsilon \times (0, \infty)$, we have $v_t \equiv 0$ on $\partial\Omega_\epsilon \times (0, \infty)$. Then the condition $H^\epsilon \ge \sigma$ at $t = 0$, Lemma 7.1 and the maximum principle imply that $\frac{\partial v^\epsilon}{\partial t} = yw^\epsilon(H^\epsilon - \sigma) \ge 0$. □

Theorem 7.3. *Let Γ, Γ_ϵ and Σ_0^ϵ's be as in Theorem 1.1 and suppose Σ_0^ϵ has mean curvature $H^\epsilon \ge \sigma$ for all $\epsilon > 0$ sufficiently small. Then Σ_t converge* *uniformly for all t to a unique smooth complete star-shaped hypersurface $\Sigma_\infty \in C^\infty(\mathbb{S}^n_+) \cap C^{1+1}(\overline{\mathbb{S}^n_+})$ with constant hyperbolic mean curvature σ and boundary Γ.*

Proof. The subconvergence of the flow follows from Theorem 6.2. Corollary 7.2 then yields $\frac{\partial v}{\partial t} \geq 0$, where v is the regular solution to the MMCF (2.11) for radial graphs. This monotonicity of v implies that the regular solution Σ_t to the MMCF (1.2) with initial hypersurface Σ_0 converges uniformly for all t to Σ_∞. □

This completes the proof of Theorem 1.1.

8. Proof of Theorem 1.2 and "good" initial hypersurfaces

In this section, we will prove Theorem 1.2 and give an example of "good" initial hypersurfaces for the Dirichlet problems (2.12) and (2.11).

Proof. (of Theorem 1.2) Note that since for any $\epsilon > 0$ sufficiently small, we have $H^\epsilon \geq \sigma$, Σ_0^ϵ (as a radial graph of the function $\mathrm{e}^{v_0^\epsilon}$ over Ω_ϵ) is a subsolution to the AMMCF (2.12). Therefore Σ_0^ϵ serves as a natural lower barrier for the AMMCF. Combining this with the uniform exterior local ball condition yields the same proof as the one of Theorem 1.1 given in the previous sections, except the C^{1+1} boundary regularity of the flow. The C^{1+1} boundary regularity of the limiting hypersurface Σ_∞ follows from an elliptic version of the argument given in Section 6, see also Section 4.3 of [9]. □

To find an example of "good" initial hypersurfaces in Theorem 1.2, for any $\epsilon > 0$ sufficiently small we will restrict ourselves to looking for an initial smooth (C^2-) hypersurface $\Sigma_0^\epsilon = \mathbf{F}(\Omega_\epsilon, 0)$ that can be represented as a radial graph of the function $\mathrm{e}^{v_0^\epsilon}$ over $\Omega_\epsilon \subset \mathbb{S}^n_+$ and has hyperbolic mean curvature $H^\epsilon \geq \sigma$ and Γ_ϵ as its boundary. Moreover, Σ_0^ϵ's satisfy the uniform exterior local ball condition and $|\nabla v_0^\epsilon|(\mathbf{z}) \leq C$ for all $\mathbf{z} \in \overline{\Omega_\epsilon}$, where C is a constant independent of ϵ. To do this, we will simply apply the implicit function theorem to construct a smooth hypersurface $\Sigma_0^\epsilon \in \mathbb{H}^{n+1}$ that is of constant hyperbolic mean curvature close to 1 and has boundary Γ_ϵ to serve as such "good" initial hypersurface. As we shall see, the construction relies heavily on the estimates in [9] for hypersurfaces with constant mean curvature as vertical graphs.

From Equations (2.7) and (2.13), one observes that if a smooth radial graph of the function e^v over Ω_ϵ has constant hyperbolic mean curvature σ

with prescribed boundary Γ_ϵ, then v satisfies

$$\begin{cases} a^{ij}v_{ij} = \dfrac{n}{y}(\sigma w + \mathbf{e}\cdot\nabla v), & \text{in } \Omega_\epsilon, \\ v = \phi^\epsilon, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{8.1}$$

where $\phi^\epsilon \in C^{1+1}(\partial\Omega_\epsilon)$ is assumed.

It is clear that for $\sigma = 1$, the flat domain $D_\epsilon \subset \{x_{n+1} = \epsilon\}$ enclosed by Γ_ϵ (known as "horosphere") is the corresponding smooth radial graph satisfying (8.1). Therefore, there exists $\sigma_0 \in [0,1) \cap [\sigma, 1)$ with σ_0 being sufficiently close to 1 so that the implicit function theorem applies to (8.1). In this way, we can obtain a hypersurface $\Sigma_0^\epsilon = \{e^{v_0^\epsilon}\mathbf{z} : \mathbf{z} \in \overline{\Omega_\epsilon}\}$, where $v_0^\epsilon \in C^\infty(\Omega_\epsilon) \cap C^{1+1}(\overline{\Omega_\epsilon})$. Moreover, Σ_0^ϵ has hyperbolic mean curvature σ_0 and $\partial\Sigma_0^\epsilon = \Gamma_\epsilon$. By continuity, Σ_0^ϵ is close to the flat domain D_ϵ and for all $\epsilon \geq 0$ the uniform exterior local ball condition is satisfied by Σ_0^ϵ's.

With this specific construction of the initial hypersurface, we next give a preliminary C^0 estimate for the solution to the AMMCF (1.3).

Lemma 8.1. *For Σ_t^ϵ with initial hypersurface Σ_0^ϵ given above, there holds the height estimate*

$$u^\epsilon(\mathbf{z}, t) < \frac{d(D)}{2}\sqrt{\frac{1-\sigma}{1+\sigma}} + \epsilon, \quad (\mathbf{z}, t) \in \Omega_\epsilon \times [0, T_\epsilon), \tag{8.2}$$

where $d(D)$ is the Euclidean diameter of D (the flat domain enclosed by Γ).

Proof. This is a direct parabolic generalization of [9, Lemma 3.2]. Let B be a ball of radius R with center on the plane $\{x_{n+1} = -\sigma R\}$ such that the n-ball $B \cap \{x_{n+1} = \epsilon\}$ has radius $r = d(D)/2$ and contains D_ϵ. By continuity, we can choose σ_0 so small that $\Sigma_0^\epsilon \subseteq B$ as well. By (i) of Lemma 3.4, Σ_t^ϵ is contained in $B \cap \mathbb{H}^{n+1}$ for any $t \in [0, T_\epsilon)$, and therefore

$$u^\epsilon(\mathbf{z}, t) < (1-\sigma)R, \quad (\mathbf{z}, t) \in \Omega_\epsilon \times [0, T_\epsilon).$$

Moreover, $R^2 = (\epsilon + \sigma R)^2 + r^2$, which implies

$$\frac{r}{\sqrt{1-\sigma^2}} + \frac{\sigma}{1-\sigma^2}\epsilon \leq R \leq \frac{r}{\sqrt{1-\sigma^2}} + \frac{1+\sigma}{1-\sigma^2}\epsilon. \tag{8.3}$$

This completes the proof. □

Remark 8.2. In particular, on Σ_0^ϵ there holds the height estimate

$$u_0^\epsilon < \frac{d(D)}{2}\sqrt{\frac{1-\sigma_0}{1+\sigma_0}} + \epsilon. \tag{8.4}$$

The only thing left to show is $|\nabla v_0^\epsilon|(\mathbf{z}) \le C$ for some constant C that is independent of $\epsilon \ge 0$ and $\mathbf{z} \in \overline{\Omega_\epsilon}$. The first step is to obtain a good barrier for v_0^ϵ at any point $\mathbf{z}_0 \in \partial\Omega_\epsilon$ corresponding to $P_0 = \mathrm{e}^{\phi^\epsilon(\mathbf{z}_0)}\mathbf{z}_0 \in \Gamma_\epsilon$. For convenience, we choose a coordinate system around P_0 so that the exterior normal to Γ_ϵ at P_0 is $\mathbf{e}_1^\epsilon$. Let $\delta_1 > 0$ (respectively δ_2) be such that for each point $P \in \Gamma_\epsilon$, a ball of radius δ_1 (respectively δ_2) is internally (respectively externally) tangent to Γ_ϵ at P. Let $B_i^\epsilon = B_i^\epsilon(\sigma_0)$, $i = 1, 2$ be the (Euclidean) balls of radius R_i centered at $C_i = P_0 + (-1)^i\delta_i\mathbf{e}_1^\epsilon + (a_i - \epsilon)\mathbf{e}$, where

$$R_i = \frac{-(-1)^i\epsilon\sigma_0 + \sqrt{\epsilon^2 + \delta_i^2(1-\sigma_0^2)}}{1-\sigma_0^2} \quad \text{and} \quad a_i = (-1)^i R_i\sigma_0. \tag{8.5}$$

Recall that $S_1^\epsilon(\sigma_0) = \partial B_1^\epsilon \cap \mathbb{H}^{n+1}$ has constant (hyperbolic) mean curvature σ_0 with respect to its outward normal while $S_2^\epsilon(\sigma_0) = \partial B_2^\epsilon \cap \mathbb{H}^{n+1}$ has constant mean curvature σ_0 with respect to its inward normal. Moreover, by our construction, B_1^ϵ and B_2^ϵ are tangent at P_0, $B_1^\epsilon \cap \{x_{n+1} = \epsilon\}$ is internally tangent to Γ_ϵ at P_0, and $B_2^\epsilon \cap \{x_{n+1} = \epsilon\}$ is externally tangent to Γ_ϵ at P_0.

Lemma 8.3 [9]. *Locally $S_1^\epsilon(\sigma_0)$ is interior to $\Sigma_0^\epsilon(\sigma_0)$ and S_2^ϵ is exterior to Σ_0^ϵ.*

Proof. This follows from the maximum principle for Equation (2.7). □

Similar to Equation (4.7), we see that $S_1^\epsilon(\sigma_0)$ and $S_2^\epsilon(\sigma_0)$ serve as good local barriers of Σ_0^ϵ around P_0 and we obtain that

$$|\nabla v_0^\epsilon|(P_0) \le C, \tag{8.6}$$

where C is independent of ϵ and $P_0 \in \Gamma_\epsilon$.

The next step is to obtain a uniform interior gradient bound for v_0^ϵ and one observes that we only need to bound

$$\mathbf{X}_0^\epsilon \cdot \nu_{\mathrm{E}}^\epsilon = \frac{\mathrm{e}^{v_0^\epsilon}}{\sqrt{1 + |\nabla v_0^\epsilon|^2}}$$

from below uniformly in ϵ. This can be done as follows. Firstly note that, since D_ϵ is a vertical graph over D and by continuity (induced from the implicit function theorem used in the construction of Σ_0^ϵ), Σ_0^ϵ is a vertical graph of the function u_0^ϵ over D as well. And similar to Lemma 8.1, we have another height estimate for vertical graphs.

Lemma 8.4 ([9], Lemma 3.5). *On Σ_0^ϵ (that has constant mean curvature σ_0) there holds*

$$u_0^\epsilon(x') \geq d(x')\sqrt{\frac{1-\sigma_0}{1+\sigma_0}} + \frac{\sigma_0\epsilon}{1+\sigma_0}, \quad x' \in D, \tag{8.7}$$

where $d(x')$ is the distance from x' to ∂D.

Proof. For $x' \in D$, let $r = d(x')$ and $R > 0$ satisfy $R^2 = (\epsilon + \sigma_0 R)^2 + d^2(x')$. Note that $B_R(x', -\sigma_0 R) \cap \{x_{n+1} = \epsilon\} \subset D_\epsilon$ and $\partial B_R(x', -\sigma_0 R) \cap \mathbb{H}^{n+1}$ has constant hyperbolic mean curvature σ_0. Then by (ii) of Lemma 3.3,

$$u_0^\epsilon(x') > (1-\sigma_0)R.$$

Now the first inequality in (8.3) gives (8.7). □

Moreover, there exists $\epsilon_1 > 0$ such that, for any $\sigma_0 \in [1-\epsilon_1, 1)$, there exists $\delta_1 = \delta_1(\epsilon_1)$ so that in the δ_1-neighborhood of Γ in D one has $|\nabla v_0^\epsilon| \leq \frac{C}{2}$, where C is the uniform gradient bound of v_0^ϵ on Γ_ϵ as in Equation (8.6). Away from the δ_1-neighborhood, by Lemma 8.4

$$\begin{aligned} \mathbf{X}_0^\epsilon \cdot \nu_{\mathrm{E}}^\epsilon &= \mathbf{X}_0^\epsilon \cdot \mathbf{e} - \mathbf{X}_0^\epsilon \cdot (\mathbf{e} - \nu_{\mathrm{E}}^\epsilon) \\ &\geq \delta_1\sqrt{\frac{1-\sigma_0}{1+\sigma_0}} - \mathrm{e}^{v_0^\epsilon}\sqrt{2 - \frac{2}{\sqrt{1+|\widetilde{\nabla} u_0^\epsilon|^2}}}, \end{aligned} \tag{8.8}$$

where $\widetilde{\nabla}$ is the Levi–Civita connection on $\mathbb{R}^{n+1}$ and we used that

$$\nu_{\mathrm{E}}^\epsilon = \left(\frac{-\widetilde{\nabla} u_0^\epsilon}{\sqrt{1+|\widetilde{\nabla} u_0^\epsilon|^2}}, \frac{1}{\sqrt{1+|\widetilde{\nabla} u_0^\epsilon|^2}}\right),$$

since Σ_0^ϵ is a vertical graph.

Now using the fact that $H_{\mathrm{E}}^{\epsilon}$ is subharmonic on the constant mean curvature hypersurface Σ_0^{ϵ} (see Theorem 2.2 of [9]), we have

Lemma 8.5 ([9], Corollary 2.3). *For any* $\lambda \in (0,1)$,

$$\sqrt{1+|\widetilde{\nabla} u_0^{\epsilon}|^2} \leq \frac{1}{(1-\lambda)\sigma_0} \quad \text{in } \Omega_{\lambda}, \tag{8.9}$$

where $\Omega_{\lambda} = \left\{ x \in D : u_0^{\epsilon} \leq \frac{\lambda \sigma_0}{\sup_{\Gamma_{\epsilon}} H_{\mathrm{E}}^{\epsilon}} \right\}$.

To make use of Lemma 8.5, we also need the following estimate on the Euclidean mean curvature $H_{\mathrm{E}}^{\epsilon}$ of Σ_0^{ϵ} on $\partial\Sigma_0^{\epsilon} = \Gamma_{\epsilon}$. For $x \in \partial D = \Gamma$, denote by $r_1(x)$ and $r_2(x)$ the radius of the largest exterior and interior spheres to ∂D at x, respectively, and let $r_1 = \min_{x\in\partial D} r_1(x), r_2 = \min_{x \in \partial D} r_2(x)$. Then we have

Lemma 8.6 ([9], Lemma 3.3). *For* $\epsilon > 0$ *sufficiently small,*

$$\begin{aligned} -\frac{\sqrt{1-\sigma_0^2}}{r_2} - \frac{\epsilon(1-\sigma_0)}{r_2^2} &< \frac{\sigma_0 - \mathbf{e}\cdot\nu_E^{\epsilon}}{u} \\ &= H_{\mathrm{E}}^{\epsilon} < \frac{\sqrt{1-\sigma_0^2}}{r_1} + \frac{\epsilon(1+\sigma_0)}{r_1^2} \quad \text{on } \Gamma_{\epsilon}. \end{aligned}$$

In particular, $\mathbf{e}\cdot\nu_E^{\epsilon} \to \sigma_0$ *on* Γ_{ϵ} *as* $\epsilon \to 0$, *provided that* ∂D *is* C^{1+1}.

Note that in (8.8), if $|\widetilde{\nabla} u_0^{\epsilon}|$ is sufficiently small then $\mathbf{X}_0^{\epsilon}\cdot\nu_{\mathrm{E}}^{\epsilon}(x') \geq C(\delta_1)$ for any $x' \in D$ that is away from the δ_1-neighborhood of Γ. In the other case, if $|\widetilde{\nabla} u_0^{\epsilon}|$ is uniformly bounded from below, then by combing the estimates in Remark 8.2 and Lemmas 8.4 to 8.6, we can choose σ_0 sufficiently close to 1 (for fixed ϵ_1) such that we still have

$$\mathbf{X}_0^{\epsilon}\cdot\nu_{\mathrm{E}}^{\epsilon}(x') \geq C(\delta_1) \quad \text{(uniformly in } \epsilon),$$

for any $x' \in D\backslash\delta_1$-neighborhood.

Now we can conclude

Theorem 8.7. *There exist constants* $\epsilon_0 > 0$ *and* $\sigma_0 \in (0,1)\cap[\sigma,1)$ *that is sufficiently close to* 1 *such that for all* $0 \leq \epsilon \leq \epsilon_0$, *there exists a smooth hypersurface* Σ_0^{ϵ} *with* $\partial\Sigma_0^{\epsilon} = \Gamma_{\epsilon} \subset \{x_{n+1} = \epsilon\}$ *and whose hyperbolic mean* *curvature is* σ_0. *Additionally,* Σ_0^ϵ *can be represented as a radial graph of a function* $e^{v_0^\epsilon}$ *over* $\Omega_\epsilon \subset \mathbb{S}^n_+$ *and*

$$|\nabla v_0^\epsilon|(\mathbf{z}) \leq C, \quad \mathbf{z} \in \overline{\Omega_\epsilon}, \tag{8.10}$$

where C *is a constant independent of* ϵ. *Moreover, the* Σ_0^ϵ*'s satisfy the uniform exterior local ball condition.*

9. Interior gradient bounds and continuous boundary data

9.1. Interior gradient bounds

We will next provide a version of a priori interior gradient estimate for the regular solution to the MMCF (2.11), which is essential for the existence result of the MMCF with less regular (e.g., continuous) boundary data. The idea follows closely the work of Evans and Spruck [7].

Lemma 9.1. *Let* v *be a* $C^{3,\frac{3}{2}}$ *function satisfying Equation* (2.11) *in* $B_\rho(P) \times (0, 2T)$ *for some* $T > 0$, *where* $B_\rho(P) \subset \{y \geq \varepsilon\}$. *Then*

$$\sqrt{1 + |\nabla v|^2}(P, T) = w(P, T) \leq C_1 e^{\frac{C_2}{\rho^2}},$$

where C_1, C_2 *are non-negative constants depending only on* $n, \sigma, \varepsilon, T$ *and* $\|v\|_{L^\infty}$.

Proof. Define

$$\mathcal{L} = \frac{\partial}{\partial t} - L,$$

where L is the linear elliptic operator from Lemma 4.1. Without loss of generality we may assume (by adding a constant to v) $1 \leq v \leq C_0$. We will derive a maximum principle for the function $h = \eta(\mathbf{z}, t, v(\mathbf{z}, t))w$ by computing $\mathcal{L}h$ in $B_\rho(P) \times (0, 2T)$, where η is non-negative, vanishes on the set $\{t(\rho^2 - (d_P(\mathbf{z})^2) = 0\}$, and is smooth where it is positive. Here $d_P(\mathbf{z})$ is the distance function (on the sphere) from P, the center of the geodesic ball $B_\rho(P)$. Then h is non-negative and vanishes on the parabolic boundary of $B_\rho(P) \times (0, 2T)$.

Choose

$$\eta \equiv g(\varphi(\mathbf{z}, t, v(\mathbf{z}, t))); \quad g(\varphi) = \mathrm{e}^{K\varphi} - 1,$$

with the constant $K > 0$ to be determined and

$$\varphi(\mathbf{z}, t, v(\mathbf{z}, t)) = \left[\frac{-v(\mathbf{z}, t)}{2v(P, T)} + \frac{t}{T}\left(1 - \left(\frac{d_P(\mathbf{z})}{\rho}\right)^2\right)\right]^+.$$

By Lemma 4.1, we have

$$\begin{aligned}\mathcal{L}h &= \eta\mathcal{L}w + w\mathcal{L}\eta - \frac{2y^2}{n}a^{ij}\eta_i w_j \\ &= \eta\mathcal{L}w + w\left(\eta_t - \frac{y^2}{n}M\eta\right) \le w\left(2\eta + \eta_t - \frac{y^2}{n}M\eta\right),\end{aligned} \tag{9.1}$$

where

$$M = a^{ij}\nabla_{ij} - \frac{n}{y}\left(\sigma\frac{\nabla v}{w} + \mathbf{e}\right)\cdot\nabla.$$

We will choose K so that $2\eta + \eta_t - \frac{y^2}{n}M\eta \le 0$ on the set where $h > 0$ and w is large.

A straightforward computation gives that on the set where $h > 0$ (using Equation (2.10))

$$\begin{aligned}M\eta &= g'(\varphi)\left(a^{ij}\nabla_{ij}\varphi - \frac{n}{y}\left(\sigma\frac{\nabla v}{w} + \mathbf{e}\right)\cdot\nabla\varphi\right) + g''(\varphi)a^{ij}\nabla_i\varphi\nabla_j\varphi \\ &= K\mathrm{e}^{K\varphi}\Bigg[\frac{-nv_t}{2y^2 v(P,T)} - \frac{n\sigma}{2ywv(P,T)} \\ &\quad - \frac{2t}{\rho^2 T}\left(a^{ij}\nabla_i d_P\nabla_j d_P + d_P a^{ij}\nabla_{ij}d_P\right) + \frac{2nt}{\rho^2 yT}\left(\sigma\frac{\nabla v}{w} + \mathbf{e}\right)\cdot d_P\nabla d_P\Bigg] \\ &\quad + K^2\mathrm{e}^{K\varphi}a^{ij}\left(\frac{v_i}{2v(P,T)} + \frac{2t}{\rho^2 T}d_P\nabla_i d_P\right)\left(\frac{v_j}{2v(P,T)} + \frac{2t}{\rho^2 T}d_P\nabla_j d_P\right).\end{aligned}$$

Using the definition of a^{ij} we find

$$\begin{aligned}&a^{ij}\left(\frac{v_i}{2v(P,T)} + \frac{2t}{\rho^2 T}d_P\nabla_i d_P\right)\left(\frac{v_j}{2v(P,T)} + \frac{2t}{\rho^2 T}d_P\nabla_j d_P\right) \\ &\quad = \frac{|\nabla v|^2}{4(v(P,T))^2 w^2} + \frac{2td_P}{Tv(P,T)\rho^2 w^2}\langle\nabla v, \nabla d_P\rangle \\ &\qquad + \frac{4t^2 d_P^2}{T^2\rho^4}\left(1 - \left\langle\frac{\nabla v}{w}, \nabla d_P\right\rangle^2\right),\end{aligned}$$

where $\langle , \rangle$ denotes the inner product with respect to the induced Euclidean metric on Σ_t. Therefore, we have

$$
\begin{aligned}
&2\eta + \eta_t - \frac{y^2}{n} M\eta \\
&\quad = 2\eta + K\mathrm{e}^{K\varphi}\left(\frac{-v_t}{2v(P,T)} + \frac{1-\left(\frac{d_P}{\rho}\right)^2}{T}\right) - \frac{y^2}{n} M\eta \\
&\quad \le 2\eta + \frac{K\mathrm{e}^{K\varphi}}{T} - \frac{y^2}{n} M\eta - \frac{K\mathrm{e}^{K\varphi} v_t}{2v(P,T)} \\
&\quad \le -\frac{y^2}{n}\mathrm{e}^{K\varphi}\left[K^2\left(\frac{|\nabla v|^2}{4w^2(v(P,T))^2} - \frac{1}{w^2}\left(\frac{32}{\rho^2} + \frac{|\nabla v|^2}{8(v(P,T))^2}\right)\right) - \frac{CK}{\rho^2} - C\right] \\
&\quad \le -\frac{y^2}{n}\mathrm{e}^{K\varphi}\left[\frac{K^2}{32} - \frac{CK}{\rho^2} - C\right],
\end{aligned}
$$

whenever $w > \max\{\sqrt{2}, \frac{32C_0}{\rho}\} = \frac{32C_0}{\rho}$ so that $\frac{|\nabla v|^2}{w^2} > \frac{1}{2}$ and $\frac{32}{w^2\rho^2} < \frac{1}{32C_0^2}$.

Thus, the choice of $K = 32CC_0\left(1 + \frac{C_0}{\rho^2}\right)$ gives

$$
\mathcal{L}h \le w\left[2\eta + \eta_t - \frac{y^2}{n} M\eta\right] < 0 \tag{9.2}
$$

on the set where $h > 0$ and $w > \frac{32C_0}{\rho}$. Then by the maximum principle, (9.2) gives

$$
h(P,T) = \left(\mathrm{e}^{\frac{K}{2}} - 1\right) w(P,T) \le \max h \le \left(\mathrm{e}^{2K} - 1\right)\frac{32C_0}{\rho} \tag{9.3}
$$

and hence

$$
w(P,T) \le C_1 \mathrm{e}^{\frac{CC_0}{\rho^2}}
$$

for a slightly larger constant C. This completes the proof. □

9.2. Continuous boundary data

By standard modulus of continuity estimates (see e.g., [20, theorem 10.18]) and with the aid of the a priori interior gradient estimate (see Lemma 9.1) proved in the previous section, one can further relax the regularity of the boundary data to be only continuous via an approximation argument. We have

Theorem 9.2. *Let Γ be the boundary of a continuous star-shaped domain in $\{x_{n+1} = 0\}$ and $\Sigma_0 = \lim_{\epsilon \to 0} \Sigma_0^\epsilon$ be as in Theorem 1.1 or Theorem 1.2. Then there exists a unique solution $\mathbf{F}(\mathbf{z}, t) \in C^\infty(\mathbb{S}^n_+ \times (0, \infty) \cap C^0(\overline{\mathbb{S}^n_+} \times [0, \infty))$ to the MMCF* (1.2)*. Moreover, there exist $t_i \nearrow \infty$ such that $\Sigma_{t_i} = F(\mathbb{S}^n_+, t_i)$ converges to a unique stationary smooth complete hypersurface $\Sigma_\infty \in C^\infty(\mathbb{S}^n_+) \cap C^0(\overline{\mathbb{S}^n_+})$ (as a radial graph over $\mathbb{S}^n_+$) which has constant hyperbolic mean curvature σ and $\partial\Sigma_\infty = \Gamma$ asymptotically.*

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