On the Neuwirth conjecture for knots

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Neuwirth asked if any non-trivial knot in the three-sphere can be embedded in a closed surface so that the complement of the surface is a connected essential surface for the knot complement. In this paper, we examine some variations on this question and prove it for all knots up to 11 crossings except for two examples. We also establish the conjecture for all Montesinos knots and for all generalized arborescently alternating knots. For knot exteriors containing closed incompressible surfaces satisfying a simple homological condition, we establish that the knots satisfy the Neuwirth conjecture. If there is a proper degree one map from knot K to knot K' and K' satisfies the Neuwirth conjecture then we prove the same is true for knot K. Algorithms are given to decide if a knot satisfies the various versions of the Neuwirth conjecture and also the related conjectures about whether all non-trivial knots have essential surfaces at integer boundary slopes.

1. Conjectures

In 1964, Neuwirth conjectured in [23, 24] that the fundamental group of the complement of a non-trivial knot in the three-sphere is a non-trivial free product with amalgamation, and the amalgamating subgroup is free. In 1984, this conjecture was solved by Culler–Shalen, by realizing such an algebraic splitting via a suitable properly embedded surface in the exterior of the knot.

Theorem 1.1 (Weak Neuwirth conjecture, [6]). *For any non-trivial knot K, there exists a properly embedded separating, orientable, incompressible and boundary incompressible surface in the exterior $E(K)$.*

However, a geometrical conjecture which is an original source of the Weak Neuwirth conjecture has not been solved. The following Neuwirth conjecture asserts that any non-trivial knot can be embedded in a closed surface, similarly to the way a torus knot can be embedded in an unknotted

torus. There are not so many geometrical properties satisfied by all non-trivial knots.

Conjecture 1.2 (Neuwirth conjecture, [23]). *For any non-trivial knot K, there exists a closed surface F containing K such that $F \cap E(K)$ is connected, incompressible and boundary incompressible.*

The following knot classes are known to satisfy the Neuwirth conjecture.

- Torus knots and cable knots

- Two-bridge knots

- Alternating knots ([2, Theorem 9.8], [21, Proposition 2.3])

- Generalized alternating knots ([28, Theorem 2])

- Non-positive +-adequate knots ([30, Corollary 2.2])

- Crosscap number two knots ([15, Theorem 6])

- Tunnel number two knots which can be non-separatingly embedded in a genus 2 Heegaard surface ([25, Lemma 2.3], [27, Lemma 1])

- Knots with accidental surfaces with non-separating accidental peripherals ([16, Theorem 2])

We will show in Corollary 4.4, that if a knot K satisfies the Neuwirth conjecture then so does any satellite knot obtained from K and any composite knot obtained by summing a knot with K. This implies that to prove the Neuwirth conjecture, it suffices to consider only simple knots and satellite knots obtained from the trivial knot.

Almost all known examples of knots satisfying the Neuwirth conjecture, are obtained by taking boundaries of regular neighbourhoods of algebraically incompressible and boundary incompressible non-orientable spanning surfaces, except for torus knots and the last two classes in the above list. Therefore, the following strong Neuwirth conjecture is plausible.

Conjecture 1.3 (Strong Neuwirth conjecture, [15, Question 5]). *For any prime non-torus knot K, there exists a non-orientable spanning surface F for K such that $F \cap E(K)$ is algebraically incompressible and boundary incompressible.*

It seems to be unknown whether Conjecture 1.3 holds even if the condition weakened.

Conjecture 1.4 (Weakly strong Neuwirth conjecture). *For any prime non-torus knot K, there exists a non-orientable spanning surface F for K such that $F \cap E(K)$ is geometrically incompressible and boundary incompressible.*

Remark 1.5. It can be observed that a composite knot bounds an algebraically (geometrically) incompressible and boundary incompressible non-orientable spanning surface if and only if at least one of the factor knots also bounds an algebraically (geometrically) incompressible and boundary incompressible non-orientable spanning surface.

The existence of an algebraically incompressible and boundary incompressible non-orientable spanning surface in Conjecture 1.3 implies the following Conjecture 1.7. It seems to be unknown whether Conjecture 1.7 holds even if the boundary slope is non-integer.

Conjecture 1.6 (Even boundary slope conjecture). *For any prime non-torus knot K, there is a properly embedded orientable incompressible and boundary incompressible surface, which is not a Seifert surface, in the exterior $E(K)$ with boundary slope an even rational number (i.e., an irreducible fraction with an even numerator).*

Conjecture 1.7 (Strong even boundary slope conjecture). *For any prime non-torus knot K, there is a properly embedded orientable incompressible and boundary incompressible surface, which is not a Seifert surface, in the exterior $E(K)$ with boundary slope an even integer.*

We will show in Corollary 4.8, that if a knot K satisfies the (strong) even boundary slope conjecture then so does any satellite knot obtained from K and any composite knot obtained by summing a knot with K. This implies that to prove the (strong) even boundary slope conjecture, it suffices to consider only simple knots and satellite knots obtained from the trivial knot.

Remark 1.8. Miyazaki [22] showed that for any integer $m \geq 0$, there is a hyperbolic knot which has $m + 1$ accidental surfaces with accidental slopes $0, 1, \ldots, m$. Tsutsumi [34] showed that for any finite set of even integers $\{a_1, \ldots, a_n\}$ and for any closed connected three-manifold M, there exists an excellent knot in M which bounds excellent non-orientable spanning surfaces $F_1, \ldots, F_n$ such that the boundary slope of F_i is a_i. Each of these two constructions shows that there exists a hyperbolic knot with finitely many Neuwirth surfaces at finitely many integer boundary slopes.

In this paper, we approach these conjectures by three methods; Murasugi sum, pre-essential surface and degree one map, which are dealt with in Sections 2, 3 and 4 respectively.

2. Murasugi sum

2.1. Geometrically incompressible and algebraically incompressible surfaces

We review the definition of essential surfaces in both the geometric and algebraic senses.

Let M be an orientable compact three-manifold, F a compact surface properly embedded in M, possibly with boundary, except for a two-sphere or disk, and let i denote the inclusion map $F \to M$. We say that F is *algebraically incompressible* if the induced map $i_* : \pi_1(F) \to \pi_1(M)$ is injective, and that F is *algebraically boundary incompressible* if the induced map $i_* : \pi_1(F, \partial F) \to \pi_1(M, \partial M)$ is injective for every choice of two base points in ∂F.

A disk D embedded in M is a *compressing disk* for F if $D \cap F = \partial D$ and ∂D is an essential loop in F. A disk D embedded in M is a *boundary compressing disk* for F if $D \cap F \subset \partial D$ is an essential arc in F and $D \cap \partial M = \partial D - \mathrm{int}(D \cap F)$. We say that F is *geometrically incompressible* (resp. *geometrically boundary incompressible*) if there exists no compressing disk (resp. boundary compressing disk) for F.

2.2. Murasugi sum and knot minors

Let F be a spanning surface for a knot or link K. Suppose that there exists a two-sphere S decomposing S^3 into two three-balls B_1, B_2 such that S intersects K transversely and $F \cap S$ consists of a disk. Put $F_i = F \cap B_i$ for $i = 1, 2$. Then we say that F has a *Murasugi decomposition* into F_1 and F_2 and we denote by $F = F_1 * F_2$. Conversely, we say that F is obtained from F_1 and F_2 by a *Murasugi sum* along a disk $F \cap S$. We say that a Murasugi sum (or Murasugi decomposition) is *plumbing* (or *deplumbing*) if S intersects K in four points.

The Murasugi sum is a natural geometric operation [9]. In fact, Gabai proved that geometric incompressibility for Seifert surfaces is preserved under Murasugi sums, and the first author showed that algebraic incompressibility for spanning surfaces is also preserved under Murasugi sums.
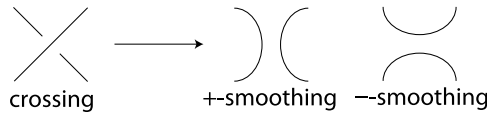
Figure 1: Two smoothings of a crossing.

Theorem 2.1 ([9, Theorem 1], [30, Lemma 3.4]). *If F_1 and F_2 are algebraically incompressible and boundary incompressible, then $F = F_1 * F_2$ is also algebraically incompressible and boundary incompressible.*

We say that a knot or link K has *minors* K_i if there exists algebraically incompressible and boundary incompressible spanning surfaces F_i for K_i such that $F = F_1 * F_2$. In the proof of Theorem 2.17, this concept plays a central role and it turns out that any Montesinos knot except for torus knots has a pretzel knot or link minor.

2.3. State surfaces

In the following, we review the definitions and results on state surfaces, which are introduced in [30].

Let K be a knot or link in the three-sphere S^3 and D a connected diagram of K on the two-sphere S^2 which separates S^3 into two three-balls, say B_+, B_-. Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ be the set of crossings of D. A map $\sigma : \mathcal{C} \to \{+, -\}$ is called a *state* for D. For each crossing $c_i \in \mathcal{C}$, we take a +-smoothing or −-smoothing according whether to $\sigma(c_i) = +$ or $-$. See figure 1. Then, we have a collection of loops $l_1, \ldots, l_m$ on S^2 and call these *state loops*. Let $\mathcal{L}_\sigma = \{l_1, \ldots, l_m\}$ be the set of state loops.

Each state loop l_i bounds a unique disk d_i in B_- close to S^2, and we may assume that these disks are mutually disjoint. For each crossing c_j and state loops l_i, l_k whose subarcs replaced c_j by a $\sigma(c_j)$-smoothing, we attach a half-twisted band b_j to d_i, d_k so that c_j is recovered. See figure 2 for $\sigma(c_j) = +$. In this way, we obtain a spanning surface which consists of disks $d_1, \ldots, d_m$ and half-twisted bands $b_1, \ldots, b_n$ and call this a *σ-state surface F_σ*.

Remark 2.2. Although we chose a disk d_i in B_- close to S^2, we could instead have chosen a disk in B_+ for each state loop which is not innermost in the two-sphere S^2. Therefore, in general, there are many state surfaces for a given state.
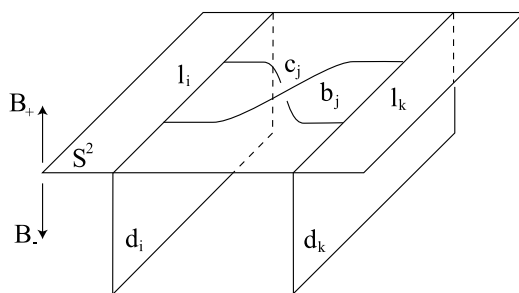
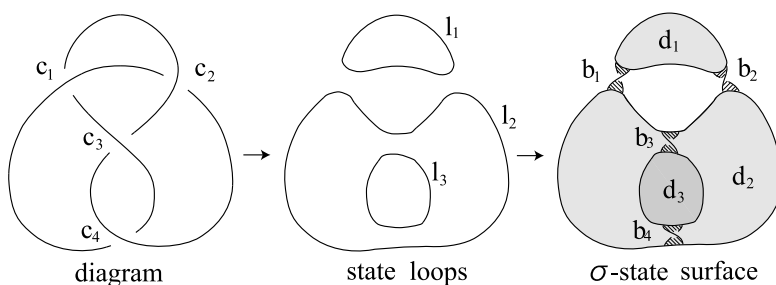Figure 2: Recovering a crossing by a half-twisted band.

Figure 3: An example of making a σ-state surface.

We construct a graph G_σ with signs on edges from F_σ by regarding a disk d_i as a vertex v_i and a band b_j as an edge e_j which has the sign $\sigma(c_j)$. We call the graph G_σ a σ-*state graph*. In general, a graph is called a *block* if it is connected and has no cut vertex. It is known that any graph has a unique decomposition into maximal blocks. Following [5, 20], we say that a diagram D is σ-*adequate* if G_σ has no loops, and that D is σ-*homogeneous* if in each block of G_σ, all edges have the same sign.

Example 2.3. Let D be a diagram of the figure eight knot which has four crossings c_1, c_2, c_3, c_4 as in figure 3. To make a σ-state surface, let $\sigma(c_1) = \sigma(c_2) = -$ and $\sigma(c_3) = \sigma(c_4) = +$, for example. Since the σ-state graph G_σ has no loops and all edges in each block have the same sign as in figure 4, D is σ-adequate and σ-homogeneous. Moreover, the block decomposition of G_σ corresponds to a Murasugi decomposition of F_σ. See figure 5.

Example 2.4. A diagram D with an orientation is said to be *positive* if all crossings have a positive sign. For any positive diagram D, there exists a state σ such that D is σ-adequate and σ-homogeneous. Indeed, we can take
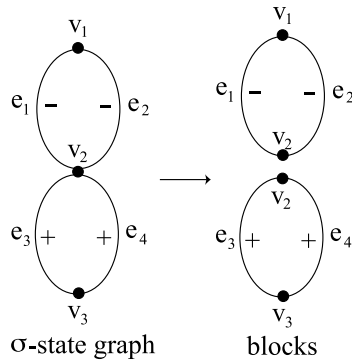
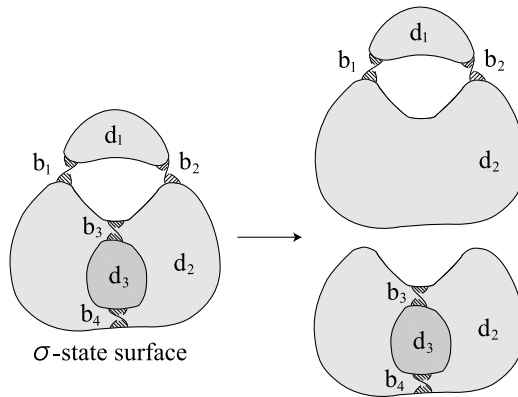Figure 4: The corresponding σ-state graph and its block decomposition.

Figure 5: The corresponding Murasugi decomposition.

σ so that $\sigma(c_j) = +$ for all c_j, namely, the *positive state* σ_+. Also we can take σ so that it yields a canonical Seifert surface F_σ, namely, the *Seifert state* $\vec{\sigma}$. Note that these states σ_+ and $\vec{\sigma}$ coincide only on a positive diagram.

Example 2.5. We say that a diagram D is +-*adequate* (or −-*adequate*) [20] if D is σ-adequate for the positive state σ_+ (or the negative state σ_-). Note that D is automatically $\sigma_\pm$-homogeneous since $\sigma_\pm(c_j) = \pm$ for all j. Furthermore, we say that a diagram D is *adequate* [33] if D is both +-adequate and −-adequate.

The Hasse diagram of various classes of knots and links is illustrated in figure 6. The set of algebraically alternating knots and links [29] includes both sets of alternating and algebraic knots and links.
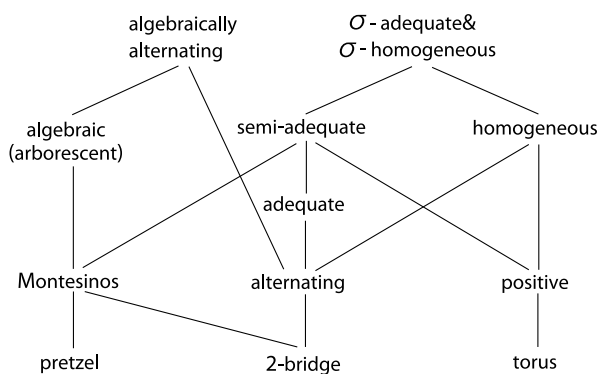
Figure 6: The Hasse diagram for the set of knot diagrams partially ordered by inclusion.

Theorem 2.6 [30]. *If a diagram is both σ-adequate and σ-homogeneous for some state σ, then the σ-state surface is algebraically incompressible and boundary incompressible.*

A knot is said to be *adequate* if it has an adequate diagram [33], that is, both σ_+-adequate and σ_--adequate. Note that at least one of σ_+ and σ_- differs from $\vec{\sigma}$.

Corollary 2.7 [30]. *If a diagram is σ-adequate and σ-homogeneous for a state σ other than the Seifert state $\vec{\sigma}$, then the knot satisfies the strong Neuwirth conjecture. In particular, adequate knots satisfy the strong Neuwirth conjecture.*

Theorem 2.6 is obtained from Theorems 2.1 and 2.8. Indeed, a knot satisfying the condition of Theorem 2.6 has an alternating knot or link minor.

Theorem 2.8 ([2, Theorem 9.8], [21, Proposition 2.3], [28, Theorem 2]). *If a diagram is reduced and alternating, then both of the checkerboard surfaces are algebraically incompressible and boundary incompressible.*

2.4. Rational tangles and Montesinos knots

We recall that a *rational tangle* is a two-tangle that is homeomorphic to the trivial two-tangle as a map of pairs consisting of the three-ball and two arcs. Usually, a rational tangle is inductively drawn as in figure 7 which consists of a_i-twists. We say that such a rational tangle diagram is *standard* if all a_i
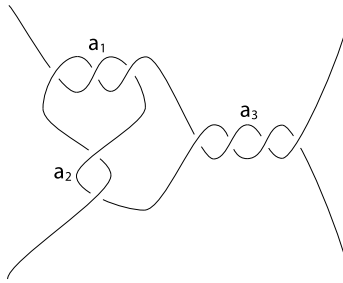
Figure 7: A rational tangle $(3, 2, 4)$ with the fraction $[4, 2, 3] = 31/7$.
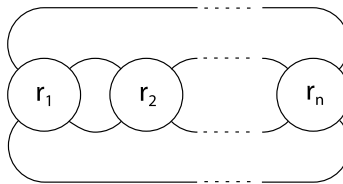
Figure 8: A Montesinos knot or link $M(r_1, \ldots, r_n)$.
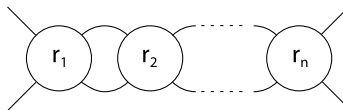
Figure 9: A Montesinos tangle $\mathrm{MT}(r_1, \ldots, r_n)$.

are positive or all a_i are negative, namely it is alternating. It is known that any rational tangle has a standard form.

The *fraction* (or *slope*) of a rational tangle $(a_1, a_2, \ldots, a_n)$, where $a_1, \ldots a_{n-1}$ are non-zero integers and a_n is an integer, is defined as the number given by the continued fraction $[a_n, a_{n-1}, \ldots, a_1]$. Conway [4] proved that the fraction is well defined and completely determines the rational tangle up to tangle equivalence.

A knot or link K is called *Montesinos* if K has a form which is obtained by summing n rational tangles T_i with slope r_i as in figure 8, and we write the result as $K = M(r_1, \ldots, r_n)$. A Montesinos knot or link $K = M(r_1, \ldots, r_n)$ is *pretzel* if all numerators of r_i are ± 1, namely $r_i = \pm 1/p_i$, and then we denote $K = P(p_1, \ldots, p_n)$.

Similarly, a tangle T is called *Montesinos* if T has a form which is obtained by summing n rational tangles T_i with the slope r_i as in figure 9, and we denote the result as $T = \mathrm{MT}(r_1, \ldots, r_n)$.

Figure 10: A deformation from $T(-r_1, r_2)$ to $T(1 - r_1, r_2 - 1)$.

Lemma 2.9. *Let* $\mathrm{MT}(-r_1, r_2)$ *be a Montesinos tangle with two rational tangles with slopes* $-r_1, r_2$, *where* $0 < r_i < 1$ *for* $i = 1, 2$. *Then* $\mathrm{MT}(-r_1, r_2)$ *can be deformed into a Montesinos tangle* $\mathrm{MT}(1 - r_1, r_2 - 1)$.
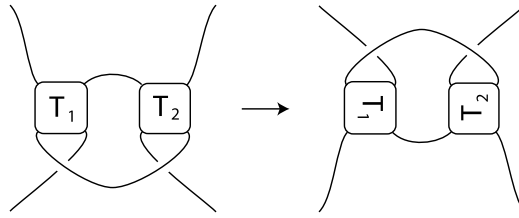
Proof. See figure 10.

We remark that the slope of T_1 is $-r_1/(1 - r_1)$ and that of T_2 is $r_2/(1 - r_2)$. The slope $\phi(T)$ of a rational tangle T can be calculated by the formula $\phi(T_1 + T_2) = \phi(T_1) + \phi(T_2)$ and $\phi(T_1)\phi(T_1^*) = -1$, where $+$ denotes the tangle sum and T^* denotes the rotation of T. See [29, Theorem 2.4] for example. $\qquad\square$

2.5. Pretzel surfaces and checkerboard surfaces

A pretzel knot or link $P(p_1, \ldots, p_n)$ bounds a checkerboard surface which consists of two disks and p_i half-twisted bands, and we call it a *pretzel surface*. The following theorem determines which pretzel surfaces for some pretzel knots or links are algebraically incompressible and boundary incompressible.

Theorem 2.10. *Let* K *be a pretzel knot or link* $P(-p_1, p_2, \ldots, p_n)$ *with* $p_i \geq 2$ *and* $n \geq 3$. *Then the pretzel surface for* K *is algebraically incompressible and boundary incompressible if and only if* $(-p_1, p_2, \ldots, p_n) \neq (-2, 3, 3)$, $(-2, 3, 4)$, $(-2, 3, 5)$ *nor* $(-2, 2, p_3)$, *where* p_3 *is odd.*

Proof. Suppose that pretzel surface F is not algebraically incompressible and boundary incompressible. By taking a regular neighbourhood of F, we obtain a genus $n - 1$ Heegaard splitting $V_1 \cup_S V_2$, where $V_1 = N(F)$ and $S = \partial N(F)$. We assume that V_1 intersects the two-sphere S^2, which includes a $(-p_1, p_2, \ldots, p_n)$-pretzel diagram of K, in an n-punctured sphere. Thus V_2 intersects S^2 in n disks $D_1, D_2, \ldots, D_n$, where D_1 is sandwiched between the $-p_1$ half-twisted band and the p_2 half-twisted band, D_2 is sandwiched between the p_2 half-twisted band and the p_3 half-twisted band, and
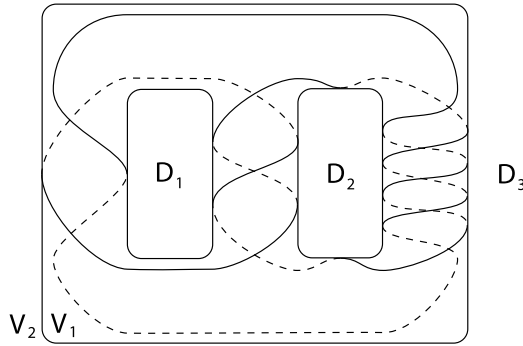
Figure 11: $(-2, 3, 5)$-pretzel knot on a genus 2 Heegaard surface S.
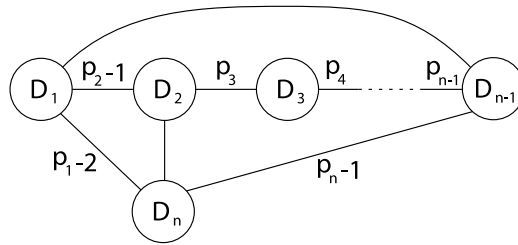
Figure 12: The graph G_i for $K = P(-p_1, p_2, \ldots, p_n)$.

D_n is an outside disk between the p_n half-twisted band and the $-p_1$ half-twisted band. The knot K can be isotoped onto the genus $n - 1$ Heegaard surface S so that $|K \cap \partial D_1| = (p_1 - 1) + (p_2 - 1)$, $|K \cap \partial D_2| = p_2 + p_3$ and $|K \cap \partial D_n| = (p_n - 1) + (p_1 - 1)$. See figure 11 for $K = P(-2, 3, 5)$.

If we cut V_2 along $D_1 \cup D_2 \cup \cdots \cup D_n$, then we have two three-balls B_1 and B_2. On ∂B_i, we have a graph G_i with fat vertices as copies of $D_1, D_2, \ldots, D_n$ and edges from $K \cap \partial B_i$. Figure 12 illustrates the graph G_i for $K = P(-p_1, p_2, \ldots, p_n)$, where the number indicated at each edge denotes the number of multiple edges.

Claim 2.11. $p_1 = 2$ and $n = 3$.

Proof. Since F is not algebraically incompressible and boundary incompressible, $S \cap E(K)$ is not geometrically incompressible and boundary incompressible. If we have a boundary compressing disk E for $S \cap E(K)$, then E can be extended a properly embedded disk E' in V_2 such that $\partial E'$ intersects K in one point. Then by taking two copies of E' and banding along K, we obtain a compressing disk for $S - K$ which cuts off a once punctured torus

from S. Therefore, without loss of generality, let D be a compressing disk for $S - K$. We assume that $|D \cap (D_1 \cup D_2 \cup \cdots \cup D_n)|$ is minimal up to isotopy of D. If $|D \cap (D_1 \cup D_2 \cup \cdots \cup D_n)| = 0$, then one of the graphs G_i is disconnected, a contradiction. Hence there exists an outermost disk δ in D with respect to $D \cap (D_1 \cup D_2 \cup \cdots \cup D_n)$. Then the boundary of δ shows that a graph G_i has a cut vertex. Since $p_i \geq 2$, it follows that $p_1 - 2 = 0$ and $n = 3$. In this case, D_2 is a cut vertex. $\qquad\square$

Without loss of generality, we may assume that $p_2 \leq p_3$.

Claim 2.12. $p_2 \leq 3$.

Proof. In general, there may be outermost disks of D in both B_1 and B_2. However, we know where the boundaries of any such disks lie, and in particular we know the position of the arc of intersection with D_2. When $p_2 \geq 4$ these two arcs necessarily cross, which shows that there can be outermost disks of D in only one of B_1 and B_2. In particular, there is more than one intersection arc on D.

Now consider the disk D, marked by the arcs of intersection with $D_1 \cup D_2 \cup D_3$. Label each such arc with i if it lies in D_i. Any outermost arc is labelled 2. Furthermore, by examining where these arcs lie in D_2, we find that as we follow ∂D away from this arc, in one direction the first arc we meet is labelled 1, while in the other direction the first arc we meet is labelled 3.

Therefore, any two outermost arcs cannot be next to each other, and hence there exists a second outermost arc which is parallel to an outermost arc. However, the second outermost arc cannot exist since one end of it is labelled 1 and the other end is labelled 3. $\qquad\square$

Claim 2.13. *If $p_2 = 3$, then $p_3 \leq 5$.*

Proof. Since $p_1 = -2$ and $p_2 = 3$, it follows that $|K \cap \partial D_1| = 3$. Therefore, by exchanging D for a disk coming from an outermost disk of $D - D_1$ if necessary, there exists a compressing or boundary compressing disk D which is disjoint from D_1. We cut V_2 along D_1, and obtain a solid torus, say V_2', and the complement of V_2' in S^3 is a solid torus, say V_1'. On the Heegaard torus $S' = \partial V_i'$, we have a theta-curve graph with two fat vertices as two copies D_1', D_1'' of D_1 and three edges k_1, k_2, k_3 as in figure 13.

If $p_3 \geq 6$, then the constituent knot $k_1 \cup k_3$ goes around V_2' at least twice. However this shows that ∂D intersects $k_1 \cup k_3$ in at least two points, and contradicts that D is a compressing or boundary compressing disk.
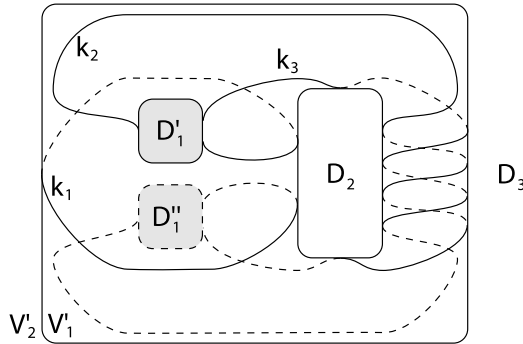
Figure 13: A theta-curve graph on a Heegaard torus S'.

Otherwise, if $p_3 \leq 5$, then there exists a compressing or boundary compressing disk for $S' - (k_1 \cup k_2 \cup k_3)$ in V_2'. □

Claim 2.14. *If $p_2 = 2$, then p_3 is odd.*

Proof. Since $p_1 = -2$ and $p_2 = 2$, it follows that $|K \cap \partial D_1| = 2$. Therefore, by exchanging D if necessary, there exists a compressing or boundary compressing disk D which is disjoint from D_1. As in the previous claim, we cut V_2 along D_1, and obtain a solid torus, say V_2', and the rest is a solid torus V_1'. On the Heegaard torus $S' = \partial V_i'$, we have a loop corresponding to k_1 and a two-cycle with two fat vertices as two copies D_1', D_1'' of D_1 and two edges k_2, k_3 similarly as in figure 13. We note that the loop k_1 goes around V_2' exactly once.

If p_3 is even, then the two-cycle $k_2 \cup k_3$ is parallel to the loop k_1 in the Heegaard torus S', hence it also goes around V_2' exactly once. Therefore, in this case, ∂D intersects $k_1 \cup k_2 \cup k_3$ in at least two points, and contradicts that D is a compressing or boundary compressing disk.

Otherwise, in the case that p_3 is odd, the two-cycle $k_2 \cup k_3$ bounds a disk in the Heegaard torus S' and this shows that $S' - (k_1 \cup k_2 \cup k_3)$ is compressible in V_2'. □

The above claims show that if the pretzel surface for K is not algebraically incompressible and boundary incompressible, then $(-p_1, p_2, \ldots, p_n) = (-2, 3, 3), (-2, 3, 4), (-2, 3, 5)$ or $(-2, 2, p_3)$, where p_3 is odd. Conversely, if $(-p_1, p_2, \ldots, p_n) = (-2, 3, 3), (-2, 3, 4), (-2, 3, 5)$ or $(-2, 2, p_3)$, where p_3 is odd, then there exists a compressing or boundary compressing disk for $S - K$ in V_2 as shown in Claims 2.13 and 2.14. This completes the proof of Theorem 2.10. □

Theorem 2.10 on pretzel surfaces can be generalized to checkerboard surfaces as follows. Let G be a planar graph in the two-sphere S^2 whose edges $e_1, \ldots, e_n$ have weights $w_1, \ldots, w_n \in \mathbb{Z}$. We replace each vertex v_i of G with a disk d_i, and replace each edge e_i with a w_i half-twisted band b_i. Then we obtain a surface F_G and a knot or link $K_G = \partial F_G$. Naturally, K_G has a diagram on S^2 and F_G can be regarded as a checkerboard surface for K_G. We remark that any knot or link can be obtained from a suitable weighted planar graph in this manner. For example, a pretzel knot or link $P(p_1, \ldots, p_n)$ can be obtained as K_G from a theta-curve graph G whose multiple edges $e_1, \ldots, e_n$ have weights $p_1, \ldots, p_n$, and the pretzel surface coincides with F_G.

Theorem 2.15. *Let G be a two-connected planar graph in S^2 with edges $e_1, \ldots, e_n$ having weights $w_1, \ldots, w_n \in \mathbb{Z}$.*

(1) *If $|w_i| \geq 3$ for all i, then the surface F_G is algebraically incompressible and boundary incompressible.*

(2) *If $w_1 \leq -2$ and $w_i \geq 2$ for $i = 2, \ldots, n$ and the surface F_G is not algebraically incompressible and boundary incompressible, then G has an edge, say e_2, that is parallel to e_1 such that $w_1 = -2$ and $w_2 = 2$ or 3.*

Proof. We can prove Theorem 2.15 by the argument similar to Theorem 2.10, so we omit the details. In the same was as the proof of Theorem 2.10, we obtain a graph G_i on the three-ball B_i and if the condition in (1) is satisfied, then G_i is two-connected. This shows that there exists no compressing disk for $\partial N(F_G) - K_G$.

As in Claim 2.11, it follows from the existence of an outermost disk δ that $w_1 = -2$ and there exists a region of $S^2 - N(F_G)$, denoted D_2 as in the argument for Theorem 2.10, containing the outermost arc. Namely, D_2 can be enlarged into a region D_2' of $S^2 - G$ which contacts the two end points of e_1 but does not contact the interior of e_1.

As in Claim 2.12, it follows from the existence of outermost disks in both B_1 and B_2 that an edge adjacent to e_1, say e_2, must be parallel to e_1 and the weight w_2 is less than or equal to 3. Otherwise, two outermost arcs which are coming from two outermost disks in both B_1 and B_2 must intersect in D_2. Thus we obtain two edges e_1 and e_2 satisfying the condition in (2). $\square$

Remark 2.16. Theorem 2.15 (2) shows that the diagram of K_G contains a Montesinos tangle $MT(-1/2, 1/2)$ or $MT(-1/2, 1/3)$.

2.6. Montesinos knots

Theorem 2.17. *Montesinos knots satisfy the strong Neuwirth conjecture.*

Proof. Let $K = M(r_1, \ldots, r_n)$ be a Montesinos knot with n rational tangles of slope r_i. We may assume that $r_i \neq 0/1, 1/0$ and $n \geq 3$ since a composite knot with two-bridge knot summands satisfies the strong Neuwirth conjecture. Moreover, we may assume that $R_- \leq R_+$, where R_- denotes the number of negative r_i's and R_+ denotes the number of positive r_i's. Suppose that K is not a torus knot. Hence, by [3], $K \neq M(-1/2, 1/3, 1/3)$ nor $M(-1/2, 1/3, 1/5)$. We put each rational tangle in a standard form.

If $R_- = 0$, then K is alternating and at least one of the two checkerboard surfaces is non-orientable, algebraically incompressible and boundary incompressible.

Hereafter, we assume that $R_- \geq 1$. Under such a condition, we can take each slope r_i so that $|r_i| < 1$.

If $R_- \geq 2$, then K is $+$-adequate and $-$-adequate. Then, by Theorem 2.6, both of the state surfaces F_{σ_+} and F_{σ_-} are algebraically incompressible and boundary incompressible, and at least one of them is non-orientable.

Henceforth, we assume that $R_- = 1$. By exchanging r_i, we can put $K = M(-r_1, r_2, \ldots, r_n)$, where $r_i > 0$ for $i = 1, 2, \ldots, n$. Then K is $+$-adequate, and the state surface F_{σ_+} is algebraically incompressible and boundary incompressible. Hence if K is not positive, then F_{σ_+} is non-orientable. Here, we remark that K is positive if and only if F_{σ_+} is orientable.

From now on, we assume that K is positive. Therefore, both of the checkerboard surfaces are non-orientable since they can be obtained as state surfaces other than the positive state σ_+.

By using a deformation as in figure 14 inductively, one of the two checkerboard surfaces for $K = M(-r_1, r_2, \ldots, r_n)$ can be deplumbed to a pretzel surface for a pretzel knot or link $K' = P(-\lceil 1/r_1 \rceil, \lceil 1/r_2 \rceil, \ldots, \lceil 1/r_n \rceil)$, where $\lceil x \rceil$ denotes the ceiling function of x which is the smallest integer not less than x.

If K' is neither $P(-2, 3, 3)$, $P(-2, 3, 4)$, $P(-2, 3, 5)$ nor $P(-2, 2, p)$, where p is odd, then the checkerboard surface for K is non-orientable, algebraically incompressible and boundary incompressible.

Therefore, hereafter we may assume that $K = M(-r_1, r_2, r_3)$ and we need to consider the following two cases.

(1) $K' = P(-2, 3, 3)$, $P(-2, 3, 4)$ or $P(-2, 3, 5)$.
   (a) $r_1 \neq 1/2$.
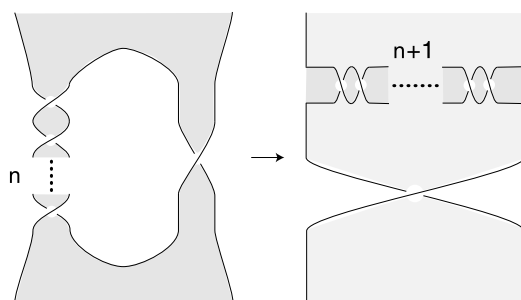
Figure 14: Deplumbing a checkerboard surface.
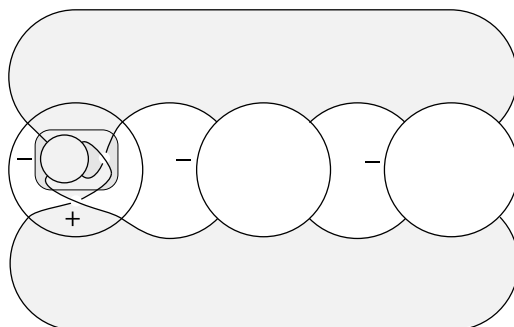
Figure 15: A state surface F_σ for $K = M(-r_1, r_2, r_3)$.

    (b) $r_1 = 1/2$.

  (2) $K' = P(-2, 2, p)$, where p is odd.
    (a) $r_2 \neq 1/2$.
    (b) $r_2 = 1/2$.

In Case (1)-(a), let σ be a state which has all negative signs except for one crossing in the rational tangle with slope $-r_1$. Then, we have a non-orientable state surface F_σ for K as in figure 15 which can be obtained from a pretzel surface for $P(-1, 3, 3)$, $P(-1, 3, 4)$ or $P(-1, 3, 5)$. Since the pretzel surface is a genus 1 Seifert surface for 3_1, 4_1 or 5_2 up to mirror image respectively, F_σ is algebraically incompressible and boundary incompressible.

In Case (1)-(b), first we assume that $r_3 \neq 1/3$, $1/4$ nor $1/5$. By using Lemma 2.9, we can deform the Montesinos knot $K = M(-1/2, r_2, r_3)$ into $K = M(1/2, r_2 - 1, r_3)$.
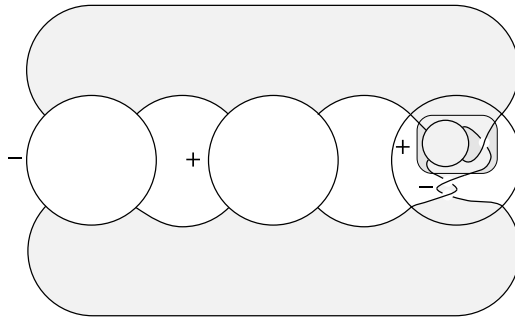
Figure 16: A state surface F_σ for $K = M(1/2, r_2 - 1, r_3)$.

In the case that $K' = P(-2, 3, 4)$, then the deformed knot $K = M(1/2, r_2 - 1, r_3)$ has a minor $P(2, -2, 4)$. By Theorem 2.10, the pretzel surface for $P(2, -2, 4)$ is algebraically incompressible and boundary incompressible and hence the checkerboard surface for $K = M(1/2, r_2 - 1, r_3)$ is also algebraically incompressible and boundary incompressible.

Otherwise, for the case that $K' = P(-2, 3, 3)$ or $P(-2, 3, 5)$, let σ be a state as in figure 16. Then, we have a non-orientable state surface F_σ for K which can be obtained by plumbings from a pretzel surface for $P(2, -2, 2)$ or $P(2, -2, 4)$.

In Case (2)-(a), by using Lemma 2.9, we can deform the Montesinos knot $K = M(-r_1, r_2, r_3)$ into $K = M(1 - r_1, r_2 - 1, r_3)$. Then, one of the two checkerboard surfaces for $K = M(1 - r_1, r_2 - 1, r_3)$ can be deplumbed into a pretzel surface for $P(p_1, -p_2, p_3)$, where $p_i \geq 2$. Since $r_2 \neq 1/2$ and the state surface F_{σ_+} for $K = M(-r_1, r_2, r_3)$ is orientable, for the corresponding slope $t_2 = -r_2/(r_2 - 1)$ of a subtangle T_2 in Lemma 2.9, $\lfloor t_2 \rfloor$ is an even integer greater than or equal to 2. Hence, $p_2 \geq 3$ and the checkerboard surface for $K = M(1 - r_1, r_2 - 1, r_3)$ is algebraically incompressible and boundary incompressible.

In Case (2)-(b), by using Lemma 2.9, we can deform the Montesinos knot $K = M(-r_1, 1/2, r_3)$ into $K = M(1 - r_1, -1/2, r_3)$. Since K is connected, $r_1 \neq -1/2$. Hence, we have $r_1 > 1/2$, $1 - r_1 < 1/2$ and $\lceil 1/(1 - r_1) \rceil \geq 3$. Therefore, the checkerboard surface for K cannot be deplumbed to one for $K' = P(q, -2, p)$, where $q \geq 3$ and p is odd. Eventually, we arrive at Case (1). $\qquad\square$

2.7. Knots with 11 crossings or fewer

Theorem 2.18. *All knots with 11 crossings or fewer except for $K11_n118$ and $K11_n126$ satisfy the Neuwirth conjecture (figure 17).*
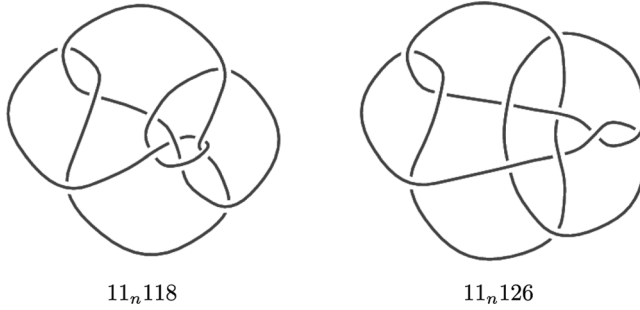
11_n118 11_n126

Figure 17: Knot diagrams from KnotInfo.

Proof. It has been confirmed in [30] that every knot diagram in the Rolfsen table [31] with at most 10 crossings except for the positive knots 8_{19}, 10_{124}, 10_{128}, 10_{134}, 10_{139} and 10_{142} is σ-adequate and σ-homogeneous for a positive or negative state σ distinct from the Seifert state $\vec{\sigma}$.

8_{19} and 10_{124} are equivalent to the $(3,4)$ and $(3,5)$-torus knot respectively and hence these knots satisfy the Neuwirth conjecture.

10_{128} and 10_{139} are Montesinos knots and by Theorem 2.17, these knots satisfy the Neuwirth conjecture.

10_{142} is the $(-4,3,3)$-pretzel knot and by Theorem 2.10, it bounds an algebraically incompressible and boundary incompressible non-orientable pretzel surface. Hence 10_{142} satisfies the Neuwirth conjecture.

It has been be confirmed in [30] that every 11 crossing knot diagram in the Hoste–Thistlethwaite knot table [14] except for $K11_n93$, $K11_n95$, $K11_n118$, $K11_n126$, $K11_n136$, $K11_n169$, $K11_n171$, $K11_n180$ and $K11_n181$ is also σ-adequate and σ-homogeneous for a positive or negative state σ distinct from the Seifert state $\vec{\sigma}$.

Furthermore, it can be checked that $K11_n93$, $K11_n95$, $K11_n136$, $K11_n$ 169, $K11_n171$, $K11_n180$ and $K11_n181$ bound algebraically incompressible and boundary incompressible non-orientable checkerboard surfaces. (You might need to deform the diagram by a Reidemeister move of type III.) □

Remark 2.19. It would be interesting to prove the Neuwirth conjecture for positive knots since the σ_+-state surface for a positive state σ_+ is orientable. Moreover, it seems to be not straightforward to prove the Neuwirth conjecture for positive knots, since all torus knots are positive knots.

2.8. Generalized arborescently alternating links

In this subsection, we generalize the classes of generalized alternating links and arborescent links, and show that any link in this huge link class bounds an algebraically incompressible and boundary incompressible generalized state surface.

First, we recall generalized alternating knots and links. Let F be a closed surface embedded in S^3 and K a knot or link contained in $F \times [-1, 1]$. Suppose that $p(K)$ is a regular projection on F, where $p : F \times [-1, 1] \to F \times \{0\} = F$ is the projection. Then, we have a regular diagram on F obtained from $p(K)$ by adding the over/under information to each double point, and we denote it by the same symbol $p(K)$ in this paper. As usual, a diagram $p(K)$ on F is said to be *alternating* if it has alternating over- and under-crossings as the diagram $p(K)$ is traversed on F. We say that a diagram $p(K)$ on F is *reduced* if there is no disk region of $F - p(K)$ which meets only one crossing. We say that a diagram $p(K)$ on F is *prime* if it contains at least one crossing and for any loop l intersecting $p(K)$ in two points except for crossings, there exists a disk D in F bounded by l such that $D \cap p(K)$ consists of an embedded arc.

Theorem 2.20 [28]. *Let F be a closed surface embedded in S^3, K a knot or link contained in $F \times [-1, 1]$ which has a reduced, prime, alternating diagram on F. Then, both checkerboard surfaces for K are algebraically incompressible and boundary incompressible.*

Next, we introduce generalized arborescently alternating knots and links. Let S be a closed surface embedded in S^3 and K a knot or link contained in $S \times [-1, 1]$. Let $p(K)$ be a connected diagram for K on S. The closed surface S separates S^3 into two submanifolds, say V_+, V_-. Let $\sigma : \mathcal{C} \to \{+, -\}$ be a state for $p(K)$, and let $\mathcal{C} = \{c_1, \ldots, c_n\}$ be the set of crossings of $p(K)$. For each crossing $c_i \in \mathcal{C}$, we take a $+$-smoothing or $-$-smoothing according to $\sigma(c_i) = +$ or $-$. See figure 1. Then, we have the set of state loops $\mathcal{L}_\sigma = \{l_1, \ldots, l_m\}$ on S.

Suppose that each state loop l_i bounds two disks d_i^+, d_i^- in V_+, V_- respectively, and we may assume that these disks are mutually disjoint. For each crossing c_j and state loops l_i, l_k whose subarcs replaced c_j by $\sigma(c_j)$-smoothing, we attach a half-twisted band b_j to d_i^-, d_k^- so that it recovers c_j. See figure 2 for $\sigma(c_j) = +$. In this way, we obtain a spanning surface which consist of disks $d_1^-, \ldots, d_m^-$ and half-twisted bands $b_1, \ldots, b_n$ and call this a σ-*state surface* F_σ.

Remark 2.21. Similarly to Remark 2.2, there are two options to choose a disk $d_i^{\pm}$ in $B_{\pm}$ bounded by each state loop l_i which is not innermost in the closed surface S. Therefore, in general, there are many state surfaces for a given state.

We construct a graph G_{σ} with signs on edges from F_{σ} by regarding a disk d_i^- as a vertex v_i and a band b_j as an edge e_j which has the same sign $\sigma(c_j)$. We call the graph G_{σ} the σ-*state graph*. Let $G_{\sigma} = G_1 * \cdots * G_r$ be the block decomposition of G_{σ}. Following [5, 20], we say that a diagram $p(K)$ is σ-*adequate* if each block G_k has no loop, and that $p(K)$ is σ-*homogeneous* if in each block G_k, all edges have the same sign.

Furthermore, the block decomposition of $G_{\sigma} = G_1 * \cdots * G_r$ corresponds to a Murasugi decomposition of $F_{\sigma} = F_1 * \cdots * F_r$. See figure 5. By cutting the closed surface S along each state loop l_i and pasting in the disk d_i^-, we have closed surfaces $S_1, \ldots, S_r$ which include knot or link diagrams $p(K_1), \ldots, p(K_r)$ respectively, where $K_i = \partial F_i$.

Theorem 2.22. *Let S be a closed surface embedded in S^3, K a knot or link contained in $S \times [-1, 1]$. Suppose that there exists a state σ for $p(K)$ such that:*

(1) *each state loop $l_i \in \mathcal{L}_{\sigma}$ bounds disks in both sides of S,*

(2) *the diagram $p(K)$ is σ-adequate,*

(3) *the diagram $p(K)$ is σ-homogeneous,*

(4) *each diagram $p(K_i)$ is reduced and prime on S_i.*

Then, the state surface F_{σ} for K is algebraically incompressible and boundary incompressible. Furthermore, if σ can be taken so that it is not the Seifert state $\vec{\sigma}$, then the knot K satisfies the strong Neuwirth conjecture.

Proof. The proof is similar to [30]. By Theorem 2.20, each spanning surface F_i for K_i is algebraically incompressible and boundary incompressible since $p(K_i)$ is a reduced, prime, alternating diagram on S_i. And by Theorem 2.1, the σ-state surface $F_{\sigma} = F_1 * \cdots * F_r$ is also algebraically incompressible and boundary incompressible. $\qquad \square$

We call a link satisfying the conditions of Theorem 2.22 a *generalized arborescently alternating link* since it can be obtained from generalized

alternating links on closed surfaces by taking Murasugi sums of those checker-board surfaces. More precisely, a link K is a *generalized arborescently alternating link* if K has a diagram $p(K)$ on a closed surface S and there exists a state σ for $p(K)$ which satisfy conditions (1)–(4) in Theorem 2.22.

Remark 2.23. The existence of knots or links which are not generalized arborescently alternating is unknown. In [30, Problem 1], we have proposed a problem for showing the existence of a knot which has no diagram which is both σ-adequate and σ-homogeneous.

3. Pre-essential surfaces

3.1. Definition of pre-essential surfaces

Definition 3.1. Suppose (S^3, K) is a knot and let $E(K)$ be the exterior of K in S^3.

A *Neuwirth surface* F for K is a compact, orientable, geometrically incompressible and boundary incompressible surface properly embedded in $E(K)$ such that the number of boundary components of F is two and the boundary slope of F is an integer.

A *pre-essential surface* S for K is a compact non-orientable surface properly embedded in $E(K)$ with the property that the boundary $\partial N(S)$ of a regular neighbourhood of S can be compressed to a properly embedded incompressible and boundary incompressible surface F for K. Note that the boundary slope of the resulting surface F is the same as the boundary slope of the pre-essential surface S. (This slope must be even rational).

A *quasi-spanning surface* is a compact non-orientable surface properly embedded in $E(K)$ with a single boundary component. We note that any even rational boundary slope has such a quasi-spanning surface at this slope. If a quasi-spanning pre-essential surface S has an integer boundary slope, namely it is spanning pre-essential, then $\partial N(S)$ compresses to a Neuwirth surface. Examples of spanning pre-essential surfaces are given in Examples 3.3 and 3.4.

Remark 3.2. A pre-essential surface must have an odd number of boundary curves and their slope is an even rational number. The reason for this is that if we attach an annulus along the boundary torus to two adjacent boundary curves of a pre-essential surface, then we can obtain a new non-orientable surface with two fewer boundary components. Since there are no closed embedded non-orientable surfaces in S^3, it immediately follows that
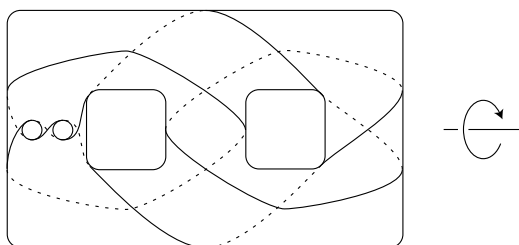
Figure 18: A regular neighbourhood of a checkerboard surface for $10_{128} = M(3/7, -1/2, 1/3)$.
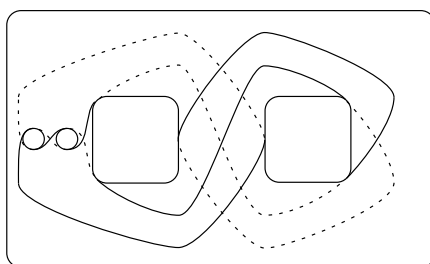
Figure 19: 10_{128} on a genus 4 closed surface of boundary slope 16.

there must be an odd number of boundary curves. Reducing to one boundary curve, we see that the boundary slope must be even since it is zero in homology with coefficients in $\mathbb{Z}_2$.

3.2. Examples of pre-essential surfaces

Example 3.3. 10_{128} bounds a pre-essential checkerboard surface. Figure 18 shows a regular neighbourhood of a checkerboard surface for $10_{128} = M(3/7, -1/2, 1/3)$.

After twisting the right-hand half, a compressing disk appears in the outside region as in figure 19.

By compressing the boundary of a regular neighbourhood of the checkerboard surface, we can obtain a Neuwirth surface for 10_{128}. Therefore, a checkerboard surface for a Montesinos knot $M(3/7, -1/2, 1/3)$ is pre-essential. Similarly, we can observe that a $(-2, 3, 3)$-pretzel knot, equivalently the $(3, 4)$-torus knot, bounds a pre-essential checkerboard surface.
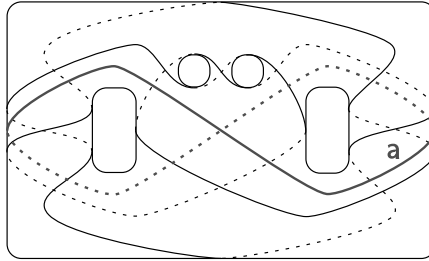
Figure 20: 10_{139} on a genus 4 closed surface.
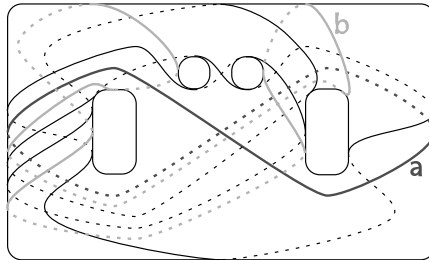
Figure 21: 10_{139} on a genus 4 closed surface.

Example 3.4. 10_{139} bounds a two-pre-essential checkerboard surface, namely, the boundary of a regular neighbourhood of the checkerboard surface becomes essential after twice compressions. Let F be a checkerboard surface for $10_{139} = M(1/3, -3/4, 1/3)$ and V_1 be a regular neighbourhood of F as in figure 20. Put $S = \partial V_1$ and $V_2 = S^3 - \text{int}V_1$. Then, similarly to the case of 10_{128}, there exists a compressing disk A for $S - K$ in V_2 whose boundary is denoted by a.

After compressing S along A, we obtain a genus 3 closed surface S' which separates S^3 into V_1' and V_2' respectively. Then, there exists a second compressing disk B for $S' - K$ in V_1' whose boundary is denoted by b. We remark that this compressing disk B cannot be found in the original pair (S, K), however it appears after isotoping a portion of K as in figure 21. This isotopy can be done by sliding a portion of K along the compressing disk A in $S \cup A$.

After compressing S along the two disks A and B, we obtain a genus 2 closed surface which is a regular neighbourhood of a non-orientable, algebraically incompressible and boundary incompressible spanning surface F' as in figure 22.
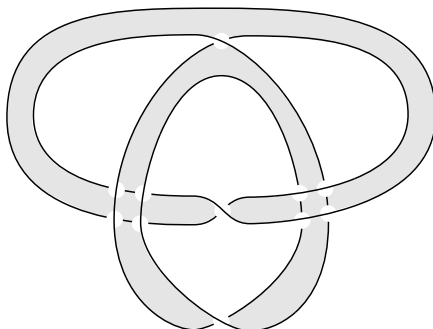
Figure 22: A non-orientable, algebraically incompressible and boundary incompressible spanning surface F' for 10_{139}.

3.3. Pre-essential surfaces and algebraic boundary incompressibility

A convenient criterion for a surface to be pre-essential is given by the following result.

Lemma 3.5. *Suppose (S^3, K) is a knot. A compact properly embedded non-orientable surface S for K is pre-essential if S is algebraically boundary incompressible.*

Proof. Assume that S is a compact properly embedded non-orientable surface in $E(K)$ which is algebraically boundary incompressible. There are two possibilities. If a series of compressions of $\partial N(S)$ results in a properly embedded orientable incompressible and boundary incompressible surface F for K then S is pre-essential and we are done. So we conclude that there must be a collection of compressions of $\partial N(S)$ which produces only trivial components. Such components must be either two-spheres or boundary parallel annuli. Then it is easy to see there must be at least one annulus component after the compressions. Consequently, there is an essential arc λ on such an annulus which is homotopic into $\partial E(K)$ keeping its ends fixed. But then λ is also an essential arc on $\partial N(S)$ which is homotopic into $\partial E(K)$ implying that S is not algebraically boundary incompressible. So this completes the proof. □

We next show that if a knot has a pre-essential surface at a non-longitude boundary slope, then it has a geometrically incompressible and boundary incompressible pre-essential surface.

Lemma 3.6. *Suppose that a knot (S^3, K) has a pre-essential surface S at a non-longitude boundary slope. Then the knot admits a geometrically incompressible and geometrically boundary incompressible pre-essential surface S_0 with the same boundary slope as S. Moreover, S_0 can be chosen to be algebraically boundary incompressible.*

Proof. Let F be the properly embedded, orientable, incompressible and boundary incompressible surface obtained by compressing $\partial N(S)$. Suppose that compressing disks $D_1, D_2, \ldots, D_k$ are used to transform $\partial N(S)$ to F. Let $F_1, F_2, \ldots, F_k = F$ be the sequence of surfaces obtained by these compressions. (Any closed components of these surfaces can be discarded). Clearly D_1 must be on the side of $\partial N(S)$ away from S. However after several compressions, we may reach a compression disk D_j for F_{j-1} which is on the same side as S and in fact which may meet S. However using a sequence of innermost disks of intersection of D_j with S, we can geometrically compress S to a new compact properly embedded non-orientable surface S' that is disjoint from both F_{j-1} and the compressing disk D_j, and hence also from F_j. (Note that compressions of S cannot produce an orientable surface, since the boundary slope of S is not a longitude). So by repeatedly changing S by compressions, we eventually obtain a new compact properly embedded non-orientable surface S^* which is disjoint from F.

Next, if we geometrically compress S^*, then this can be achieved in the complement of F, since any compressing disk for S^* can be isotoped off of F. Then since the boundary slope of S^* is not a longitude, after a finite number of steps we end up with a non-orientable geometrically incompressible surface S_0 with the same boundary curve as S and which is disjoint from F. Consider next a singular boundary compression of S_0. This is an immersed disk with one boundary arc on S_0 and one boundary arc γ on $\partial N(K)$. By making the map of the disk transverse to F, we can assume that the inverse image of F is a collection of arcs $\mathcal{A}$ with ends on γ and simple closed curves. The latter loops can be removed by homotoping subdisks off F, leaving only arcs.

If an innermost arc of $\mathcal{A}$ has image which is homotopic into ∂F, then the subdisk it cuts off has image which can be homotoped off F. But since F has no non-trivial boundary compressions, by repeating this step for all the arcs of $\mathcal{A}$, we eventually obtain a new immersed boundary compression disk which is disjoint from F. But this is impossible, since the arc γ of the boundary compression disk must run from one side of ∂S_0 to the other side along $\partial N(K)$. This completes the proof that S_0 is geometrically incompressible and geometrically boundary incompressible, since it is algebraically

boundary incompressible. Finally since S_0 is also algebraically boundary incompressible, it is pre-essential by Lemma 3.5. $\qquad\square$

Putting together Lemmas 3.5 and 3.6, we have the following theorem.

Theorem 3.7. *Let (S^3, K) be a knot and assume that the boundary slope r is not longitudinal. Then the following conditions are equivalent.*

(1) *There exists a pre-essential surface S with boundary slope r.*

(2) *There exists an algebraically boundary incompressible non-orientable surface S_0 with boundary slope r.*

Moreover we can also assume that S_0 is geometrically incompressible (and geometrically boundary incompressible).

3.4. Pre-essential surfaces and a homological property

The following results Theorems 3.9 and 3.12 give criteria producing an essential surface at an even boundary slope and the Neuwirth conjecture respectively, using a homological argument. We first give a key lemma required for both proofs.

Lemma 3.8. *Suppose that (S^3, K) is a non-trivial knot and X is a connected two-subcomplex of a triangulation $\mathcal{T}$ of $E(K)$. Assume that $X \cap \partial E(K) = \mathcal{C}$ is a non-empty collection of disjoint simple closed curves, all at a non-zero rational even boundary slope and the mapping induced by inclusion $H_1(X, \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2)$ has image zero. Then there is a properly embedded compact connected non-orientable surface F which is disjoint from X and meets $\partial E(K)$ at a collection of simple closed curves $\mathcal{C}'$ with the same boundary slope as the loops of $\mathcal{C}$. Moreover if X is non-separating, then the number of curves in $\mathcal{C}'$ is at most the number in $\mathcal{C}$. If X is separating, the number of curves in $\mathcal{C}'$ is at most half the number in $\mathcal{C}$.*

Proof. See [19] for a discussion of basic concepts in normal surface theory.

We use a method from [18]. As X is a subcomplex of the triangulation $\mathcal{T}$, the vertices and edges of X are in the one-skeleton $\mathcal{T}^1$ of $\mathcal{T}$. Take a maximal tree in each loop of $X \cap \partial E(K) = C$. Extend this forest to a maximal tree in the 1-skeleton of X. Next extend this to a maximal tree in the one-skeleton of $X \cup \partial E(K)$. Finally, extend this to a maximal tree T in T^1. Note that $T \cap \partial E(K)$ is a maximal forest in the one-skeleton of $\partial E(K)$, and every tree in this forest contains all except one of the edges of a loop in C.
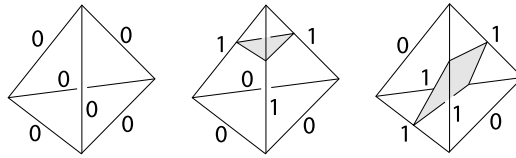
Figure 23: Labelling edges of a triangulation $\mathcal{T}$.

Now each edge E of $\mathcal{T} \setminus T$ can be labelled by either 0 or 1 depending on whether the cycle defined by connecting the ends of E along a path in T is 0 or 1 in $H_1(E(K), \mathbb{Z}_2) = \mathbb{Z}_2$. We also label all the edges in T by 0. It is easy to see that every face of $\mathcal{T}$ is then labelled either by all edges 0 or two edges 1 and one edge 0 since the sum of the labels must be 0 in $\mathbb{Z}_2$. Consequently each tetrahedron has all edges labelled 0, or three edges at a common vertex labelled 1 and the other three edges in a common face labelled 0, or two opposite edges labelled 0 and the other four edges labelled 1 (figure 23). In the first case, we choose the empty surface inside the tetrahedron, in the second a triangular normal disk with one corner on each edge labelled 1 and in the third case a quadrilateral normal disk with one corner on each edge labelled 1. See figure 11. Doing this for every tetrahedron in $\mathcal{T}$ gives a properly embedded normal surface F which is clearly non-empty.

We claim that F is connected, disjoint from X and non-orientable, and has a family of boundary curves $\mathcal{C}'$ satisfying the conditions in the theorem. This will complete the proof. First of all, note that any component of F is non-separating, since by definition, an edge containing a corner of a normal disk of F meets F in one point and the ends can be connected along T in the complement of F. Hence we see that F cannot have any closed components since there are no closed non-separating surfaces in S^3. Next, $F \cap X = \emptyset$. For by assumption, the map $H_1(X, \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2)$ has image zero. Hence any edge E of $\mathcal{T} \setminus T$ which is in X has label 0, since $T \cap X$ consists of a maximal tree in X and all these edges are also labelled 0. But then all the normal disks of F are disjoint from all the edges and hence triangular faces of the subcomplex X. It now follows immediately that $F \cap X = \emptyset$.

The next step is to show that F has a collection of boundary curves $\mathcal{C}'$ parallel to the loops of $\mathcal{C}$. Once this is established, it follows that the components of F must be non-orientable, because the boundary slope of the loops of $\mathcal{C}$ is not a longitude and the components of F are non-separating. The labels of the edges of $\mathcal{C}$ are all 0 since all these edges are in X. There must be edges labelled 1 in $\partial E(K)$ since $H_1(\partial E(K), \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2)$ is onto. Hence we see that $\partial F \neq \emptyset$, and that each component C of ∂F is

non-separating and disjoint from the loops of $X \cap \partial E(K)$ by the previous argument. Every component or set of components of F has an odd number of boundary components, since otherwise pairs of boundary components could be connected by annuli in $\partial E(K)$ producing a closed non-separating surface in S^3, a contradiction. Hence F is connected.

Finally, we want to bound the number of curves in $\partial F = \mathcal{C}'$. Recall that the maximal tree T contains a forest $\mathcal{F}$ of edges in $\partial E(K)$ with the property that each tree in $\mathcal{F}$ contains all the edges except one of a loop C of $\mathcal{C}$. We claim this implies that there cannot be two parallel edges of ∂F in any annulus component of $\partial E(K) \setminus \mathcal{C}$. Deducing the bound on the number of curves in ∂F is then easy.

Suppose on the contrary to the claim that there are parallel curves C, C' of ∂F in an annulus component A of $\partial E(K) \setminus \mathcal{C}$. All the edges of $\mathcal{T}$ meeting $C \cup C'$ are labelled 1 by construction. Hence none of these edges are in the forest $\mathcal{F} \subset T$. The trees of the forest meet the annulus $\overline{A}$ in subtrees T_1, T_2 where T_1 (respectively T_2) contains all the edges except one of the boundary component C_1 (respectively C_2) of $\overline{A}$, where $C_1 \cup C_2 = \partial \overline{A}$. But then we see that both the trees T_1, T_2 cannot contain any of the edges which cross C, C' since the edges in the trees are labelled 0. Hence it is obvious that the forest $\mathcal{F}$ is not maximal, since it will not reach any vertices of $\partial E(K)$ trapped between C, C' in A. This contradiction establishes the claim.

To complete the bound, if X is non-separating, since there is at most one curve of ∂F in any annulus of $\partial E(K) \setminus \mathcal{C}$, the number of curves in $\partial F = \mathcal{C}'$ is at most the number of curves in $X \cap \partial E(K) = \mathcal{C}$. If X is separating, then since F is on one side of X, we see that there is at most one curve of ∂F in any annulus of $\partial E(K) \setminus \mathcal{C}$ on the same side of X as F, i.e., half of all the annuli. So this gives the required bounds and completes the proof. $\qquad \square$

Theorem 3.9. *Suppose that (S^3, K) is a non-trivial knot. Then K has a pre-essential surface at a non-longitudinal boundary slope if and only if K has a properly embedded orientable incompressible and boundary incompressible surface G at an even slope, so that the inclusion $H_1(G, \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2)$ has image zero. Moreover if a surface G can be found with two boundary curves, then the pre-essential surface can be chosen with a single boundary curve (i.e., a quasi-spanning surface) and vice versa.*

Proof. If K has a pre-essential surface S, then by definition there is a properly embedded orientable, incompressible and boundary incompressible surface G at the boundary slope of S. Note that the surface G is disjoint from a compact non-orientable properly embedded surface S_0 by Lemma 3.6. Hence

every loop C on G is disjoint from S_0 and so the homology class of C is zero in $H_1(E(K), \mathbb{Z}_2)$ since the unique non-zero class of $H_1(E(K), \mathbb{Z}_2)$ is dual to the homology class $[S_0]$ in $H_2(E(K), \partial E(K), \mathbb{Z}_2)$. It is obvious that if S has a single boundary curve then G has two boundary curves.

Conversely suppose there is a properly embedded orientable incompressible and boundary incompressible surface G at an even boundary slope, with the property that $H_1(G, \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2)$ has image zero. Using Lemma 3.8, we can construct a properly embedded non-orientable surface S satisfying $S \cap G = \emptyset$, since $H_1(G, \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2)$ has image zero. So we conclude that there is a compact properly embedded non-orientable surface S in $E(K)$ which is disjoint from G. Note that S has a single boundary curve if G has two boundary curves.

But now it is easy to deduce that S is pre-essential. As in Lemma 3.5, we can perform a series of compressions of $\partial N(S)$ to obtain a new incompressible surface S_0 which is disjoint from G. Next, any boundary compression of S_0 can be isotoped off of G, as in the proof of Lemma 3.5. It is easy to see that there cannot be such a boundary compression of S_0 disjoint from G, since the boundary curves of S_0 and G are parallel. Hence S_0 is incompressible and boundary incompressible and we conclude that S is pre-essential as required. $\qquad \square$

Remark 3.10. Theorem 3.9 shows that the even boundary slope conjecture is very closely related to the existence of a pre-essential surface. If a knot has a pre-essential surface at a non-longitudinal slope, then it satisfies the even boundary slope conjecture and the additional assumption about $\mathbb{Z}_2$ homology implies the converse statement.

Example 3.11. A nice class of examples is the (p, q)-torus knots $K_{p,q}$. These knots all satisfy the Neuwirth conjecture with Neuwirth surface given by an essential annulus. Since the boundary slope of an essential annulus for $K_{p,q}$ is pq, it follows that the boundary slope of the annulus is even if one of p, q are even and is odd if both p, q are odd. We can therefore apply Theorem 3.9 to deduce that even torus knots have pre-essential surfaces, since it is obvious that the annulus has image zero in $\mathbb{Z}_2$ homology.

It is also not difficult to show that the odd torus knots do not have pre-essential surfaces. One way of proving this is to observe that if there was such a surface, then there would have to be a properly embedded orientable incompressible and boundary incompressible surface at an even boundary slope, by Lemma 3.5. But torus knots have no orientable incompressible surfaces other than annuli and the unique minimal genus Seifert surfaces, which

have longitudinal boundary slopes. So we conclude that the only separating essential surfaces are annuli and so no pre-essential surface is possible in the odd case.

3.5. Pre-essential surfaces and closed surfaces

In this subsection, we show that for large knots, the Neuwirth conjecture is satisfied if a closed incompressible surface can be found in the knot complement with suitable homological properties.

Theorem 3.12. *Suppose that (S^3, K) is a knot and there is an embedded closed incompressible surface Σ in $E(K)$. Suppose that $i : \Sigma \to E(K)$ is the inclusion map and the induced map $i_* : H_1(\Sigma, \mathbb{Z}) \to H_1(E(K), \mathbb{Z})$ has non-zero image, but the induced map $\tilde{i}_* : H_1(\Sigma, \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2)$ has zero image. Then K bounds a pre-essential surface and satisfies the Neuwirth conjecture.*

Proof. The first step is to construct a properly embedded orientable incompressible and boundary incompressible surface S with the property that $\partial S \subset \partial E(K) \cup \Sigma$ and $\text{int}(S) \cap \Sigma = \emptyset$. We also require that the boundary slope of $S \cap \partial E(K)$ is a non-zero even integer.

Consider the three-manifold M obtained by splitting $E(K)$ open along Σ and discarding the component P that only has boundary given by Σ. Hence $\partial M = \partial E(K) \cup \Sigma$. Let W_1, W_2 be the images of $j_* : H_1(\partial E(K), \mathbb{Q}) \to H_1(M, \mathbb{Q})$ and of $k_* : H_1(\Sigma, \mathbb{Q}) \to H_1(M, \mathbb{Q})$ respectively, where j, k are the inclusion maps. Using the Mayer–Vietoris sequence applied to the decomposition $E(K) = M \cup P$, we see that the rank of $H_1(M, \mathbb{Q})$ is $g + 1$ where g is the genus of Σ.

There are two possibilities for the rank of W_1, namely $1, 2$. In the former case, as is well known we could construct an orientable spanning surface Σ^* in M with one essential boundary curve on $\partial E(K)$, using e.g., the Stallings argument [32]. (See the next paragraph for more details on this technique). But then Σ^* would be an orientable spanning surface for K and hence would have longitudinal boundary slope. But by the assumption that $i_* : H_1(\Sigma, \mathbb{Z}) \to H_1(E(K), \mathbb{Z})$ has non-zero image, again using the Mayer–Vietoris sequence, we see that W_2 must contain a non-zero multiple of the homology class of the meridian $[C]$, which is homologically dual to the relative homology class of Σ^*. This is a contradiction since Σ^* is disjoint from Σ. We conclude that the rank of W_1 is 2.

On the other hand, the assumption that $\tilde{i}_* \colon H_1(\Sigma, \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2)$ has zero image, implies that any multiple of the homology class of the meridian $[C]$ contained in W_2 must be even.

Next, we can construct a map $f : M \to S^1$ using the cohomology class in $H^1(M, \mathbb{Z})$ of the meridian C. Making f transverse to a base point $x_0 \in S^1$, we get that $f^{-1}(x_0)$ is a properly embedded orientable surface S' with $\partial S' \cap \partial E(K) = C'$. We can surger the map f as in the classical method of Stallings [1, 32]. After a finite number of homotopies of f, the result is that S' is replaced by an incompressible and boundary incompressible surface, which still satisfies $\partial S' \cap \partial E(K) = C'$. We again denote this surface by S'. Notice that since W_2 contains an even multiple of the homology class of the meridian $[C]$, we see that $S' \cap \Sigma \neq \emptyset$. For by the same argument as in the previous paragraph, if $S' \cap \Sigma = \emptyset$, then the two-chain carried by S' would have non-zero intersection number with $[C]$, which gives a contradiction. But then the sum of the homology classes of the loops of $S' \cap \Sigma$ equals $[C']$ in $H_1(M, \mathbb{Z})$. This shows that W_2 contains a subgroup of rank 2, with basis consisting of $[C']$ and a non-zero even multiple $m[C]$ of $[C]$.

We can now complete the construction of S. Since W_1, W_2 both contain the homology class $m[C] + [C']$ there is a two-chain c^* with boundary given by curves on $\partial E(K)$ and Σ which represent $m[C] + [C']$. We can then build a new cohomology class in $H^1(M, \mathbb{Z})$ which is dual to this chain. Construct a new map $f^* : M \to S^1$ using this cohomology class, arranging that $f^{*-1}(x_0)$ is a properly embedded orientable surface S with $\partial S \cap \partial E(K) = C^*$, where C^* has slope $m[C] + [C']$. This surface also has some boundary curves on Σ. By surgering the map f^*, we can homotope f^* until the inverse image $f^{*-1}(x_0)$ is incompressible and boundary incompressible. So this completes the construction of S.

To finish the proof, we use Lemma 3.8. Namely we build a pre-essential surface S' which is disjoint from S. Consider the homomorphism $\alpha \colon H_1(M, \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2) = \mathbb{Z}_2$ induced by the inclusion map of $M \subset E(K)$. Clearly kernel(α) contains all the homology classes of loops in S and in Σ. Therefore we can choose $X = S \cup \Sigma$ and apply Lemma 3.8 giving a compact properly embedded non-orientable surface S' with a connected essential boundary disjoint from X. It is easy to see that S' is pre-essential. For any possibly singular boundary compression of S' can be pulled off the incompressible surface Σ and then off the incompressible and boundary incompressible surface S.

The final step is to compress $\partial N(S')$. Note that this can be done in the complement of $S \cup \Sigma$ but we do not actually need this. We end up with a properly embedded incompressible and boundary incompressible surface F

with two boundary curves at non-zero even integer boundary slope. So this implies the knot satisfies the Neuwirth conjecture. $\qquad\square$

Remark 3.13. In [26], it is shown that if (S^3, K) is a large knot and it contains a closed embedded incompressible surface Σ so that the image $H_1(\Sigma, \mathbb{Z}) \to H_1(E(K), \mathbb{Z})$ is zero, then there is a Seifert surface F for K disjoint from Σ. Conversely if a knot (S^3, K) has a Seifert surface F which is not free, then there is a closed embedded incompressible surface Σ disjoint from F with the property that $H_1(\Sigma, \mathbb{Z}) \to H_1(E(K), \mathbb{Z})$ is zero. The method of Theorem 3.12 gives a similar result for non-orientable spanning surfaces.

Example 3.14. We can construct, as follows, many examples of knots (S^3, K) with embedded incompressible surfaces Σ, satisfying the image of $H_1(\Sigma, \mathbb{Z}_2) \to H_1(E(K), \mathbb{Z}_2)$ is zero but the image of $H_1(\Sigma, \mathbb{Z}) \to H_1(E(K), \mathbb{Z})$ is non-zero. Construct a knotted handlebody Y in S^3, i.e., an embedding so that $\overline{S^3 \setminus Y} = M$ is a three-manifold with incompressible boundary $\Sigma = \partial Y$. Choose an embedded knot $K \subset Y$ with the following three properties. Firstly, K is disk busting in Y, i.e., meets every compressing disk in Y. Secondly, the induced map by inclusion $H_1(K, \mathbb{Z}_2) \to H_1(Y, \mathbb{Z}_2)$ has image zero. Finally, the inclusion induces a map $H_1(K, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ with non-zero image. Then it is easy to see that Σ is a closed incompressible surface in $E(K)$ satisfying the required properties as in Theorem 3.12. Note that there are many such knots which are parallel into Σ, i.e., are embedded in Y so that the knot is isotopic into $\Sigma = \partial Y$.

4. Degree one maps

Definition 4.1. A degree one map ϕ between knots $(S^3, K) \to (S^3, K')$ is a continuous proper map between knot complements $E(K) \to E(K')$ which induces an isomorphism $H_3(E(K), \partial E(K)) \to H_3(E(K'), \partial E(K'))$.

Theorem 4.2. *If there exists a degree one map $(S^3, K) \to (S^3, K')$ and K' satisfies the Neuwirth conjecture, then K also satisfies the Neuwirth conjecture.*

Proof. We follow the method of surgering maps as in Theorem 3.12. Choose a Neuwirth surface F' for the knot K' and assume that ϕ is a degree one map $(S^3, K) \to (S^3, K')$. We first homotope ϕ so that it is transverse to F'. We can also assume after a homotopy that ϕ restricts to a homeomorphism $\partial E(K) \to \partial E(K')$ since it is easy to check that the induced map on the

boundary tori is degree one. Then the inverse image F_0 of F' under ϕ is a properly embedded compact orientable surface with two boundary curves, which have the same boundary slope as F, since as is well known, a degree one map maps the longitude, meridian pair for K to the longitude, meridian pair for K'. Although F_0 need not be connected, it must have a connected component F_1 containing the two boundary curves. For suppose there are two components F_2, F_3 each with one boundary curve. The map ϕ restricted to F_2 gives an induced map $H_1(F_2) \to H_1(F')$ which maps the class $[\partial F_2] = 0$ in $H_1(F_2)$ to a non-zero class in $H_1(F')$, which is of the form $[C]$ where C is one of the components of $\partial F'$. This is a contradiction.

If F_1 is incompressible and boundary incompressible, it is clearly a Neuwirth surface for K. Since F_1 has two boundary components, if it is boundary compressible then it is compressible. Choose a compressing disk D for F. We can now perform Stallings technique of surgering the map ϕ. Namely a homotopy of ϕ can be performed so that the inverse image of F' is the result of compressing F along D. After a finite number of surgeries, we must end up with an incompressible component F^* of $\phi^{-1}(F')$ which has the same boundary as F_1. But then F^* is incompressible and boundary incompressible and hence is the required Neuwirth surface for K. $\square$

Corollary 4.3. *If there exists a degree one map* $(S^3, K) \to (S^3, K')$ *and* K' *has a pre-essential surface, then* K *also has a pre-essential surface.*

Proof. Suppose that S' is a pre-essential surface for K'. By definition, if G' is the boundary of $N(S')$ in $E(K')$, then compressing G' produces a properly embedded orientable incompressible and boundary incompressible surface F' for K'. Applying Theorem 4.2, we see that the degree one map $\phi : (S^3, K) \to (S^3, K')$ can be homotoped until the inverse image of F' contains a component which is a properly embedded orientable incompressible and boundary incompressible surface F for K.

Now it is easy to also arrange that the inverse image of S' contains a non-orientable quasi-spanning surface S for K. In fact, recall from the proof of Theorem 4.2, ϕ can be assumed to restrict to a homeomorphism between the peripheral tori preserving the longitude, meridian pair. Therefore, the inverse image of S' will contain exactly one boundary curve at the same slope as that of S'. Hence if the boundary slope of S' is not the longitude, then S must be a non-orientable surface. But then S is disjoint from F and so if we compress the boundary of $N(S)$ in $E(K)$, all the compressions can be isotoped off F. We see that S is pre-essential, since compressing $\partial N(S)$

in $E(K)$ will produce a properly embedded orientable incompressible and boundary incompressible surface (not necessarily the same as F).

Finally in the case that the pre-essential surface S' has longitudinal boundary slope, then the inverse image must still contain a non-orientable spanning surface S. For the map induced by ϕ from S' to S must have degree one, so that the induced map $H_2(S', \partial S') \to H_2(S, \partial S)$ is an isomorphism. But this is impossible if S is non-orientable and S' is orientable. So this completes the proof. □

Corollary 4.4. *Suppose a knot K satisfies the Neuwirth conjecture or has a pre-essential surface, respectively. Then any satellite knot K^* obtained from K and any sum $K \# K'$ satisfy the Neuwirth conjecture or has a pre-essential surface, respectively.*

Proof. By Theorem 4.2 together with the fact that there is clearly a degree one map from a satellite knot K^* to the knot K, if K satisfies the Neuwirth conjecture then so does K^*. Similarly by Corollary 4.3, if K has a pre-essential surface then so does the satellite knot K^* using the degree one map $K^* \to K$.

The argument for a sum $K \# K'$ is similar. We just observe that there is a degree one map $K \# K' \to K$. □

It is a fundamental problem whether a topological invariant becomes "smaller" under a degree one map between knots ([35, Question 3.1]). It is known that the rank of the fundamental group, Gromov's simplicial volume, the Haken number of incompressible surfaces, the knot genus, the homology group and the Alexander polynomial behave in this way under degree one maps. Theorem 4.5 is an addition to this list of invariants and it can be used to show that there does not exist a degree one map between two knots, using the calculation of boundary slopes of knots in [7, 11, 13] and recently [8].

Theorem 4.5. *If there exists a degree one map $(S^3, K) \to (S^3, K')$, then the set of boundary slopes of K includes the set of boundary slopes of K'.*

Proof. By the methods of Theorem 4.2 and Corollary 4.3, if there is a properly embedded orientable incompressible and boundary incompressible surface with a given boundary slope in $E(K')$, then by pulling this back under the degree one map and performing surgery of the map, we obtain a similar surface in $E(K)$ with the same boundary slope. Hence the set of boundary slopes of K' is a subset of the set of boundary slopes of K. □

Remark 4.6. In [12, Corollary 2.5], a similar result was obtained for two-bridge knots and epimorphisms.

Corollary 4.7. *If there exists a degree one map $(S^3, K) \to (S^3, K')$ and K' satisfies the (strong) even boundary slope conjecture, then K also satisfies the (strong) even boundary slope conjecture.*

Corollary 4.8. *If K satisfies the (strong) even boundary slope conjecture, then any satellite knot or sum of a knot with K also satisfies the (strong) even boundary slope conjecture.*

Proof. The proof follows along the same lines as Corollary 4.4.   □

Example 4.9. Take a two-component boundary link $L \subset S^3$ with the property that an arc λ between the two components of L can be found which misses a choice of two disjoint spanning surfaces for the components of L. We also assume that the handlebody $Y = N(L \cup \lambda)$ is knotted, i.e., ∂Y is incompressible in $\overline{S^3 \setminus Y}$. Next pick any knot K embedded in Y which is disk busting in Y. We claim that the knot (S^3, K) has a degree one map to the knot (S^3, K') where the pair (Y', K') is homeomorphic to the pair (Y, K) and $Y' \subset S^3$ is unknotted.

The reason is that the two disjoint spanning surfaces in $\overline{S^3 \setminus Y}$ can be mapped to disks and then the closure of the complement of these spanning surfaces mapped to a ball, so that there is a degree one map $\overline{S^3 \setminus Y} \to Y_1$ which is a genus 2 handlebody with meridian disks with the same boundaries as the spanning surfaces. This map clearly extends to a degree one map $E(K) \to E(K')$ as claimed.

5. Algorithms

We show that there are algorithms to decide if a knot satisfies all of the different versions of the Neuwirth conjecture, except for the weakly strong one. The key techniques are obtained from [17, 19].

Theorem 5.1. *Suppose that (S^3, K) is a knot. Then there are algorithms to decide if K satisfies the Neuwirth conjecture, the strong Neuwirth conjecture, the even boundary slope conjecture and the strong even boundary slope conjecture.*

Proof. We first consider the even and strong even boundary slope conjectures. These follow from [17, 19] as follows. By Corollaries 3.9, 3.10 and 3.11

of [19], there is an algorithm to construct the finite set of boundary slopes of all embedded normal surfaces and hence in particular of properly embedded orientable incompressible and boundary incompressible surfaces, which can be isotoped to be normal. These occur as slopes of the finite set of vertices of the projective solution space of normal surfaces in a one vertex triangulation of $E(K)$.

Next, by doubling $E(K)$, we obtain a closed manifold $2E(K)$ with a one vertex triangulation which is symmetric under the involution interchanging the two copies of $E(K)$. We claim that if there is an incompressible and boundary incompressible orientable surface S at some slope α of K, then there is a closed orientable incompressible surface S^* at a vertex of the projective solution space of $2E(K)$, where $S^* \cap E(K)$ is similarly incompressible and boundary incompressible in $E(K)$ and has the same slope α as S. Since there are finitely many vertex solutions and these are constructible, each can be checked to see if it has an even rational or even integer boundary slope of intersections with the torus $\partial E(K)$ and if it is incompressible, using standard techniques as discussed in either [17] or [19]. Hence the algorithm to check if a knot satisfies the even or strong even boundary slope conjectures is complete, once we have verified the claim above.

The proof of the claim is also straightforward. Given S, it is obvious that $2S$ is a closed orientable incompressible surface in $2E(K)$ meeting $\partial E(K)$ in essential curves at slope α. As usual, we can isotope $2S$ to be least weight normal and then using [17] write a multiple of the normal class of $2S$ as a sum of vertex solutions, each of which is incompressible. So it remains to see why we can assume that these vertex solutions give at least one surface of the form S^* which meets $\partial E(K)$ at slope α. But this follows by [19]. For there, a key result is that if a properly embedded normal surface in $E(K)$ at slope α is written as a Haken sum, then the summands must either have the same slope α or a unique associated slope β. The slope β has the property that the sum of normal curves representing α and β represents a multiple of the trivial curve in a one-vertex triangulation of a torus. But in our case, it is easy to see that no associated slopes can occur since clearly our least weight normal surface in the isotopy class of $2S$ has no trivial curves of intersection with $\partial E(K)$. Hence, we conclude that all the vertex solutions obtained by decomposing a multiple of $2S$ are either disjoint from $\partial E(K)$ or meet $\partial E(K)$ with slope α and there must be at least one surface S^* of the latter type. Note that $S^* \cap E(K)$ is boundary incompressible, since it is orientable and therefore a boundary compression would imply the existence of a compression. So this completes the discussion of the even and strong even boundary slope algorithms.

We now consider the Neuwirth conjecture and strong Neuwirth conjectures for a knot (S^3, K). Suppose that there is an orientable incompressible and boundary incompressible surface $S \subset E(K)$ with ∂S consisting of two curves with integer slope. The key idea is to use fundamental normal solutions rather than vertex normal solutions. Recall that a fundamental normal surface F has the property that F cannot be written as a Haken sum of two non-empty normal surfaces. Haken [10] proved that there is a finite set of constructible fundamental normal surfaces in any closed triangulated three-manifold.

The procedure is similar to the case of the boundary slope conjectures. Namely given S, we can assume that $2S$ is a closed orientable incompressible least weight surface in its isotopy class in $2E(K)$. Moreover, we still have the key property that $2S \cap \partial E(K)$ consists of two copies of an essential curve with integer boundary slope. Now if we write $2S$ as a sum of least weight fundamental normal surfaces, using [17], it follows that each of these is again a closed orientable incompressible surface in $2E(K)$. But we can also deduce from [19] that each of these summands either is disjoint from $\partial E(K)$ or meets $\partial E(K)$ in essential curves at the same slope as S. (As usual, since all these surfaces are least weight, no trivial curves of intersection with $\partial E(K)$ can occur). Moreover, it is easy to see using an elementary homology argument on $\partial E(K)$ that the total number of curves of intersection of all the summands with $\partial E(K)$ must be two. Hence we see that either one summand has two such curves or two summands have one curve of intersection each. The first case gives us the result required — by checking all fundamental solutions we search for one with two intersection curves with $\partial E(K)$ which is incompressible. It then splits along $\partial E(K)$ into two surfaces each of which is a Neuwirth surface for K. In the second case, splitting a surface with one curve of intersection with $\partial E(K)$ gives two spanning surfaces for K which must be either orientable Seifert surfaces, in which case the boundary slope of S is longitudinal, or are non-orientable. In the latter case, as usual, a boundary of a regular neighbourhood of such a surface is the required Neuwirth surface. Hence, we have proved that there is an algorithm to verify the Neuwirth conjecture in all cases except where the boundary slope is longitudinal.

For the final case, suppose that the Neuwirth surface S we seek is longitudinal. In this case, we have to complete the argument in case $2S$ splits as a sum of fundamental solutions so that two of them have one curve of intersection with $\partial E(K)$ and split along $\partial E(K)$ into orientable Seifert surfaces for K. It is easy to see that these two fundamental surfaces have different normal classes, so that when they are combined by a Haken sum, a

connected normal surface is obtained. But then there is a simple algorithm to complete the problem. Namely search amongst fundamental normal surfaces in $2E(K)$ for ones which meet $\partial E(K)$ in one longitudinal curve. For pairs of such surfaces check whether their Haken sum is incompressible and connected. Such a surface will then split along $\partial E(K)$ into two longitudinal Neuwirth surfaces as required.

It remains to discuss the strong Neuwirth conjecture. This is very similar to the argument for the Neuwirth conjecture. Assume that S is a non-orientable surface at an even integer boundary slope in $E(K)$, so that $S' = \partial N(S)$ is incompressible and boundary incompressible in $E(K)$. By [10], we can write $[2S] = \Sigma n_i[F_i]$ where $F_1, F_2, \ldots, F_k$ are fundamental normal surfaces in $2E(K)$ and n_i are positive integers. By the arguments in the previous paragraphs, it follows that there must be some i_0 so that $n_{i_0} = 1$ and F_{i_0} meets $\partial E(K)$ in a single curve at the same boundary slope as S. Moreover each F_i is disjoint from $\partial E(K)$ for each $i \neq i_0$. (This follows since the results of [19] on boundary slopes apply to normal surfaces, not just incompressible normal surfaces). We see that if F_{i_0} is split along $\partial E(K)$, the result is either non-orientable spanning surfaces at the same boundary slope as S, or Seifert surfaces if the slope of S is longitudinal. However, if all the surfaces F_i are orientable, then so is $[2S]$, which is a contradiction. On the other hand, each F_i, for $i \neq i_0$, is a closed surface in $E(K)$ so must be orientable. Hence we conclude that F_{i_0} must be non-orientable and must split along $\partial E(K)$ into at least one non-orientable spanning surface for K.

To complete the argument, we want to apply [17]. The equation $[2S] = \Sigma n_i[F_i]$ implies $[2S'] = 2[2S] = \Sigma 2n_i[F_i]$. But then by [17], we deduce that all the surfaces F_i are π_1-injective and so in particular F_{i_0} is π_1-injective. Therefore $\partial N(F_{i_0})$ meets $E(K)$ in an incompressible and boundary incompressible surface. So this shows that $F_{i_0} \cap E(K)$ is an essential non-orientable spanning surface for K. We can now verify the strong Neuwirth conjecture, by searching amongst fundamental normal surfaces in $2E(K)$ for a solution F_{i_0} with the property that $\partial N(F_{i_0})$ is incompressible in $2E(K)$ and $F_{i_0} \cap \partial E(K)$ has even integer boundary slope. $\qquad\square$

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