

# Schoen–Yau–Gromov–Lawson theory and isoparametric foliations

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Motivated by the celebrated Schoen–Yau–Gromov–Lawson surgery theory on metrics of positive scalar curvature, we construct a double manifold associated with a minimal isoparametric hypersurface in the unit sphere. The resulting double manifold carries a metric of positive scalar curvature and an isoparametric foliation as well. To investigate the topology of the double manifolds, we use  $K$ -theory and the representation of the Clifford algebra for the FKM-type, and determine completely the isotropy subgroups of singular orbits for homogeneous case.

## 1. Introduction

One of the simplest invariants of a Riemannian manifold is its scalar curvature function. Here, we say an  $n$ -dimensional manifold  $M$  carries a metric of positive scalar curvature  $R_M$  if  $R_M \geq 0$  and  $R_M(p) > 0$  for some point  $p \in M$ . Then a natural question to raise is “Which manifolds admit Riemannian metrics of positive scalar curvature?” In recent decades, this subject has been the focus of lively research. The first important contribution to this subject was made by Lichnerowicz [14] in 1962, who showed that a compact spin manifold with non-vanishing  $\hat{A}$ -genus cannot carry a Riemannian metric of positive scalar curvature. Hitchin [11] generalized this result. More precisely, he used a ring homomorphism  $\alpha$ , constructed by Milnor, from  $\Omega_*^{\text{spin}}$ , the spin cobordism ring, to  $KO^{-*}(\text{pt})$ , and proved that  $\alpha(M)$  vanishes if  $M$  carries a metric of positive scalar curvature. When  $\dim M \equiv 0 \pmod{4}$ ,  $\alpha(M)$  can be identified with  $\hat{A}(M)$  (up to a factor), so this recovers the result of Lichnerowicz. One surprising and beautiful result of this study was that half of the exotic spheres in dimensions  $8k + 1$  and  $8k + 2$  cannot carry metrics of positive scalar curvature. Another remarkable step toward answering the question above was made when Schoen and Yau [20], and independently, Gromov and Lawson [6] established the following “surgery theorem” on metrics of positive scalar curvature:

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Dedicated to Professor Yuanlong Xin on his 70th birthday.

**Theorem.** *Let  $X$  be a compact manifold which carries a Riemannian metric of positive scalar curvature. Then any manifold which can be obtained from  $X$  by performing surgeries in codimension  $\geq 3$  also carries a metric of positive scalar curvature.*

Inspired by Schoen–Yau–Gromov–Lawson’s surgery theory, we will construct a new manifold with rich geometrical properties from a Riemannian manifold with an embedding hypersurface. In particular, we implement this construction on a unit sphere with a minimal isoparametric hypersurface, finding that the new manifold admits not only complicated topology, but also a metric of positive scalar curvature. Moreover the isoparametric foliation is kept. Details of the construction are given in the following.

Given a compact, connected manifold  $X^n$  ( $n \geq 3$ ) without boundary. Let  $Y^{n-1} \hookrightarrow X^n$  be a connected embedding hypersurface with a trivial normal bundle, and  $\pi_0(X - Y) \neq 0$ , i.e., the complement of  $Y$  in  $X$  is not connected. Then  $Y^{n-1}$  separates  $X^n$  into two components, say  $X_+^n$  and  $X_-^n$ , with the same boundary  $Y^{n-1}$ . (The assumption  $\pi_0(X - Y) \neq 0$  is necessary. For example,  $T^2 - S^1$ , removing a latitude circle from the torus, is connected.) Since  $Y$  has a trivial normal bundle in  $X$ , we can choose a unit normal vector field  $\xi$  on  $Y$ , which is an interior normal direction with respect to  $X_+$ .

Define a continuous function  $r : X^n \rightarrow \mathbb{R}$

$$x \mapsto \begin{cases} \text{dist}(x, Y), & \text{if } x \in X_+, \\ -\text{dist}(x, Y), & \text{if } x \in X_-, \end{cases}$$

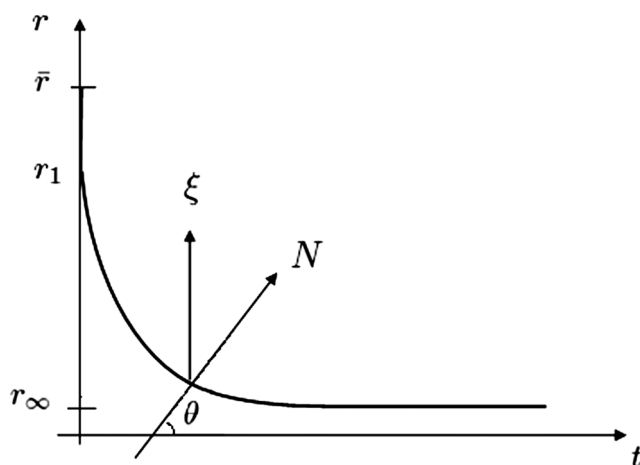
where  $\text{dist}(x, Y)$  means the distance from  $x$  to  $Y$ . Clearly,  $X_+$  ( $X_-$ ) is just the subset that  $r \geq 0$  (resp.  $r \leq 0$ ). Let  $Y_r := \{x \in X \mid r(x) = r\}$  for  $|r|$  so small that  $Y_r$  is still an embedding hypersurface. We extend  $\xi$  to a unit vector field in a neighborhood of  $Y$  such that  $\xi$  is normal to  $Y_r$ .

From now on, without loss of generality, we only deal with  $X_+^n$ . Concerning with the Riemannian product space  $X_+^n \times \mathbb{R}$  with coordinates  $(x, t)$ , for a small number  $\bar{r} > 0$ , we can define a hypersurface  $M^n$  in  $X_+^n \times \mathbb{R}$  as [6]

$$M^n := \{(x, t) \in X_+^n \times \mathbb{R} : (r(x), t) \in \gamma, r(x) \leq \bar{r}\},$$

where  $\gamma$  is a curve in the  $(r, t)$ -plane as pictured below:

$$N : \text{unit “exterior” normal vector field on } M, \sin \theta := \langle N, \xi \rangle$$



The curve  $\gamma$  begins at one end with a vertical line segment  $t = 0$ ,  $r_1 \leq r \leq \bar{r}$ , and ends with a horizontal line segment  $r = r_\infty > 0$ , with  $r_\infty$  small enough as we will require.

Now fix a point  $q = (x, t) \in M$  corresponding to  $(r, t) \in \gamma$ . Choose an orthonormal basis  $e_1, e_2, \dots, e_{n-1}$  of  $T_x Y_r$  such that the shape operator  $A_\xi$  is expressed as  $A_\xi e_i = \mu_i(r) e_i$  for  $i = 1, \dots, n - 1$ . Then the associated principal curvatures of  $M$  at  $q$  are of the form  $\lambda_i = \mu_i(r) \sin \theta$  for  $i = 1, \dots, n - 1$ . As observed by Gromov–Lawson [6], the tangent vector of  $M \cap (l \times \mathbb{R})$  is also a principal direction for the second fundamental form of  $M$  in  $X_+^n \times \mathbb{R}$ , where  $l$  is a geodesic ray in  $X_+$  emanating from  $Y$ . We denote this tangent vector by  $e_n$ . Thus the  $n$ th principal curvature at  $q$  is  $\lambda_n := k$ , the (nonnegative) curvature of  $\gamma$  at  $(r, t)$ .

Look at the Gauss equation:

$$K_{ij}^M = K_{ij}^{X \times \mathbb{R}} + \lambda_i \lambda_j, \quad 1 \leq i, j \leq n,$$

where  $K_{ij}^M$  is the sectional curvature of  $M$  of the plane  $e_i \wedge e_j$ , and  $K_{ij}^{X \times \mathbb{R}}$  is the corresponding sectional curvature of  $X_+ \times \mathbb{R}$ . Since the metric of  $X_+ \times \mathbb{R}$  is the product metric, we see,

$$\begin{aligned} K_{ij}^{X \times \mathbb{R}} &= K_{ij}^X, & 1 \leq i, j \leq n - 1, \\ K_{n,j}^{X \times \mathbb{R}} &= K_{\xi,j}^X \cos^2 \theta, \end{aligned}$$

where  $K^X$  is the sectional curvature of  $X_+$ .

It follows immediately that the scalar curvature of  $M$  with the induced metric can be expressed as

$$(1.1) \quad R_M = \sum_{i \neq j}^n K_{ij}^M = R_X + 2A \sin^2 \theta + 2kH(r) \sin \theta,$$

where

$$A := \sum_{i < j \leq n-1} \mu_i(r) \mu_j(r) - \text{Ric}^X(\xi, \xi);$$

$$H(r) = \sum_{i=1}^{n-1} \mu_i(r), \text{ the mean curvature of } Y_r.$$

**Remark 1.1.** Since  $\gamma$  ends with a horizontal line segment,  $M$  has the standard product metric as  $t$  goes to infinity. This guarantees that we can glue  $X_+$  (resp.  $X_-$ ) smoothly in metric with a copy of itself along  $Y$  to get a new manifold called the double of  $X_+$  (resp.  $X_-$ ), and denoted by  $D(X_+)$  (resp.  $D(X_-)$ ). The double of a manifold with boundary, as a topological concept, appeared in 1930's. Gromov–Lawson [7] studied the geometric property of the double of a manifold, and showed an interesting theorem which states that if  $X$  carries a metric of positive scalar curvature, and  $Y$  is a minimal hypersurface, then the double manifold  $D(X_+)$  (resp.  $D(X_-)$ ) also carries a metric of positive scalar curvature. But in their construction of the double manifold, they “bent” too much near the boundary of  $X_+$ , inducing some singularities or creases. However in our method, an explicit construction of  $D(X_+)$  (resp.  $D(X_-)$ ) with satisfactory properties is given.

**Remark 1.2.** Formula (1.1) is the expression of the scalar curvature of  $M$  in  $X_+ \times \mathbb{R}$ . It also holds in  $X_- \times \mathbb{R}$  although  $\xi$  is the exterior normal vector field of  $X_-$ . Gromov and Lawson [6] studied the scalar curvature of  $M$ . Their formula is expressed in form of the estimate of principal curvatures, while ours is an explicit expression. Their main result on surgery is of course correct although in their formula (1) [6] they lost a factor 2. In addition, in (1') they missed one item related to the second fundamental form of the submanifold. But this mistake would result in the missing of the item  $H(r)$  in our formula (1.1), which is, however, essential for our research. Rosenberg and Stolz [19] modified Gromov–Lawson's expression, but they also lost the principal curvatures or the second fundamental form of the submanifold.

From now on, we will be concerned with  $X^n = S^n(1)$ . Suppose that  $Y^{n-1}$  is a compact minimal isoparametric hypersurface, i.e., a compact

hypersurface with vanishing mean curvature and constant principal curvatures  $\mu_1, \mu_2, \dots, \mu_{n-1}$  (cf. [4]). In fact, in every isoparametric family in the unit sphere, there does exist one and only one minimal hypersurface (cf. [8]). Recall an elegant result of Münzner [17] that the number  $g$  of distinct principal curvatures must be 1, 2, 3, 4 or 6; and the multiplicities  $m_i$  ( $i = 1, 2, \dots, g$ ) of distinct principal curvatures satisfy  $m_k = m_{k+2}$  (subscripts mod  $g$ ). We will denote them by  $m_+$  and  $m_-$ , respectively. More precisely, as it is well known that every isoparametric hypersurface in the unit sphere corresponds to an isoparametric function  $f$  with image  $[-1, 1]$ . Denote the focal submanifolds by  $M_+ := f^{-1}(1)$  and  $M_- := f^{-1}(-1)$  so that  $\text{codim}(M_+) = m_+ + 1$ ,  $\text{codim}(M_-) = m_- + 1$ .

One of the main results of the present paper is:

**Theorem 1.1.** Let  $Y^{n-1}$  be a compact minimal isoparametric hypersurface in  $S^n(1)$ ,  $n \geq 3$ , which separates  $S^n$  into  $S_+^n$  and  $S_-^n$ . Then each of doubles  $D(S_+^n)$  and  $D(S_-^n)$  admits a metric of positive scalar curvature. Moreover, there is still an isoparametric foliation in  $D(S_+^n)$  (or  $D(S_-^n)$ ).

**Remark 1.3.** As a direct result, we get the  $KO$ -characteristic numbers

$$\alpha(D(S_+^n)) = 0, \quad \alpha(D(S_-^n)) = 0.$$

Furthermore, we have:

**Proposition 1.1.**  $D(S_+^n)$  (resp.  $D(S_-^n)$ ) is a  $\pi$ -manifold, i.e., a stably parallelizable manifold. In particular, it is an orientable, spin manifold with vanishing Stiefel–Whitney classes and Pontrjagin classes.

It is worth pointing out that the condition “ $D(S_+^n)$  is stably parallelizable” does not imply the conclusion  $\alpha(D(S_+^n)) = 0$ . For instance, as Kervaire–Milnor (Theorem 3.1 in [13]) proved, every homotopy sphere is a  $\pi$ -manifold. But as we stated before, there do exist some  $8k + 1$ - and  $8k + 2$ -dimensional exotic spheres with non-vanishing  $KO$ -characteristic number  $\alpha$ .

For isoparametric hypersurfaces in unit spheres, taking the different values of  $g$  into account, we know:

When  $g = 1$ , an isoparametric hypersurface must be a great or small sphere. Thus the double construction is trivial, namely,  $D(S_+^n) \cong S^n$ .

When  $g = 2$ , an isoparametric hypersurface must be a standard product of two spheres  $S^k(r) \times S^{n-k-1}(s)$  with  $r^2 + s^2 = 1$ . Thus  $D(S_+^n) \cong S^k \times S^{n-k}$  or  $S^{k+1} \times S^{n-k-1}$ .

When  $g = 3$ , E. Cartan has classified the isoparametric hypersurfaces. In fact, they are all homogeneous (see, for example, [4]).

When  $g = 4$ , except for the unknown case  $(m_+, m_-) = (7, 8)$  (or  $(8, 7)$ ), all the isoparametric hypersurfaces are either of FKM-type or homogeneous (cf. [2, 3]).

When  $g = 6$ , all the isoparametric hypersurfaces must be homogeneous (see, for example, [15]).

Given all these classifications, in order to study the properties of the double manifold  $D(S_+^n)$ , it suffices to consider the cases that  $Y$  is either homogeneous or of FKM-type, except for the case  $(g, m_+, m_-) = (4, 7, 8)$ . Therefore, we divide our research into two parts, one is on the homogeneous case, and the other is on the FKM-type.

We begin by recalling a well-known result that homogeneous hypersurfaces in  $S^n$  are isoparametric since they have constant principal curvatures. They have been characterized as principal orbits of the isotropy representation of rank two symmetric spaces, and are classified completely by Hsiang and Lawson (cf. [12, 23]). From the corresponding cohomogeneity one action on  $S^n$  with a certain slice representation of the normal disc, we derive a cohomogeneity one action on  $D(S_+^n)$ . In terms of the isotropy subgroup  $K_0$  of the principal orbit and  $K_\pm$  of the singular orbits (focal submanifolds)  $M_\pm$ , we classify  $D(S_+^n)$  in Section 3 with respect to the homogeneous hypersurface  $Y$ .

In particular, we investigate  $D(S_+^4)$  in the case  $(g, m_+, m_-) = (3, 1, 1)$ , finding an interesting phenomenon that  $D(S_+^4) \cong S^2 \times S^2/\sigma$ , where  $\sigma$  is an involution different from that of the oriented Grassmannian  $G_2(\mathbb{R}^4) \cong S^2 \times S^2/\sim$ .

Next, we turn to the FKM-type. For every orthogonal representation of the Clifford algebra  $\mathcal{C}_{m-1}$  on  $\mathbb{R}^l$ , Ferus, Karcher and Münzner [5] constructed a homogeneous polynomial  $F$  on  $\mathbb{R}^{2l}$ . The level hypersurfaces of  $F|_{S^{2l-1}}$  are isoparametric in  $S^{2l-1}$  with  $g = 4$  and multiplicities of distinct principal curvatures  $(m_+, m_-, m_+, m_-) = (m, l - m - 1, m, l - m - 1)$ . If  $m \not\equiv 0 \pmod{4}$ ,  $F$  is determined by  $m$  and  $l$  up to a rigid motion of  $S^{2l-1}$ ; if, however  $m \equiv 0 \pmod{4}$ , there are inequivalent representations of  $\mathcal{C}_{m-1}$  on  $\mathbb{R}^l$  parameterized by an integer  $q$ , the index of the representation (cf. [24]). In the second case, denote by  $M_+(m, l, q)$ ,  $M_-(m, l, q)$  the corresponding focal submanifolds, respectively. According to Ferus *et al.* [5],  $M_+$  has a trivial normal bundle, while  $M_-$  is diffeomorphic to an  $S^{l-1}$  bundle over  $S^m$ . Thus  $D(S_+^{2l-1}) \cong M_+ \times S^{m+1}$ . As for  $D(S_-^{2l-1})$ , a delicate calculation of the topology on a sphere bundle over  $M_-$  leads to the following.

**Theorem 1.2.** Given an odd prime  $p$ . If  $q_1 \not\equiv \pm q_2 \pmod{p}$ , then  $D(S_+^n)$  and  $D(S_-^n)$  have different homotopy types.

As we claimed in Proposition 1.1,  $D(S_+^n)$  is a  $\pi$ -manifold with vanishing Stiefel–Whitney classes and Pontrjagin classes. Without the aid of characteristic classes, it is usually not easy to distinguish the homeomorphism classes. Our Theorem 1.2 is established by using mod  $p$  cohomology operators.

## 2. Geometry of the double manifold $D(S_+^n)$

This section will be devoted to the proof of Theorem 1.1. We prefer to prove this result by making use of fundamental properties of isoparametric hypersurfaces and some straightforward verifications.

Let  $Y^{n-1}$  be a minimal isoparametric hypersurface in the unit sphere  $S^n(1)$ . It is well known that  $Y$  is a level hypersurface with vanishing mean curvature of an isoparametric function  $f$  on  $S^n(1)$ . By an isoparametric function on  $S^n(1)$ , we mean a function  $f : S^n(1) \rightarrow \mathbb{R}$  satisfying:

$$(2.1) \quad \begin{cases} |\nabla f|^2 = b(f), \\ \Delta f = a(f), \end{cases}$$

where  $\nabla f$  is the gradient of  $f$ ,  $\Delta f$  is the Laplacian of  $f$ ,  $b$  is a smooth function on  $\mathbb{R}$ , and  $a$  is a continuous function on  $\mathbb{R}$  (see [22], for an excellent survey). We require that the isoparametric function is proper (cf. [9]) so that both focal submanifolds have codimensions greater than 1.

Recall that an isoparametric hypersurface  $Y^{n-1}$  in  $S^n(1)$  has constant principal curvatures, which we denote by  $\mu_1(0), \mu_2(0), \dots, \mu_{n-1}(0)$  as before corresponding to the unit normal vector field  $\xi = \frac{\nabla f}{|\nabla f|}$ . A key reason for choosing  $Y^{n-1}$  to be minimal isoparametric is that, as we will see, its induced metric from  $S^n(1)$  has positive scalar curvature.

By Gauss equation, for a closed minimal hypersurface  $N$  in a unit sphere  $S^n(1)$ ,

$$S = (n-1)(n-2) - R_N,$$

where  $S$  is square of the length of the second fundamental form. If, in addition,  $N$  is a minimal isoparametric hypersurface on  $S^n$ , Peng and Terng [18] asserted that

$$S = (g-1)(n-1),$$

which implies  $R_N \geq 0$ , and “=” is achieved if and only if  $(m_+, m_-) = (1, 1)$  since  $n-1 = \frac{g}{2}(m_+ + m_-)$ .

It follows immediately that the minimal isoparametric hypersurface  $Y$  has  $R_Y \geq 0$ ,  $H(0) = \sum_{i=1}^{n-1} \mu_i(0) = 0$ , and  $S = \sum_{i=1}^{n-1} \mu_i^2(0) = (g-1)(n-1)$ , which imply that

$$2 \sum_{i < j}^{n-1} \mu_i \mu_j |_Y = H(0)^2 - \sum_{i=1}^{n-1} \mu_i^2 |_Y = -(g-1)(n-1).$$

By definition in formula (1.1), we see  $A = \sum_{i < j \leq n-1} \mu_i \mu_j - (n-1)$ . In order to simplify the calculation of  $R_M$ , we set

$$a(r) := 2 \sum_{i < j}^{n-1} \mu_i \mu_j |_{Y_r} - 2 \sum_{i < j}^{n-1} \mu_i \mu_j |_Y = 2 \sum_{i < j}^{n-1} \mu_i \mu_j |_{Y_r} + (g-1)(n-1).$$

Since

$$\lim_{r \rightarrow 0} 2 \sum_{i < j}^{n-1} \mu_i \mu_j |_{Y_r} = \lim_{r \rightarrow 0} \left( H(r)^2 - \sum_{i=1}^{n-1} \mu_i^2 |_{Y_r} \right) = -(g-1)(n-1),$$

we have

$$(2.2) \quad \lim_{r \rightarrow 0} a(r) = 0$$

In fact, according to the Bochner–Weitzenböck formula:

$$\frac{1}{2} \Delta |\nabla f|^2 = |\text{Hess} f|^2 + \langle \nabla f, \nabla(\Delta f) \rangle + \text{Ric}(\nabla f, \nabla f),$$

by virtue of the expression of Hessian of  $f$  (cf. [10]):

$$\text{Hess} f = \text{diag} \left( -\sqrt{b(f)} \mu_1, \dots, -\sqrt{b(f)} \mu_{n-1}, b'(f)/2 \right),$$

with  $b(f) = g^2(1 - f^2)$  in formula (2.1) by the famous Cartan–Münzner equations, we obtain

$$(2.3) \quad \sum_{i=1}^{n-1} \mu_i^2 |_{Y_r} = \frac{(n-1)(g-1) - cf + (n-1)f^2}{1 - f^2} \quad \text{with} \quad c = \frac{g^2(m_- - m_+)}{2}.$$



Hence we can express  $H(r)$  explicitly as

$$(2.4) \quad H(r) = (n-1) \frac{f|_{Y_r}}{\sqrt{1-f^2|_{Y_r}}} - \frac{c}{g\sqrt{1-f^2|_{Y_r}}}.$$

It follows immediately that  $H(0) = 0$  and  $H(r) > 0$  for any  $r > 0$ .

Consequently, from the definition of  $a(r)$  and (2.3), (2.4), it follows that

$$(2.5) \quad \begin{aligned} a(r) &= H^2(r) - \sum_{i=1}^{n-1} \mu_i^2|_{Y_r} + (g-1)(n-1) \\ &= (g-1)(n-1) + \frac{1}{1-f^2} \left\{ (n-1)^2 f^2 - \frac{2c}{g}(n-1)f + \frac{c^2}{g^2} - (n-1)f^2 \right. \\ &\quad \left. + cf - (g-1)(n-1) \right\}, \end{aligned}$$

Substituting all these equalities in (1.1), we get immediately

$$(2.6) \quad \begin{aligned} R_M|_{Y_r} &= n(n-1) \cos^2 \theta + (n-g-1)(n-1) \sin^2 \theta + a(r) \sin^2 \theta \\ &\quad + 2kH(r) \sin \theta \end{aligned}$$

with  $H(r)$  and  $a(r)$  in (2.4), (2.5), respectively.

Since we have the dimension relation  $n-1 = \frac{g}{2}(m_+ + m_-)$ , it suffices to analyze the following two cases for our destination.

(A):  $(m_+, m_-) = (1, 1)$ .

This is just the case that  $n-g-1 = 0$ . Since  $a(r)$  is identically 0 in this case, (2.6) becomes

$$R_M = n(n-1) \cos^2 \theta + 2kH(r) \sin \theta.$$

By controlling the “bending” angle of the curve  $\gamma$ , we can assume  $0 \leq k \leq \frac{1}{2}$  so that  $R_M|_{Y_r} = n(n-1) \cos^2 \theta + 2kH(r) \sin \theta \geq 0$ , and “=” is achieved if and only if  $r = 0$ .

(B):  $\text{Max}\{m_+, m_-\} \geq 2$ .

In this case,  $n-g-1 > 0$ . For  $\theta \in [0, \frac{\pi}{2}]$ , it is easily seen that

$$\text{Min}\{n(n-1) \cos^2 \theta + (n-g-1)(n-1) \sin^2 \theta\} = (n-g-1)(n-1),$$

thus by (2.6),

$$R_M \geq (n-g-1)(n-1) + a(r) \sin^2 \theta + 2kH(r) \sin \theta.$$

With the same assumption on  $k$  as in case (A),  $R_M$  has a positive lower bound.

Up to now, we changed only the metric near the minimal isoparametric hypersurface  $Y$  along the curve  $\gamma$  into a product metric while preserving the positive scalar curvature, as desired. In this way, gluing two copies of  $S_+^n$ , we get the double manifold of positive scalar curvature. More importantly, there is still an isoparametric foliation on  $D(S_+^n)$ , remaining the same with that in  $S^n(1)$  as  $r \geq r_1$ . In a neighborhood of  $Y$  with diameter  $2r_1$ , the principal curvatures turn out to be  $\mu_1 \cos \theta, \mu_2 \cos \theta, \dots, \mu_{n-1} \cos \theta$ .

The proof is now complete. □

### 3. Topology of the double manifold $D(S_+^n)$

First of all, we compute the cohomology groups.

**Proposition 3.1.** Let the ring of coefficient  $R = \mathbb{Z}$  if  $M_+$  and  $M_-$  are both orientable and  $R = \mathbb{Z}_2$  otherwise. Then

$$\begin{cases} H^0(D(S_+^n)) = R, \\ H^1(D(S_+^n)) = H^1(M_+), \\ H^q(D(S_+^n)) = H^{q-1}(M_-) \oplus H^q(M_+), \quad \text{for } 2 \leq q \leq n-2 \\ H^{n-1}(D(S_+^n)) = H^{n-2}(M_-), \\ H^n(D(S_+^n)) = R. \end{cases}$$

For  $D(S_-^n)$ , analogous identities hold. □

**Remark 3.1.** By Morse theory, we see that if  $m_+ > 1$  (resp.  $m_- > 1$ ), then  $M_-$  (resp.  $M_+$ ) is orientable. In fact, we define a spherical distance function on the focal submanifold  $M_-$ .

$$\begin{aligned} L_p : M_- &\longrightarrow \mathbb{R} \\ x &\mapsto \cos^{-1}\langle p, x \rangle, \end{aligned}$$

where  $p$  belongs to the complement of  $M_\pm$  in  $S^n$ . The Morse index theorem states that the index of  $L_p$  at a non-degenerate critical point  $x$  equals the number of focal points (counting multiplicities) of  $(M_-, x)$  on the shortest geodesic segment from  $p$  to  $x$ . Immediately, we obtain, for example when  $g = 4$ , the index of non-degenerate critical points are  $0, m_+, m_+ + m_-$  and  $2m_+ + m_-$ , respectively. Consequently, we have the cell decomposition

$M_- = S^{m_+} \cup e^{m_++m_-} \cup e^{2m_++m_-}$ . Thus if  $m_+ > 1$ ,  $M_-$  is simply connected. Similar results hold for other values of  $g$ .

In order to prove Proposition 3.1, we recall a topological theorem of Münzner (cf. [17]) stated as

**Theorem.** *Let  $N$  be a compact connected hypersurface in  $S^n$  such that:*

- (a)  $S^n$  is divided into two manifolds  $B_+$  and  $B_-$  with the same boundary  $N$ .
- (b)  $B_+$  (resp.  $B_-$ ) has the structure of a disc bundle over a compact manifold  $M_+$  (resp.  $M_-$ ) of dimension  $n - 1 - m_+$  (resp.  $n - 1 - m_-$ ).

Let the ring of coefficient  $R = \mathbb{Z}$  if  $M_+$  and  $M_-$  are both orientable and  $R = \mathbb{Z}_2$ , otherwise. Let  $\nu = m_+ + m_-$ . Then

$$H^q(M_{\pm}) = \begin{cases} R, & \text{for } q \equiv 0 \pmod{\nu}, \quad 0 \leq q < n - 1, \\ R, & \text{for } q \equiv m_{\mp} \pmod{\nu}, \quad 0 \leq q < n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Further,

$$H^q(N) = \begin{cases} R, & \text{for } q = 0, \quad n - 1, \\ H^q(M_+) \oplus H^q(M_-), & \text{for } 1 \leq q \leq n - 2. \end{cases}$$

□

To complete the proof of Proposition 3.1, we observe that a minimal isoparametric hypersurface  $Y$  in  $S^n$  satisfies the hypotheses of the previous theorem, getting the cohomology groups  $H^q(M_{\pm})$ , equivalently,  $H^q(S_{\pm}^n)$ . Finally, by the Mayer–Vietoris sequence of  $(D(S_+^n), S_+^n, S_+^n)$ , we arrive at the conclusion immediately. □

Next, we give a proof of

**Proposition 1.1.**  $D(S_+^n)$  (resp.  $D(S_-^n)$ ) is a  $\pi$ -manifold, i.e., a stably parallelizable manifold. In particular, it is an orientable, spin manifold with vanishing Stiefel–Whitney classes and Pontrjagin classes.

Suppose we are now given a (minimal) isoparametric hypersurface in  $S^n$ . As Münzner asserted (cf. [4] p. 283),  $S_+^n$  has the structure of a differential disc bundle over  $M_+$ . In fact, it is the normal disc bundle over  $M_+$ . More

precisely, we have

$$(3.1) \quad \begin{array}{c} B^{m+1} \hookrightarrow S_+^n = B(\nu_+) \\ \downarrow \pi \\ M_+ \end{array}$$

where  $\nu_+$  is the normal bundle over  $M_+$ ,  $B^{m+1}$  is the fiber disc.

Since  $S_+^n$  has a metric, we can define a homeomorphism as:

$$\begin{aligned} B_1^n \sqcup_{id} B_2^n &\longrightarrow S(\nu_+ \oplus \mathbf{1}) \\ e &\longmapsto \begin{cases} (e, \sqrt{1 - |e|^2}), & \text{for } e \in B_1^n, \\ (e, -\sqrt{1 - |e|^2}), & \text{for } e \in B_2^n, \end{cases} \end{aligned}$$

where  $B_1^n, B_2^n$  are two copies of  $S_+^n = B(\nu_+)$ ,  $S(\nu_+ \oplus \mathbf{1})$  is a sphere bundle of the Whitney sum  $\nu_+ \oplus \mathbf{1}$ , here  $\mathbf{1}$  is a trivial line bundle over  $M_+$ .

As a result, we get a new bundle

$$\begin{array}{c} S^{m+1} \hookrightarrow D(S_+^n) \\ \downarrow \rho \\ M_+ \end{array}$$

and a correspondence  $D(S_+^n) \cong S(\nu_+ \oplus \mathbf{1})$ . It follows immediately that

$$\begin{aligned} T(D(S_+^n)) \oplus \mathbf{1} &\cong T(S(\nu_+ \oplus \mathbf{1})) \oplus \mathbf{1} \\ &\cong \rho^* TM_+ \oplus \rho^*(\nu_+ \oplus \mathbf{1}) \\ &\cong \rho^* j_+^* T S^n \oplus \mathbf{1} \cong (\mathbf{n} + \mathbf{1}), \end{aligned}$$

where  $j_+ : M_+ \rightarrow S^n$  is an inclusion. In other words,  $D(S_+^n)$  is stably parallelizable, i.e., a  $\pi$ -manifold. This completes the proof.  $\square$

As indicated in Introduction, we will be mainly concerned with the minimal isoparametric hypersurface  $Y$  in the following two cases: homogeneous hypersurface and FKM-type.

### 3.1. Homogeneous hypersurface

Let  $Y$  be a homogeneous hypersurface in  $S^n$ , as Hsiang and Lawson [12] showed:  $Y$  can be characterized as a principal orbit of the isotropy representation of some rank two symmetric space  $U/K$ .

To begin with, we provide a brief description of the corresponding rank two symmetric spaces. Again, let  $g$  be the number of distinct principal curvatures of the homogeneous hypersurface  $Y$ . As mentioned before,  $g$  can only be 1, 2, 3, 4 or 6.

When  $g = 1$ ,  $Y$  is a hypersphere in  $S^n$ , the corresponding rank two symmetric space is

$$(S^1 \times SO(n+1))/SO(n) = S^1 \times S^n.$$

When  $g = 2$ ,  $Y$  is a Riemannian product of two spheres  $S^k(r) \times S^{n-k-1}(s)$  with  $r^2 + s^2 = 1$ ,  $1 \leq k \leq n-2$ , the corresponding rank two symmetric space is

$$(SO(k+2) \times SO(n-k+1))/(SO(k+1) \times SO(n-k)) = S^{k+1} \times S^{n-k}.$$

When  $g = 3$ ,  $Y$  is congruent to a tube of constant radius around the Veronese embedding of real projective plane  $\mathbb{R}P^2$  into  $S^4$ , or complex projective plane  $\mathbb{C}P^2$  into  $S^7$ , or quaternionic projective plane  $\mathbb{H}P^2$  into  $S^{13}$ , or Cayley projective plane  $\mathbb{O}P^2$  into  $S^{25}$ . The corresponding rank two symmetric spaces are

$$SU(3)/SO(3); \quad SU(3) \times SU(3)/SU(3); \quad SU(6)/Sp(3); \quad E_6/F_4.$$

When  $g = 4$ ,  $Y$  is a principal orbit of the isotropy representation of

$$SO(5) \times SO(5)/SO(5); \quad SO(10)/U(5); \quad E_6/T \cdot \text{Spin}(10);$$

or of two-plane Grassmannians

$$\begin{aligned} SO(k+2)/SO(k) \times SO(2) \quad (k \geq 3), \\ SU(k+2)/S(U(k) \times U(2)) \quad (k \geq 3), \\ Sp(k+2)/Sp(k) \times Sp(2) \quad (k \geq 2). \end{aligned}$$

When  $g = 6$ ,  $Y$  is a principal orbit of the isotropy representation of

$$G_2/SO(4) \quad \text{or} \quad G_2 \times G_2/G_2.$$

Now let  $G$  be a closed subgroup of the isometry group of  $S^n$  acting on  $S^n$  with cohomogeneity one. We equip the orbit space  $S^n/G$  with the quotient topology relative to the canonical projection  $S^n \rightarrow S^n/G$ . Since  $n > 1$ ,  $S^n$  is simply connected and compact, for topological reasons  $S^n/G$  must be

homeomorphic to  $[-1, 1]$  and each singular orbit has codimension greater than 1.

We denote the singular orbits corresponding to  $\pm 1$  by  $M_{\pm}$ , and their isotropy subgroups by  $K_{\pm}$ , respectively. Naturally, there is a diffeomorphism  $M_{\pm} \cong G/K_{\pm}$ . The other orbits are congruent to each other, and they are all principal orbits. It makes sense to fix an orbit  $Y$  corresponding to a certain value in  $(-1, 1)$  so that  $Y$  is minimal, and denote its isotropy subgroup by  $K_0$ . The existence of such  $Y$  is clear. Similarly,  $Y \cong G/K_0$ .

Based on the bundle structure of  $S_{\pm}^n$  over  $M_{\pm}$  and the following group action of  $K_{\pm}$ :

$$\begin{aligned} K_{\pm} \times (G \times B_{\pm}^{m_{\pm}+1}) &\longrightarrow G \times B_{\pm}^{m_{\pm}+1} \\ (k, g, x) &\longmapsto (gk^{-1}, k \bullet x) \end{aligned}$$

where  $\bullet$  is a slice representation (for details, see for example, [1]), we decompose  $S^n$  into

$$S^n = G \times_{K_+} B_+^{m_++1} \cup_Y G \times_{K_-} B_-^{m_-+1}.$$

Next, by gluing two copies of  $S_+^n$ , we define a new action of the isotropy group  $K_+$  on  $G \times S^{m_++1}$ :

$$\begin{aligned} K_+ \times (G \times S^{m_++1}) &\longrightarrow G \times S^{m_++1} \\ (k, g, (x, t)) &\longmapsto (gk^{-1}, k \star (x, t)) \end{aligned}$$

where  $k \star (x, t) := (k \bullet x, t)$ ,  $t = \pm\sqrt{1 - |x|^2}$ ,  $(x, t) \in S^{m_++1}$ .

Consequently, we have the diffeomorphism

$$D(S_+^n) \cong G \times_{K_+} B_+^{m_++1} \cup_Y G \times_{K_+} B_+^{m_++1} \cong G \times S^{m_++1} / K_+.$$

In conclusion,  $D(S_+^n)$  can be determined by  $K_+$  and its action on  $G \times S^{m_++1}$ . Similarly, we can also express  $D(S_-^n)$  in this way.

A series of delicate calculations lead us to a complete list of the isotropy subgroups  $K_0$  and  $K_{\pm}$  of homogeneous hypersurfaces and focal submanifolds as follows. To the best of our knowledge, the determinations of  $K_+$  and  $K_-$  have not previously appeared in the literature. The main difficulty occurred in the calculation of exceptional Lie groups.

In the following, we first illustrate the calculations of  $K_0, K_+$  and  $K_-$  in the case of the symmetric pair  $(E_6, T \cdot \text{Spin}(10))$  with  $(g, m_+, m_-) = (4, 6, 9)$ , and then give an example of the case  $(SU(3), SO(3))$  with  $(g, m_+, m_-) = (3, 1, 1)$ .

Homogeneous (isoparametric) hypersurfaces in the unit sphere.

$g$	$(m_+, m_-)$	$(U, K)$	$K_0$	$K_+$	$K_-$
1	$n-1$	$(S^1 \times SO(n+1), SO(n))$	$SO(n-1)$	$SO(n)$	$SO(n)$
2	$(p, q)$	$n \geq 2$ $(SO(p+2) \times SO(q+2), SO(p+1) \times SO(q+1))$	$SO(p) \times SO(q)$	$SO(p+1) \times SO(q)$	$SO(p) \times SO(q+1)$
3	$(1, 1)$	$p, q \geq 1$ $(SU(3), SO(3))$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$S(O(2) \times O(1))$	$S(O(1) \times O(2))$
3	$(2, 2)$	$(SU(3) \times SU(3), SU(3))$	$T^2$	$S(U(2) \times U(1))$	$S(U(1) \times U(2))$
3	$(4, 4)$	$(SU(6), Sp(3))$	$Sp(1)^3$	$Sp(2) \times Sp(1)$	$Sp(2) \times Sp(1)$
3	$(8, 8)$	$(E_6, F_4)$	$Spin(8)$	$Spin(9)$	$Spin(9)$
4	$(2, 2)$	$(SO(5) \times SO(5), SO(5))$	$T^2$	$SO(2) \times SO(3)$	$U(2)$
4	$(4, 5)$	$(SO(10), U(5))$	$SU(2)^2 \times U(1)$	$Sp(2) \times U(1)$	$SU(2) \times U(3)$
4	$(6, 9)$	$(E_6, T \cdot Spin(10))$	$U(1) \cdot Spin(6)$	$U(1) \cdot Spin(7)$	$S^1 \cdot SU(5)$
4	$(1, m-2)$	$(SO(m+2), SO(m) \times SO(2))$	$SO(m-2) \times \mathbb{Z}_2$	$SO(m-2) \times SO(2)$	$O(m-1)$
4	$(2, 2m-3)$	$m \geq 3$ $(SU(m+2), S(U(m) \times U(2)))$	$S(U(m-2) \times T^2)$	$S(U(m-2) \times U(2))$	$S(U(m-1) \times T^2)$
4	$(4, 4m-5)$	$m \geq 3$ $(Sp(m+2), Sp(m) \times Sp(2))$	$Sp(m-2) \times Sp(1)^2$	$Sp(m-2) \times Sp(2)$	$Sp(m-1) \times Sp(1)^2$
6	$(1, 1)$	$m \geq 2$ $(G_2, SO(4))$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$O(2)$	$O(2)$
6	$(2, 2)$	$(G_2 \times G_2, G_2)$	$T^2$	$U(2)$	$U(2)$

**Example 3.1.** The calculation of  $K_+, K_-$  of the symmetric pair  $(E_6, T \cdot \text{Spin}(10))$  with  $(g, m_+, m_-) = (4, 6, 9)$ .

At the beginning, we introduce some notations and operations on the division Cayley algebra  $\mathbb{O}$ , which is generated by  $\{e_0 = 1, e_1, \dots, e_7\}$  and satisfies

- 1) for  $i > 0, e_i^2 = -1$ ;
- 2) for  $i, j > 0, i \neq j, e_i e_j = -e_j e_i$ ;
- 3)  $e_1 e_2 = e_4$ ;
- 4) if  $e_i e_j = e_k$  for some  $i, j, k > 0$ , then  $e_{i+1} e_{j+1} = e_{k+1}$  and  $e_{2i} e_{2j} = e_{2k}$  (subscripts mod 7).

Let  $M_3(\mathbb{O})$  be the set of  $3 \times 3$  matrices with entries in  $\mathbb{O}$ ,  $\mathcal{H}_3$  the set of Hermitian matrices in  $M_3(\mathbb{O})$ , namely,

$$\mathcal{H}_3 = \{X \in M_3(\mathbb{O}) \mid {}^t \bar{X} = X\},$$

where the conjugate of any element  $x = \sum_{i=0}^7 x_i e_i \in \mathbb{O}$  is defined by

$$\bar{x} = x_0 e_0 - \sum_{i=1}^7 x_i e_i.$$

In the following, we will always denote an element  $X \in \mathcal{H}_3$  of the form

$$X = X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \text{for } \xi_i \in \mathbb{R}, x_i \in \mathbb{O}$$

by

$$X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(x_1) + F_2(x_2) + F_3(x_3).$$

The Jordan product, as a basic operation in  $\mathcal{H}_3$ , is a multiplication defined by

$$X \circ Y = \frac{1}{2}(XY + YX), \quad \text{for } X, Y \in \mathcal{H}_3.$$

Usually,  $(\mathcal{H}_3, \circ)$  is called the exceptional Jordan algebra. Moreover, the trace  $\text{Tr}(X)$ , the inner product  $(X, Y)$  and the determinant  $\det X$  can be defined



respectively by

$$\begin{aligned} \text{Tr}(X) &= \xi_1 + \xi_2 + \xi_3, \quad \text{for } X = X(\xi, x), \\ (X, Y) &= \text{Tr}(X \circ Y), \\ \det X &= \xi_1 \xi_2 \xi_3 + \text{Re}(x_1 x_2 x_3) - \xi_1 |x_1|^2 - \xi_2 |x_2|^2 - \xi_3 |x_3|^2. \end{aligned}$$

Let  $\mathcal{H}_3^{\mathbb{C}} = \{X_1 + \sqrt{-1}X_2 \mid X_1, X_2 \in \mathcal{H}_3\}$  be the complexification of Jordan algebra  $\mathcal{H}_3$ . In the same manner, we have the Jordan product, the trace, the  $\mathbb{C}$ -linear form  $(, )$  and the determinant in  $\mathcal{H}_3^{\mathbb{C}}$ . A Hermitian inner product  $\langle , \rangle$  on  $\mathcal{H}_3^{\mathbb{C}}$  is given by

$$\langle X, Y \rangle = (X, \tau Y), \quad \text{for } X, Y \in \mathcal{H}_3^{\mathbb{C}},$$

where  $\tau$  is the complex conjugate of  $\mathcal{H}_3^{\mathbb{C}}$ .

With all these notations, an equivalent definition of the group  $E_6$  can be given by (cf. [25])

$$E_6 = \{\alpha \in GL(\mathcal{H}_3^{\mathbb{C}}, \mathbb{C}) \mid \det(\alpha X) = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}.$$

Set

$$\mathcal{SH}_3 = \{A \in M_3(\mathbb{O}) \mid {}^t \bar{A} = -A, \text{Tr}(A) = 0\}.$$

As above, we denote an element  $A \in \mathcal{SH}_3$  of the form

$$A = \begin{pmatrix} a_1 & x_3 & -\bar{x}_2 \\ -\bar{x}_3 & a_2 & x_1 \\ x_2 & -\bar{x}_1 & a_3 \end{pmatrix}, \quad a_i, x_i \in \mathbb{O}, \quad \bar{a}_i = -a_i, \quad a_1 + a_2 + a_3 = 0.$$

by

$$A = a_1 E_1 + a_2 E_2 + a_3 E_3 + A_1(x_1) + A_2(x_2) + A_3(x_3).$$

Notice that  $[\mathcal{SH}_3, \mathcal{SH}_3] = \mathcal{SH}_3$ ,  $[\mathcal{SH}_3, \mathcal{H}_3] = \mathcal{H}_3$ . Thus any  $A \in \mathcal{SH}_3$  induces a map  $\tilde{A} : \mathcal{H}_3 \rightarrow \mathcal{H}_3$  expressed as

$$\tilde{A}(X) = \frac{1}{2}[A, X].$$

Let  $\mathfrak{t}'$  be the subalgebra of  $gl(\mathcal{H}_3)$  generated by  $\{\tilde{A} \mid A \in \mathcal{SH}_3\}$ . Then  $\mathfrak{t}'$  is isomorphic to the (compact) Lie algebra of  $F_4$ .

Furthermore, observing that any  $X \in \mathcal{H}_3$  also induces a map  $\tilde{X} : \mathcal{H}_3 \rightarrow \mathcal{H}_3$  defined by

$$\tilde{X}(Y) = X \circ Y, \quad Y \in \mathcal{H}_3,$$

we set  $\mathfrak{p}' = \{\tilde{X} | X \in \mathcal{H}_3, \text{Tr}(X) = 0\}$ , then the Lie algebra  $\mathfrak{t}' + \sqrt{-1}\mathfrak{p}'$  is just the (compact) Lie algebra of  $E_6$ , denoted by  $\mathfrak{e}_6$ . In the following discussions, we will omit the symbol “ $\sim$ ” for simplicity.

Let  $\mathfrak{d}_4$  be the subalgebra of  $\mathfrak{t}'$  generated by  $\{\Sigma a_i E_i | a_i \in \mathbb{O}, \bar{a}_i = -a_i, \Sigma a_i = 0\}$ . Then for any  $D \in \mathfrak{d}_4, X = X(\xi, x)$ ,

$$D \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3 x_3 & \overline{D_2 x_2} \\ \overline{D_3 x_3} & 0 & D_1 x_1 \\ D_2 x_2 & \overline{D_1 x_1} & 0 \end{pmatrix},$$

where  $D_1, D_2, D_3$  are elements of Lie algebra  $so(8)$  and satisfy the principle of triality:

$$(D_1 x)y + x(D_2 y) = \overline{D_3(\bar{x}y)}, \quad x, y \in \mathbb{O},$$

which implies that  $D_2, D_3$  are uniquely determined by  $D_1$ . Hence the map defined by  $D \mapsto D_1$  is an isomorphism from  $\mathfrak{d}_4$  to  $so(8)$ .

Furthermore, setting

$$\begin{aligned} \mathfrak{D}_i &= \{A_i(x) \mid x \in \mathbb{O}\}, \quad i = 1, 2, 3, \\ \mathfrak{R}_i &= \{F_i(x) \mid x \in \mathbb{O}\}, \quad i = 1, 2, 3, \\ \mathfrak{R}_0 &= \{\Sigma \xi_i E_i \mid \xi_i \in \mathbb{R}, \Sigma \xi_i = 0\}, \end{aligned}$$

we can decompose the Lie algebra  $\mathfrak{e}_6$  as

$$\mathfrak{e}_6 = \mathfrak{d}_4 + \mathfrak{D}_1 + \mathfrak{D}_2 + \mathfrak{D}_3 + \sqrt{-1}\mathfrak{R}_0 + \sqrt{-1}\mathfrak{R}_1 + \sqrt{-1}\mathfrak{R}_2 + \sqrt{-1}\mathfrak{R}_3.$$

Since there is a transformation  $\sigma$  of  $\mathcal{H}_3^{\mathbb{C}}$  expressed as

$$\sigma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

(obviously,  $\sigma^2 = id$ ), an involution  $\gamma$  of  $E_6$  can be naturally defined by  $\gamma(\alpha) = \sigma\alpha\sigma$ , for  $\alpha \in E_6$ . Thus the decomposition of  $\mathfrak{e}_6$  corresponding to  $\gamma$

can be written as  $\mathfrak{e}_6 = \mathfrak{t} + \mathfrak{p}$ , where

$$\begin{aligned} \mathfrak{t} &= \{\delta \in \mathfrak{e}_6 \mid \sigma\delta = \delta\sigma\} = \mathfrak{d}_4 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{A}_0 + \sqrt{-1}\mathfrak{A}_1, \\ \mathfrak{p} &= \{\delta \in \mathfrak{e}_6 \mid \sigma\delta = -\delta\sigma\} = \mathfrak{D}_2 + \mathfrak{D}_3 + \sqrt{-1}\mathfrak{A}_2 + \sqrt{-1}\mathfrak{A}_3. \end{aligned}$$

Choosing a maximal Abelian subspace of  $\mathfrak{p}$  as  $\mathfrak{h} = \{A_2(\lambda_1 e_0) + \sqrt{-1}F_2(\lambda_2 e_1) \mid \lambda_1, \lambda_2 \in \mathbb{R}\}$ , and denoting by  $\Delta$  the set of restricted positive roots with respect to  $\mathfrak{h}$ , we have

$$(3.2) \quad \mathfrak{e}_6 = \mathfrak{m} + \mathfrak{h} + \sum_{\lambda \in \Delta} \{\mathfrak{t}_\lambda + \mathfrak{p}_\lambda\},$$

where

$$\begin{aligned} \mathfrak{m} &= \{A \in \mathfrak{t} \mid [A, H] = 0, \text{ for } H \in \mathfrak{h}\}, \\ \mathfrak{t}_\lambda &= \{A \in \mathfrak{t} \mid \text{ad}(H)^2 A = -\lambda(H)^2 A, \text{ for } H \in \mathfrak{h}\}, \\ \mathfrak{p}_\lambda &= \{A \in \mathfrak{p} \mid \text{ad}(H)^2 A = -\lambda(H)^2 A, \text{ for } H \in \mathfrak{h}\}. \end{aligned}$$

Set  $\tilde{e}_i = e_1 e_i$ , for  $i > 1$  and  $G_{ij} = E_{ij} - E_{ji}$ , for  $i, j = 0, 1, \dots, 7$ , where  $E_{ij}$  is the matrix with  $(i, j)$  entry 1 and all others 0. By a direct computation, we can express  $\mathfrak{m}$  explicitly as

$$\mathfrak{m} = \text{span}\{\sqrt{-1}(E_1 - 2E_2 + E_3), D \mid D_2 = G_{ij}, i, j > 1\} \cong so(6) \oplus \mathbb{R}.$$

Moreover, we calculate  $\mathfrak{t}_\lambda$  and  $\mathfrak{p}_\lambda$  in (3.2) with respect to the root system  $\Delta$ , and list them in the following table.

Let  $K = \{\alpha \in E_6 \mid \alpha\sigma = \sigma\alpha\}$ , which acts on  $\mathfrak{p}$  by the adjoint representation. The orbits can only be of the following three types:

1. If  $H_0 \in \mathfrak{h}$  with  $\lambda_1(H_0) \cdot \lambda_2(H_0) \neq 0$  and  $\lambda_1(H_0) \neq \pm\lambda_2(H_0)$ , the Lie algebra of the isotropy subgroup  $K_0$  at  $H_0$  is  $\mathfrak{m}$  and  $K_0 \cong U(1) \cdot \text{Spin}(6)$ .

2. If  $H_+ \in \mathfrak{h}$  with either  $\lambda_1(H_+) = 0$  or  $\lambda_2(H_+) = 0$ . Without loss of generality, assume  $\lambda_2(H_+) = 0$ . According to the previous table, the Lie algebra of the isotropy subgroup  $K_+$  is  $\mathfrak{m} \oplus \mathfrak{k}_{\lambda_2} \cong so(7) \oplus \mathbb{R}$ . Then it is not difficult to see that

$$K_+ \cong U(1) \cdot \text{Spin}(7).$$

3. If  $H_- \in \mathfrak{h}$  with either  $\lambda_1(H_-) = \lambda_2(H_-)$  or  $\lambda_1(H_-) = -\lambda_2(H_-)$ . Without loss of generality, assume  $\lambda_1(H_-) = \lambda_2(H_-)$ . According to the previous table, the Lie algebra  $\mathfrak{t}_-$  of the isotropy subgroup  $K_-$  is given by

$$\mathfrak{t}_- = \mathfrak{m} + \mathfrak{t}_\mu + \mathfrak{t}_{\mu/2}, \quad \text{for } \mu = \lambda_1 - \lambda_2.$$

Dimension	$\lambda \in \Delta$	Basis of $\mathfrak{t}_\lambda$	Basis of $\mathfrak{p}_\lambda$
6	$\lambda_1$	$D \in \mathfrak{d}_4 : D_2 = G_{0i},$ $i > 1$	$A_2(e_i) : i > 1$
6	$\lambda_2$	$D \in \mathfrak{d}_4 : D_2 = G_{1i},$ $i > 1$	$\sqrt{-1}F_2(e_i) : i > 1$
1	$\lambda_1 - \lambda_2$	$D + \sqrt{-1}(E_1 - E_3) :$ $D_2 = G_{01}$	$A_2(e_1) + \sqrt{-1}F_2(e_0)$
1	$\lambda_1 + \lambda_2$	$D - \sqrt{-1}(E_1 - E_3) :$ $D_2 = G_{01}$	$A_2(e_1) - \sqrt{-1}F_2(e_0)$
8	$\frac{1}{2}(\lambda_1 - \lambda_2)$	$A_1(e_0) - \sqrt{-1}F_1(e_1)$	$A_3(e_0) - \sqrt{-1}F_3(e_1)$
		$A_1(e_1) + \sqrt{-1}F_1(e_0)$	$A_3(e_1) + \sqrt{-1}F_3(e_0)$
8	$\frac{1}{2}(\lambda_1 + \lambda_2)$	$A_1(e_i) + \sqrt{-1}F_1(\tilde{e}_i) :$ $i > 1$	$A_3(e_i) - \sqrt{-1}F_3(\tilde{e}_i) :$ $i > 1$
		$A_1(e_0) + \sqrt{-1}F_1(e_1)$	$A_3(e_0) + \sqrt{-1}F_3(e_1)$
8	$\frac{1}{2}(\lambda_1 + \lambda_2)$	$A_1(e_1) - \sqrt{-1}F_1(e_0)$	$A_3(e_1) - \sqrt{-1}F_3(e_0)$
		$A_1(e_i) - \sqrt{-1}F_1(\tilde{e}_i) :$ $i > 1$	$A_3(e_i) + \sqrt{-1}F_3(\tilde{e}_i) :$ $i > 1$

Let  $V^{10}$  be a ten-dimensional vector space defined by

$$V^{10} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau(\xi) \end{pmatrix} \mid \xi \in \mathbb{C}, x \in \mathbb{O} \right\} \subset \mathcal{H}_3^{\mathbb{C}}.$$

It is well known that  $K \cong T^1 \cdot \text{Spin}(10)$ , and the representation  $\phi : \text{Spin}(10) \rightarrow SO(V^{10})$  is just the vector representation.

We finally introduce a complex structure  $J$  on  $V^{10}$  as follows:

$$J \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau(\xi) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{-1}\xi & x \cdot e_1 \\ 0 & \bar{x} \cdot e_1 & -\sqrt{-1}\tau(\xi) \end{pmatrix}, \text{ for } \xi \in \mathbb{C}, x \in \mathbb{O}.$$

By a direct computation, we find that elements of the subalgebra  $[\mathfrak{t}_-, \mathfrak{t}_-] \subset so(10)$  commute with the complex structure  $J$ , specifically,  $[\mathfrak{t}_-, \mathfrak{t}_-] \cong su(5)$ . Moreover, the center  $\mathfrak{c}(\mathfrak{t}_-) \cong \mathbb{R}$  is not contained in  $so(10)$ . Therefore, we can conclude

$$\mathfrak{t}_- = \mathfrak{c}(\mathfrak{t}_-) \oplus [\mathfrak{t}_-, \mathfrak{t}_-] \cong \mathbb{R} \oplus su(5),$$

and via the representation  $\phi$  the corresponding isotropy subgroup is

$$K_- \cong S^1 \cdot SU(5),$$

where  $S^1$  is a group generated by the center  $\mathfrak{c}(\mathfrak{t}_-)$ .

**Example 3.2. An explicit description of  $D(S^4_+)$  with  $(g, m_+, m_-) = (3, 1, 1)$ .**

Firstly, recall a result of Cartan that the isoparametric hypersurface in this case must be a tube of constant radius over a standard Veronese embedding of  $\mathbb{R}P^2$  into  $S^4$ .

Let  $\nu$  be the normal bundle of  $\mathbb{R}P^2 \hookrightarrow S^4$ , which is non-orientable since  $T\mathbb{R}P^2 \oplus \nu = \mathbf{4}$ , a four-dimensional trivial bundle. Let  $\eta$  be the Hopf line bundle over  $\mathbb{R}P^2$ . It is well known that  $T\mathbb{R}P^2 \oplus \mathbf{1} = 3\eta$ . Thus  $3\eta \oplus \nu = T\mathbb{R}P^2 \oplus \mathbf{1} \oplus \nu = \mathbf{5}$ . Hence  $4\eta \oplus \nu = \mathbf{5} \oplus \eta$ .

**Assertion 1.**  $4\eta \cong \mathbf{4}$ .

It follows at once that  $\nu \oplus \mathbf{4} = \eta \oplus \mathbf{5}$ . Then we deduce by obstruction theory that  $\nu \oplus \mathbf{1} = \eta \oplus \mathbf{2}$ , and thus  $D(S^4_+) \cong S(\nu \oplus \mathbf{1}) \cong S(\eta \oplus \mathbf{2})$ . Furthermore, we show

**Assertion 2.**  $D(S^4_+) \cong S^2 \times S^2 / \sigma$ , where  $\sigma$  is an involution.

*Proof of Assertion 2.* Again, let  $\eta$  be a Hopf line bundle over  $\mathbb{R}P^n$ ,  $E(\eta)$  be the total space of  $\eta$ , then

$$(3.3) \quad \begin{aligned} E(\eta) &\cong S^n \times \mathbb{R} / (x, t) \sim (-x, -t), \\ &\downarrow \\ S^n / x &\sim -x = \mathbb{R}P^n, \end{aligned}$$

where  $x \in S^n$ ,  $t \in \mathbb{R}$ . This interpretation deduces that for  $x \in S^n$ ,  $(t_1, \dots, t_p) \in \mathbb{R}^p$  and  $(s_1, \dots, s_q) \in \mathbb{R}^q$ ,

$$(3.4) \quad \begin{aligned} &S^n \times \mathbb{R}^{p+q} / (x, t_1, \dots, t_p, s_1, \dots, s_q) \sim (-x, t_1, \dots, t_p, -s_1, \dots, -s_q) \\ &\cong E(\mathfrak{p} \oplus q\eta) \end{aligned}$$

In particular,

$$D(S^4_+) \cong S(\eta \oplus \mathbf{2}) \cong S^2 \times S^2 / (x, y_1, y_2, y_3) \sim (-x, -y_1, y_2, y_3).$$

where  $x \in S^2, (y_1, y_2, y_3) \in S^2$ . □

Assertion 1 should be well known. However, we would like to give an interesting and simple proof.

*Proof of Assertion 1.* By (3.4), it suffices to define a point-wise isomorphism  $\Phi$

$$(3.5) \quad \begin{array}{ccc} S^2 \times \mathbb{R}^4 / (x, t) \sim (-x, -t) & \xrightarrow{\Phi} & S^2 / \mathbb{Z}_2 \times \mathbb{R}^4 \\ \downarrow & & \downarrow \\ \mathbb{R}P^2 & \xrightarrow{id} & \mathbb{R}P^2 \end{array}$$

where  $\mathbb{R}^4$  is identified with the quaternions  $\mathbb{H}$ , and  $x \in S^2 = \{x \in \mathbb{H} \mid |x| = 1, Re\ x = 0\}, t \in \mathbb{H}$ .

Define  $\Phi(x, t) := (x, xt)$ . Obviously,  $\Phi$  is well defined, and for a fixed  $x$ , it is a linear isomorphism. □

It is worth remarking that  $D(S^4_+)$  is not diffeomorphic to the oriented Grassmannian

$$G_2(\mathbb{R}^4) \cong S^2 \times S^2 / (x, y) \sim (-x, -y),$$

where  $x, y \in S^2$ . To show this remark, firstly, recall that (cf. [21])

$$H^*(G_2(\mathbb{R}^4); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_1, a_2] / a_1^3 = 0, a_1^2 a_2 + a_2^2 = 0,$$

where  $a_1 \in H^1(G_2(\mathbb{R}^4); \mathbb{Z}_2), a_2 \in H^2(G_2(\mathbb{R}^4); \mathbb{Z}_2)$ . By this cohomology ring structure, it is not difficult to conclude the total Stiefel–Whitney class

$$W(G_2(\mathbb{R}^4)) = 1 + w_1^2.$$

This means that the first Stiefel–Whitney class vanishes, and the second Stiefel–Whitney class  $w_2(G_2(\mathbb{R}^4)) \neq 0$ . In other words,  $G_2(\mathbb{R}^4)$  is an orientable manifold, but not spin, while  $D(S^4_+)$  is spin as mentioned in Proposition 1.1.

### 3.2. FKM-type

In this subsection, we investigate the FKM-type isoparametric hypersurfaces in spheres with four distinct principal curvatures (cf. [2, 5]).

According to Ferus *et al.* [5], for a symmetric Clifford system  $\{P_0, \dots, P_m\}$  on  $\mathbb{R}^{2l}$ , i.e.,  $P_i$ 's are symmetric matrices satisfying  $P_i P_j + P_j P_i = 2\delta_{ij} I_{2l}$ , there is a homogeneous polynomial on  $\mathbb{R}^{2l}$  defined by

$$(3.6) \quad F : \mathbb{R}^{2l} \rightarrow \mathbb{R}$$

$$F(z) = |z|^4 - 2 \sum_{i=0}^m \langle P_i z, z \rangle^2.$$

It can be shown that if  $l - m - 1 > 0$ , then the level sets of the restriction  $f = F|_{S^{2l-1}}$  constitute a family of isoparametric hypersurfaces with  $g = 4$  distinct principal curvatures with multiplicities  $m_+ = m$ ,  $m_- = l - m - 1$ . The focal submanifolds are  $M_+ = f^{-1}(1)$ ,  $M_- = f^{-1}(-1)$ , with codimensions  $m + 1$  and  $l - m$  in  $S^{2l-1}$ , respectively.

Clearly, the  $+1$  eigenspace of  $P_0$ , say  $E_+(P_0)$ , is invariant under the transformations  $E_1 = P_1 P_2, \dots, E_{m-1} = P_1 P_m$ . As usual, let  $\delta(m)$  be the dimension of the irreducible Clifford algebra  $\mathcal{C}_{m-1}$ -modules (e.g.,  $\delta(4) = 4$ ,  $\delta(8) = 8$ ,  $\delta(m + 8) = 16\delta(m)$ ). Then  $l = k\delta(m)$ , for some positive integer  $k$ . As is known in representation theory, when  $m \equiv 0 \pmod{4}$ , there exist exactly two irreducible  $\mathcal{C}_{m-1}$ -modules  $\Delta_m^+$  and  $\Delta_m^-$  distinguished by  $E_1 E_2 \cdots E_{m-1} = Id$  or  $-Id$ . If we write  $E_+(P_0) = a\Delta_m^+ \oplus b\Delta_m^-$  as  $\mathcal{C}_{m-1}$ -modules, then

$$\text{tr}(P_0 P_1 \cdots P_m) = 2q\delta(m),$$

where  $q = a - b$ . On the other hand, noticing  $k = a + b$ , we see

$$(3.7) \quad q \equiv k \pmod{2}.$$

By [24], two symmetric Clifford systems with the same index  $q$  give rise to equivalent isoparametric functions on  $S^{2l-1}$ . Therefore, when  $m \equiv 0 \pmod{4}$ ,  $f$  is determined by  $m$ ,  $l = k\delta(m)$ , as well as  $q$  up to a rigid motion of  $S^{2l-1}$ . However, when  $m \not\equiv 0 \pmod{4}$ ,  $q$  is always zero, so  $f$  is determined only by  $m$  and  $l$  up to a rigid motion of  $S^{2l-1}$ .

In the rest of this subsection, we focus on the case when  $m \equiv 0 \pmod{4}$ .

First, we denote the corresponding isoparametric hypersurface by  $M(m, l, q)$ , and focal submanifolds by  $M_{\pm}(m, l, q)$ . Next, recall the following conclusions shown in [5]:

(a) The normal bundle  $\nu_+$  is trivial, in particular,  $M(m, l, q)$  is diffeomorphic to the product  $M_+(m, l, q) \times S^m$ ;

(b)  $M_-(m, l, q)$  is diffeomorphic to an  $S^{l-1}$  bundle over  $S^m$ .

Therefore, the property (a), together with the proof of Proposition 1.1, implies that  $D(S_+^{2l-1}) \cong D(B(\nu_+)) \cong S(\nu_+ \oplus \mathbf{1}) \cong M_+ \times S^{m+1}$ . In the remaining part, we will be concerned with the topology of  $D(S_-^{2l-1})$ . On this account, we prove Theorem 1.2 in two steps.

*Proof. Step 1.* As stated before,  $D(S_-^{2l-1}) \cong D(B(\nu_-)) \cong S(\nu_- \oplus \mathbf{1})$  is the total space of a sphere bundle. Set  $\zeta := \nu_- \oplus \mathbf{1}$ , then  $D(S_-^{2l-1}) = S(\zeta)$ , where  $\zeta$  is a vector bundle over  $M_-$  with disc bundle  $B(\zeta)$  and sphere bundle  $S(\zeta)$ , respectively.

$$(3.8) \quad \begin{array}{c} \zeta : \mathbb{R}^{l-m+1} \hookrightarrow E(\zeta) \supset B(\zeta) \supset S(\zeta) = D(B(\nu_-)) \\ \downarrow \pi \\ M_-^{m+l-1} \end{array}$$

As mentioned in (b) above,  $M_-(m, l, q)$  is diffeomorphic to an  $S^{l-1}$  bundle over  $S^m$ , that is:  $M_-(m, l, q) \cong S(\xi)$ , where  $\xi$  is a vector bundle over  $S^m$  so that

$$(3.9) \quad \begin{array}{c} S^{l-1} \hookrightarrow M_- = S(\xi) \\ \downarrow \pi_1 \\ S^m \end{array}$$

**Lemma 3.1.** The Pontrjagin class

$$p_{\frac{m}{4}}(\zeta) = -\pi_1^* p_{\frac{m}{4}}(\xi) = -q \cdot \beta(m) \cdot \left(\frac{m}{2} - 1\right)! \cdot \pi_1^* \gamma,$$

where  $\beta(m) = \begin{cases} 1, & m \equiv 0 \pmod{8}, \\ 2, & m \equiv 4 \pmod{8}, \end{cases}$   $\gamma \in H^m(S^m; \mathbb{Z})$  is a suitable generator.

*Proof.* Denoting the total Pontrjagin class of an arbitrary vector bundle  $\eta$  by  $P(\eta) = 1 + p_1(\eta) + p_2(\eta) + \dots$ , from [16], we know that  $P(\eta \oplus \mathbf{1}) = P(\eta)$ .

Firstly, since  $\nu_- \oplus TM_- \cong TS^{2l-1}|_{M_-}$  is stably parallelizable (in fact it is trivial for the dimension reason), we have  $P(\nu_- \oplus TM_-) = 1$ . On the other hand, since  $m \geq 4$ ,  $l = k\delta(m) \equiv 0 \pmod{4}$ , thus  $l - 1$  cannot be divided by 4. By reason of rank of the sphere bundle  $S(\xi)$ , we can deduce that  $P(M_-) := P(TM_-) = 1 + p_{\frac{m}{4}}(M_-)$ .



Consequently, since in this case the cohomology of  $M_-$  with coefficients  $\mathbb{Z}$  has no torsion (Proposition 3.1), it follows from  $P(\zeta) = P(\nu_- \oplus \mathbf{1}) = P(\nu_-)$  that

$$(3.10) \quad p_{\frac{m}{4}}(\zeta) = -p_{\frac{m}{4}}(M_-).$$

Next, for the tangent bundle of  $M_-$ , we have  $TM_- \oplus \mathbf{1} \cong \pi_1^*TS^m \oplus \pi_1^*\xi$ . As a direct result,  $TM_- \oplus \mathbf{2} = (\mathbf{m} + \mathbf{1}) \oplus \pi_1^*\xi$ , which implies that  $p_{\frac{m}{4}}(M_-) = p_{\frac{m}{4}}(\pi_1^*\xi)$ .

Since  $l - 1 > m$ , there exists a section of the sphere bundle  $S(\xi)$ , thus its Euler class  $e$  vanishes. Hence, from the Gysin cohomology sequence with coefficients  $\mathbb{Z}$  associated with  $S(\xi)$ :

$$(3.11) \quad \rightarrow H^i(S^m) \xrightarrow{e} H^{i+l}(S^m) \xrightarrow{\pi_1^*} H^{i+l}(M_-) \rightarrow H^{i+1}(S^m) \rightarrow \dots,$$

we deduce that

$$(3.12) \quad \pi_1^* : H^m(S^m; \mathbb{Z}) \longrightarrow H^m(M_-; \mathbb{Z}) \quad \text{is an isomorphism.}$$

Thus by (3.10)

$$(3.13) \quad p_{\frac{m}{4}}(\zeta) = -\pi_1^*p_{\frac{m}{4}}(\xi) \quad \text{under the isomorphism } \pi_1^*.$$

At the mean time,  $\xi - \text{rank } \xi \in \widetilde{KO}(S^m)$ , which will be abbreviated as  $\xi$ . Let us consider the complexification homomorphism

$$(3.14) \quad \begin{aligned} \mathbb{C} : \widetilde{KO}(S^m) &\rightarrow \widetilde{K}(S^m) \\ \xi &\mapsto \xi \otimes \mathbb{C} \\ 1 &\mapsto \begin{cases} 1, & m \equiv 0 \pmod{8}, \\ 2, & m \equiv 4 \pmod{8}. \end{cases} \end{aligned}$$

Recall a well-known result that the Chern character  $Ch : \widetilde{K}(S^m) \rightarrow H^m(S^m; \mathbb{Z})$  is an isomorphism for even  $m$ , namely, the top Chern class of the generator of  $\widetilde{K}(S^m)$  is equal to a generator of  $H^m(S^m; \mathbb{Z})$  multiplied by  $(\frac{m}{2} - 1)!$ . By the isomorphisms  $\xi \cong a\Delta^+ + b\Delta^- \cong (a - b)\Delta^+ + b(\Delta^+ + \Delta^-) \cong q\Delta^+ + \mathbf{b}$ , (since  $\Delta^+ + \Delta^-$  is trivial) where  $\Delta^+ \in \widetilde{KO}(S^m)$  is a generator, we finally arrive at

$$(3.15) \quad p_{\frac{m}{4}}(\xi) = q \cdot \beta(m) \cdot \left(\frac{m}{2} - 1\right)! \cdot \gamma \in H^m(S^m; \mathbb{Z}) \cong \mathbb{Z},$$

where  $\gamma \in H^m(S^m; \mathbb{Z})$  is a suitable generator, and

$$\beta(m) = \begin{cases} 1, & m \equiv 0 \pmod{8}, \\ 2, & m \equiv 4 \pmod{8}. \end{cases}$$

In summary,  $p_{\frac{m}{4}}^m(\zeta) = -\pi_1^* p_{\frac{m}{4}}^m(\xi) = -q \beta(m) (\frac{m}{2} - 1)! \pi_1^* \gamma$ . □

**Step 2.** We mainly make use of Wu Square  $\mathcal{P}^1$  (cf. [16]).

Let  $p := 2r + 1 \geq 3$  be an odd prime, and  $m = 2(p - 1) = 4r \equiv 0 \pmod{4}$ . We will need the following fundamental criterion from number theory.

**Wilson’s Theorem.** *p is a prime, if and only if  $(p - 1)! \equiv -1 \pmod{p}$ .*

We recall the Wu Squares with coefficients  $\mathbb{Z}_p$ , which are generalized from Steenrod Squares with coefficients  $\mathbb{Z}_2$  by Wu Wen–Tsün:

$$\mathcal{P}^i : H^j(X; \mathbb{Z}_p) \rightarrow H^{j+4ri}(X; \mathbb{Z}_p).$$

For  $(B(\zeta), S(\zeta)) \supset B(\zeta) \supset S(\zeta)$ , one has the following commutative diagram of the cohomology sequences with coefficients  $\mathbb{Z}_p$ :

$$(3.16) \quad \begin{array}{ccccccc} H^j(B(\zeta), S(\zeta)) & \rightarrow & H^j(B(\zeta)) & \rightarrow & H^j(S(\zeta)) & \xrightarrow{\delta} & H^{j+1}(B(\zeta), S(\zeta)) \rightarrow \dots \\ \downarrow \mathcal{P}^i & & \downarrow \mathcal{P}^i & & \downarrow \mathcal{P}^i & & \downarrow \mathcal{P}^i \\ H^{j+4ri}(B(\zeta), S(\zeta)) & \rightarrow & H^{j+4ri}(B(\zeta)) & \rightarrow & H^{j+4ri}(S(\zeta)) & \xrightarrow{\delta} & H^{j+1+4ri}(B(\zeta), S(\zeta)) \rightarrow \dots \end{array}$$

in particular,  $\mathcal{P}^i$  satisfies  $\delta \mathcal{P}^i = \mathcal{P}^i \delta$ .

Define  $q_i(\zeta) := \Phi^{-1} \cdot \mathcal{P}^i \cdot \Phi(1)$ , where  $1 \in H^0(M_-; \mathbb{Z}_p)$ , and  $\Phi$  is the Thom isomorphism

$$\Phi : H^i(M_-; \mathbb{Z}) \longrightarrow H^{i+l-m+1}(B(\zeta), S(\zeta); \mathbb{Z}).$$

By Wu theorem (cf. [16]),  $q_i(\zeta)$  can be expressed in form of a combination of Pontrjagin classes  $p_1(\zeta), p_2(\zeta), \dots, p_{ri}(\zeta)$ . Observing  $p_0(\zeta) = 1, p_1(\zeta) = 0, \dots, p_{r-1}(\zeta) = 0, p_r(\zeta) \neq 0$ , we want to represent  $q_1(\zeta)$  by  $p_r(\zeta)$ .

By Newton’s identities and Lemma 3.1,

$$\begin{aligned} q_1(\zeta) & \equiv (-1)^{r+1} \cdot r \cdot p_r(\zeta) \pmod{p} \end{aligned}$$

$$\begin{aligned} &\equiv (-1)^{\frac{m}{4}+1} \cdot r \cdot (-1) \cdot q \cdot \beta(m) \cdot \left(\frac{m}{2} - 1\right)! \pi_1^*(\gamma) \pmod{p} \\ &\equiv (-1)^{\frac{m}{4}} \cdot q \cdot \beta(m) \cdot \frac{p-1}{2} \cdot (p-2)! \pi_1^*(\gamma) \pmod{p}. \end{aligned}$$

When  $m \equiv 4 \pmod{8}$ , a direct application of Wilson Theorem gives  $q_1(\zeta) \equiv (-1)^{\frac{m}{4}+1} \cdot q \cdot \pi_1^*(\gamma) \pmod{p}$ ; When  $m \equiv 0 \pmod{8}$ , by Wilson Theorem, it is not difficult to show  $\frac{1}{2}(p-1)! \equiv \frac{1}{2}(p-1) \pmod{p}$ , which yields that  $q_1(\zeta) \equiv (-1)^{\frac{m}{4}} \cdot q \cdot \frac{p-1}{2} \cdot \pi_1^*(\gamma) \pmod{p}$ .

In summary,

$$(3.17) \quad q_1(\zeta) \equiv \begin{cases} (-1)^{\frac{m}{4}} \cdot q \cdot \frac{p-1}{2} \pi_1^*(\gamma) \pmod{p}, & m \equiv 0 \pmod{8}, \\ (-1)^{\frac{m}{4}+1} \cdot q \cdot \pi_1^*(\gamma) \pmod{p}, & m \equiv 4 \pmod{8}. \end{cases}$$

Fix  $j = l - m, i = 1$  in (3.16). Since  $m < l - m < l - 1, H^{l-m}(B) = 0, H^l(B) = 0$ , so both  $\delta$ 's are injective:

$$(3.18) \quad \begin{array}{ccc} H^{l-m}(S) & \xrightarrow{\delta} & H^{l-m+1}(B, S) \xleftarrow{\Phi} H^0(M_-) \cong \mathbb{Z}_p \\ & \text{inj.} & \cong \\ & \downarrow \mathcal{P}^1 & \downarrow \mathcal{P}^1 \\ H^l(S) & \xrightarrow{\delta} & H^{l+1}(B, S) \xleftarrow{\Phi} H^m(M_-) \cong \mathbb{Z}_p \\ & \text{inj.} & \cong \end{array}$$

On the other hand, by Gysin cohomology sequence of  $\zeta$  with coefficient  $\mathbb{Z}_p$ :

$$(3.19) \quad \rightarrow H^k(M_-) \xrightarrow{\cup e} H^{k+l-m+1}(M_-) \xrightarrow{\pi^*} H^{k+l-m+1}(S(\zeta)) \rightarrow H^{k+1}(M_-) \rightarrow \dots,$$

we get  $H^{l-m}(S(\zeta)) \cong \mathbb{Z}_p, H^l(S(\zeta)) \cong \mathbb{Z}_p$ , thus  $\delta$  are isomorphisms since they are injective as we stated above. At last, by the definition  $q_1(\zeta) := \Phi^{-1} \cdot \mathcal{P}^1 \cdot \Phi(1)$ , different values of  $q$  give rise to different Wu Squares  $\mathcal{P}^1$ , as we desired.

The proof of Theorem 1.2 is now complete! □

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## References

- [1] G.E. Bredon, *Introduction to compact transformation groups*, Academic Press, INC, Orlando, FL, 1972.
- [2] T.E. Cecil, Q.S. Chi and G.R. Jensen, *Isoparametric hypersurfaces with four principal curvatures*, Ann. Math. **166** (2007), 1–76.
- [3] Q.S. Chi, *Isoparametric hypersurfaces with four principal curvatures, III*, preprint, 2011, [arXiv:1104.3249v3](#).
- [4] T.E. Cecil and P.T. Ryan, *Tight and taut immersions of manifolds*, Res. Notes Math. **107**, Pitman, London, 1985.
- [5] D. Ferus, H. Karcher and H.F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502, for an english version, see [arXiv:1112.2780](#).
- [6] M. Gromov and H.B. Lawson Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. **111** (1980), 423–434.
- [7] M. Gromov and H.B. Lawson Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. Inst. Hautes Etudes Sci. **58** (1983), 83–196.
- [8] J.Q. Ge and Z.Z. Tang, *Geometry of isoparametric hypersurfaces in Riemannian manifolds*, to appear in Asian J. Math., [arXiv:1006.2577](#).
- [9] J.Q. Ge and Z.Z. Tang, *Isoparametric functions and exotic spheres*, J. Reine Angew. Math., doi: 10.1515/crelle-2012-0005, March 2012.
- [10] J.Q. Ge, Z.Z. Tang and W.J. Yan, *A filtration for isoparametric hypersurfaces in Riemannian manifolds*, preprint, 2011, [arXiv:1102.1126](#).
- [11] N. Hitchin, *Harmonic spinors*, Adv. Math. **14** (1974), 1–55.

- [12] W.Y. Hsiang and H.B. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Differential Geom. **5** (1971), 1–38.
- [13] M.A. Kervaire and J.W. Milnor, *Groups of homotopy spheres. I*, Ann. Math. **77** (1963), 504–537.
- [14] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris A–B **257** (1963), 7–9.
- [15] R. Miyaoka, *Isoparametric hypersurfaces with  $(g, m) = (6, 2)$* , Ann. Math., (2012) <http://annals.math.princeton.edu/toappear>.
- [16] J.W. Milnor and J.D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, NJ, 1974.
- [17] H.F. Münzner, *Isoparametric hyperflächen in sphären, I*, Math. Ann. **251** (1980), 57–71  
H.F. Münzner, *Isoparametric hyperflächen in sphären, II*, Math. Ann. **256** (1981), 215–232.
- [18] C.K. Peng and C.L. Terng, *Minimal hypersurfaces of spheres with constant scalar curvature*, Ann. of Math. Stud. (103), Princeton University Press, Princeton, NJ, 1983, 177–198.
- [19] J. Rosenberg and S. Stolz, *Metrics of positive scalar curvature and connections with surgery*, Surveys on Surgery Theory, Vol. 2, Ann. of Math. Stud. (149), Princeton University Press, Princeton, NJ, 2001.
- [20] R. Schoen and S.T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), 159–183.
- [21] Z.Z. Tang, *Nonexistence of weakly almost complex structures on Grassmannians*, Proc. Amer. Math. Soc. **121** (1994), 1267–1270.
- [22] G. Thorbergsson, *A survey on isoparametric hypersurfaces and their generalizations*, in handbook of Differential Geometry, Vol. I, North-Holland, Amsterdam, 963–995, 2000.
- [23] R. Takagi and T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a sphere*, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972.
- [24] Q.M. Wang, *On the topology of Clifford isoparametric hypersurfaces*, J. Differential Geom. **27** (1988), 55–66.
- [25] I. Yokota, *Exceptional Lie groups*, preprint, 2009, [arXiv:0902.0431v1](https://arxiv.org/abs/0902.0431v1).

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