

Stability of coassociative conical singularities

JASON D. LOTAY

We study the stability of coassociative 4-folds with conical singularities under perturbations of the ambient G_2 structure by defining an integer invariant of a coassociative cone which we call the stability index. The stability index of a coassociative cone is determined by the spectrum of the curl operator acting on its link. We explicitly calculate the stability index for cones on group orbits. We also describe the stability index for cones fibred by 2-planes over algebraic curves using the degree and genus of the curve and the spectrum of the Laplacian on the link. Finally, we apply our results to construct the first known examples of coassociative 4-folds with conical singularities in compact manifolds with G_2 holonomy.

1. Introduction

Coassociative 4-folds are *calibrated*, hence *minimal*, submanifolds of 7-manifolds with G_2 structures, first defined in [10]. Of particular interest are coassociative 4-folds in manifolds with G_2 holonomy, of which few examples are known. Coassociative submanifolds with conical singularities have been previously studied by the author in [25, 26], building upon the work on special Lagrangian submanifolds with conical singularities by Joyce in [14–18].

We continue to generalize the work of Joyce to the coassociative setting by defining the notion of *stability index* for coassociative cones. The stability index is a non-negative integer invariant for a coassociative cone, the vanishing of which guarantees that coassociative 4-folds with a singularity modelled on that cone will have a smooth moduli space of deformations: in particular, they are *stable* under small perturbations of the G_2 structure on the 7-manifold.

We calculate the stability index for certain types of coassociative cones and use our results to construct the first examples of coassociative 4-folds with conical singularities in compact manifolds with G_2 holonomy. This is

an essential step in the proposed construction given in [21] of coassociative fibrations of compact G_2 manifolds.

1.1. Motivation

Even though there has been a wealth of research devoted to calibrated submanifolds with conical singularities in manifolds with special holonomy, particularly in [14–18] and [25, 26], there were no known examples of such submanifolds. Given this well-developed theory, we were motivated to construct coassociative examples. As far as the author is aware, it is unknown whether there are special Lagrangian m -folds with conical singularities in Calabi–Yau m -folds for $m \geq 3$.

The stability index of a special Lagrangian cone is defined in [15] in terms of the spectrum of the Laplacian on the link, and is calculated for cones over flat tori originally given in [10, Section III]. The stability index for special Lagrangian cones over certain homogeneous spaces is calculated in [34] and Haskins [11] shows that the only stable special Lagrangian T^2 -cone is the cone over the flat torus.

In contrast, the stability index for a coassociative cone is determined by the spectrum of the *curl operator* acting on 1-forms on the 3-dimensional link of the cone. Although still a natural geometric object, there is relatively little material on the spectrum of the curl operator in the literature, and it is certainly less straightforward to analyse than the spectrum of the Laplacian which has received so much attention. We are thus required to undertake fundamental elementary calculations to describe the stability index even in simple cases.

Eigenforms for the curl operator naturally define contact structures and are dual to certain Beltrami fields, which are important in hydrodynamics. Beltrami fields are also equivalent to Reeb vector fields by the work in [8], so are of particular interest in contact geometry. We hope therefore that our spectral calculations may be of wider benefit.

Motivated by the SYZ conjecture (see [38]), one would hope to construct coassociative fibrations of compact manifolds with G_2 holonomy. An elementary argument shows that some of the fibres must necessarily be singular, so it is natural to assume that they have the simplest type of singularity, namely conical singularities. As already mentioned, the stability results in this paper are essential for the proposed construction of a coassociative $K3$ fibration in [21]. Moreover, our results will almost certainly be useful for any other construction of a coassociative fibration with conically singular fibres.

1.2. Summary

We begin in Section 2 with the basic definitions we need, including concepts from calibrated geometry and Geometric Measure Theory. We also discuss the relationship between complex and coassociative geometry.

In Section 3 we mainly review the material in [25], defining coassociative 4-folds with conical singularities and describing their deformation theory. Our definition of coassociative conical singularities (Definition 3.2) on the face of it looks rather strong. However, we prove the following regularity result, generalizing results in [15, 37], which shows that the definition is more applicable than it appears.

Theorem 1.1. *If a coassociative integral current has a multiplicity one, Jacobi integrable tangent cone with isolated singularity at an interior point p , then it has a conical singularity at p .*

Remarks. The link of a coassociative cone is a *Lagrangian* (or totally real) submanifold of the nearly Kähler 6-sphere \mathcal{S}^6 . A coassociative cone is *Jacobi integrable* if every infinitesimal variation of its link as a Lagrangian in \mathcal{S}^6 is integrable.

In Section 4, we begin by briefly reviewing the known examples of coassociative cones. Then, by refining some of the results in [25], we define the notion of \mathcal{C} -stability index for a coassociative cone C in a family \mathcal{C} . We simply call the \mathcal{C} -stability index of C the stability index when \mathcal{C} is the family of cones generated by G_2 transformations and translations of C .

Sections 5 and 6 are devoted to calculating the stability index for certain coassociative cones. In Section 5, we analyse the curl operator acting on Berger 3-spheres and their quotients by finite groups, and thus determine the stability index for all homogeneous coassociative cones. We deduce the following.

Theorem 1.2. *The only stable homogeneous coassociative cones are coassociative 4-planes and the $\mathrm{Sp}(1)$ -invariant coassociative cone given in Example 4.2.*

Remark. The cone in Example 4.2 was originally constructed in [22, Section 7] and identified as coassociative in [10, Section IV].

In Section 6 we study coassociative cones which are fibred by 2-planes over algebraic curves: either *holomorphic curves* in $\mathbb{C}\mathbb{P}^2$, where the corresponding coassociative cone is complex, or *null-torsion pseudoholomorphic curves* in \mathcal{S}^6 . In both cases, we express the stability index in terms of the spectrum of the Laplacian acting on functions on the link and algebro-geometric data from the curve. These results allow us to examine the behaviour of the stability index under deformations of these types of coassociative cones.

Finally, in Section 7, after discussing the construction in [20] of compact manifolds with G_2 holonomy, we apply the results of Sections 5 and 6 to prove the following.

Theorem 1.3. *Given a pair of maximal deformation families of Fano 3-folds, one can construct a one-parameter family of compact manifolds with G_2 holonomy, which contain coassociative K3 surfaces with conical singularities.*

Remark. The key ingredients in the proof are Theorem 1.1, the \mathcal{C} -stability index for a homogeneous complex cone C in a deformation family \mathcal{C} and the invariance of the \mathcal{C} -stability index under deformations of C in \mathcal{C} .

Given the proof of Theorem 1.3, it is clear that this result will naturally extend to give coassociative 4-folds with conical singularities in the compact holonomy G_2 manifolds constructed by Corti *et al.* [6], where one replaces Fano 3-folds by a more general class of complex 3-folds.

Notes.

- (a) Manifolds are taken to be non-singular and submanifolds to be embedded, for convenience, unless stated otherwise.
- (b) We use the convention that $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z}^+ = \mathbb{N} \setminus \{0\}$.

2. Coassociative 4-folds

In this section, we cover all of the basic definitions and theory we need.

2.1. Calibrated geometry and Geometric Measure Theory

We will need some general theory from *calibrated geometry* later, so we start with the following definition.

Definition 2.1. Let (M, g) be a Riemannian manifold. An *oriented tangent m -plane* W on M is an oriented m -dimensional vector subspace W of $T_x M$, for some x in M . Given an oriented tangent m -plane W on M , $g|_W$ is a Euclidean metric on W and hence, using $g|_W$ and the orientation on W , there is a natural volume form, vol_W , which is an m -form on W .

A closed m -form ϕ on M is a *calibration* on M if $\phi|_W \leq \text{vol}_W$ for all oriented tangent m -planes W on M , where $\phi|_W = \kappa \cdot \text{vol}_W$ for some $\kappa \in \mathbb{R}$, so $\phi|_W \leq \text{vol}_W$ if $\kappa \leq 1$. An oriented m -dimensional submanifold S of M is a *calibrated submanifold* or *ϕ -submanifold* if $\phi|_S = \text{vol}_S$.

We shall need some ideas from Geometric Measure Theory. A good introduction to this theory, which we only need in a superficial way, is given in [32].

Definition 2.2. Let (M, g) be a complete Riemannian manifold and let \mathcal{H}^m be m -dimensional Hausdorff measure on M . By [32, Proposition 3.11], one can define an \mathcal{H}^m -measurable subset S of M to be an *m -dimensional rectifiable set* if S has finite \mathcal{H}^m -measure and \mathcal{H}^m -almost all of S is covered by the disjoint union of a countable number of compact C^1 -submanifolds of M . Let $\text{vol}(S)$ be the \mathcal{H}^m -measure of S . By [32, Proposition 3.12], an m -dimensional rectifiable set S in M has a well-defined m -dimensional tangent plane \mathcal{H}^m -almost everywhere, and so is orientable almost everywhere. If S is an oriented rectifiable set, let $\mathbf{s}(x)$ be the unit m -vector to S at x given by the choice of orientation, when this is well-defined.

Let $\mathcal{D}^m(M)$ be the space of smooth compactly supported m -forms on M . An *m -dimensional current* on M is an element of the dual space $\mathcal{C}^m(M) = \mathcal{D}^m(M)^*$ and we define the support of $T \in \mathcal{C}^m(M)$, $\text{supp } T$, to be the smallest closed set in M such that, for any $\xi \in \mathcal{D}^m(M)$, $\text{supp } \xi \cap \text{supp } T = \emptyset$ implies that $T(\xi) = 0$. We define the *boundary* $\partial T \in \mathcal{C}^{m-1}(M)$ of $T \in \mathcal{C}^m(M)$ via the formula $\partial T(\xi) = T(d\xi)$ for $\xi \in \mathcal{D}^{m-1}(M)$. We also define the *interior* T° of T to be the set $\text{supp } T \setminus \text{supp } \partial T$.

Given an m -dimensional oriented rectifiable set S and a function $\nu : S \rightarrow \mathbb{Z}^+$ such that $\int_S \nu(x) d\mathcal{H}^m < \infty$, we define an associated element T_S of $\mathcal{C}^m(M)$ via

$$T_S(\xi) = \int_S \mathbf{s}(x) \cdot \xi(x) \nu(x) d\mathcal{H}^m.$$

If $\text{supp } T_S$ is compact, we say that T_S (or simply S , since T_S is defined by S) is an *m -dimensional rectifiable current* and we denote the set of m -dimensional rectifiable currents by $\mathcal{R}^m(M)$. We also let $\mathcal{I}^m(M) = \{S \in \mathcal{R}^m(M) : \partial S \in \mathcal{R}^{m-1}(M)\}$ be the set of m -dimensional *integral currents*.

Finally, we define the set of m -dimensional *locally integral currents* by

$$\mathcal{I}_{\text{loc}}^m(M) = \{T \in \mathcal{C}^m(M) : \forall x \in M \exists S \in \mathcal{I}^m(M) \text{ with } x \notin \text{supp}(T - S)\}$$

and similarly define $\mathcal{R}_{\text{loc}}^m(M)$.

The idea behind a rectifiable current is to generalize the notion of a compact C^1 -submanifold with boundary to include multiplicities (given by the function ν) and to allow for singular behaviour. Since the boundary of a rectifiable current may be very badly behaved, we often require that the rectifiable current be integral. As we shall need to deal with planes and cones which are definitely not compact, we expand our notation to include the “local” versions of the integral and rectifiable currents.

The majority of this paper will be dedicated to the study of cones, so we make some formal definitions for convenience.

Definition 2.3. Recall the notation of Definition 2.2, let W be a normed vector space and let $\mathcal{S}(W)$ be the unit sphere in W with respect to the norm. An element $C \in \mathcal{R}_{\text{loc}}^m(W)$ is a *cone* in W if $tC = C$ for all $t > 0$, and we call $C \cap \mathcal{S}(W)$ the *link* of C .

We formally define convergence in the space of currents as follows.

Definition 2.4. Recall the notation of Definition 2.2. We say that a sequence (S_j) in $\mathcal{R}_{\text{loc}}^m(M)$ converges to $S \in \mathcal{R}_{\text{loc}}^m(M)$ if $S_j \rightarrow S$ in the weak topology in $\mathcal{C}^m(M)$; that is,

$$S_j \rightarrow S \quad \text{if and only if} \quad \int_{S_j} \xi \rightarrow \int_S \xi \quad \text{as } j \rightarrow \infty,$$

for all $\xi \in \mathcal{D}^m(M)$, where integration is carried out with respect to \mathcal{H}^m and includes multiplicities.

In the seminal work on calibrated geometry [10], the relationship between calibrated geometry and Geometric Measure Theory is discussed at length. We note some of the observations originally presented there.

Definition 2.5. Let (M, g) be a complete Riemannian manifold, let ϕ be an m -form which is a calibration on M and recall the notation of Definition 2.2. By the work in [10], we can define $S \in \mathcal{I}^m(M)$ to be an *integral ϕ -current* if S is calibrated with respect to ϕ ; that is, $\int_S \phi = \text{vol}(S)$. Then integral ϕ -currents are volume-minimizing in their homology class. We can also define a *locally integral ϕ -current* in M in the obvious manner.

One of the key ideas in Geometric Measure Theory is the concept of a *tangent cone*, which we now define.

Definition 2.6. Let (M, g) be a complete Riemannian n -manifold and recall the notation of Definition 2.2. Let $S \in \mathcal{R}_{\text{loc}}^n(M)$ and let $x \in S^\circ$. Choose a diffeomorphism $v : V \rightarrow B$, where B is an open neighbourhood of the origin in \mathbb{R}^n and V is an open neighbourhood of x in M . Let $U = V \cap S$ and let $\Upsilon = dv|_x$, which is an isomorphism between $T_x M$ and \mathbb{R}^n . A *tangent cone* for S at x is a cone C in $T_x M$ such that there exists a strictly decreasing positive sequence (r_j) , converging to zero as $j \rightarrow \infty$, such that

$$r_j^{-1}v(U \setminus \{x\}) \rightarrow \Upsilon(C) \quad \text{as } j \rightarrow \infty$$

in the sense of Definition 2.4.

In Geometric Measure Theory there are two notions of “tangent cone”: one is a set and the other is a current. We have defined a current in Definition 2.6 and so it is, strictly speaking, an *oriented tangent cone* in the sense of Geometric Measure Theory. Oriented tangent cones can be defined for more general currents than locally rectifiable ones, but then one only requires a weaker form of convergence in the definition.

We conclude with an important result that follows from [19, Theorem 4.4.4].

Proposition 2.7. *Let ϕ be a calibration on a complete Riemannian manifold M and let S be an integral ϕ -current in M . There exists a tangent cone to S at each $x \in S^\circ$, and it is a locally integral $\phi|_x$ -current in $T_x M$.*

2.2. Calibrated geometry in \mathbb{R}^7

We define coassociative 4-folds in \mathbb{R}^7 by introducing a distinguished 3-form.

Definition 2.8. Let (x_1, \dots, x_7) be coordinates on \mathbb{R}^7 and write $\mathbf{dx}_{ij\dots k}$ for the form $dx_i \wedge dx_j \wedge \dots \wedge dx_k$. Define a 3-form φ_0 on \mathbb{R}^7 by:

$$(2.1) \quad \varphi_0 = \mathbf{dx}_{123} + \mathbf{dx}_{145} + \mathbf{dx}_{167} + \mathbf{dx}_{246} - \mathbf{dx}_{257} - \mathbf{dx}_{347} - \mathbf{dx}_{356}.$$

The Hodge dual of φ_0 is a 4-form given by:

$$(2.2) \quad *\varphi_0 = \mathbf{dx}_{4567} + \mathbf{dx}_{2367} + \mathbf{dx}_{2345} + \mathbf{dx}_{1357} - \mathbf{dx}_{1346} - \mathbf{dx}_{1256} - \mathbf{dx}_{1247}.$$

The forms φ_0 and $*\varphi_0$ are calibrations by [10, Theorems IV.1.4 and IV.1.16]. Submanifolds calibrated with respect to φ_0 and $*\varphi_0$ are called *associative 3-folds* and *coassociative 4-folds* respectively. We can also characterize the coassociative 4-folds as the oriented 4-dimensional submanifolds N in \mathbb{R}^7 satisfying $\varphi_0|_N \equiv 0$, oriented such that $*\varphi_0|_N > 0$, by [10, Proposition IV.4.5 and Theorem IV.4.6].

Remark. The form φ_0 is sometimes called the “ G_2 3-form” because the exceptional Lie group G_2 is the stabilizer of φ_0 in $GL(7, \mathbb{R})$.

A straightforward calculation yields the following lemma.

Lemma 2.9. *Identify \mathbb{R}^7 with $\mathbb{R} \oplus \mathbb{C}^3$ so that x_1 is the coordinate on \mathbb{R} and $z_1 = x_2 + ix_3$, $z_2 = x_4 + ix_5$ and $z_3 = x_6 + ix_7$ are coordinates on \mathbb{C}^3 . If ω_0 and Ω_0 are the standard Kähler and holomorphic forms on \mathbb{C}^3 , then:*

$$(2.3) \quad \varphi_0 = dx_1 \wedge \omega_0 + \operatorname{Re} \Omega_0,$$

$$(2.4) \quad *\varphi_0 = \frac{1}{2} \omega_0 \wedge \omega_0 - dx_1 \wedge \operatorname{Im} \Omega_0,$$

where φ_0 and $*\varphi_0$ are given in (2.1) and (2.2).

Since $\operatorname{Re} \Omega_0$ and $\operatorname{Im} \Omega_0$ are both calibrations on \mathbb{C}^3 we have the following definition, again due to Harvey and Lawson [10].

Definition 2.10. Let (z_1, \dots, z_m) be coordinates on \mathbb{C}^m and let ω_0 and Ω_0 be the Kähler and holomorphic volume forms on \mathbb{C}^m . Then $\cos \theta \operatorname{Re} \Omega_0 + \sin \theta \operatorname{Im} \Omega_0$ is a calibration on \mathbb{C}^m for all real constants θ , and its corresponding calibrated submanifolds are real m -dimensional submanifolds of \mathbb{C}^m called *special Lagrangian m -folds* (with *phase $e^{i\theta}$*). Moreover, special Lagrangian m -folds with phase $e^{i\theta}$ are the oriented real m -dimensional submanifolds L of \mathbb{C}^m such that $\omega_0|_L \equiv 0$ and $(\sin \theta \operatorname{Re} \Omega_0 - \cos \theta \operatorname{Im} \Omega_0)|_L \equiv 0$, up to a choice of orientation.

Examination of (2.3) and (2.4) immediately yields the following elementary result.

Corollary 2.11. *In the notation of Lemma 2.9, $\mathbb{R} \times L \subseteq \mathbb{R} \oplus \mathbb{C}^3$ and $N \subseteq \mathbb{C}^3$ are coassociative in \mathbb{R}^7 if and only if L is a special Lagrangian 3-fold with phase $-i$ and N is a complex surface in \mathbb{C}^3 respectively.*

Since we will be concerned with coassociative cones, we make the following convenient definition.

Definition 2.12. The 6-sphere \mathcal{S}^6 inherits a *nearly Kähler* structure from the standard G_2 structure on \mathbb{R}^7 . In particular, if r is the radial coordinate and \mathbf{e}_r is the radial vector field on \mathbb{R}^7 , $\omega = (\mathbf{e}_r \cdot \varphi_0)|_{r=1}$ is a non-degenerate 2-form on \mathcal{S}^6 which is *not* closed. Using this 2-form and the round metric g on \mathcal{S}^6 we can define an almost complex structure J by $\omega(x, y) = g(Jx, y)$ for tangent vectors x, y . The almost complex structure J is *not* integrable.

An oriented 3-dimensional submanifold $L \subseteq \mathcal{S}^6$ is the link of a coassociative cone if and only if $\omega|_L \equiv 0$. Thus, we say that the link of a coassociative cone in \mathbb{R}^7 is a *Lagrangian* submanifold of \mathcal{S}^6 .

An oriented surface $\Sigma \subseteq \mathcal{S}^6$ is a *pseudoholomorphic curve* if and only if $\omega|_\Sigma = \text{vol}_\Sigma$ or, equivalently, if $J(T_\sigma \Sigma) = T_\sigma \Sigma$ for all $\sigma \in \Sigma$. Note that Σ is the link of an associative cone if and only if Σ is a pseudoholomorphic curve.

By Corollary 2.11, any complex 2-dimensional cone in \mathbb{C}^3 is coassociative in $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$. Thus, the Hopf lift of any holomorphic curve in $\mathbb{C}\mathbb{P}^2$ to a totally geodesic \mathcal{S}^5 in \mathcal{S}^6 is Lagrangian. Since special Lagrangian 3-folds in \mathbb{C}^3 are associative in $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$, minimal Legendrian surfaces in \mathcal{S}^5 , which are the links of special Lagrangian cones in \mathbb{C}^3 , give examples of pseudoholomorphic curves in \mathcal{S}^6 .

2.3. G_2 structures

So that we may define coassociative submanifolds of more general 7-manifolds, we make the following definition.

Definition 2.13. Let M be an oriented 7-manifold and recall the 3-form φ_0 on \mathbb{R}^7 given in (2.1). For each $x \in M$ there exists an orientation preserving isomorphism $\iota_x : T_x M \rightarrow \mathbb{R}^7$. Since $\dim G_2 = 14$, $\dim \text{GL}_+(T_x M) = 49$ and $\dim \Lambda^3 T_x^* M = 35$, the $\text{GL}_+(T_x M)$ orbit of $\iota_x^*(\varphi_0)$ in $\Lambda^3 T_x^* M$, denoted $\Lambda_+^3 T_x^* M$, is open. A 3-form φ on M is *positive* if $\varphi|_x \in \Lambda_+^3 T_x^* M$ for all $x \in M$. Denote the bundle of positive 3-forms by $\Lambda_+^3 T^* M$.

A positive 3-form is identified with the G_2 3-form φ_0 on \mathbb{R}^7 at each point in M . Therefore, to each positive 3-form φ we can uniquely associate a 4-form $*_\varphi \varphi$ and a metric g_φ on M such that the triple $(\varphi, *_\varphi \varphi, g_\varphi)$ corresponds to $(\varphi_0, *\varphi_0, g_0)$ at each point. Note that since the metric g_φ depends on φ , the Hodge star $*_\varphi$ depends on φ also. This leads us to our next definition.

Definition 2.14. Let M be an oriented 7-manifold and $\varphi \in C^\infty(\Lambda_+^3 T^*M)$. If g_φ is the metric associated with φ , we call (φ, g_φ) a G_2 structure on M . If φ is closed (or $*_\varphi\varphi$ is closed) then (φ, g_φ) is a *closed* (or *coclosed*) G_2 structure. A closed and coclosed G_2 structure is called *torsion-free*.

Our choice of notation here agrees with [5].

Remark. By [35, Lemma 11.5], (φ, g_φ) is a torsion-free G_2 structure on M if and only if the holonomy of g_φ is contained in G_2 .

Definition 2.15. Let M be an oriented 7-manifold with a G_2 structure (φ, g_φ) , denoted (M, φ, g_φ) . If (φ, g_φ) is closed, we say that (M, φ, g_φ) is an *almost G_2 manifold*. If (φ, g_φ) is torsion-free, we call (M, φ, g_φ) a G_2 manifold.

Note. By [19, Proposition 11.2.1], the metric g_φ on a compact G_2 manifold M has G_2 holonomy if and only if the fundamental group $\pi_1(M)$ is finite.

We are now able to complete our definitions regarding coassociative 4-folds.

Definition 2.16. A 4-dimensional submanifold N of (M, φ, g_φ) is coassociative if and only if $\varphi|_N \equiv 0$ and $*_\varphi\varphi|_N > 0$.

Note. Though we may define coassociative 4-folds with respect to any G_2 structure, for deformation theory and related results to hold we need it to be closed. Therefore, we shall work with almost G_2 manifolds for greatest useful generality.

The next result, [31, cf. Proposition 4.2], is invaluable in describing the deformation theory of coassociative 4-folds.

Proposition 2.17. *Let N be a coassociative 4-fold in an almost G_2 manifold (M, φ, g_φ) . There is an isometric isomorphism between the normal bundle $\nu(N)$ of N in M and $(\Lambda_+^2)_{g_\varphi|_N} T^*N$ given by $J_N : \nu \mapsto (\nu \lrcorner \varphi)|_{TN}$. Thus, infinitesimal coassociative deformations of N are governed by closed self-dual 2-forms on N .*

Remarks. From Proposition 2.17 and some further analysis, one may deduce as in [19, Theorem 12.3.4], by following [31, Theorem 4.5], that the moduli space of deformations of a compact coassociative 4-fold N in

an almost G_2 manifold is a manifold of dimension $b_+^2(N)$. The author in [25] adapted this deformation theory result to the situation where N has *conical singularities*, which will be invaluable for the study in this article. The relevant material in [25] will be the focus of Section 3.

We shall also be briefly concerned with $SU(3)$ structures, so they form the subject of the next definition.

Definition 2.18. On \mathbb{C}^3 , let g_0 , ω_0 and Ω_0 denote the standard Euclidean metric, Kähler form and holomorphic volume form respectively. Let (Y, J, g, ω) be an almost Hermitian 6-manifold; that is, g is a Riemannian metric on the almost complex 6-manifold Y , J is an almost complex structure preserved by g and ω is the associated (non-degenerate) $(1, 1)$ -form on M . An $SU(3)$ structure on Y is a choice of nowhere vanishing $(3, 0)$ -form Ω on Y such that, for all $y \in Y$, there exists an orientation preserving isomorphism $\iota_y : T_y Y \rightarrow \mathbb{C}^3$ satisfying $\iota_y^*(g_0) = g|_y$, $\iota_y^*(\omega_0) = \omega|_y$ and $\iota_y^*(\Omega_0) = \Omega|_y$.

If $(Y, J, g, \omega, \Omega)$ is an almost Hermitian 6-manifold endowed with an $SU(3)$ structure, the product 7-manifold $M = \mathbb{R} \times Y$ (or $\mathcal{S}^1 \times Y$) has a “product” G_2 structure given by $\varphi = dx \wedge \omega + \text{Re } \Omega$ and $g_\varphi = dx^2 + g$, where x is the coordinate on \mathbb{R} or \mathcal{S}^1 , by [19, Proposition 11.1.9]. Moreover, $*_\varphi \varphi = \frac{1}{2} \omega \wedge \omega - dx \wedge \text{Im } \Omega$.

Notes. If (Y, g, J, ω) is a compact Kähler 3-manifold and Ω is a nowhere vanishing *holomorphic* $(3, 0)$ -form on Y , then $(Y, g, J, \omega, \Omega)$ is called an *almost Calabi–Yau 3-fold*. If, in addition, $\omega^3 = \frac{3i}{4} \Omega \wedge \bar{\Omega}$ then the $SU(3)$ structure is *torsion-free* and $(Y, J, g, \omega, \Omega)$ is called a *Calabi–Yau 3-fold*. These are the natural “ $SU(3)$ analogues” of the manifolds defined in Definition 2.15. In particular, the Calabi–Yau condition is equivalent to saying that the compact Kähler manifold has metric g with holonomy contained in $SU(3)$.

3. Conical singularities

In this section, we review some of the theory of conical singularities of coassociative 4-folds as described in [25]. We also prove an important new result that shows that singular coassociative integral currents with particularly “nice” tangent cones have conical singularities.

3.1. Coassociative 4-folds with conical singularities

We first define a preferred choice of local coordinates on an almost G_2 manifold near a finite set of points, which is an analogue of one given for almost

Calabi–Yau manifolds in [15, Definition 3.6]. We let $B(0; \delta) \subseteq \mathbb{R}^7$ denote the open ball about 0 with radius $\delta > 0$.

Definition 3.1. Let (M, φ, g_φ) be an almost G_2 manifold and let $z_1, \dots, z_s \in M$ be distinct points. There exist a constant $\epsilon_M \in (0, 1)$, an open set $V_i \ni z_i$ in M with $V_i \cap V_j = \emptyset$ for $j \neq i$ and a diffeomorphism $\chi_i : B(0; \epsilon_M) \rightarrow V_i$ with $\chi_i(0) = z_i$, for $i = 1, \dots, s$, such that $\zeta_i = d\chi_i|_0 : \mathbb{R}^7 \rightarrow T_{z_i}M$ is an isomorphism identifying the standard G_2 structure (φ_0, g_0) on \mathbb{R}^7 with the pair $(\varphi|_{z_i}, g_\varphi|_{z_i})$. We call the set $\{\chi_i : B(0; \epsilon_M) \rightarrow V_i : i = 1, \dots, s\}$ a G_2 coordinate system near z_1, \dots, z_s .

We say that two G_2 coordinate systems near z_1, \dots, z_s , with maps χ_i and $\tilde{\chi}_i$ for $i = 1, \dots, s$, respectively, are *equivalent* if $d\tilde{\chi}_i|_0 = d\chi_i|_0 = \zeta_i$ for all i .

Definition 3.2. Let (M, φ, g_φ) be an almost G_2 manifold, let $N \subseteq M$ be compact and connected and let $z_1, \dots, z_s \in N$ be distinct. We also let $\{\chi_i : B(0; \epsilon_M) \rightarrow V_i : i = 1, \dots, s\}$ be a G_2 coordinate system near z_1, \dots, z_s , as in Definition 3.1. We say that N is a 4-fold in M with *conical singularities at z_1, \dots, z_s with rate μ* , denoted a *CS 4-fold*, if $\hat{N} = N \setminus \{z_1, \dots, z_s\}$ is a (non-singular) 4-dimensional submanifold of M and there exist constants $0 < \epsilon < \epsilon_M$ and $\mu \in (1, 2)$, a compact 3-dimensional Riemannian submanifold (L_i, h_i) of $\mathcal{S}^6 \subseteq \mathbb{R}^7$, where h_i is the restriction of the round metric on \mathcal{S}^6 to L_i , an open set $U_i \ni z_i$ in N with $U_i \subseteq V_i$ and a smooth map $\Phi_i : (0, \epsilon) \times L_i \rightarrow B(0; \epsilon_M) \subseteq \mathbb{R}^7$, for $i = 1, \dots, s$, such that $\Psi_i = \chi_i \circ \Phi_i : (0, \epsilon) \times L_i \rightarrow U_i \setminus \{z_i\}$ is a diffeomorphism, and Φ_i satisfies

$$(3.1) \quad \Phi_i(r_i, x_i) - \iota_i(r_i, x_i) \in (T_{r_i x_i} \iota_i(C_i))^\perp \quad \text{for all } (r_i, x_i) \in (0, \epsilon) \times L_i$$

and

$$(3.2) \quad |\nabla_i^j(\Phi_i(r_i, x_i) - \iota_i(r_i, x_i))| = O(r_i^{\mu-j}) \quad \text{for } j \in \mathbb{N} \text{ as } r_i \rightarrow 0,$$

where $\iota_i(r_i, x_i) = r_i x_i \in B(0; \epsilon_M)$, ∇_i is the Levi-Civita connection of the cone metric $g_i = dr_i^2 + r_i^2 h_i$ on $C_i = (0, \infty) \times L_i$ coupled with partial differentiation on \mathbb{R}^7 , and $|\cdot|$ is calculated with respect to g_i .

We call C_i the *cone* at the singularity z_i and L_i the *link* of the cone C_i . We may write N as the disjoint union $N = K_N \sqcup \bigsqcup_{i=1}^s U_i$, where K_N is compact.

If \hat{N} is coassociative in M , we say that N is a *CS coassociative 4-fold*.

Remark. If N is a CS 4-fold, \hat{N} is *non-compact*.

Suppose N is a CS 4-fold at z_1, \dots, z_s with rate μ in (M, φ, g_φ) and use the notation of Definition 3.2. The induced metric on \hat{N} , $g_\varphi|_{\hat{N}}$, makes \hat{N} into a Riemannian manifold. Moreover, it is clear from (3.2) that, as long as $\mu < 2$, the maps Ψ_i satisfy

$$(3.3) \quad |\nabla_i^j(\Psi_i^*(g_\varphi|_{\hat{N}}) - g_i)| = O(r_i^{\mu-1-j}) \quad \text{for } j \in \mathbb{N} \text{ as } r_i \rightarrow 0.$$

Consequently, the condition $\mu > 1$ guarantees that the induced metric on \hat{N} genuinely converges to the conical metric on C_i .

Note. As shown on [25, p. 6], since $\mu \in (1, 2)$, Definition 3.2 is independent of the choice of G_2 coordinate system near the singularities, up to equivalence.

Definition 3.3. Let N be a CS coassociative 4-fold in an almost G_2 manifold and use the notation of Definition 3.2. A *radius function* on \hat{N} is a smooth map $\rho : \hat{N} \rightarrow (0, 1]$ such that there exist positive constants $c_1 < 1$ and $c_2 > 1$ with $c_1 r_i < \Psi_i^*(\rho) < c_2 r_i$ on $(0, \epsilon) \times L_i$ for $i = 1, \dots, s$.

It is clear how we may construct such a function.

We now make a definition that also depends only on equivalence classes of G_2 coordinate systems near the singularities.

Definition 3.4. Let N be a CS coassociative 4-fold at z_1, \dots, z_s in an almost G_2 manifold. Use the notation of Definitions 3.1 and 3.2. For $i = 1, \dots, s$, define a cone \hat{C}_i in $T_{z_i}M$ by $\hat{C}_i = (\zeta_i \circ \iota_i)(C_i)$. We call \hat{C}_i the *tangent cone* at z_i .

Using (3.2), one sees that \hat{C}_i is a tangent cone at z_i in the sense of Definition 2.6. Since the tangent cone has multiplicity one, [37, Theorem 5.7] implies that \hat{C}_i is the unique tangent cone to N at z_i . It is still an open question whether a general calibrated integral current has a unique tangent cone at each point.

We conclude with a straightforward result that follows from Proposition 2.7 or, by more elementary means, from [25, Proposition 3.6].

Proposition 3.5. *Let N be a CS coassociative 4-fold at z_1, \dots, z_s in an almost G_2 manifold. The tangent cones at z_1, \dots, z_s are coassociative.*

3.2. Weighted Banach spaces

For this subsection we let (P, g) denote the non-singular part of a CS coassociative 4-fold in an almost G_2 manifold with the induced metric and let ρ be a radius function on P , as given in Definitions 3.2 and 3.3. We also let ∇ denote the Levi-Civita connection of g . We define *weighted* Banach spaces of forms on P as in [2, Section 1] so as to implement the analytic framework of [25].

We shall use the notation and definition of the usual “unweighted” Banach spaces of forms as in [13, Section 1.2]; that is, Sobolev and Hölder spaces are denoted by L_k^p and $C^{k,a}$, respectively, where $p \geq 1$, $k \in \mathbb{N}$ and $a \in (0, 1)$. Recall that, by the Sobolev Embedding Theorem, L_k^p embeds continuously in L_l^q if $l \leq k$ and $l - \frac{4}{q} \leq k - \frac{4}{p}$, and L_k^p embeds continuously in $C^{l,a}$ if $k - \frac{4}{p} \geq l + a$. We also introduce the notation C_{loc}^k for the space of forms ξ such that $f\xi$ lies in C^k for every smooth compactly supported function f , and similarly define spaces $L_{k,loc}^p$ and $C_{loc}^{k,a}$.

Definition 3.6. Let $p \geq 1$, $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. The *weighted Sobolev space* $L_{k,\lambda}^p(\Lambda^m T^*P)$ of m -forms ξ on P is the subspace of $L_{k,loc}^p(\Lambda^m T^*P)$ such that the norm

$$\|\xi\|_{L_{k,\lambda}^p} = \sum_{j=0}^k \left(\int_P |\rho^{j-\lambda} \nabla^j \xi|^p \rho^{-4} dV_g \right)^{\frac{1}{p}}$$

is finite. Then $L_{k,\lambda}^p(\Lambda^m T^*P)$ is a Banach space.

We now define the *dual* weighted Sobolev space that shall be useful later.

Definition 3.7. Use the notation from Definition 3.6. Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, let $k, l \in \mathbb{N}$ and let $\lambda \in \mathbb{R}$. We then define $\langle \cdot, \cdot \rangle : L_{k,\lambda}^p(\Lambda^m T^*P) \times L_{l,-4-\lambda}^q(\Lambda^m T^*P) \rightarrow \mathbb{R}$ by

$$\langle \xi, \eta \rangle = \int_P \xi \wedge * \eta.$$

We shall refer to this as the *dual pairing*. For our purposes, we take the dual space of $L_{k,\lambda}^p(\Lambda^m T^*P)$ to be $L_{l,-4-\lambda}^q(\Lambda^m T^*P)$, with linear functionals represented by dual pairings.

Definition 3.8. Let $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$. The *weighted C^k -space* $C_\lambda^k(\Lambda^m T^*P)$ of m -forms ξ on P is the subspace of $C_{\text{loc}}^k(\Lambda^m T^*P)$ such that the norm

$$\|\xi\|_{C_\lambda^k} = \sum_{j=0}^k \sup_P |\rho^{j-\lambda} \nabla^j \xi|$$

is finite. We define $C_\lambda^\infty(\Lambda^m T^*P) = \bigcap_{k \geq 0} C_\lambda^k(\Lambda^m T^*P)$. Then $C_\lambda^k(\Lambda^m T^*P)$ is a Banach space, but $C_\lambda^\infty(\Lambda^m T^*P)$ is not in general.

Definition 3.9. Let E be a vector bundle on P endowed with Euclidean metrics on its fibres and a connection preserving these metrics. Let $d(x, y)$ be the geodesic distance between points $x, y \in P$, let $a \in (0, 1)$, let $k \in \mathbb{N}$ and let $\lambda \in \mathbb{R}$. Let

$$H = \{(x, y) \in P \times P : x \neq y, c_1 \rho(x) \leq \rho(y) \leq c_2 \rho(x) \text{ and} \\ \text{there exists a geodesic in } P \text{ of length } d(x, y) \text{ from } x \text{ to } y\},$$

where $0 < c_1 < 1 < c_2$ are constant. A section s of E is *Hölder continuous* (with *exponent* a) if

$$[s]^a = \sup_{(x,y) \in H} \frac{|s(x) - s(y)|_E}{d(x, y)^a} < \infty.$$

We understand the quantity $|s(x) - s(y)|_E$ as follows. Given $(x, y) \in H$, there exists a geodesic γ of length $d(x, y)$ connecting x and y . Parallel translation along γ using the connection on E identifies the fibres over x and y and the metrics on them. Thus, with this identification, $|s(x) - s(y)|_E$ is well-defined.

The *weighted Hölder space* $C_\lambda^{k,a}(\Lambda^m T^*P)$ is defined to be the subspace of $C_{\text{loc}}^{k,a}(\Lambda^m T^*P)$ such that the norm

$$\|\xi\|_{C_\lambda^{k,a}} = \|\xi\|_{C_\lambda^k} + [\xi]_\lambda^{k,a}$$

is finite, where

$$[\xi]_\lambda^{k,a} = [\rho^{k+a-\lambda} \nabla^k \xi]^a.$$

Then $C_\lambda^{k,a}(\Lambda^m T^*P)$ is a Banach space. It is clear that we have embeddings $C_\lambda^{k,a}(\Lambda^m T^*P) \hookrightarrow C_\lambda^l(\Lambda^m T^*P)$ and $C_\lambda^{k+1}(\Lambda^m T^*P) \hookrightarrow C_\lambda^{l,a}(\Lambda^m T^*P)$ if $l \leq k$.

Remark. The set H in Definition 3.9 is introduced so that $[\xi]_\lambda^{k,a}$ is well-defined.

Finally, we shall need the analogue of the Sobolev Embedding Theorem for weighted spaces, which is adapted from [23, Lemma 7.2] and [2, Theorem 1.2].

Theorem 3.10 (Weighted Sobolev Embedding Theorem). *Let $p, q \geq 1$, $a \in (0, 1)$, $\lambda, \nu \in \mathbb{R}$ and $k, l \in \mathbb{N}$.*

- (a) *If $k \geq l$, $k - \frac{4}{p} \geq l - \frac{4}{q}$, and either $p \leq q$ and $\lambda \geq \nu$, or $p > q$ and $\lambda > \nu$, there is a continuous embedding $L_{k, \lambda}^p(\Lambda^m T^* P) \hookrightarrow L_{l, \nu}^q(\Lambda^m T^* P)$.*
- (b) *If $k - \frac{4}{p} \geq l + a$, there is a continuous embedding $L_{k, \lambda}^p(\Lambda^m T^* P) \hookrightarrow C_\lambda^{l, a}(\Lambda^m T^* P)$.*

3.3. Deformation theory

We now review and discuss the key deformation theory results for CS coassociative 4-folds from [25]. We begin by recalling the linear differential operator governing infinitesimal deformations of CS coassociative 4-folds.

Definition 3.11. Let N be a CS coassociative 4-fold in an almost G_2 manifold. Let $p > 4$, $k \geq 2$ and let $\lambda \in \mathbb{R}$. Define

$$(3.4) \quad (d_+ + d^*)_\lambda : L_{k+1, \lambda}^p(\Lambda_+^2 T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) \rightarrow L_{k, \lambda-1}^p(\Lambda^3 T^* \hat{N})$$

by $(d_+ + d^*)_\lambda(\alpha, \beta) = d\alpha + d^*\beta$.

Notes. We use the operator (3.4) rather than simply the exterior derivative on self-dual 2-forms since the former operator is elliptic, whereas the latter is not. The choice of $p > 4$ and $k \geq 2$ ensures that $L_{k+1, \lambda}^p$ -solutions to the (non-linear) deformation problem are in fact smooth.

Definition 3.12. Let N be a CS coassociative 4-fold in an almost G_2 manifold and use the notation of Definition 3.2. Let

$$D(\lambda, i) = \{(\alpha_i, \beta_i) \in C^\infty(\Lambda^2 T^* L_i \oplus \Lambda^3 T^* L_i) : d_i \alpha_i = -\lambda \beta_i, \\ d_i *_{i} \alpha_i + d_i^* \beta_i = -(\lambda + 2)\alpha_i\},$$

where $*_i$, d_i and d_i^* are the Hodge star, the exterior derivative and its formal adjoint on L_i . By [25, Propositions 5.1 and 5.2], the set \mathcal{D} of real numbers

such that (3.4) is not Fredholm is countable and discrete and given by

$$\mathcal{D} = \bigcup_{i=1}^s \{\lambda : D(\lambda, i) \neq 0\}.$$

We also set $d(\lambda) = \sum_{i=1}^s \dim D(\lambda, i)$.

Note. The forms $(\alpha_i, \beta_i) \in D(\lambda, i)$ correspond to homogeneous forms $(\alpha, \beta) \in C^\infty(\Lambda^2_+ T^* C_i \oplus \Lambda^4 T^* C_i)$ of order λ which satisfy $d\alpha + d^*\beta = 0$ on the cone C_i .

In [25] the author studied deformations of CS coassociative 4-folds, allowing the singularities and tangent cones to vary, and permitting changes in the ambient G_2 structure. However, for our purposes, we require a slightly more general theory, where we allow the underlying cones on which the singularities are modelled to vary as well. We now define the moduli space of deformations.

Definition 3.13. Let N be a CS coassociative 4-fold in an almost G_2 manifold (M, φ, g_φ) and let \mathcal{D} be given by Definition 3.12. Suppose further that N has singularities at z_1, \dots, z_s with rate $\mu \in (1, 2) \setminus \mathcal{D}$, having cone C_i and tangent cone \hat{C}_i at z_i for all i . Let $\mathfrak{C} = \prod_{i=1}^s \mathcal{C}_i$ where \mathcal{C}_i is a smooth, connected family of coassociative cones in \mathbb{R}^7 , closed under the natural action of $G_2 \ltimes \mathbb{R}^7$, such that $C_i \in \mathcal{C}_i$ for all i .

The moduli space of deformations $\mathcal{M}(N, \mu, \mathfrak{C})$ is the set of CS coassociative 4-folds N' in M such that:

- (a) N' has a singularity at z'_i with rate μ and cone in \mathcal{C}_i for $i = 1, \dots, s$; and
- (b) there exists a diffeomorphism $h : M \rightarrow M$, isotopic to the identity, such that $h(z_i) = z'_i$ for all i , $h|_N : N \rightarrow N'$ is a homeomorphism and $h|_{\hat{N}} : \hat{N} \rightarrow N' \setminus \{z'_1, \dots, z'_s\}$ is a diffeomorphism.

We state the deformation theory result we require.

Theorem 3.14. *Use the notation of Definition 3.13, let $p > 4$ and $k \geq 2$. There exist finite-dimensional vector spaces $\mathcal{I}(N, \mu, \mathfrak{C})$ and $\mathcal{O}(N, \mu, \mathfrak{C})$, with $\mathcal{O}(N, \mu, \mathfrak{C})$ contained in $L^p_{k, \mu-1}(\Lambda^3 T^* \hat{N})$, and there exist*

- (a) a smooth manifold $\hat{\mathcal{M}}(N, \mu, \mathfrak{C})$, which is an open neighbourhood of 0 in $\mathcal{I}(N, \mu, \mathfrak{C})$, and

(b) a smooth map $\pi : \hat{\mathcal{M}}(N, \mu, \mathfrak{C}) \rightarrow \mathcal{O}(N, \mu, \mathfrak{C})$, with $\pi(0) = 0$,

such that an open neighbourhood of zero in $\text{Ker } \pi$ is homeomorphic to an open neighbourhood of N in $\mathcal{M}(N, \mu, \mathfrak{C})$.

Furthermore, if $\mathcal{O}(N, \mu, \mathfrak{C}) = \{0\}$, then $\mathcal{M}(N, \mu, \mathfrak{C})$ is a smooth manifold of dimension equal to that of $\mathcal{I}(N, \mu, \mathfrak{C})$.

We can actually say more about the spaces $\mathcal{I}(N, \mu, \mathfrak{C})$ and $\mathcal{O}(N, \mu, \mathfrak{C})$.

Proposition 3.15. *Use the notation of Definitions 3.11–3.13 and Theorem 3.14.*

- (a) $\mathcal{I}(N, \mu, \mathfrak{C})$ contains a subspace isomorphic to $\text{Ker}(d_+ + d^*)_\mu$.
- (b) $\mathcal{O}(N, \mu, \mathfrak{C})$ is transverse to the space $\text{Image}(d_+ + d^*)_\mu$, it is contained in $d(L_{k+1, \mu}^p(\Lambda^2 T^* \hat{N})) \subseteq L_{k, \mu-1}^p(\Lambda^3 T^* \hat{N})$ and satisfies

$$(3.5) \quad \dim \mathcal{O}(N, \mu, \mathfrak{C}) \leq \sum_{\lambda \in (-2, \mu) \cap \mathcal{D}} d(\lambda) - \sum_{i=1}^s \dim C_i.$$

These results essentially follow from [25, Theorem 7.9 and Proposition 8.10], the only difference being that we allow the cones C_i on which the singularities are modelled to deform in the families \mathcal{C}_i , which may be larger than the families given simply by translations and G_2 transformations of C_i . Rather than repeating the entire analysis in [25] with this change, we appeal to the similar discussion in [16, Section 8.3]; that is, the *infinitesimal deformation space* $\mathcal{I}(N, \mu, \mathfrak{C})$ is unchanged but the dimension of the *obstruction space* $\mathcal{O}(N, \mu, \mathfrak{C})$ is reduced by the dimension of the families in which the cones vary.

Notes. In (3.5), the sum is over $(-2, \mu) \cap \mathcal{D}$, rather than $(-1, \mu) \cap \mathcal{D}$ as in [25, Proposition 8.10]. This ability to improve to the latter smaller set was based on the claim in [25, Proposition 5.3], which followed from applying [23, Theorem 10.2], that $(-2, -1] \cap \mathcal{D} = \emptyset$. Although $(-2, -1) \cap \mathcal{D} = \emptyset$ for almost all examples of interest, often $-1 \in \mathcal{D}$ so the claim is erroneous. However, we shall show in Proposition 4.11 that not only is [25, Proposition 8.10] valid, but the estimate in (3.5) can be improved even further.

In [25, Theorem 7.13] we proved a result for deformations of N where the ambient G_2 structure on M also varies. We see that we have to restrict our choice of perturbations of the G_2 structure as follows.

Definition 3.16. Use the notation of Definition 3.13. By [25, Proposition 6.19] there exists a neighbourhood T_N of N in M , which retracts onto N and $H_{\text{dR}}^3(T_N) \cong H_{\text{cs}}^3(\hat{N})$, the third compactly supported cohomology group of \hat{N} . Thus, any closed positive 3-form φ' on M defines a cohomology class $[\varphi'|_{\hat{N}}] \in H_{\text{cs}}^3(\hat{N})$ and, moreover, if $[\varphi'|_{\hat{N}}] \neq 0$ in $H_{\text{cs}}^3(\hat{N})$ there are no nearby coassociative deformations of N . Therefore, we let $\mathcal{F} = \{(\varphi^f, g_{\varphi^f}) : f \in B(0; \delta) \subseteq \mathbb{R}^r\}$ be a smooth r -dimensional family of closed G_2 structures on M , with $\varphi^0 = \varphi$ and $[\varphi^f|_{\hat{N}}] = 0$ for all $f \in B(0; \delta)$.

The obstruction space for the deformation problem where the ambient G_2 structure deforms in the family \mathcal{F} , suitably generalized to include the possibility that the cones at the singularities vary in the family \mathfrak{C} , is contained in $\mathcal{O}(N, \mu, \mathfrak{C})$. Therefore, if $\mathcal{O}(N, \mu, \mathfrak{C}) = \{0\}$, for any sufficiently small perturbation of the G_2 structure (φ, g_φ) in \mathcal{F} we obtain a corresponding CS deformation of N which is coassociative with respect to the new G_2 structure. Thus, we have the following result using [25, Theorem 7.13].

Theorem 3.17. *Use the notation of Definitions 3.13 and 3.16 and Theorem 3.14. If $\mathcal{O}(N, \mu, \mathfrak{C}) = \{0\}$, there exists $\delta_N \in (0, \delta)$ such that, if $f \in B(0; \delta_N)$, then there exists a CS deformation N^f of N in T_N with s singularities with rate μ , modelled on cones in \mathfrak{C} , which is coassociative with respect to φ^f .*

Note. By the method of proof of [25, Theorem 7.13], a “sufficiently small” perturbation $(\varphi', g_{\varphi'})$ of (φ, g_φ) is one for which $\|\varphi' - \varphi\|_{C^1} < \epsilon_{C^1}$ and $\|\varphi' - \varphi\|_{L_2^p} < \epsilon_{L_2^p}$ for some $p > 4$, where the norms are calculated in T_N with respect to g_φ , and ϵ_{C^1} and $\epsilon_{L_2^p}$ are small constants determined by the geometry of (M, φ, g_φ) near N (i.e., in T_N). Thus δ_N is chosen so that these C^1 and L_2^p norms are smaller than the appropriate constants.

3.4. Multiplicity one tangent cones

In this subsection, we study coassociative integral currents with multiplicity one tangent cones, motivated by the work on special Lagrangian integral currents in [15, Section 6]. The key condition is that the underlying cone is *Jacobi integrable*, which we define for a coassociative cone following [15, Definition 6.7].

Definition 3.18. Let C be a coassociative cone in \mathbb{R}^7 with compact link L in \mathcal{S}^6 such that $C \setminus \{0\}$ is non-singular. We say that $v \in C^\infty(\nu_{\mathcal{S}^6}(L))$

is a *Lagrangian Jacobi field* if $\alpha_v = *(v \lrcorner \varphi_0)|_{TL} \in C^\infty(T^*L)$ satisfies $d\alpha_v = -3*\alpha_v$, where $*$ is the Hodge star on L . Notice that v is a Lagrangian Jacobi field if and only if $r^3*\alpha_v + r^2\alpha_v \wedge dr$ is a closed self-dual 2-form on C of order $O(r)$, and so defines an infinitesimal deformation of C as a coassociative cone; i.e., an infinitesimal deformation of L as a Lagrangian in S^6 in the sense of Definition 2.12. Since Lagrangians in S^6 are minimal, Lagrangian Jacobi fields are Jacobi fields in the usual sense.

We say that C is *Jacobi integrable* if every Lagrangian Jacobi field v on L defines a smooth one-parameter family $\{L_t = \exp_{tv}(L) \subseteq S^6 : t \in (-\tau, \tau)\}$, for some $\tau > 0$, of Lagrangian submanifolds of S^6 .

Our next result proves Theorem 1.1; namely, that interior singular points of coassociative integral currents, with multiplicity one tangent cones modelled on Jacobi integrable cones, are conical singularities in the sense of Definition 3.2.

Theorem 3.19. *Let N be a coassociative integral current in an almost G_2 manifold M and let $z \in N^\circ$ be a singular point of N . Let $\{\chi : B(0; \epsilon_M) \rightarrow V\}$ be a G_2 coordinate system near z in the sense of Definition 3.1, with $\zeta = d\chi|_0$. Suppose that C is a cone in \mathbb{R}^7 with compact link L such that $C \setminus \{0\}$ is non-singular and $\hat{C} = \zeta(C) \subseteq T_z M$ is a multiplicity one tangent cone for N at z . Then C is coassociative and \hat{C} is the unique tangent cone for N at z .*

Suppose further that C is Jacobi integrable in the sense of Definition 3.18 and let $U = N \cap V$. Then there exist $\epsilon \in (0, \epsilon_M)$ and an embedding $\Phi : (0, \epsilon) \times L \rightarrow B(0; \epsilon_M)$ such that $U \setminus \{z\} = \chi \circ \Phi((0, \epsilon) \times L)$ as an embedded submanifold of M and Φ satisfies (3.1) and (3.2) for some $\mu \in (1, 2)$.

Proof. The fact that C is coassociative follows from Proposition 2.7. By [37, Theorem 5.7], the tangent cone \hat{C} is unique and $U \setminus \{z\}$ can be realized as a C^2 -embedding Ψ of $(0, \epsilon) \times L$, for some $\epsilon > 0$. Moreover, $\Psi = \chi \circ \Phi$ where Φ is a C^2 -embedding of $(0, \epsilon) \times L$ into $B(0; \epsilon_M)$ which satisfies (3.1). This last point is not explicit in the statement of the theorem, but does follow from the proof. Moreover, if $\iota : C \setminus \{0\} \cong (0, \infty) \times L \rightarrow \mathbb{R}^7$ is the inclusion map, Φ also satisfies:

$$(3.6) \quad |\Phi(r, x) - \iota(r, x)| = o(r),$$

$$(3.7) \quad |\nabla(\Phi(r, x) - \iota(r, x))| = o(1),$$

$$(3.8) \quad |\nabla^2(\Phi(r, x) - \iota(r, x))| = O(1)$$

as $r \rightarrow 0$. Equation (3.8) again is not stated explicitly in [37, Theorem 5.7] but follows from the minimality of C and N as discussed in the proof of [1, Theorem 1]. The aforementioned theorem [1, Theorem 1] and its proof imply that there exists $\mu \in (1, 2)$ such that the estimates (3.6) and (3.7) can be improved to:

$$(3.9) \quad |\Phi(r, x) - \iota(r, x)| = O(r^\mu),$$

$$(3.10) \quad |\nabla(\Phi(r, x) - \iota(r, x))| = O(r^{\mu-1})$$

as $r \rightarrow 0$, since C is Jacobi integrable. We now only need a regularity argument to complete the proof, similar to [27, Proposition 4.17], which we now detail.

Let $P = \iota((0, \epsilon) \times L) \subseteq \mathbb{R}^7$ and recall that, since P is coassociative, we have an isomorphism $J_P : \nu(P) \rightarrow \Lambda_+^2 T^*P$ by Proposition 2.17. For $\alpha \in C_{\text{loc}}^2(\Lambda_+^2 T^*P)$ and $v = J_P^{-1}(\alpha)$, define

$$F(\alpha) = \exp_v^*(\varphi_0|_{\exp_v(P)}) \in C_{\text{loc}}^1(\Lambda^3 T^*P)$$

and

$$G(\alpha) = \pi_{\Lambda_+^2}(d^*F(\alpha)) \in C_{\text{loc}}^0(\Lambda_+^2 T^*P),$$

where $\pi_{\Lambda_+^2}$ is the projection from 2-forms to self-dual 2-forms on P . Clearly, if $F(\alpha) = 0$, the deformation $\exp_v(P)$ of P is coassociative and $G(\alpha) = 0$. Moreover, $dF|_0(\alpha) = d\alpha$ by [31, p. 731] and thus $dG|_0(\alpha) = d_+^*d\alpha$, where $d_+^* = \pi_{\Lambda_+^2} \circ d^*$ acting on 3-forms on P . We deduce that $G(\alpha) = 0$ is a non-linear *elliptic* equation for α at 0; that is, its linearization at 0 is elliptic.

Since Φ is a C^2 -coassociative embedding satisfying (3.1), it defines a C^2 -normal vector field v_Φ on P and hence a C^2 -self-dual 2-form $\alpha_\Phi = J_P(v_\Phi)$ on P which satisfies $F(\alpha_\Phi) = 0$. Moreover, $\alpha_\Phi \in C_\mu^2(\Lambda_+^2 T^*P)$ by (3.9)–(3.10) and (3.8), since $\mu - 2 < 0$. Thus, $\alpha_\Phi \in C_\mu^{1,a}(\Lambda_+^2 T^*P)$ for any $a \in (0, 1)$.

Let ∇ be the Levi-Civita connection of the conical metric on P . We can write, for $\alpha \in C_{\text{loc}}^2(\Lambda_+^2 T^*P)$ and $p \in P$,

$$G(\alpha)(p) = R(p, \alpha(p), \nabla\alpha(p))\nabla^2\alpha(p) + E(p, \alpha(p), \nabla\alpha(p)),$$

where R and E are smooth functions of their arguments, since $G(\alpha)$ is linear in $\nabla^2\alpha$ and a smooth function of α . This leads us to define a new operator

on $\beta \in C_{\text{loc}}^2(\Lambda_+^2 T^*P)$ by

$$S_\Phi(\beta)(p) = R(p, \alpha_\Phi(p), \nabla\alpha_\Phi(p))\nabla^2\beta(p).$$

Note that S_Φ is *not* the linearization of G of 0, but it is a *linear second-order elliptic* operator on β with coefficients in $C_{\text{loc}}^{0,a}$.

Since $G(\alpha_\Phi) = 0$, $S_\Phi(\alpha_\Phi) = -E(\alpha_\Phi)$, where we set

$$E(\beta)(p) = E(p, \beta(p), \nabla\beta(p))$$

for $p \in P$ and $\beta \in C_{\text{loc}}^2(\Lambda_+^2 T^*P)$. As argued in [27, Proposition 4.17], $E(\beta) \in C_{\mu-1}^{k,a}$ if $\beta \in C_\mu^{k+1,a}$ since E is no worse than quadratic in β and $\nabla\beta$. Thus, using the Schauder regularity estimates as given in [30] and the fact that $\alpha_\Phi \in C_\mu^{1,a}$, we deduce that $\alpha_\Phi \in C_\mu^{2,a}(\Lambda_+^2 T^*P)$. A standard inductive argument leads us to the conclusion that $\alpha_\Phi \in C_\mu^{k,a}$ for all $k \in \mathbb{N}$. Thus the corresponding map Φ satisfies (3.2) as claimed. \square

From Definition 3.2, we have an immediate corollary to Theorem 3.19.

Corollary 3.20. *Let N be a connected coassociative integral current in an almost G_2 manifold with $\partial N = \emptyset$. If N has multiplicity one tangent cones at its singular points modelled on Jacobi integrable coassociative cones, then N is a CS coassociative 4-fold in the sense of Definition 3.2.*

4. Coassociative cones and stability index

In this section, we define the notion of *stability* of conical singularities of coassociative 4-folds using a numerical invariant for a coassociative cone which we call the *stability index*. This stability index is calculated using the spectrum of the curl operator acting on 1-forms on the link of the cone. We begin by giving a brief survey of the known examples of coassociative cones.

4.1. Examples

For this subsection it is convenient to identify \mathbb{R}^7 with the imaginary octonions $\text{Im } \mathbb{O}$. Let $\{\varepsilon_1, \dots, \varepsilon_7\}$ be a basis for $\text{Im } \mathbb{O}$ satisfying the multiplication

law below.

	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
ε_1	-1	ε_3	$-\varepsilon_2$	ε_5	$-\varepsilon_4$	ε_7	$-\varepsilon_6$
ε_2	$-\varepsilon_3$	-1	ε_1	ε_6	$-\varepsilon_7$	$-\varepsilon_4$	ε_5
ε_3	ε_2	$-\varepsilon_1$	-1	$-\varepsilon_7$	$-\varepsilon_6$	ε_5	ε_4
ε_4	$-\varepsilon_5$	$-\varepsilon_6$	ε_7	-1	ε_1	ε_2	$-\varepsilon_3$
ε_5	ε_4	ε_7	ε_6	$-\varepsilon_1$	-1	$-\varepsilon_3$	$-\varepsilon_2$
ε_6	$-\varepsilon_7$	ε_4	$-\varepsilon_5$	$-\varepsilon_2$	ε_3	-1	ε_1
ε_7	ε_6	$-\varepsilon_5$	$-\varepsilon_4$	ε_3	ε_2	$-\varepsilon_1$	-1

If we identify $\text{Im } \mathbb{O} \cong \mathbb{R}^7$ so that $\{\varepsilon_1, \dots, \varepsilon_7\}$ corresponds to the standard oriented orthonormal basis in \mathbb{R}^7 , then the multiplication law we have chosen is consistent with the G_2 structure on \mathbb{R}^7 given by φ_0 in (2.1).

The coassociative cones invariant under a closed 3-dimensional Lie subgroup of G_2 were classified in [29]. We now review these examples starting with the degenerate example of a coassociative cone, namely a coassociative 4-plane.

Example 4.1 (Planes). By [19, Proposition 12.1.2], G_2 acts transitively on the set of coassociative 4-planes with isotropy $\text{SO}(4)$. Therefore, any coassociative 4-plane is equivalent up to G_2 transformation to

$$C_0 = \{x_0\varepsilon_1 + x_1\varepsilon_3 + x_2\varepsilon_5 + x_3\varepsilon_7 \in \text{Im } \mathbb{O} : (x_0, x_1, x_2, x_3) \in \mathbb{R}^4\}.$$

The link L_0 of C_0 is trivially a totally geodesic $\text{SO}(4)$ -invariant \mathcal{S}^3 in \mathcal{S}^6 .

Our next example was first introduced in [22, Section 7] and shown to be coassociative in [10, Theorem IV.3.2].

Example 4.2 (U(2) symmetry). The cone C_1 in $\text{Im } \mathbb{O}$, with link

$$L_1 = \left\{ \frac{\sqrt{5}}{3} \bar{q}\varepsilon_1 q + \frac{2}{3} q\varepsilon_5 : q \in \langle 1, \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle_{\mathbb{R}} \text{ with } |q| = 1 \right\},$$

is coassociative and invariant under a $\text{U}(2)$ subgroup of G_2 . The Lagrangian L_1 is realized as an $\text{Sp}(1)$ -orbit in \mathcal{S}^6 .

Recall that, by Corollary 2.11, any complex 2-dimensional cone in \mathbb{C}^3 embedded in \mathbb{R}^7 is coassociative.

Example 4.3 (Complex SO(3) symmetry). Identify $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ as in Lemma 2.9. The cone

$$C_2 = \{(0, z_1, z_2, z_3) \in \mathbb{R} \oplus \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 0\}$$

is coassociative and invariant under the standard SO(3) action on \mathbb{C}^3 . The real link L_2 of C_2 in \mathcal{S}^6 is diffeomorphic to SO(3) and is the Hopf lift of the constant curvature degree 2 $\mathbb{C}\mathbb{P}^1$ in $\mathbb{C}\mathbb{P}^2$.

Our final symmetric examples are most easily described using homogeneous harmonic cubics on \mathbb{R}^3 .

Example 4.4 (SO(3) symmetry). Identify $\text{Im } \mathbb{O}$ with the homogeneous harmonic cubics $\mathcal{H}^3(\mathbb{R}^3)$ on \mathbb{R}^3 by:

$$\begin{aligned} \varepsilon_1 &\mapsto \frac{\sqrt{10}}{10} x(2x^2 - 3y^2 - 3z^2); & \varepsilon_3 &\mapsto \frac{\sqrt{6}}{2} x(y^2 - z^2); \\ \varepsilon_2 &\mapsto -\sqrt{6}xyz; & \varepsilon_5 &\mapsto -\frac{\sqrt{15}}{10} z(4x^2 - y^2 - z^2); \\ \varepsilon_4 &\mapsto -\frac{\sqrt{15}}{10} y(4x^2 - y^2 - z^2); & \varepsilon_7 &\mapsto -\frac{1}{2} z(z^2 - 3y^2). \\ \varepsilon_6 &\mapsto \frac{1}{2} y(y^2 - 3z^2); \end{aligned}$$

The standard SO(3) action on \mathbb{R}^3 induces an action on $\mathcal{H}^3(\mathbb{R}^3)$, hence on $\text{Im } \mathbb{O}$.

Let L_3 be the orbit through ε_6 of this SO(3) action on $\text{Im } \mathbb{O}$ and let L_4 be the orbit through ε_2 . By [29, Theorem 4.3] and observations in [28, Examples 6.6 and 6.15], $L_3 \cong \text{SO}(3)/\text{S}_3$ and $L_4 \cong \text{SO}(3)/\text{A}_4$ are Lagrangian. Thus the cones C_3 and C_4 on L_3 and L_4 respectively are coassociative and SO(3)-invariant.

Note. The SO(3)-orbit through ε_1 is a constant curvature $\frac{1}{6}$ pseudoholomorphic curve in \mathcal{S}^6 often called the *Borůvka sphere* in \mathcal{S}^6 .

All of the coassociative cones introduced so far have links which are fibred by oriented circles of constant radius over a surface. Lagrangians in \mathcal{S}^6 with this property were classified in [28]. This classification describes all of the known links of coassociative cones and it turns out that the circles have radius either $\frac{2}{3}$ or 1. To describe these examples we make two definitions.

Definition 4.5. Let $\mathbf{u} : \Sigma \rightarrow \mathcal{S}^6$ be a surface, let Π be a 2-plane subbundle of $\mathbf{u}^*(T\mathcal{S}^6)$ and let $\mathcal{U}(\Pi) = \{\mathbf{v} \in \Pi : |\mathbf{v}| = 1\}$. For $\gamma \in (0, \frac{\pi}{2}]$ we define $\mathbf{x}_\gamma : \mathcal{U}(\Pi) \rightarrow \mathcal{S}^6$ by $\mathbf{x}_\gamma(\mathbf{v}) = \cos \gamma \mathbf{u} + \sin \gamma \mathbf{v}$. We say that the image of \mathbf{x}_γ is a *tube of radius γ* (in Π) about Σ .

Definition 4.6. If $\mathbf{u} : \Sigma \rightarrow \mathcal{S}^6$ is a non-totally geodesic pseudoholomorphic curve in the sense of Definition 2.12, there is an orthogonal decomposition $\mathbf{u}^*(T^{1,0}\mathcal{S}^6) = T^{1,0}\Sigma \oplus N_1\Sigma \oplus N_2\Sigma$, where $N_1\Sigma$ and $N_2\Sigma$ are holomorphic line bundles such that the second fundamental form of \mathbf{u} takes values in $N_1\Sigma$. We call $N_1\Sigma$ and $N_2\Sigma$ the *first* and *second normal bundles*, respectively.

If $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is a moving orthonormal frame for $T^{1,0}\Sigma \oplus N_1\Sigma \oplus N_2\Sigma$ and θ_1 is the $(1, 0)$ -form dual to \mathbf{f}_1 , then the structure equations for Σ are:

$$\begin{aligned} d\mathbf{u} &= -2i\mathbf{f}_1\theta_1 + 2i\bar{\mathbf{f}}_1\bar{\theta}_1; & d\mathbf{f}_1 &= -i\mathbf{u}\bar{\theta}_1 + \mathbf{f}_1\kappa_{11} + \mathbf{f}_2\kappa_{21}; \\ d\mathbf{f}_2 &= -\mathbf{f}_1\bar{\kappa}_{21} + \mathbf{f}_2\kappa_{22} + \mathbf{f}_3\kappa_{32} - \bar{\mathbf{f}}_3\theta_1; & d\mathbf{f}_3 &= -\mathbf{f}_2\bar{\kappa}_{32} + \mathbf{f}_3\kappa_{33} + \bar{\mathbf{f}}_2\theta_1; \\ d\theta_1 &= -\kappa_{11} \wedge \theta_1; & d\kappa_{11} &= -\kappa_{21} \wedge \bar{\kappa}_{21} + 2\theta_1 \wedge \bar{\theta}_1; \\ d\kappa_{22} &= \kappa_{21} \wedge \bar{\kappa}_{21} - \kappa_{32} \wedge \bar{\kappa}_{32} - \theta_1 \wedge \bar{\theta}_1; & d\kappa_{33} &= \kappa_{32} \wedge \bar{\kappa}_{32} - \theta_1 \wedge \bar{\theta}_1; \\ d\kappa_{21} &= (\kappa_{11} - \kappa_{22}) \wedge \kappa_{21}; & d\kappa_{32} &= (\kappa_{22} - \kappa_{33}) \wedge \kappa_{32}, \end{aligned}$$

for some imaginary-valued 1-forms $\kappa_{11}, \kappa_{22}, \kappa_{33}$ such that $\kappa_{11} + \kappa_{22} + \kappa_{33} = 0$ and complex-valued 1-forms κ_{21} and κ_{32} . Moreover, by the work in [4, Section 4], there exist holomorphic functions K and T such that $\kappa_{21} = K\theta_1$ and $\kappa_{32} = T\theta_1$. We identify K with the second fundamental form of Σ , so $K \neq 0$, and we call T the *torsion* of Σ . Of particular interest are the pseudoholomorphic curves with *null-torsion*, i.e., with $T \equiv 0$, since they may be viewed as certain algebraic curves in the 5-quadric in $\mathbb{C}\mathbb{P}^6$ and every compact Riemann surface can be realized as such a curve in \mathcal{S}^6 by the work in [4, Section 4].

Remark. The Borůvka sphere Σ in \mathcal{S}^6 has null-torsion.

The coassociative cones with links which are fibred by circles of radius $\frac{2}{3}$ are locally cones over tubes of radius $\sin^{-1}(\frac{2}{3})$ in $N_2\Sigma$ about null-torsion pseudoholomorphic curves $\Sigma \subseteq \mathcal{S}^6$. This includes C_1 given in Example 4.2, where Σ is a totally geodesic \mathcal{S}^2 (which is the degenerate case of a null-torsion curve), and C_4 given in Example 4.4, where Σ is the Borůvka sphere.

We shall be more concerned with the case where the link is fibred by oriented geodesic circles so we make the following definition as in [24].

Definition 4.7. A 4-dimensional submanifold N of \mathbb{R}^n is *2-ruled* if there is a surface Σ and a smooth fibration $\pi : N \rightarrow \Sigma$ whose fibres are affine 2-planes in \mathbb{R}^n . In addition, we say that N is *r-framed* if there is a choice of oriented frame for each 2-plane $\pi^{-1}(\sigma)$ which varies smoothly with $\sigma \in \Sigma$.

The 2-ruled coassociative 4-folds in \mathbb{R}^7 were studied in [24], and there was a further focus on the conical case in [9]. By [28, Theorems 1.1–1.2] we can extend the work in [9] and describe all r-framed 2-ruled coassociative cones. We refer the reader to [28] for full details but give a brief description here.

Example 4.8 (The 2-ruled family). Complex 2-dimensional cones in \mathbb{C}^3 embedded in \mathbb{R}^7 are 2-ruled coassociative cones by Corollary 2.11.

Recall Definition 4.5. The general r-framed 2-ruled coassociative cone C has link L such that, for all p in an open dense subset L^* of L , there exist an open set $U \ni p$, a non-totally geodesic pseudoholomorphic curve $\mathbf{u} : \Sigma \rightarrow \mathcal{S}^6$, and a holomorphic line subbundle Π of $\mathbf{u}^*(T^{1,0}\mathcal{S}^6)$ such that $U \cap L^*$ is a tube of radius $\frac{\pi}{2}$ in Π about Σ . There are restrictions on the choice of line subbundle Π (see [28, Example 7.4]), however, in particular, we may always choose $\Pi = N_2\Sigma$.

We conclude this subsection with the following important examples of 2-ruled coassociative cones.

Example 4.9. Let C be a coassociative cone with link L admitting a Killing vector field whose integral curves are geodesic circles in \mathcal{S}^6 . By [41, Theorem 2], either C is a complex 2-dimensional cone in \mathbb{C}^3 or L is locally a tube of radius $\frac{\pi}{2}$ in $N_1\Sigma$ or $N_2\Sigma$ about a *null-torsion* pseudoholomorphic curve Σ in \mathcal{S}^6 .

Cones as in Example 4.9 are given trivially by C_2 in Example 4.3 but also by C_3 in Example 4.4, since L_3 is a tube of radius $\frac{\pi}{2}$ in $N_2\Sigma$ about the Borůvka sphere Σ .

4.2. Stability index and the curl operator

For this subsection, use the notation of Definitions 3.2 and 3.13 and Theorem 3.14. In particular, N is a CS coassociative 4-fold where the s singularities have rate $\mu \in (1, 2) \setminus \mathcal{D}$, where \mathcal{D} is given in Definition 3.12, and the s -tuple of cones at the singularities (C_1, \dots, C_s) lies in $\mathfrak{C} = \prod_{i=1}^s \mathcal{C}_i$.

To get an effective notion of stability for the conical singularities of N we need to improve the estimate (3.5) for the dimension of the obstruction space $\mathcal{O}(N, \mu, \mathfrak{C})$ for the deformation problem for N . To understand this we need to compare the maps $(d_+ + d^*)_\lambda$ given by (3.4) and

$$(4.1) \quad (d + d^*)_\lambda : L^p_{k+1, \lambda}(\Lambda^2 T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) \rightarrow L^p_{k, \lambda-1}(\Lambda^3 T^* \hat{N})$$

$$(\alpha, \beta) \mapsto d\alpha + d^* \beta$$

for $\lambda \in (-2, \mu]$. From Theorem 3.14, we see that $\mathcal{O}(N, \mu, \mathfrak{C})$ is a subspace of the cokernel of $(d_+ + d^*)_\mu$ which is transverse to the cokernel of $(d + d^*)_\mu$. Moreover, from the work in [25, Section 8], the sum over $\lambda \in (-2, \mu) \cap \mathcal{D}$ in (3.5) is the upper bound of the dimension of the space of forms which adds to the cokernel of $(d_+ + d^*)_\lambda$ as the rate λ increases from -2 to μ . Hence, we can improve the upper bound by considering the forms which add to the cokernel of $(d_+ + d^*)_\lambda$ but *not* the cokernel of $(d + d^*)_\lambda$. These cokernels are isomorphic to the annihilators of the images of $(d_+ + d^*)_\lambda$ and $(d + d^*)_\lambda$ under the dual pairing given in Definition 3.7. Thus, by comparing these annihilators we may obtain our estimate.

We begin with the following.

Proposition 4.10. *Let $\mathcal{A}_+(\lambda)$ and $\mathcal{A}(\lambda)$ denote the annihilators of the images of (3.4) and (4.1) via the dual pairing given in Definition 3.7. For $\lambda \leq -1$, $\mathcal{A}_+(\lambda) = \mathcal{A}(\lambda)$.*

Proof. As observed in the proof of [25, Proposition 8.6], if $\lambda \notin \mathcal{D}$, there exist finite-dimensional spaces $\mathcal{C}_+(\lambda)$ and $\mathcal{C}(\lambda)$ of smooth compactly supported 3-forms on \hat{N} such that

$$L^p_{k, \lambda-1}(\Lambda^3 T^* \hat{N}) = \text{Image}(d_+ + d^*)_\lambda \oplus \mathcal{C}_+(\lambda) = \overline{\text{Image}(d + d^*)_\lambda} \oplus \mathcal{C}(\lambda)$$

and the dual pairings between $\mathcal{C}_+(\lambda)$ and $\mathcal{A}_+(\lambda)$, and between $\mathcal{C}(\lambda)$ and $\mathcal{A}(\lambda)$, are non-degenerate. Let $\mathcal{C}'(\lambda)$ be such that $\mathcal{C}_+(\lambda) = \mathcal{C}(\lambda) \oplus \mathcal{C}'(\lambda)$.

Notice by Definition 3.7 that if $q > 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$ and $l \in \mathbb{N}$ then, for any $\lambda \in \mathbb{R}$,

$$\mathcal{A}_+(\lambda) = \{ \gamma \in L^q_{l+1, -3-\lambda}(\Lambda^3 T^* \hat{N}) : \langle d\alpha + d^* \beta, \gamma \rangle = 0$$

$$\text{for all } (\alpha, \beta) \in L^p_{k+1, \lambda}(\Lambda^2_+ T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) \}$$

$$\begin{aligned} &= \{ \gamma \in L^q_{l+1, -3-\lambda}(\Lambda^3 T^* \hat{N}) : \langle \alpha, d^* \gamma \rangle = 0, \langle \beta, d\gamma \rangle = 0 \\ &\quad \text{for all } (\alpha, \beta) \in L^p_{k+1, \lambda}(\Lambda^2_+ T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) \} \\ &= \{ \gamma \in L^q_{l+1, -3-\lambda}(\Lambda^3 T^* \hat{N}) : d\gamma = 0, d^* \gamma \in L^q_{l, -4-\lambda}(\Lambda^2_- T^* \hat{N}) \}, \end{aligned}$$

where the integration by parts is justified by the choice of weight for the dual weighted Sobolev space. Similarly,

$$\mathcal{A}(\lambda) = \{ \gamma \in L^q_{l+1, -3-\lambda}(\Lambda^3 T^* \hat{N}) : d\gamma = d^* \gamma = 0 \}.$$

If $\lambda \leq -1$, then $-3 - \lambda \geq \lambda - 1$ so, by Theorem 3.10, $L^q_{l+1, -3-\lambda} \hookrightarrow L^p_{k, \lambda-1}$ for sufficiently large l . Since $\mathcal{A}_+(\lambda)$ consists of smooth forms, as it is the kernel of an elliptic operator, we can choose l arbitrarily large and see that $\mathcal{A}_+(\lambda) \subseteq L^p_{k, \lambda-1}(\Lambda^3 T^* \hat{N})$ and the same is clearly true for $\mathcal{A}(\lambda)$. Moreover, if additionally $\lambda \notin \mathcal{D}$, since $\mathcal{A}_+(\lambda)$ and $\mathcal{A}(\lambda)$ are of equal dimension to $\mathcal{C}_+(\lambda)$ and $\mathcal{C}(\lambda)$, and the annihilators are by construction orthogonal to the closures of the images of (3.4) and (4.1), we deduce that $\mathcal{A}_+(\lambda) = \mathcal{C}_+(\lambda)$ and $\mathcal{A}(\lambda) = \mathcal{C}(\lambda)$.

Suppose that $\gamma \in \mathcal{C}'(\lambda)$. Then γ is compactly supported and lies in $\mathcal{A}_+(\lambda)$. Therefore, since $d^* \gamma$ is anti-self-dual,

$$\|d^* \gamma\|_{L^2}^2 = \int_{\hat{N}} -d^* \gamma \wedge d^* \gamma = \int_{\hat{N}} -d^* \gamma \wedge d\gamma = \int_{\hat{N}} -d(*\gamma \wedge d^* \gamma) = 0,$$

where the integration by parts is valid since γ is compactly supported. Thus, $\gamma \in \mathcal{A}(\lambda) = \mathcal{C}(\lambda)$, so $\gamma = 0$. We deduce that $\mathcal{C}(\lambda) = \mathcal{C}_+(\lambda) = \mathcal{A}(\lambda) = \mathcal{A}_+(\lambda)$ for $\lambda \leq -1$, $\lambda \notin \mathcal{D}$. The annihilators are well-defined for $\lambda \in \mathcal{D}$ and hence, since the dimension of $\mathcal{A}_+(\lambda)$ is lower semi-continuous at $\lambda = -1$, as remarked in the proof of [25, Proposition 8.4], we see that $\mathcal{A}(\lambda) = \mathcal{A}_+(\lambda)$ for all $\lambda \leq -1$. □

Proposition 4.11. *Recall the notation of Definition 3.12. Let*

$$\check{D}(\lambda, i) = \{ \gamma_i \in C^\infty(T^* L_i) : d_i^* \gamma_i = 0, d_i \gamma_i = -(\lambda + 2) *_i \gamma_i \}$$

and let $\check{d}(\lambda) = \sum_{i=1}^s \dim \check{D}(\lambda, i)$. Then

$$(4.2) \quad \dim \mathcal{O}(N, \mu, \mathfrak{E}) \leq \sum_{\lambda \in (-1, \mu) \cap \mathcal{D}} \check{d}(\lambda) - \sum_{i=1}^s \dim \mathcal{C}_i.$$

Proof. By Proposition 4.10, there are no forms which add to the cokernel of (3.4) but not the cokernel of (4.1) for $\lambda \leq -1$. Therefore, we can first improve the estimate (3.5) by restricting to the range of rates $(-1, \mu) \cap \mathcal{D}$.

The map $(d + d^*)_\lambda$, given in (4.1), is not elliptic, but forms part of the map

$$d + d^* : L^p_{k+1, \lambda}(\Lambda^{\text{even}}T^*\hat{N}) \rightarrow L^p_{k, \lambda-1}(\Lambda^{\text{odd}}T^*\hat{N}),$$

which is elliptic. Therefore, we can apply the theory of [23] and deduce that there is a countable discrete set of rates $\mathcal{E} \supseteq \mathcal{D}$ for which $d + d^*$ is Fredholm and, moreover, we can calculate which forms on L_i correspond to forms on \hat{N} which subtract from the kernel or add to the cokernel as λ increases.

The cokernel of $d + d^*$ is isomorphic to the annihilator of the image of $d + d^*$, using the dual pairing given in Definition 3.7. The work in [23] identifies the forms which add to this annihilator as the rate crosses elements $\lambda \in \mathcal{E}$. To calculate the changes in the annihilator, we need to consider homogeneous forms on the cone C_i of the following type:

$$(\gamma^1, \gamma^3) = (r^{-\lambda-2}\alpha^1 + r^{-\lambda-3}\alpha^0 \wedge dr, r^{-\lambda}\alpha^3 + r^{-\lambda-1}\alpha^2 \wedge dr),$$

for $(\alpha^0, \alpha^1, \alpha^2, \alpha^3) \in C^\infty(\oplus_{m=0}^3 \Lambda^m T^*L_i)$, which satisfy $d^*\gamma^1 = d\gamma^1 + d^*\gamma^3 = d\gamma^3 = 0$, where d^* is calculated using the cone metric. We therefore have the following conditions which define \mathcal{E} and the changes to the kernel or cokernel of $d + d^*$ acting on even forms, using the notation of Definition 3.12:

$$(4.3) \quad d_i^*\alpha^1 = -\lambda\alpha^0, \quad d_i\alpha^0 + d_i^*\alpha^2 = -(\lambda + 2)\alpha^1,$$

$$(4.4) \quad d_i\alpha^2 = -\lambda\alpha^3, \quad d_i\alpha^1 + d_i^*\alpha^3 = -(\lambda + 2)\alpha^2.$$

Thus, by (4.3) and (4.4), forms $(\alpha, \beta) \in D(\lambda, i)$ giving rise to cokernel forms for $(d_+ + d^*)_\lambda$, $\lambda \in \mathcal{D}$, which can lie in $\mathcal{O}(N, \mu, \mathfrak{C})$ must be transverse to forms $(\alpha', \beta') \in C^\infty(\Lambda^2 T^*L_i \oplus \Lambda^3 T^*L_i)$ satisfying:

$$(4.5) \quad d_i\alpha' = -\lambda\beta', \quad d_i^*\beta' = -(\lambda + 2)\alpha', \quad d_i^*\alpha' = 0.$$

For $\lambda \neq -2, 0$, solutions to (4.5) are equivalent to giving exact forms $\beta' \in C^\infty(\Lambda^3 T^*L_i)$ with $\Delta_i\beta' = \lambda(\lambda + 2)\beta'$, where Δ_i is the Laplacian on L_i , and setting $\alpha' = -(\lambda + 2)^{-1}d_i^*\beta'$. For $\lambda = 0$, β' is locally constant and $\alpha' = 0$. Since β is also an exact eigenform of Δ_i with eigenvalue $\lambda(\lambda + 2)$ for $\lambda \neq 0$ and is constant if $\lambda = 0$, we quickly deduce that $\beta = 0$. Note that for the case $\lambda = -2$, β must be zero since it is harmonic and exact.

Observe that α is automatically orthogonal to coexact forms since $d_i\alpha = 0$ and $-(\lambda + 2)\alpha = d_i *_i \alpha$. Setting $\gamma_i = *_i \alpha$, we see that we can replace the quantities $d(\lambda)$ by $\check{d}(\lambda)$ in our estimate (4.2) for the dimension of $\mathcal{O}(N, \mu, \mathfrak{C})$ as claimed. \square

Theorem 3.14 and Proposition 4.11 invite us to define the stability index of a coassociative cone in a similar manner to [16, Definition 3.6].

Definition 4.12. Let C be a coassociative cone in \mathbb{R}^7 with compact link $L \subseteq S^6$ such that $C \setminus \{0\}$ is non-singular. Let \mathcal{C} be a smooth, connected family of coassociative cones in \mathbb{R}^7 which contains C and is closed under the natural action of $G_2 \times \mathbb{R}^7$. Finally, for $\lambda \in \mathbb{R}$, let

$$(4.6) \quad \check{D}(\lambda) = \{\gamma \in C^\infty(T^*L) : d^*\gamma = 0, d\gamma = -(\lambda + 2) * \gamma\}.$$

We define the \mathcal{C} -stability index of C by

$$(4.7) \quad \text{ind}_{\mathcal{C}}(C) = \sum_{\lambda \in (-1, 1]} \dim \check{D}(\lambda) - \dim \mathcal{C}.$$

If the family \mathcal{C} consists solely of the $G_2 \times \mathbb{R}^7$ transformations of C , we simply write $\text{ind}_{\mathcal{C}}(C) = \text{ind}(C)$ and call $\text{ind}(C)$ the *stability index* of C .

Note that the sum in (4.7) is well-defined because the set of λ for which $\dim \check{D}(\lambda) \neq 0$ is countable and discrete by the observations in Definition 3.12. Moreover, $\dim \check{D}(1)$ is the dimension of the space of Lagrangian Jacobi fields on L by Definition 3.18, so the space of all infinitesimal deformations of C as a coassociative cone, and $\check{D}(0)$ corresponds to the $O(1)$ closed self-dual 2-forms on C , so $\dim \check{D}(0)$ is at least as large as the space of translations of C . Thus, $\text{ind}_{\mathcal{C}}(C) \geq 0$.

We say that the cone C is \mathcal{C} -stable if $\text{ind}_{\mathcal{C}}(C) = 0$. If $\text{ind}(C) = 0$ we say that the cone C is *stable*. We also say that C is *rigid* if $\dim \check{D}(1) = 14 - \dim G$, where G is the Lie subgroup of G_2 preserving C .

Notes.

- (a) It is clear that stability of C implies rigidity. It is also clear that if C is rigid then it is Jacobi integrable, since then every Lagrangian deformation of L in S^6 comes from G_2 transformations. However, we shall see that one may have cones which are Jacobi integrable but neither stable nor rigid.
- (b) By Theorem 3.14 and Proposition 4.11, if all the cones C_i at the singularities of a CS coassociative 4-fold N are \mathcal{C}_i -stable and $\mu \in (1, 2)$ is

such that $(1, \mu] \cap \mathcal{D} = \emptyset$, then the obstruction space $\mathcal{O}(N, \mu, \mathfrak{C}) = \{0\}$ and so the moduli space of deformations $\mathcal{M}(N, \mu, \mathfrak{C})$ is a smooth manifold near N . Moreover, N is “stable” under deformations of the ambient closed G_2 structure by Theorem 3.17.

We see from (4.6) that to determine the stability index for a coassociative cone we need to study the equation

$$*d\gamma = -(\lambda + 2)\gamma,$$

for 1-forms γ on a compact Riemannian 3-manifold L for $\lambda \in (-1, 1]$. The operator $*d : C^\infty(T^*L) \rightarrow C^\infty(T^*L)$ is a natural self-adjoint operator on L which we call the *curl operator*. Thus, our problem is to calculate the negative eigenvalues of the curl operator and their multiplicities. For convenience we make the following definition.

Definition 4.13. Let (L, g_L) be a compact Riemannian 3-manifold and let $c_L = -*d : C^\infty(T^*L) \rightarrow C^\infty(T^*L)$. Denote by $\sigma_L(\nu)$ the multiplicity of a non-zero eigenvalue ν of c_L . Note that $\dim \check{D}(\lambda) = \sigma_L(\lambda + 2)$ for $\lambda \neq -2$.

Finding the positive spectrum of c_L and the multiplicities is an extremely complicated problem and in general there is no hope to solve it. However, since we need only consider eigenvalues in $(1, 3)$, the problem is tractable in special cases. For possible further applications to coassociative geometry it is of greatest practical use to study eigenvalues in the range $(0, 4)$.

5. Homogeneous cones

In this section, we explicitly determine the stability index for coassociative cones whose links are orbits of closed 3-dimensional subgroups of G_2 . We achieve this by calculating the small eigenvalues of the curl operator on Berger 3-spheres and their quotients using elementary methods.

5.1. Berger 3-spheres

Definition 5.1. Let \mathbb{H} denote the quaternions with standard basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Identify $\mathcal{S}^3 \cong \text{Sp}(1)$ and let $\mathbf{x} : \text{Sp}(1) \rightarrow \mathbb{H}$ denote the inclusion map of $\text{Sp}(1)$ as unit quaternions. Then $d\mathbf{x} = \mathbf{x}\omega$ for a 1-form ω taking values in the Lie algebra of $\text{Sp}(1)$, which here is represented by $\text{Im } \mathbb{H}$. Therefore,

we can write

$$\mathbf{x} = x_0\mathbf{1} + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \quad \text{and} \quad \omega = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}.$$

Since the Maurer–Cartan form ω satisfies the equations $d\omega + \omega \wedge \omega = 0$ and $d\mathbf{x} = \mathbf{x}\omega$, we have that

$$(5.1) \quad dx_0 = -x_1\omega_1 - x_2\omega_2 - x_3\omega_3, \quad dx_1 = x_0\omega_1 - x_3\omega_2 + x_2\omega_3,$$

$$(5.2) \quad dx_2 = x_3\omega_1 + x_0\omega_2 - x_1\omega_3, \quad dx_3 = -x_2\omega_1 + x_1\omega_2 + x_0\omega_3$$

and

$$(5.3) \quad d\omega_1 = -2\omega_2 \wedge \omega_3, \quad d\omega_2 = -2\omega_3 \wedge \omega_1, \quad d\omega_3 = -2\omega_1 \wedge \omega_2.$$

We define a 1-parameter family of metrics on \mathcal{S}^3 by $g_{\tau^2} = \tau^2\omega_1^2 + \omega_2^2 + \omega_3^2$ for $\tau > 0$. The Riemannian manifolds $(\mathcal{S}^3, g_{\tau^2})$ are the *Berger 3-spheres*.

It is immediately clear that finding the eigenvalues of $*d$ on a Berger 3-sphere will involve the Laplacian acting on \mathcal{S}^3 , thus homogeneous harmonic polynomials on \mathbb{R}^4 . We are thus lead to make the following definitions.

Definition 5.2. Use the notation of Definition 5.1. For $m = 1, 2, 3$, we define operators ∂_m on $f \in C^\infty(\mathcal{S}^3)$ by the expression: $df = \partial_1 f \omega_1 + \partial_2 f \omega_2 + \partial_3 f \omega_3$. For a unit imaginary quaternion $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ we define $\partial_{\mathbf{q}} = q_1\partial_1 + q_2\partial_2 + q_3\partial_3$. We say that $f \in C^\infty(\mathcal{S}^3)$ has \mathbf{q} -weight $w \geq 0$ if $\partial_{\mathbf{q}}^2 f = -w^2 f$.

Observe that the Laplacian Δ_{τ^2} on $(\mathcal{S}^3, g_{\tau^2})$ acting on functions is given by $\Delta_{\tau^2} = -\frac{1}{\tau^2}\partial_1^2 - \partial_2^2 - \partial_3^2$.

Of course, ∂_m is just Lie derivative along the vector field dual to ω_m .

Definition 5.3. Use the notation of Definition 5.1. For $k \in \mathbb{N}$, let \mathcal{P}_k be the space of homogeneous polynomials in x_0, x_1, x_2, x_3 of degree k on \mathcal{S}^3 . Let \mathcal{Q}_k be the subspace of \mathcal{P}_k consisting of polynomials which are eigenfunctions of the standard Laplacian Δ_1 ; that is, restrictions of homogeneous harmonic polynomials in 4 real variables to \mathcal{S}^3 . We let $\mathcal{A}_k = \{p_1\omega_1 + p_2\omega_2 + p_3\omega_3 : p_1, p_2, p_3 \in \mathcal{P}_k\}$ and $\mathcal{B}_k = \{p_1\omega_1 + p_2\omega_2 + p_3\omega_3 : p_1, p_2, p_3 \in \mathcal{Q}_k\}$.

It is often clearer to work with the representation of $\text{Sp}(1) = \text{SU}(2)$ on \mathbb{C}^2 rather than \mathbb{H} , so we make the following useful definition.

Definition 5.4. Recall the notation of Definition 5.1. We define complex coordinates on $\mathbb{H} \cong \mathbb{C}^2$ by $z_1 = x_0 + ix_1$ and $z_2 = x_2 + ix_3$. For $\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \text{Sp}(1)$, its action on \mathbb{C}^2 is given by

$$\mathbf{q} : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} q_0 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

For $k \in \mathbb{N}$, let $\mathcal{P}_k^{\mathbb{C}}$ denote the space of homogeneous polynomials in $z_1, z_2, \bar{z}_1, \bar{z}_2$ of degree k restricted to \mathcal{S}^3 and let $\mathcal{Q}_k^{\mathbb{C}}$ be the subspace of $\mathcal{P}_k^{\mathbb{C}}$ consisting of polynomials which are harmonic on \mathbb{C}^2 . We also let $\mathcal{R}_m^{\mathbb{C}}$, for $m \in \mathbb{Z}$, be the space of homogeneous polynomials p in $z_1, z_2, \bar{z}_1, \bar{z}_2$, restricted to \mathcal{S}^3 , such that under the action of $\cos \theta \mathbf{1} + \sin \theta \mathbf{i}$, p maps to $e^{im\theta} p$. Note that $p \in \mathcal{R}_m^{\mathbb{C}}$ if and only if $\bar{p} \in \mathcal{R}_{-m}^{\mathbb{C}}$.

We now recall the following well-known facts concerning eigenfunctions of the Laplacian on \mathcal{S}^3 .

Theorem 5.5. *Use the notation of Definitions 5.1–5.3.*

- (a) *For each $k \in \mathbb{N}$, there is a direct sum decomposition $\mathcal{Q}_k = \bigoplus_{l=0}^{\lfloor k/2 \rfloor} \mathcal{Q}_{k,k-2l}$ such that the elements of $\mathcal{Q}_{k,k-2l}$ have \mathbf{i} -weight $k - 2l$. Moreover,*

$$\dim \mathcal{Q}_{k,k-2l} = 2k + 2 \text{ if } l < \left\lfloor \frac{k}{2} \right\rfloor, \quad \dim \mathcal{Q}_{2l,0} = 2l + 1$$

and $\dim \mathcal{Q}_k = (k + 1)^2$.

- (b) *The eigenvalues of Δ_{τ^2} are of the form $k(k + 2) + \left(\frac{k-2l}{\tau}\right)^2(1 - \tau^2)$ for $k \in \mathbb{N}$ and $l \leq \lfloor \frac{k}{2} \rfloor$, and the corresponding eigenspace is $\mathcal{Q}_{k,k-2l}$.*

Proof. The results of [39, Lemmas 3.1 and 4.1] give the decomposition in (a) and all of (b). We can determine the dimensions of the $\mathcal{Q}_{k,k-2l}$ by explicitly identifying the functions as the real and imaginary parts of elements in $\mathcal{Q}_k^{\mathbb{C}} \cap \mathcal{R}_{k-2l}^{\mathbb{C}}$, in the notation of Definition 5.4. Clearly $\mathcal{Q}_{2l,0}$ is the space of lifts of eigenfunctions of the Laplacian on $\mathbb{C}\mathbb{P}^1$ with eigenvalue $4l(l + 1)$, which has dimension $2l + 1$. For the remaining spaces, it is straightforward to see that each $\mathcal{Q}_{k,k-2l}$ has the same dimension for fixed k , and since $\dim \mathcal{Q}_k = (k + 1)^2$, as it is the multiplicity of the eigenvalue $k(k + 2)$ for Δ_1 on \mathcal{S}^3 , it is an elementary calculation to find that $\dim \mathcal{Q}_{k,k-2l} = 2k + 2$ if $l \neq \lfloor \frac{k}{2} \rfloor$. \square

The proofs of the quoted results from [39] rest on the fact that ∂_1 commutes with Δ_{τ^2} for any $\tau > 0$. This is certainly not true of ∂_2 and ∂_3 if $\tau \neq 1$.

Proposition 5.6. *Use the notation of Definitions 4.13 and 5.1 and Theorem 5.5. The positive eigenvalues of c_L on $(L, g_L) = (\mathcal{S}^3, g_{\tau^2})$ are*

$$\nu_{k,k-2l} = \tau + \sqrt{\tau^2 + k(k+2) + \left(\frac{k-2l}{\tau}\right)^2 (1-\tau^2)} \quad \text{and} \quad \nu_k = \frac{k+2}{\tau}$$

for $k, l \in \mathbb{N}$ with $l \leq \lfloor \frac{k}{2} \rfloor$. Moreover,

$$\sigma_L(\nu_{k,k-2l}) = \dim \mathcal{Q}_{k,k-2l} \quad \text{and} \quad \sigma_L(\nu_k) = 2k + 2.$$

Note. For the multiplicity count here we regard the $\nu_{k,k-2l}$ and ν_k as distinct. If they agree then we add the multiplicities.

Proof. From (5.1)–(5.3) we see that $c_L = -*d$ sends \mathcal{A}_k , given in Definition 5.3, to itself. Since $\cup_{k \in \mathbb{N}} \mathcal{A}_k$ is dense in $C^\infty(T^*L)$ we need only consider $c_L \alpha = \nu \alpha$ for $\alpha \in \mathcal{A}_k$ to determine the eigenvalues ν of c_L .

The equation $c_L \alpha = \nu \alpha$ for $\alpha = p_1 \omega_1 + p_2 \omega_2 + p_3 \omega_3$ and $\nu > 0$ is equivalent to the following system, using the notation of Definition 5.2:

$$(5.4) \quad \left(2 - \frac{\nu}{\tau}\right) p_1 = \partial_2 p_3 - \partial_3 p_2,$$

$$(5.5) \quad (2 - \nu\tau) p_2 = \partial_3 p_1 - \partial_1 p_3,$$

$$(5.6) \quad (2 - \nu\tau) p_3 = \partial_1 p_2 - \partial_2 p_1.$$

Moreover, since $*d\alpha = -\nu\alpha$ for $\nu \neq 0$, we have that $d^*\alpha = 0$, which is equivalent to the condition:

$$(5.7) \quad \frac{1}{\tau} \partial_1 p_1 + \tau \partial_2 p_2 + \tau \partial_3 p_3 = 0.$$

From (5.1)–(5.2) we see that, if ϵ_{abc} is the standard permutation symbol,

$$(5.8) \quad [\partial_a, \partial_b] = 2\epsilon_{abc} \partial_c.$$

Using (5.4)–(5.8) we calculate:

$$\begin{aligned} (5.9) \quad \Delta_{\tau^2} p_1 &= - \left(\frac{1}{\tau^2} \partial_1^2 + \partial_2^2 + \partial_3^2 \right) p_1 \\ &= ([\partial_1, \partial_2] - (2 - \nu\tau) \partial_3) p_2 + ([\partial_1, \partial_3] + (2 - \nu\tau) \partial_2) p_3 \\ &= -\nu\tau (\partial_2 p_3 - \partial_3 p_2) \\ &= \nu(\nu - 2\tau) p_1. \end{aligned}$$

Thus p_1 is a $\nu(\nu - 2\tau)$ -eigenfunction of Δ_{τ^2} if $p_1 \neq 0$ and hence

$$(5.10) \quad \nu(\nu - 2\tau) = k(k + 2) + \left(\frac{k - 2l}{\tau}\right)^2 (1 - \tau^2),$$

from which the formula for $\nu_{k,k-2l}$ follows.

If $p_1 = 0$ then (5.4)–(5.8) imply that

$$(5.11) \quad -\partial_1^2 p_j = (\nu\tau - 2)^2 p_j \quad \text{and} \quad -(\partial_2^2 + \partial_3^2) p_j = 2(\nu\tau - 2) p_j,$$

for $j = 2, 3$. Thus, $\Delta_1 p_j = \nu\tau(\nu\tau - 2) p_j$, so $\nu\tau(\nu\tau - 2) = k(k + 2)$ from which the formula for ν_k follows.

To determine the multiplicity of $\nu_{k,k-2l}$ we make the following observations. First, using (5.5)–(5.6) we see that

$$(5.12) \quad (\partial_1^2 + (\nu\tau - 2)^2) p_2 = (\partial_1 \partial_2 - (\nu\tau - 2) \partial_3) p_1,$$

$$(5.13) \quad (\partial_1^2 + (\nu\tau - 2)^2) p_3 = (\partial_1 \partial_3 + (\nu\tau - 2) \partial_2) p_1.$$

Second, from (5.4)–(5.7), we calculate

$$(5.14) \quad (\partial_2^2 + \partial_3^2 + 2(\nu\tau - 2)) p_2 = -\left(\frac{1}{\tau^2} \partial_2 \partial_1 + \left(4 - \frac{\nu}{\tau}\right) \partial_3\right) p_1,$$

$$(5.15) \quad (\partial_2^2 + \partial_3^2 + 2(\nu\tau - 2)) p_3 = -\left(\frac{1}{\tau^2} \partial_3 \partial_1 - \left(4 - \frac{\nu}{\tau}\right) \partial_2\right) p_1.$$

Combining (5.12)–(5.15) we see that p_1 determines p_2 and p_3 unless they satisfy (5.11), which happens if and only if $\nu_{k,k-2l} = \nu_k$. Thus, the multiplicity of $\sigma_L(\nu_{k,k-2l})$ is determined by the number of choices for p_1 , so $\sigma_L(\nu_{k,k-2l}) = \dim \mathcal{Q}_{k,k-2l}$.

For the multiplicity of ν_k , we may take $p_1 = 0$ and see that (5.6) determines p_3 given p_2 unless $k = 0$. If $k \neq 0$, $\sigma_L(\nu_k)$ is the number of choices for p_2 . By (5.11), we see that $p_2 \in \mathcal{Q}_{k,k}$, so $\sigma_L(\nu_k) = \dim \mathcal{Q}_{k,k} = 2k + 2$ by Theorem 5.5. For $k = 0$, p_2 and p_3 are arbitrary constants so $\sigma_L(\nu_0) = 2$. \square

Notes. We have some important elementary observations from the proof of Proposition 5.6, which will be important later.

- (a) There is a basis for the $\nu_{k,k-2l}$ -eigenforms on $(\mathcal{S}^3, g_{\tau^2})$ consisting of forms $p_1 \omega_1 + p_2 \omega_2 + p_3 \omega_3$ with $p_1 \in \mathcal{Q}_{k,k-2l}$ and $p_2, p_3 \in \mathcal{Q}_k$ determined by p_1 .
- (b) There is a basis for the ν_k -eigenforms on $(\mathcal{S}^3, g_{\tau^2})$ consisting of forms $p_2 \omega_2 + p_3 \omega_3$ such that $p_2, p_3 \in \mathcal{Q}_{k,k}$ and p_3 is determined by p_2 if $k > 0$.

We have an immediate corollary to Proposition 5.6 which builds on an already well-known result.

Corollary 5.7. *Use the notation of Definitions 4.13 and 5.1. The positive eigenvalues of c_{L_0} on $(L_0, g_{L_0}) = (\mathcal{S}^3, g_1)$ are $k + 2$ for $k \in \mathbb{N}$ and we have $\sigma_{L_0}(k + 2) = (k + 2)^2 - 1$. Hence, a coassociative 4-plane in \mathbb{R}^7 is stable.*

Proof. Clearly if $\tau = 1$ then $\nu_{k, k-2l} = 1 + \sqrt{(k+1)^2} = k + 2 = \nu_k$. If $k = 2n$, then the multiplicity of the eigenvalue $2n + 2$ is

$$\begin{aligned} & \sum_{l=0}^{n-1} \sigma_{L_0}(\nu_{2n,l}) + \sigma_{L_0}(\nu_{2n,n}) + \sigma_{L_0}(\nu_{2n}) \\ &= n(4n + 2) + 2n + 1 + 4n + 2 = (2n + 2)^2 - 1. \end{aligned}$$

Similarly, if $k = 2n + 1$ the multiplicity of the eigenvalue $2n + 3$ is

$$\sum_{l=0}^n \sigma_{L_0}(\nu_{2n+1,l}) + \sigma_{L_0}(\nu_{2n+1}) = (n + 1)(4n + 4) + 4n + 4 = (2n + 3)^2 - 1.$$

The multiplicities follow.

For a coassociative 4-plane C_0 ,

$$\sum_{\lambda \in (-1,1]} \dim \check{D}(\lambda) = \dim \check{D}(0) + \dim \check{D}(1) = \sigma_{L_0}(2) + \sigma_{L_0}(3) = 3 + 8 = 11.$$

The number of non-trivial translations of C_0 is 3, and the stabilizer of C_0 in G_2 is isomorphic to $SO(4)$ as observed in Example 4.1. Thus, the dimension of the space of $G_2 \ltimes \mathbb{R}^7$ transformations of C_0 is $3 + 14 - 6 = 11$. Therefore $\text{ind}(C_0) = 0$. □

By [7, Example 5.1 and Theorem 5.1], the Lagrangian L_1 given in Example 4.2 is isometric to \mathcal{S}^3 with the metric $\frac{8}{3}g_{\frac{1}{6}}$. We may thus calculate the stability index of C_1 as follows.

Corollary 5.8. *Use the notation of Definitions 4.13 and 5.1 and Example 4.2. The eigenvalues of c_{L_1} on $(L_1, g_{L_1}) = (\mathcal{S}^3, \frac{8}{3}g_{\frac{1}{6}})$ which lie in $(0, 4)$ are*

given by $\{\frac{1}{2}, 2, 3, \frac{1+\sqrt{145}}{4}, \frac{7}{2}\}$. Moreover,

$$\begin{aligned} \sigma_{L_1}\left(\frac{1}{2}\right) &= 1, & \sigma_{L_1}(2) &= 7, & \sigma_{L_1}(3) &= 10, \\ \sigma_{L_1}\left(\frac{1+\sqrt{145}}{4}\right) &= 5, & \sigma_{L_1}\left(\frac{7}{2}\right) &= 6. \end{aligned}$$

Hence, the coassociative cone C_1 is stable.

Proof. Let $\lambda_{k,k-2l} = \sqrt{\frac{3}{8}}\nu_{k,k-2l}$ and let $\lambda_k = \sqrt{\frac{3}{8}}\nu_k$, using Proposition 5.6 with $\tau = \frac{1}{\sqrt{6}}$. Then $\lambda_{k,k-2l}$ and λ_k are the eigenvalues of c_{L_1} .

We notice that $\lambda_{5,1} = \frac{1}{4}(1 + \sqrt{241}) > 4$, so we need only consider $\lambda_{k,k-2l}$ for $k \leq 4$ for eigenvalues in $(0, 4)$. We may calculate:

$$\begin{aligned} \lambda_{0,0} &= \frac{1}{2}, & \lambda_{1,1} &= 2, & \lambda_{2,2} &= \frac{7}{2}, & \lambda_{2,0} &= 2, & \lambda_{3,3} &= 5, & \lambda_{3,1} &= 3, \\ \lambda_{4,4} &= \frac{13}{2}, & \lambda_{4,2} &= \frac{1}{4}(1 + \sqrt{265}) > 4, & \lambda_{4,0} &= \frac{1}{4}(1 + \sqrt{145}) < 4. \end{aligned}$$

Since $\lambda_k = \frac{3k}{2} + 3$, only $\lambda_0 = 3$ is relevant here. By Proposition 5.6,

$$\begin{aligned} \sigma_{L_1}\left(\frac{1}{2}\right) &= \dim \mathcal{Q}_{0,0}, \\ \sigma_{L_1}(2) &= \dim \mathcal{Q}_{1,1} + \dim \mathcal{Q}_{2,0}, & \sigma_{L_1}(3) &= \dim \mathcal{Q}_{3,1} + 2, \\ \sigma_{L_1}\left(\frac{1+\sqrt{145}}{4}\right) &= \dim \mathcal{Q}_{4,0}, & \sigma_{L_1}\left(\frac{7}{2}\right) &= \dim \mathcal{Q}_{2,2}. \end{aligned}$$

The multiplicities now follow from Theorem 5.5.

Since C_1 is non-planar and the stabilizer of C_1 under G_2 transformations is $U(2)$, the dimension of the family of $G_2 \times \mathbb{R}^7$ transformations of C_1 is $7 + 14 - 4 = 17$. Further,

$$\sum_{\lambda \in (-1,1]} \dim \check{D}(\lambda) = \dim \check{D}(0) + \dim \check{D}(1) = \sigma_{L_1}(2) + \sigma_{L_1}(3) = 7 + 10 = 17,$$

so $\text{ind}(C_1) = 0$. □

Remark. We may observe, as in [40, Theorem 5], that $\tau^2 = \frac{1}{6}$ is the critical value at which the multiplicity of the first eigenvalue of Δ_{τ^2} on \mathcal{S}^3 “jumps”.

5.2. Quotients of the 3-sphere

We now wish to consider quotients of \mathcal{S}^3 by finite groups. The possible finite groups are the cyclic groups \mathbb{Z}_n , the binary dihedral groups D_n^* , the binary tetrahedral group A_4^* , the binary octahedral group S_4^* and the binary icosahedral group A_5^* . We know, from Section 4.1, that the only groups we must consider are \mathbb{Z}_2 , D_3^* and A_4^* . We describe the actions of the finite subgroups we need explicitly.

Definition 5.9. Let unit quaternions act on \mathbb{H} by left-multiplication.

- (a) The cyclic group of order $n \geq 2$ acts as $\mathbb{Z}_n = \{\cos(\frac{2k\pi}{n})\mathbf{1} + \sin(\frac{2k\pi}{n})\mathbf{i} : k = 0, 1, \dots, n - 1\}$.
- (b) The binary dihedral group of order $4n$, for $n \geq 1$, acts as $D_n^* = \mathbb{Z}_{2n} \cup \mathbf{j}\mathbb{Z}_{2n}$.
- (c) The binary tetrahedral group acts as $A_4^* = D_2^* \cup \{\frac{1}{2}(\pm\mathbf{1} \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})\}$, where any combination of signs is permissible.

To understand the spectrum of c_L on quotients of \mathcal{S}^3 we need the following.

Proposition 5.10. Use the notation of Definitions 4.13 and 5.1. Let ξ_m be the vector field on $(L, g_L) = (\mathcal{S}^3, g_{\tau^2})$ dual to ω_m . Then c_L and the Lie derivative \mathcal{L}_{ξ_1} commute for all $\tau > 0$, and c_L commutes with \mathcal{L}_{ξ_m} for all m if $\tau = 1$.

Proof. Using Cartan’s formula, we see that

$$[c_L, \mathcal{L}_{\xi_1}]\alpha = \xi_1 \lrcorner d * d\alpha + d(\xi_1 \lrcorner * d\alpha) - * d(\xi_1 \lrcorner d\alpha)$$

for 1-forms α . Let $\alpha = p_1\omega_1 + p_2\omega_2 + p_3\omega_3$ and let ∂_j be the operator given in Definition 5.2 for $j = 1, 2, 3$. We calculate:

$$\begin{aligned} *d\alpha &= \tau(\partial_2 p_3 - \partial_3 p_2 - 2p_1)\omega_1 \\ &\quad + \frac{1}{\tau}(\partial_3 p_1 - \partial_1 p_3 - 2p_2)\omega_2 + \frac{1}{\tau}(\partial_1 p_2 - \partial_2 p_1 - 2p_3)\omega_3, \end{aligned}$$

hence

$$\begin{aligned} \xi_1 \lrcorner d*\alpha &= \left(\left(\frac{1}{\tau} \partial_1 \partial_3 + 2 \left(\tau + \frac{1}{\tau} \right) \partial_2 \right) p_1 + \left(\tau \partial_2 \partial_3 - \frac{4}{\tau} \partial_1 \right) p_2 \right. \\ &\quad \left. + \left(\frac{4}{\tau} - \frac{1}{\tau} \partial_1^2 - \tau \partial_2^2 \right) p_3 \right) \omega_2 \\ &\quad + \left(\left(-\frac{1}{\tau} \partial_1 \partial_2 + 2 \left(\tau + \frac{1}{\tau} \right) \partial_3 \right) p_1 + \left(-\frac{4}{\tau} + \frac{1}{\tau} \partial_1^2 + \tau \partial_3^2 \right) p_2 \right. \\ &\quad \left. + \left(-\tau \partial_3 \partial_2 - \frac{4}{\tau} \partial_1 \right) p_3 \right) \omega_3 \end{aligned}$$

and

$$\begin{aligned} d(\xi_1 \lrcorner *d\alpha) &= \tau(-2\partial_1 p_1 - \partial_1 \partial_3 p_2 + \partial_1 \partial_2 p_3) \omega_1 \\ &\quad + \tau(-2\partial_2 p_1 - \partial_2 \partial_3 p_2 + \partial_2^2 p_3) \omega_2 \\ &\quad + \tau(-2\partial_3 p_1 - \partial_3^2 p_2 + \partial_3 \partial_2 p_3) \omega_3. \end{aligned}$$

Since

$$\xi_1 \lrcorner d\alpha = (\partial_1 p_2 - \partial_2 p_1 - 2p_3) \omega_2 - (\partial_3 p_1 - \partial_1 p_3 - 2p_2) \omega_3,$$

we have that:

$$\begin{aligned} *d(\xi_1 \lrcorner d\alpha) &= \tau([\partial_3, \partial_2] p_1 + (-\partial_3 \partial_1 + 2\partial_2) p_2 + (\partial_2 \partial_1 + 2\partial_3) p_3) \omega_1 \\ &\quad + \frac{1}{\tau} ((\partial_1 \partial_3 + 2\partial_2) p_1 - 4\partial_1 p_2 + (4 - \partial_1^2) p_3) \omega_2 \\ &\quad + \frac{1}{\tau} ((-\partial_1 \partial_2 + 2\partial_3) p_1 - (4 - \partial_1^2) p_2 - 4\partial_1 p_3) \omega_3. \end{aligned}$$

Combining these formulae and using (5.8) shows that $[c_L, \mathcal{L}_{\xi_1}] \alpha = 0$. Clearly this argument will extend to \mathcal{L}_{ξ_2} and \mathcal{L}_{ξ_3} in the case $\tau = 1$. □

Proposition 5.11. *Use the notation of Definitions 4.13 and 5.9 and Proposition 5.6. Let $n \in \mathbb{Z}^+ \setminus \{1\}$. The positive eigenvalues of c_L on $(L, g_L) = (\mathcal{S}^3/\mathbb{Z}_n, g_{\tau^2})$ are*

$$\begin{aligned} \nu_{rn+2l, rn} &= \tau + \sqrt{\tau^2 + (rn + 2l)(rn + 2l + 2) + \left(\frac{rn}{\tau}\right)^2 (1 - \tau^2)} \quad \text{and} \\ \nu_{sn+n-2} &= \frac{(s + 1)n}{\tau} \end{aligned}$$

for $l, r, s \in \mathbb{N}$. Moreover,

$$\begin{aligned} \sigma_L(\nu_{rn+2l, rn}) &= 2rn + 4l + 2 \quad (r > 0), & \sigma_L(\nu_{2l, 0}) &= 2l + 1, \\ \sigma_L(\nu_{sn+n-2}) &= 2sn + 2n - 2. \end{aligned}$$

Proof. Let ξ_1 be the vector field dual to ω_1 . By Proposition 5.10, c_L and \mathcal{L}_{ξ_1} commute. Thus, using the notation of Definition 5.3, we may restrict attention to $\alpha \in \mathcal{B}_k \otimes \mathbb{C}$ such that

$$*d\alpha = -\nu\alpha \quad \text{and} \quad \mathcal{L}_{\xi_1}\alpha = im\alpha$$

for $\nu > 0$ and $m \in \mathbb{Z}$. For α to descend to the quotient $\mathcal{S}^3/\mathbb{Z}_n$, we must have that $m \equiv 0 \pmod{n}$. If we write $\alpha = p_1\omega_1 + p_2\omega_2 + p_3\omega_3$ then using Cartan’s formula and the notation of Definition 5.2 we find that

$$\begin{aligned} im\alpha &= \xi_1 \lrcorner d\alpha + d(\xi_1 \lrcorner \alpha) \\ &= -\nu\xi_1 \lrcorner * \alpha + d(\xi_1 \lrcorner (p_1\omega_1 + p_2\omega_2 + p_3\omega_3)) \\ &= -\nu\xi_1 \lrcorner \left(\frac{1}{\tau} p_1\omega_2 \wedge \omega_3 + \tau p_2\omega_3 \wedge \omega_1 + \tau p_3\omega_1 \wedge \omega_2 \right) + dp_1 \\ &= \partial_1 p_1\omega_1 + (\partial_2 p_1 - \nu\tau p_3)\omega_2 + (\partial_3 p_1 + \nu\tau p_2)\omega_3. \end{aligned}$$

For $p_1 \neq 0$ we see that $-\partial_1^2 p_1 = m^2 p_1$, so α is a $\nu_{k, k-2l}$ -eigenform for c_L by Proposition 5.6, where $k - 2l = rn$ for some $r \in \mathbb{N}$. For $p_1 = 0$, we observe that

$$-m^2 p_2 = im(im p_2) = im(-\nu\tau p_3) = -\nu^2 \tau^2 p_2.$$

Since $\nu\tau = k + 2$ by Proposition 5.6, we have that $k = (s + 1)n - 2$ for some $s \in \mathbb{N}$, since $k \geq 0$. Thus α has eigenvalue ν_{sn+n-2} in these cases. The eigenvalues and multiplicities now follow from Theorem 5.5 and Proposition 5.6. □

From the observations in [28, Example 6.14] we see that L_2 given in Example 4.3 is isometric to $\mathcal{S}^3/\mathbb{Z}_2 \cong \text{SO}(3)$ with metric $2g_2$. We therefore calculate the spectrum of c_{L_2} in $(0, 4)$ using Proposition 5.11 as follows.

Corollary 5.12. *Use the notation of Definitions 4.13, 5.1 and 5.9 and Example 4.3. The eigenvalues of c_{L_2} on $(L_2, g_{L_2}) = (\mathcal{S}^3/\mathbb{Z}_2, 2g_2)$ in $(0, 4)$*

are: $\{1, 2, 3, 1 + \sqrt{5}\}$. Moreover,

$$\sigma_{L_2}(1) = 2, \quad \sigma_{L_2}(2) = 7, \quad \sigma_{L_2}(3) = 16, \quad \sigma_{L_2}(1 + \sqrt{5}) = 3.$$

Let \mathcal{C}_o be the family of cones generated by $G_2 \ltimes \mathbb{R}^7$ transformations of cones

$$C_{\mathbf{a}} = \{(0, z_1, z_2, z_3) \in \mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7 : a_1 z_1^2 + a_2 z_2^2 + a_3 z_3^2 = 0\},$$

where $\mathbf{a} \in \mathcal{T} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_i > 0 \text{ for all } i \text{ and } a_1 + a_2 + a_3 = 1\}$. Then $C_2 = C_{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})}$, $\text{ind}(C_2) = 5$ and $\text{ind}_{\mathcal{C}_o}(C_2) = 0$, so C_2 is \mathcal{C}_o -stable and Jacobi integrable but not stable or rigid.

Proof. Using the notation of Proposition 5.6, let $\lambda_{2r+2l,2r} = \frac{1}{\sqrt{2}}\nu_{2r+2l,2r}$ and $\lambda_{2s} = \frac{1}{\sqrt{2}}\nu_{2s}$, calculated using $\tau = \sqrt{2}$. Then $\lambda_{2r+2l,2r}$ and λ_{2s} are the eigenvalues of c_{L_2} by Proposition 5.11.

Since $\lambda_{4,4} = 4$, we need only consider $\lambda_{2r+2l,2r}$ for $r, l \leq 1$. We also see that $\lambda_{2s} = s + 1$. We therefore calculate

$$\begin{aligned} \lambda_{0,0} &= 2, & \lambda_{2,2} &= 3, & \lambda_{2,0} &= 1 + \sqrt{5}, \\ \lambda_0 &= 1, & \lambda_2 &= 2, & \lambda_4 &= 3. \end{aligned}$$

The eigenvalues and multiplicities follow from Proposition 5.11.

Since C_2 is non-planar and the stabilizer of C_2 under G_2 transformations is $SO(3)$, the dimension of the family of $G_2 \ltimes \mathbb{R}^7$ transformations of C_2 is $7 + 14 - 3 = 18$. Further,

$$\sum_{\lambda \in (-1,1]} \dim \check{D}(\lambda) = \dim \check{D}(0) + \dim \check{D}(1) = \sigma_{L_2}(2) + \sigma_{L_2}(3) = 7 + 16 = 23,$$

so $\text{ind}(C_2) = 5$. Now, $C_2 = C_{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})} \in \mathcal{C}_o$. Moreover, if $\mathbf{a} \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $C_{\mathbf{a}}$ has trivial stabilizer in $G_2 \ltimes \mathbb{R}^7$ and has a two-parameter family of deformations up to rigid motion given by varying \mathbf{a} . Therefore, $\dim \mathcal{C}_o = 7 + 14 + 2 = 23$ and C_2 is \mathcal{C}_o -stable. Finally note that $\dim \check{D}(1) = 16$, so the space of Lagrangian Jacobi fields on L_2 is equal to the space of genuine deformations of L_2 in the family of links of cones in \mathcal{C}_o , which also shows that \mathcal{C}_o is a maximal deformation family for C_2 . □

We now consider the binary dihedral group.

Proposition 5.13. *Use the notation of Definitions 4.13, 5.1 and 5.9 and Proposition 5.6. For $n \in \mathbb{Z}^+$, the positive eigenvalues of c_L on $(L, g_L) = (\mathcal{S}^3/D_n^*, g_{\tau^2})$ are:*

$$\begin{aligned} & \nu_{2rn+2n+2l, 2rn+2n} \\ &= \tau + \sqrt{\tau^2 + 4(rn + n + l)(rn + n + l + 1) + 4\left(\frac{rn + n}{\tau}\right)^2 (1 - \tau^2)}; \\ \nu_{4l, 0} &= \tau + \sqrt{\tau^2 + 8l(2l + 1)}; \quad \text{and} \quad \nu_{2sn+2n-2} = \frac{2(s + 1)n}{\tau} \end{aligned}$$

for $l, r, s \in \mathbb{N}$. Moreover,

$$\begin{aligned} \sigma_L(\nu_{2rn+2n+2l, 2rn+2n}) &= 2rn + 2n + 2l + 1, & \sigma_L(\nu_{4l, 0}) &= 4l + 1, \\ \sigma_L(\nu_{2sn+2n-2}) &= 4sn + 4n - 2. \end{aligned}$$

Proof. Let $\alpha = p_1\omega_1 + p_2\omega_2 + p_3\omega_3$ be an eigenform of c_L of positive eigenvalue ν . From Definition 5.9(b), $\mathbb{Z}_{2n} \subseteq D_n^*$. Hence, by Proposition 5.11, the possible ν are of the form $\nu_{k, k-2l}$ where $k - 2l \equiv 0 \pmod{2n}$ or $\nu_{2sn+2n-2}$, using the notation of Proposition 5.6.

Recalling Definition 5.4, we see that \mathbf{j} sends $(z_1, z_2) \in \mathbb{C}^2$ to $(z_2, -z_1)$ and hence \mathbf{j} maps $\mathcal{R}_m^{\mathbb{C}}$ to $\mathcal{R}_{-m}^{\mathbb{C}}$ for each $m \in \mathbb{Z}$. By note (a) after Proposition 5.6 and Definition 5.9(b), the multiplicity of $\nu_{2rn+2l, 2rn}$ is the number of choices of $p_1 \in \mathcal{Q}_{2rn+2l, 2rn}$ which are \mathbf{j} -invariant. By Theorem 5.5, p_1 is the sum of real and imaginary parts of polynomials in $\mathcal{Q}_{2rn+2l}^{\mathbb{C}} \cap \mathcal{R}_{2rn}^{\mathbb{C}}$. Since \mathbf{j}^2 clearly acts as the identity on $\mathcal{Q}_{2rn+2l}^{\mathbb{C}}$, we can decompose $\mathcal{Q}_{2rn+2l, 2rn}$ into ± 1 -eigenspaces for \mathbf{j} . Thus, for $r > 0$, the subspace of $\mathcal{Q}_{2rn+2l, 2rn}$ which is \mathbf{j} -invariant is half the total dimension. For $r = 0$, it is straightforward to see that \mathbf{j} acts as $(-1)^l$ on $\mathcal{Q}_{2l}^{\mathbb{C}} \cap \mathcal{R}_0^{\mathbb{C}}$, so $p_1 \in \mathcal{Q}_{2l, 0}$ is \mathbf{j} -invariant if and only if l is even.

By note (b) after Proposition 5.6, the multiplicity of ν_k , if $k \neq 0$, is determined by the number of choices for p_2 . Now, since p_2 is only well-defined up to sign on the quotient of \mathcal{S}^3 by D_n^* , we need to calculate the number of polynomials in $\mathcal{Q}_{k, k}$ on which \mathbf{j} acts as ± 1 . However, since $k = 2sn + 2n - 2 > 0$ is even, we can find a basis for $\mathcal{Q}_{k, k}$ consisting of ± 1 -eigenfunctions for \mathbf{j} by the previous paragraph. For ν_0 the multiplicity is 2 because p_2 and p_3 are constant.

The result follows from Theorem 5.5 and Proposition 5.6. □

By [28, Example 6.15], L_3 given in Example 4.4 is isometric to $\text{SO}(3)/\text{S}_3 \cong \mathcal{S}^3/\text{D}_3^*$ with metric $6g_6$. We may thus apply Proposition 5.13 as follows.

Corollary 5.14. *Use the notation of Definitions 4.13, 5.1 and 5.9 and Example 4.4. The eigenvalues of c_{L_3} on $(L_3, g_{L_3}) = (\mathcal{S}^3/\text{D}_3^*, 6g_6)$ in $(0, 4)$ are: $\{1, 2, 3, 1 + \sqrt{5}\}$. Moreover,*

$$\sigma_{L_3}(1) = 10, \quad \sigma_{L_3}(2) = 23, \quad \sigma_{L_3}(3) = 41, \quad \sigma_{L_3}(1 + \sqrt{5}) = 5.$$

Hence, the cone C_3 is not stable and $\text{ind}(C_3) = 46$. Moreover, C_3 is not rigid.

Proof. By Proposition 5.13 applied with $\tau = \sqrt{6}$, the eigenvalues of c_{L_3} are $\lambda_{6r+6+2l,6r+6} = \frac{1}{\sqrt{6}}\nu_{6r+6+2l,6r+6}$, $\lambda_{4l,0} = \frac{1}{\sqrt{6}}\nu_{4l,0}$ and $\lambda_{6s+4} = \frac{1}{\sqrt{6}}\nu_{6s+4}$, for $l, r, s \in \mathbb{N}$. Since $\lambda_{8,0} = 1 + \frac{1}{3}\sqrt{129} > 4$, $\lambda_{8,6} = 1 + \frac{1}{3}\sqrt{84} > 4$ and $\lambda_{12,12} = 4$, we need only calculate:

$$\lambda_{0,0} = 2, \quad \lambda_{4,0} = 1 + \sqrt{5}, \quad \lambda_{6,6} = 3, \quad \lambda_{6s+4} = s + 1.$$

The multiplicities now follow from Proposition 5.13.

Since C_3 is non-planar and has $\text{SO}(3)$ stabilizer in G_2 ,

$$\begin{aligned} \text{ind}(C_3) &= \sum_{\lambda \in (-1,1]} \dim \check{D}(\lambda) - 7 - (14 - 3) = \dim \check{D}(0) + \dim \check{D}(1) - 18 \\ &= \sigma_{L_3}(2) + \sigma_{L_3}(3) - 18 = 23 + 41 - 18 = 46 \end{aligned}$$

as claimed. We also see that $\dim \check{D}(1) = 41 > 11 = \dim \text{G}_2 - \dim \text{SO}(3)$, so C_3 is not rigid. □

Lastly, we study the constant curvature 1 metric on $\mathcal{S}^3/\text{A}_4^*$.

Proposition 5.15. *Use the notation of Definitions 4.13 and 5.9. Let $(L, g_L) = (\mathcal{S}^3/\text{A}_4^*, g_1)$. The positive eigenvalues of c_L are*

$$\nu_{2r} = 2r + 2 \quad \text{and} \quad \nu_{6s+4,6s+4} = 6s + 6$$

for $r, s \in \mathbb{N}$. Moreover,

$$\sigma_L(\nu_{2r}) = (2r + 1) \left(1 + 2 \left[\frac{r}{3} \right] + \left[\frac{r}{2} \right] - r \right) \quad \text{and} \quad \sigma_L(\nu_{6s+4,6s+4}) = 12s + 10,$$

where $[q]$ denotes the integer part of a non-negative rational number q .

Proof. From Definition 5.9(c), we observe that A_4^* is generated by \mathbf{i} and \mathbf{j} , which are elements of order 4, and

$$\frac{1}{2}(\mathbf{1} + \mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{1}{2}\mathbf{1} + \frac{\sqrt{3}}{2} \left(\frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}} \right),$$

which has order 6. By Proposition 5.6 and Corollary 5.7, we know that the possible positive eigenvalues of c_L are $k + 2$ for $k \in \mathbb{N}$ with corresponding eigenforms $\alpha \in \mathcal{B}_k$, in the notation of Definition 5.3.

Suppose $\alpha = p_1\omega_1 + p_2\omega_2 + p_3\omega_3 \in \mathcal{B}_k \otimes \mathbb{C}$ is an eigenform of c_L . We may write $\alpha = P_1\Omega_1 + P_2\Omega_2 + P_3\Omega_3$ where

$$\begin{aligned} \Omega_1 &= \frac{\omega_1 + \omega_2 + \omega_3}{\sqrt{3}}, & \Omega_2 &= \frac{\omega_1 - \omega_3}{\sqrt{2}}, & \Omega_3 &= \frac{-\omega_1 + 2\omega_2 - \omega_3}{\sqrt{6}}, \\ P_1 &= \frac{p_1 + p_2 + p_3}{\sqrt{3}}, & P_2 &= \frac{p_1 - p_3}{\sqrt{2}}, & P_3 &= \frac{-p_1 + 2p_2 - p_3}{\sqrt{6}}. \end{aligned}$$

Let ξ_m be the vector field dual to ω_m and set $\xi = (\xi_1 + \xi_2 + \xi_3)/\sqrt{3}$. Since ξ commutes with c_L by Proposition 5.10, we may look for α satisfying

$$c_L\alpha = (k + 2)\alpha \quad \text{and} \quad \mathcal{L}_\xi\alpha = im\alpha$$

for $m \in \mathbb{Z}$. For α to be well-defined on L we must have that $m \equiv 0 \pmod{6}$. Using (5.3) we calculate:

$$\begin{aligned} \mathcal{L}_\xi\Omega_1 &= \frac{1}{3}(\mathcal{L}_{\xi_2+\xi_3}\omega_1 + \mathcal{L}_{\xi_3+\xi_1}\omega_2 + \mathcal{L}_{\xi_1+\xi_2}\omega_3) \\ &= \frac{1}{3}(-2\omega_3 + 2\omega_2 - 2\omega_1 + 2\omega_3 - 2\omega_2 + 2\omega_1) = 0; \\ \mathcal{L}_\xi\Omega_2 &= 2\Omega_3; \\ \mathcal{L}_\xi\Omega_3 &= -2\Omega_2. \end{aligned}$$

Recall the notation of Definitions 5.2 and 5.4. By similar methods to the proof of Proposition 5.11, P_1 has \mathbf{q} -weight $6l$ for some $l \in \mathbb{N}$, where $\mathbf{q} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$. Moreover, P_1 must be invariant under \mathbf{i} and \mathbf{j} , so P_1 is A_4^* -invariant. The A_4^* -invariant eigenfunctions of the Laplacian on \mathcal{S}^3 are effectively determined in [12, Theorem 4.4], so we deduce that $k = 2r$ and P_1 lies in a subspace of $\mathcal{Q}_{2r}^{\mathbb{C}}$ of dimension $(2r + 1)(1 + 2\lfloor \frac{r}{3} \rfloor + \lfloor \frac{r}{2} \rfloor - r)$. The formulae for ν_{2r} and $\sigma_L(\nu_{2r})$ follow.

Again following the proof of Proposition 5.11, we observe that P_1 determines P_2 and P_3 unless $P_2, P_3 \in \mathcal{Q}_{6s+4}^{\mathbb{C}}$ have \mathbf{q} -weight $6s + 4$ for some $s \in \mathbb{N}$.

Thus, we also have eigenvalues of the form $6s + 6$ with multiplicity equal to the number of choices for P_2 . As in the proof of Proposition 5.13, P_2 is only well-defined up to sign on the quotient of \mathcal{S}^3 . Since P_2 is a polynomial of even degree it has even \mathbf{i} -weight and \mathbf{j} -weight, so P_2 maps to $\pm P_2$ under the action of \mathbf{i} and \mathbf{j} . Since we may decompose the harmonic polynomials on \mathcal{S}^3 by \mathbf{q} -weight rather than \mathbf{i} -weight as in Theorem 5.5, and the metric on L has constant curvature, we deduce that $\sigma_L(6s + 6) = \dim \mathcal{Q}_{6s+4,6s+4} = 12s + 10$. \square

As observed in [28, Example 6.6], the link L_4 of C_4 , given in Example 4.4, is isometric to $\text{SO}(3)/A_4 \cong \mathcal{S}^3/A_4^*$ with constant curvature $\frac{1}{16}$. We may therefore deduce the following.

Proposition 5.16. *Use the notation of Definitions 4.13, 5.1 and 5.9 and Example 4.4. The eigenvalues of c_{L_4} on $(L_4, g_{L_4}) = (\mathcal{S}^3/A_4^*, 16g_1)$ in $(0, 4)$ are: $\{\frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\}$. Moreover,*

$$\begin{aligned} \sigma_{L_4}\left(\frac{1}{2}\right) &= 1, & \sigma_{L_4}\left(\frac{3}{2}\right) &= 10, & \sigma_{L_4}(2) &= 7, \\ \sigma_{L_4}\left(\frac{5}{2}\right) &= 9, & \sigma_{L_4}(3) &= 22, & \sigma_{L_4}\left(\frac{7}{2}\right) &= 26. \end{aligned}$$

Hence, the cone C_4 is not stable and $\text{ind}(C_4) = 30$. Moreover, C_4 is not rigid.

Proof. By Proposition 5.15, the positive eigenvalues of c_{L_4} are $\lambda_{2r} = \frac{1}{2}(r + 1)$ and $\lambda_{6s+4,6s+4} = \frac{3}{2}(s + 1)$. We are restricted to $r = 0, 3, 4, 6$ and $s = 0, 1$ for eigenvalues in $(0, 4)$ since the multiplicity of λ_{2r} for $r = 1, 2, 5$ is zero.

Hence, on C_4 ,

$$\begin{aligned} \sum_{\lambda \in (-1, 1]} \dim \check{D}(\lambda) &= \dim \check{D}\left(-\frac{1}{2}\right) + \dim \check{D}(0) + \dim \check{D}\left(\frac{1}{2}\right) + \dim \check{D}(1) \\ &= \sigma_{L_4}\left(\frac{3}{2}\right) + \sigma_{L_4}(2) + \sigma_{L_4}\left(\frac{5}{2}\right) + \sigma_{L_4}(3) \\ &= 10 + 7 + 9 + 22 = 48. \end{aligned}$$

Since C_4 is non-planar and $\text{SO}(3)$ -invariant, the dimension of the family of $G_2 \times \mathbb{R}^7$ transformations of C_4 has dimension $7 + 14 - 3 = 18$. Thus $\text{ind}(C_4) = 30$. Observe that $\dim \check{D}(1) = 22 > 11 = \dim G_2 - \dim \text{SO}(3)$, so C_4 is not rigid. \square

The stability index is a measure of the geometry complexity of the coassociative cone. We might therefore expect homogeneous cones to have the greatest chance of being stable. Since we find that the only stable homogeneous coassociative cones are 4-planes and the cone in Example 4.2, we might naively expect that these are the only stable coassociative cones.

In Geometric Measure Theory one defines the m -dimensional *density* of a set $S \subseteq \mathbb{R}^n$ at a point p by

$$\Theta(S, p) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^m(S \cap B_n(p; r))}{\text{vol}(B_m(0; r))},$$

where \mathcal{H}^m is m -dimensional Hausdorff measure and $B_n(p; r)$ is the ball in \mathbb{R}^n of radius r about p . If S is an m -dimensional submanifold then $\Theta(S, p) = 1$ for all $p \in S$. For coassociative cones C in \mathbb{R}^7 with isolated singularities at the origin and compact links L it is straightforward to calculate the 4-dimensional density as $\Theta(C, 0) = \text{vol}(L) / \text{vol}(S^3)$. We may therefore easily calculate the density for the homogeneous coassociative cones given in Examples 4.1–4.4 as follows:

$$\Theta(C_0, 0) = 1, \quad \Theta(C_1, 0) = \frac{16}{9}, \quad \Theta(C_2, 0) = 4, \quad \Theta(C_3, 0) = 36, \quad \Theta(C_4, 0) = 64.$$

The fact that C_0 and C_1 are the homogeneous cones with the lowest density and are also the only stable ones is suggestive.

6. Algebraic curves and 2-ruled cones

Since Example 4.8 gives the largest known family of coassociative cones, we are motivated to analyse the stability index for 2-ruled cones. We therefore need to understand the curl operator on geodesic S^1 -bundles over surfaces. We are particularly interested in the case where the surface is an algebraic curve.

We start with some definitions from the theory of Riemannian submersions.

Definition 6.1. Let (L, g_L) be a compact Riemannian 3-manifold which is an S^1 -bundle $\pi : L \rightarrow \Sigma$ over a compact Riemannian surface (Σ, g_Σ) . Suppose further that π is a Riemannian submersion. Let ξ be a unit vector field spanning the vertical distribution of π and let θ be the 1-form dual to ξ .

We define a form α on L to be *horizontal* if $\xi \lrcorner \alpha = 0$ and we denote the bundle of horizontal m -forms by $\Lambda_h^m T^*L$. Trivially all functions are horizontal. We can identify the horizontal m -forms with m -forms on Σ . We

define the *horizontal Hodge star* $*_h$ on horizontal forms α via the equation: $*\alpha = *_h\alpha \wedge \theta$.

By [33, Equation (2.1.2)],

$$\Lambda^{m+1}T^*L = \Lambda_h^{m+1}T^*L \oplus (\Lambda_h^m T^*L \wedge \theta),$$

so we can define a *horizontal derivative* $d_h : C^\infty(\Lambda^m T^*L) \rightarrow C^\infty(\Lambda_h^{m+1} T^*L)$ by sending α to the horizontal part of $d\alpha$. Since d_h sends any section of $\Lambda_h^m T^*L \wedge \theta$ to zero, $d_h : C^\infty(\Lambda_h^m T^*L) \rightarrow C^\infty(\Lambda_h^{m+1} T^*L)$ is an antiderivation. We can therefore define the formal adjoint d_h^* of d_h and the *horizontal Laplacian* $\Delta_h = d_h d_h^* + d_h^* d_h$. The horizontal Laplacian is not necessarily elliptic.

If Δ_L is the ordinary Laplacian on L , we call $\Delta_v = \Delta_L - \Delta_h$ the *vertical Laplacian*. Note in the case of functions that $\Delta_v = -\mathcal{L}_\xi^2$.

The situation above includes the Berger 3-spheres, tubes over pseudoholomorphic curves in S^6 and real links of complex 2-dimensional cones.

The key result for understanding the Laplacian on functions in the situation of Definition 6.1 is the following [3, Theorem 1.5].

Proposition 6.2. *Use the notation of Definition 6.1. If $\pi : L \rightarrow \Sigma$ has totally geodesic fibres then, on functions, Δ_L , Δ_h and Δ_v commute.*

For convenience we make the following definition.

Definition 6.3. Let L be a compact Riemannian 3-manifold. Let $m_L(\nu)$ denote the multiplicity of the eigenvalue ν of the Laplacian Δ_L on functions. Let $\mathcal{E}_L = \{\nu \in \mathbb{R} : m_L(\nu) \neq 0\}$, which is a countable discrete set.

Remarks. Use the notation of Definition 6.1 and suppose that $\pi : L \rightarrow \Sigma$ has totally geodesic fibres. Let $m_L(\nu_h, \nu_v)$ denote the dimension of the space of functions f such that $\Delta_h f = \nu_h f$ and $\Delta_v f = \nu_v f$. Note that, since Δ_h and Δ_v are non-negative operators, we must have $\nu_h \geq 0$ and $\nu_v \geq 0$. Moreover, by one of the main results in [42], $m_L(\nu_h, 0)$ is the multiplicity of the eigenvalue ν_h of the Laplacian Δ_Σ acting on functions on Σ . Finally, by Proposition 6.2, every eigenvalue of Δ_L is of the form $\nu_h + \nu_v$, so $m_L(\nu) \leq \sum_{\nu_h + \nu_v = \nu} m_L(\nu_h, \nu_v)$.

We now have the following result that is similar to Proposition 5.10.

Proposition 6.4. *Use the notation of Definitions 4.13 and 6.1. If the fibres of $\pi : L \rightarrow \Sigma$ are totally geodesic and ξ is a Killing vector field for g_L , then c_L and the Lie derivative \mathcal{L}_ξ commute.*

Proof. Recall Definition 6.1. By the work in [33, Section 2.1], since $\nabla_\xi \xi = 0$, $d\theta$ is horizontal. Moreover, if we write a 1-form γ on L as $\gamma = \alpha + f\theta$ with α horizontal and f a function, then

$$*d\gamma = *_h(\mathcal{L}_\xi \alpha - d_h f) + (*_h d_h \alpha + f *_h d\theta)\theta \quad \text{and} \quad \mathcal{L}_\xi \gamma = \mathcal{L}_\xi \alpha + (\mathcal{L}_\xi f)\theta.$$

Thus, since ξ is Killing and $d\theta$ is horizontal, it is straightforward to compute:

$$*d(\mathcal{L}_\xi \gamma) = (*_h \mathcal{L}_\xi^2 \alpha - *_h d_h(\mathcal{L}_\xi f)) + (*_h d_h(\mathcal{L}_\xi \alpha) + (\mathcal{L}_\xi f) *_h d\theta)\theta$$

and

$$\mathcal{L}_\xi(*d\gamma) = (*_h \mathcal{L}_\xi^2 \alpha - *_h \mathcal{L}_\xi(d_h f)) + (*_h \mathcal{L}_\xi(d_h \alpha) + (\mathcal{L}_\xi f) *_h d\theta)\theta.$$

Note that, since the fibres of π are totally geodesic, $[d_h, \mathcal{L}_\xi] = 0$ on horizontal forms by [33, Equation (2.1.6)]. We conclude that $[c_L, \mathcal{L}_\xi] = 0$ as claimed. \square

Observe that, by Example 4.9, we have two types of links of coassociative cones where Definition 6.1 and Propositions 6.2 and 6.4 apply: namely, Hopf lifts of holomorphic curves in $\mathbb{C}\mathbb{P}^2$ and tubes of radius $\frac{\pi}{2}$ in the first or second normal bundle about a null-torsion pseudoholomorphic curve in \mathcal{S}^6 . We may therefore try to describe the spectrum of the curl operator in these cases.

We begin with the links of complex 2-dimensional cones.

Theorem 6.5. *Let Σ be a compact, connected, holomorphic curve in $\mathbb{C}\mathbb{P}^2$ with degree d_Σ . Let L be the Hopf lift of Σ in \mathcal{S}^5 . Use the notation of Definitions 4.13 and 6.3. The eigenvalues of c_L in $(0, 4)$ are 1, 2, 3 and $\lambda + 2 \in (2, 4) \setminus \{3\}$ such that $\lambda(\lambda + 2) \in \mathcal{E}_L$. Moreover,*

$$\begin{aligned} \sigma_L(1) &= d_\Sigma^2 - d_\Sigma, & \sigma_L(2) &= d_\Sigma^2 + d_\Sigma + 1, \\ \sigma_L(3) &= m_L(3) + d_\Sigma^2 + 3d_\Sigma, & \sigma_L(\lambda + 2) &= m_L(\lambda(\lambda + 2)). \end{aligned}$$

Furthermore, $m_L(3) \geq 6$ if $d_\Sigma \geq 2$.

Proof. Notice that L is a $U(1)$ -bundle over Σ , and that the projection $\pi : L \rightarrow \Sigma$ is a Riemannian submersion with totally geodesic fibres when L and Σ are given the induced metrics from the standard metrics on \mathcal{S}^5 and $\mathbb{C}\mathbb{P}^2$. Therefore, Definition 6.1 and Proposition 6.2 apply. If we let ξ be the vector

field given by the $U(1)$ action then, as observed in Example 4.9, ξ is a Killing vector field for the metric on L , so Proposition 6.4 applies. Hence, to understand the positive eigenvalues ν of c_L , it is enough to study the equations

$$(6.1) \quad d\gamma = -\nu * \gamma \quad \text{and}$$

$$(6.2) \quad \mathcal{L}_\xi \gamma = im\gamma$$

for $\gamma \in C^\infty(T^*L \otimes \mathbb{C})$ and $m \in \mathbb{Z}$.

Let ω_0 be the standard Kähler form on \mathbb{C}^3 . Since Σ is holomorphic, $\omega_0|_\Sigma = \text{vol}_\Sigma$ and thus $\omega = \omega_0|_L$ is a nowhere vanishing 2-form on L . Moreover, $*\omega$ is the 1-form on L dual to the $U(1)$ vector field ξ , where $*$ is the Hodge star on L , so $\theta = *\omega$ in the notation of Definition 6.1.

By (2.3), if x_1 is the coordinate on \mathbb{R} in the decomposition $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ and C is the cone on L embedded in \mathbb{R}^7 , then $\beta = \frac{\partial}{\partial x_1} \lrcorner \varphi|_C = \omega_0|_C$ is a self-dual 2-form which corresponds to the deformation of C by translation in the x_1 direction. By Corollary 2.11 and Proposition 2.17, β must be closed. Notice that $\omega_0|_C = r^2\omega + r*\omega \wedge dr$, where r is the radial coordinate, and so $d\beta = 0$ if and only if $d*\omega = -2\omega$.

By Definition 6.1, given $\gamma \in C^\infty(T^*L \otimes \mathbb{C})$, there exist $f \in C^\infty(L)$ and $\alpha \in C^\infty(T_h^*L \otimes \mathbb{C})$ such that $\gamma = f*\omega + \alpha$. Thus,

$$d\gamma = d(f*\omega) + d\alpha = d_h f \wedge *\omega - 2f\omega + d\alpha \quad \text{and} \quad *\gamma = f\omega + *_h \alpha \wedge *\omega$$

We deduce from looking at the horizontal components of (6.1) that

$$(6.3) \quad d_h \alpha = -(\nu - 2)f\omega.$$

For convenience we set $d_v = d - d_h$. Hence, using Cartan's formula and the facts that $\xi \lrcorner \alpha = \xi \lrcorner \omega = 0$ and $\xi \lrcorner *\omega = 1$, (6.2) becomes:

$$\begin{aligned} im\gamma &= d(\xi \lrcorner \gamma) + \xi \lrcorner d\gamma = df + \xi \lrcorner d(f*\omega) + \xi \lrcorner d\alpha \\ &= df + \xi \lrcorner (d_h f \wedge *\omega) + \xi \lrcorner d_v \alpha = d_v f + \xi \lrcorner d_v \alpha. \end{aligned}$$

We can also use (6.1) in (6.2) to see that:

$$im\gamma = df - \nu \xi \lrcorner *\gamma = df - \nu f \xi \lrcorner \omega - \nu \xi \lrcorner (*_h \alpha \wedge *\omega) = df + \nu *_h \alpha.$$

Together, we deduce that

$$(6.4) \quad d_h f = im\alpha - \nu *_h \alpha,$$

$$(6.5) \quad d_v f = imf * \omega,$$

$$(6.6) \quad d_v \alpha = im * \omega \wedge \alpha.$$

Since $\nu > 0$, (6.1) forces $d^* \gamma = 0$ and thus

$$(6.7) \quad d_h *_h \alpha = -imf \omega.$$

As in [33, Section 2.2], the complex structure J on Σ defines a complex structure map J_h on horizontal forms, which agrees with J on horizontal 1-forms (viewed as lifts of 1-forms on Σ). This allows us to define $\Lambda_h^{p,q} T^* L \otimes \mathbb{C}$, where $p, q \in \mathbb{N}$, to be the bundle of horizontal $(p + q)$ -forms which have eigenvalue $i(p - q)$ under J_h , and thus give the decomposition $\Lambda_h^m T^* L \otimes \mathbb{C} = \bigoplus_{p+q=m} \Lambda_h^{p,q} T^* L \otimes \mathbb{C}$. We may therefore define operators

$$\begin{aligned} \partial_h &: C^\infty(\Lambda_h^{p,q} T^* L \otimes \mathbb{C}) \rightarrow C^\infty(\Lambda_h^{p+1,q} T^* L \otimes \mathbb{C}) \quad \text{and} \\ \bar{\partial}_h &: C^\infty(\Lambda_h^{p,q} T^* L \otimes \mathbb{C}) \rightarrow C^\infty(\Lambda_h^{p,q+1} T^* L \otimes \mathbb{C}) \end{aligned}$$

by taking appropriate components of d_h .

Since α is a horizontal complex 1-form on L , it can be decomposed into $(1, 0)$ and $(0, 1)$ components. Thus, $*_h \alpha = *_h(\alpha^{1,0} + \alpha^{0,1}) = i\alpha^{1,0} - i\alpha^{0,1}$ and so (6.4) becomes:

$$(6.8) \quad \partial_h f = i(m - \nu)\alpha^{1,0},$$

$$(6.9) \quad \bar{\partial}_h f = i(m + \nu)\alpha^{0,1}.$$

We notice that $f = 0$ forces $m^2 = \nu^2$ otherwise $\alpha = 0$ by (6.8) and (6.9). Furthermore, (6.3) for $f = 0$ becomes

$$(6.10) \quad \bar{\partial}_h \alpha^{1,0} + \partial_h \alpha^{0,1} = 0.$$

Thus, $f = 0 = m - \nu$ is equivalent to $\bar{\partial}_h \alpha^{1,0} = 0 = \alpha^{0,1}$ and $f = 0 = m + \nu$ is equivalent to $\partial_h \alpha^{0,1} = 0 = \alpha^{1,0}$.

The hyperplane bundle over $\mathbb{C}\mathbb{P}^2$ defines a complex line bundle H over Σ which is also a real line bundle over L . We can, of course, identify H over L with the cone C , and so H has a global section s over L given simply by $s(x) = x$. It is clear that $\mathcal{L}_\xi s = is$ and $\alpha \otimes s^{-m}$, by (6.6), is a $U(1)$ -invariant section of $T^* L \otimes \mathbb{C} \otimes H^{-m}$, and so pushes down to be a well-defined section of $T^* \Sigma \otimes H^{-m}$ over Σ . The condition $\bar{\partial}_h \alpha^{1,0} = 0$ given by

(6.10), when $m = \nu$, is then equivalent to saying that $\alpha^{1,0} \otimes s^{-\nu}$ is a holomorphic section of $P_\nu = T^{*1,0}\Sigma \otimes H^{-\nu}$. Since H is the bundle $\mathcal{O}_\Sigma(-1)$ and $T^{*1,0}\Sigma \cong \mathcal{O}_\Sigma(d_\Sigma - 3)$, we see that $P_\nu \cong \mathcal{O}_\Sigma(\nu + d_\Sigma - 3)$. Thus, by Riemann–Roch, the dimension of the vector space of holomorphic sections of P_ν is

$$h^0(P_\nu) = d_\Sigma(\nu + d_\Sigma - 3) + 1 - g_\Sigma.$$

(Notice that P_3 is isomorphic to the normal bundle of Σ in $\mathbb{C}\mathbb{P}^2$ and so its holomorphic sections correspond precisely to the infinitesimal deformations of Σ as a holomorphic curve.) Similarly, for $m = -\nu$, we have that $\alpha^{0,1} \otimes s^\nu$ is an anti-holomorphic section of \overline{P}_ν . Hence, for $f = 0$, we get integer eigenvalues $\nu \in \{1, 2, 3\}$ for c_L in $(0, 4)$ (since $m^2 = \nu^2$) and contributions of $2h^0(P_\nu)$ to $\sigma_L(\nu)$. Using the degree-genus formula, we calculate

$$(6.11) \quad 2h^0(P_\nu) = d_\Sigma(d_\Sigma - 3 + 2\nu).$$

It follows from (6.3)–(6.7) that

$$(6.12) \quad \begin{aligned} \Delta_L f &= *d*(imf * \omega + im\alpha - \nu *_h \alpha) \\ &= -*d(imf\omega + im *_h \alpha \wedge *\omega + \nu\alpha \wedge *\omega) \\ &= -(im(imf) \text{vol}_L + im(-imf) \text{vol}_L - \nu(\nu - 2)f \text{vol}_L) \\ &= \nu(\nu - 2)f. \end{aligned}$$

Since $\mathcal{L}_\xi f = imf$ by (6.4), (6.5), we deduce from Definition 6.1 and Proposition 6.2 that

$$(6.13) \quad \Delta_h f = (\nu(\nu - 2) - m^2)f.$$

Recall that the eigenvalues of Δ_L and Δ_h have to be non-negative.

If $\nu \in (0, 2)$ then the only solution of (6.12) is $f = 0$. Thus, $m^2 = \nu^2$ by (6.8) and (6.9), so $\nu = 1$ and $\sigma_L(1) = d_\Sigma(d_\Sigma - 1)$ from (6.11) as claimed.

If $\nu = 2$ and $f \neq 0$ then $m = 0$ by (6.13) and the solutions of (6.12) consist of constant (non-zero) functions. Since $f \neq 0$ is constant, we see from (6.8)–(6.9) that $\alpha = 0$. Thus, we have a 1-dimensional space of solutions to (6.1), (6.2) for $\nu = 2$ and $f \neq 0$. We deduce the formula for $\sigma_L(2)$ from (6.11).

If $\nu \in (2, 4)$ and $f \neq 0$ then the non-negativity of $\nu^2 - 2\nu - m^2$ by (6.13) forces $|m| \leq 2$. Moreover, solutions to (6.12) define α via (6.8) and (6.9) since $m^2 \neq \nu^2$. We deduce the remaining eigenvalues of c_L and multiplicities from (6.11) as claimed.

Now suppose $d_\Sigma > 1$ so that the cone C is non-planar. We can view the Lie algebra of $SU(3)$ as a subalgebra of \mathfrak{g}_2 and decompose $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus$

$\mathfrak{su}(3)^\perp$. If $v \in \mathfrak{su}(3)^\perp$ is identified with a tangent vector on \mathbb{R}^7 , $v|_C$ defines a normal vector w which in turn defines an infinitesimal deformation of C as a coassociative cone. Moreover, since $v \in \mathfrak{su}(3)^\perp$ and C is non-planar, w is a non-trivial infinitesimal deformation which is not complex. Thus $(w \lrcorner \varphi)|_C$ will be an order $O(r)$ self-dual 2-form, which does not arise purely from a horizontal 1-form on L , and will define a 3-eigenform of c_L with $f \neq 0$ (since, for $f = 0$, the 3-eigenforms correspond to the infinitesimal deformations of Σ as a holomorphic curve). Since $\dim \mathfrak{su}(3)^\perp = 6$, we deduce that $m_L(3) \geq 6$ as claimed. \square

It is easy to check that Theorem 6.5 applied to a totally geodesic $\mathbb{C}\mathbb{P}^1$ in $\mathbb{C}\mathbb{P}^2$ implies the stability of complex 2-planes as coassociative 4-planes as we already knew from Corollary 5.7. However, we can in fact prove the following.

Proposition 6.6. *The only complex 2-dimensional cones in \mathbb{C}^3 , with compact non-singular complex links in $\mathbb{C}\mathbb{P}^2$, which are stable as coassociative cones in \mathbb{R}^7 are complex 2-planes.*

Proof. Suppose C is a counterexample to the statement of the proposition and suppose, for simplicity, that the complex link Σ of C is connected. Embed C as a coassociative cone in \mathbb{R}^7 and let G be the Lie subgroup of G_2 preserving C . Since C is non-planar and supposed to be stable, in the notation of Proposition 4.12 we should have $\dim \check{D}(0) = 7$, since every translation of C will define an order $O(1)$ coassociative deformation, and $\dim \check{D}(1) = 14 - \dim G$. By Theorem 6.5, this occurs if and only if

$$d_\Sigma^2 + d_\Sigma = 6 \quad \text{and} \quad d_\Sigma^2 + 3d_\Sigma = 14 - \dim G - m_L(3).$$

Subtracting these equations and using the fact that $m_L(3) \geq 6$ shows that $2d_\Sigma \leq 2 - \dim G$, but this contradicts $d_\Sigma > 1$ as C is non-planar. \square

We now wish to mimic our result for holomorphic curves in $\mathbb{C}\mathbb{P}^2$ for the case of null-torsion pseudoholomorphic curves in S^6 .

Theorem 6.7. *Recall Definitions 4.5, 4.6, 4.13 and 6.3. Let $\Sigma \subseteq S^6$ be a compact, connected, null-torsion pseudoholomorphic curve of genus g_Σ and let $c_1(N_2\Sigma)$ be the first Chern number of $N_2\Sigma$. Let L be the tube of radius $\frac{\pi}{2}$ in $N_2\Sigma$ about Σ . The eigenvalues of c_L in $(0, 4)$ are 1, 2, 3 and*

$\lambda + 2 \in (2, 4) \setminus \{3\}$ such that $\lambda(\lambda + 2) \in \mathcal{E}_L$. Moreover,

$$\begin{aligned} \sigma_L(1) &= 2g_\Sigma - 2c_1(N_2\Sigma) - 2, & \sigma_L(2) &= 2g_\Sigma - 4c_1(N_2\Sigma) - 1, \\ \sigma_L(3) &= m_L(3) + 2g_\Sigma - 6c_1(N_2\Sigma) - 2, & \sigma_L(\lambda + 2) &= m_L(\lambda(\lambda + 2)). \end{aligned}$$

Proof. It is straightforward to see that Definition 6.1 applies to the natural projection $\pi : L \rightarrow \Sigma$ and that the fibres of π are totally geodesic since L is a tube of radius $\frac{\pi}{2}$. Moreover, by Example 4.9, if ξ is the unit vector field given by the \mathcal{S}^1 -fibration of L over Σ then ξ is Killing. By Proposition 6.4 it is therefore enough to study (6.1)–(6.2) for $\gamma \in C^\infty(T^*L \otimes \mathbb{C})$ and $m \in \mathbb{Z}$ to find the positive eigenvalues ν of c_L .

If we let θ be the 1-form on L dual to ξ then, since Σ has null-torsion, the structure equations for L given in [28, Section 6.4] imply that $d\theta = -2*\theta$. This formula, together with the fact that Σ is endowed with a complex structure, enables us to follow the proof of Theorem 6.5 and see that we essentially have two possible contributions to eigenvalues of c_L : either eigenfunctions of the Laplacian on L or holomorphic sections of $P_\nu = T^{*1,0}\Sigma \otimes H^{-\nu}$ for $\nu \in \mathbb{Z}^+$, where $H = N_2\Sigma$ since we can identify the cone over L with $N_2\Sigma$ over Σ .

Recall the structure equations for Σ given in Definition 4.6. Since $\kappa_{32} = 0$, the curvature form of $N_2\Sigma$ is

$$d\kappa_{33} = -\theta_1 \wedge \bar{\theta}_1,$$

so $c_1(N_2\Sigma) < 0$ (in fact, it is proportional to $-\text{vol}(\Sigma)$). Thus, by Riemann–Roch, we have that

$$h^0(P_\nu) = c_1(T^{*1,0}\Sigma) - \nu c_1(N_2\Sigma) + 1 - g_\Sigma = g_\Sigma - 1 - \nu c_1(N_2\Sigma).$$

The result now follows from the proof of Theorem 6.5. □

Remarks.

- (a) The first Chern number $c_1(N_2\Sigma)$ is the negative of the degree of Σ as a holomorphic curve in the 5-quadric in $\mathbb{C}\mathbb{P}^6$, so is just $-\frac{1}{4\pi} \text{vol}(\Sigma)$.
- (b) Theorem 6.7 will not immediately generalize to the other possible case given by Example 4.9, namely tubes L of radius $\frac{\pi}{2}$ in $N_1\Sigma$. First, the relationship between the derivative of the vertical 1-form θ and its dual $*\theta$ is not as straightforward. Second, $c_1(N_1\Sigma)$ is often positive, so it is not as easy to calculate $\sigma_L(\nu)$.

One can apply Theorem 6.7 to the Borůvka sphere Σ , for which $c_1(N_2\Sigma) = -6$, and recover the result of Corollary 5.14. Theorem 6.7 can also be applied to a totally geodesic 2-sphere, which is the degenerate case of a null-torsion curve, to prove Corollary 5.7.

We now prove the analogue of Proposition 6.6.

Proposition 6.8. *Recall Definitions 4.5 and 4.6. Let C be a coassociative cone whose link is a tube of radius $\frac{\pi}{2}$ in $N_2\Sigma$ about a compact (non-totally geodesic) null-torsion pseudoholomorphic curve Σ in \mathcal{S}^6 . Then C is not stable.*

Proof. Suppose for simplicity that Σ is connected and let G be the subgroup of G_2 preserving C . By Theorem 6.7, C is stable only if

$$2g_\Sigma - 4c_1(N_2\Sigma) = 8 \quad \text{and} \quad m_L(3) + 2g_\Sigma - 6c_1(N_2\Sigma) - 2 = 14 - \dim G,$$

since C is non-planar. These equations force

$$g_\Sigma = m_L(3) + \dim G - 4 \quad \text{and} \quad 2c_1(N_2\Sigma) = m_L(3) + \dim G - 8.$$

Therefore, $m_L(3) + \dim G \geq 4$ as $g_\Sigma \geq 0$. Hence, $c_1(N_2\Sigma) \geq -2$ so the degree of Σ as a holomorphic curve in $\mathbb{C}\mathbb{P}^6$ must be 1 or 2. We deduce that Σ must have genus zero and lie in some $\mathbb{C}\mathbb{P}^2$ in $\mathbb{C}\mathbb{P}^6$. We now show that Σ cannot be a plane curve by the structure equations in Definition 4.6, giving our required contradiction.

The embedding of Σ as a holomorphic curve in $\mathbb{C}\mathbb{P}^6$ is given by \mathbf{f}_3 . Since Σ is non-totally geodesic, $d\mathbf{f}_3$ depends on \mathbf{f}_2 and $d\mathbf{f}_2$ depends on \mathbf{f}_1 . However, $d\mathbf{f}_1$ has a non-zero component in the direction of \mathbf{u} , which is independent of $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$, since θ_1 is nowhere vanishing on the curve. It therefore follows that Σ cannot lie in some $\mathbb{C}\mathbb{P}^2$ in $\mathbb{C}\mathbb{P}^6$. □

Theorems 6.5 and 6.7 invite us to make the following definition.

Definition 6.9. For a compact Riemannian 3-manifold L let

$$\eta(L) = \sum_{\lambda \in (0,1]} m_L(\lambda(\lambda + 2)),$$

using the notation of Definition 6.3.

Remark. By Theorems 6.5 and 6.7, the stability index of certain 2-ruled cones with link L fibred over an algebraic curve Σ will be the sum of a

topological term determined by the degree and genus of Σ , and an analytic piece given by $\eta(L)$. We can therefore think of $\eta(L)$ as a sort of “ η -invariant”.

We now make an elementary observation, which we state for complex cones although it is equally valid for cones whose links are as in Theorem 6.7.

Proposition 6.10. *Let C be a complex 2-dimensional cone in \mathbb{C}^3 with compact real link L such that $C \setminus \{0\}$ is non-singular. Let \mathcal{C} consist of $G_2 \ltimes \mathbb{R}^7$ transformations of a deformation family for C as a complex cone and recall Definitions 4.12 and 6.9. Let L' be the link of $C' \in \mathcal{C}$. Then*

$$\text{ind}_{\mathcal{C}}(C) - \text{ind}_{\mathcal{C}}(C') = \eta(L) - \eta(L')$$

for all $C' \in \mathcal{C}$. Moreover, there exists an open neighbourhood \mathcal{C}' of C in \mathcal{C} such that, for all $C' \in \mathcal{C}'$, $\eta(L) - \eta(L') \geq 0$. Thus, if C is \mathcal{C} -stable then C' is \mathcal{C} -stable for all $C' \in \mathcal{C}'$.

Proof. By Definitions 4.12 and 4.13 and Theorem 6.5, the difference in the \mathcal{C} -stability indices of C and C' is determined by the spectra of L and L' as claimed because the degree of the complex link is the same for C and C' .

Deformations of L will change the spectrum, but the only way in which $\eta(L) \neq \eta(L')$, for a sufficiently small perturbation L' of L , is if a new element of the spectrum is created strictly above 3 under the deformation and the number of the elements of the spectrum in $(0, 3]$ decreases. Thus, $\eta(L) \geq \eta(L')$ for all L' in some open neighbourhood of L , proving the existence of \mathcal{C}' .

If C is \mathcal{C} -stable then $\text{ind}_{\mathcal{C}}(C') \leq \text{ind}_{\mathcal{C}}(C) = 0$ for all $C' \in \mathcal{C}'$. We deduce that $\text{ind}_{\mathcal{C}}(C') = 0$ from the non-negativity of the stability index. \square

We conclude this section with an application of Proposition 6.10.

Corollary 6.11. *Recall $C_{\mathbf{a}}$, \mathcal{C}_{\circ} and \mathcal{T} defined in Corollary 5.12. The cone $C_{\mathbf{a}}$ is Jacobi integrable and \mathcal{C}_{\circ} -stable for all $\mathbf{a} \in \mathcal{T}$.*

Proof. Recall that C_2 , with link L_2 , given by Example 4.3 satisfies $C_2 = C_{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})}$ and $\text{ind}_{\mathcal{C}_{\circ}}(C_2) = 0$ by Corollary 5.12. Let $L_{\mathbf{a}}$ be the link of $C_{\mathbf{a}}$.

Let

$$\mathcal{T}_{\circ} = \{\mathbf{a} \in \mathcal{T} : \text{ind}_{\mathcal{C}_{\circ}}(C_{\mathbf{a}}) = 0\},$$

which is non-empty since $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \mathcal{T}_{\circ}$. By Proposition 6.10 \mathcal{T}_{\circ} is open. Our aim is to show that \mathcal{T}_{\circ} is also closed in \mathcal{T} , since then $\mathcal{T}_{\circ} = \mathcal{T}$ by connectedness.

Recall that $\dim \mathcal{C}_\circ = 23$ and that $m_{L_{\mathbf{a}}}(3) \geq 6$ by Theorem 6.5. Thus

$$\text{ind}_{\mathcal{C}_\circ}(C_{\mathbf{a}}) = \sum_{\lambda \in (0,1)} m_{L_{\mathbf{a}}}(\lambda(\lambda + 2)) + (m_{L_{\mathbf{a}}}(3) - 6)$$

by Theorem 6.5. We deduce that $\mathbf{a} \in \mathcal{T}_\circ$ if and only if $m_{L_{\mathbf{a}}}(3) = 6$ and $m_{L_{\mathbf{a}}}(\lambda(\lambda + 2)) = 0$ for all $\lambda \in (0, 1)$.

Suppose, for a contradiction, that \mathcal{T}_\circ is not closed, so there exists $\mathbf{a} \in \overline{\mathcal{T}_\circ} \cap \mathcal{T} \setminus \mathcal{T}_\circ$. Therefore $m_{L_{\mathbf{a}}}(3) > 6$ or $m_{L_{\mathbf{a}}}(\lambda(\lambda + 2)) > 0$ for some $\lambda \in (0, 1)$. However, since the spectrum of the Laplacian varies continuously under deformations of the metric, the latter can occur if and only if there exists $\mathbf{a}' \in \overline{\mathcal{T}_\circ}$ for which $m_{L_{\mathbf{a}'}}(3) > 6$. Since this cannot happen for $\mathbf{a}' \in \mathcal{T}_\circ$ we may suppose therefore that $m_{L_{\mathbf{a}'}}(3) > 6$.

Hence, there is a Lagrangian Jacobi field v on $L_{\mathbf{a}}$ which is independent of the Lagrangian Jacobi fields corresponding to deformations of $C_{\mathbf{a}}$ in \mathcal{C}_\circ . For all \mathbf{a}' close to \mathbf{a} we have that $L_{\mathbf{a}'} = \exp_{v'}(L_{\mathbf{a}})$ for some Lagrangian Jacobi field v' . Moreover, since $L_{\mathbf{a}'}$ is Lagrangian and $v + v'$ is a Lagrangian Jacobi field on $L_{\mathbf{a}}$, we may view v as a Lagrangian Jacobi field on $L_{\mathbf{a}'}$ for \mathbf{a}' sufficiently close to \mathbf{a} . However, any open neighbourhood of \mathbf{a} meets \mathcal{T}_\circ as $\mathbf{a} \in \overline{\mathcal{T}_\circ}$, so there exists $\mathbf{a}' \in \mathcal{T}_\circ$ such that $L_{\mathbf{a}'}$ has a Lagrangian Jacobi field independent of those corresponding to deformations of $C_{\mathbf{a}'}$ in \mathcal{C}_\circ . Thus $m_{L_{\mathbf{a}'}}(3) > 6$ for some $\mathbf{a}' \in \mathcal{T}_\circ$, which is our required contradiction. \square

7. Examples of coassociative 4-folds with conical singularities

In this section we produce our examples of coassociative 4-folds with conical singularities. We describe the construction of compact G_2 manifolds we require and the singular 4-dimensional submanifolds which arise following [20] and [21]. Initially we will have a singular coassociative 4-fold N in a compact almost G_2 manifold M , but the ambient G_2 structure will have torsion. We then show that N has conical singularities which are stable under deformations of the G_2 structure. Finally, we deform the G_2 structure on M so that it has no torsion and simultaneously deform N to produce our CS coassociative 4-fold.

7.1. Examples of compact G_2 manifolds

Here we review the relevant material from [20]. The key ingredients will be *Fano 3-folds* and *K3 surfaces*, which we now define.

Definition 7.1. A compact complex 3-dimensional manifold X is a *Fano 3-fold* if its first Chern class is positive. Equivalently, X has ample anticanonical bundle. Fano 3-folds are simply connected and projective.

A *K3 surface* P is a simply connected, compact, complex surface with $c_1(P) = 0$. A generic divisor in the anticanonical linear system of a Fano 3-fold is a smooth K3 surface by the work of Shokurov [36].

Remark. One definition of a Calabi–Yau 3-fold is a compact Kähler 3-fold with vanishing first Chern class or, equivalently, with trivial canonical bundle.

The construction of compact G_2 manifolds in [20] proceeds via the construction of certain *non-compact* Calabi–Yau 3-folds. These non-compact Riemannian manifolds are *asymptotically cylindrical*. We define these manifolds formally.

Definition 7.2. Let (Y, g) be a Riemannian n -manifold. We say that Y is *asymptotically cylindrical* (with rate λ) if there exist constants $\lambda < 0$ and $R > 0$, a compact subset K of Y , a compact Riemannian $(n-1)$ -manifold (S, g_S) and a diffeomorphism $\Psi : (R, \infty) \times S \rightarrow Y \setminus K$ satisfying

$$|\nabla^j(\Psi^*(g) - g_{\text{cyl}})| = O(e^{\lambda t}), \quad \text{as } t \rightarrow \infty \text{ for all } j \in \mathbb{N},$$

where $g_{\text{cyl}} = dt^2 + g_S$ is the cylindrical metric on $(0, \infty) \times S$, ∇ is the Levi-Civita connection of g_{cyl} and $|\cdot|$ is calculated with respect to g_{cyl} .

We now introduce some important notation.

Definition 7.3. Let \mathcal{X} be a maximal deformation family of Fano 3-folds, let $X \in \mathcal{X}$ and let P, Q be K3 surfaces in the anticanonical linear system of X such that $P \cap Q$ is a non-singular curve in X . Let $\tilde{X}(P, Q)$ denote the blow-up of X along $P \cap Q$ and let \tilde{P} denote the proper transform of P in $\tilde{X}(P, Q)$. Finally, let $Y(X, P, Q) = \tilde{X}(P, Q) \setminus \tilde{P}$.

Since $\tilde{X}(P, Q)$ is the blow-up of X along $P \cap Q$, we have a smooth map $\varpi : \tilde{X}(P, Q) \rightarrow \mathbb{C}\mathbb{P}^1$ whose fibres are the proper transforms of the divisors in the pencil defined by P and Q . We may introduce a holomorphic coordinate ζ on $\mathbb{C}\mathbb{P}^1$ such that $\varpi^{-1}(0) = \tilde{P}$ and see that, for some open neighbourhood U of 0 in the ζ coordinate, $\varpi^{-1}(U \setminus \{0\})$ is diffeomorphic to $(0, \infty) \times P \times \mathcal{S}^1$. Thus we may view $Y(X, P, Q)$ as a manifold with a cylindrical “end” with cross-section $P \times \mathcal{S}^1$. This motivates the next result which follows from [20, Corollary 6.43].

Theorem 7.4. *Use the notation of Definition 7.3. There is a smooth complete metric g_Y on $Y = Y(X, P, Q)$ such that (Y, g_Y) is asymptotically cylindrical and the holonomy of g_Y is $SU(3)$.*

Suppose we have a pair of maximal deformation families \mathcal{X}_1 and \mathcal{X}_2 of Fano 3-folds. Using the notation of Definition 7.3, we have a pair of asymptotically cylindrical complex 3-folds $Y_1 = Y_1(X_1, P_1, Q_1)$ and $Y_2 = Y_2(X_2, P_2, Q_2)$ with holonomy $SU(3)$ by Theorem 7.4. We thus have 7-manifolds $Z_i = Y_i \times \mathcal{S}^1$, for $i = 1, 2$, which are asymptotically cylindrical to $(0, \infty) \times P_i \times \mathcal{S}^1 \times \mathcal{S}^1$. In [20, Section 4] it is explained that if P_1 and P_2 satisfy a certain “matching condition”, then one can apply a “twisted connected sum” construction to Z_1 and Z_2 to get a one-parameter family of compact almost G_2 manifolds $\{(M_T, \varphi_T, g_{\varphi_T}) : T > T_0\}$ for some $T_0 > 0$. Moreover, $(\varphi_T, g_{\varphi_T})$ is simply the product G_2 structure on Z_i , as described in Definition 2.18, away from the “interpolation region” where Z_1 and Z_2 are “glued”. The only question is whether this “matching condition” holds for P_1 and P_2 . The answer [20, Theorem 6.44] is that there always exist $X_i \in \mathcal{X}_i$ such that P_1 and P_2 can be chosen which satisfy the “matching condition”. Finally, [20, Proposition 5.32 and Theorem 5.34] imply that one can always perturb the closed G_2 structure on M_T to a torsion-free one for sufficiently large T . We can summarize these observations as a theorem.

Theorem 7.5. *Let \mathcal{X}_1 and \mathcal{X}_2 be maximal deformation families of Fano 3-folds and recall the notation of Definitions 2.13, 2.14, 2.15 and 7.3. There exist constants $T_0 > 0$ and $\lambda < 0$ and, for $i = 1, 2$, $X_i \in \mathcal{X}_i$ and a K3 surface P_i in the anticanonical linear system of X_i such that, for all $T > T_0$ and suitable Q_1, Q_2 as in Definition 7.3, the following hold.*

- (a) *There is a compact almost G_2 manifold $(M_T, \varphi_T, g_{\varphi_T})$ which, outside some compact set I_T , is diffeomorphic to the disjoint union of $Y_1(X_1, P_1, Q_1) \times \mathcal{S}^1$ and $Y_2(X_2, P_2, Q_2) \times \mathcal{S}^1$ endowed with the product G_2 structure induced from the asymptotically cylindrical $SU(3)$ structure given by Theorem 7.4.*
- (b) *Let $p > 4$. There is a smooth 2-form η_T on M_T , satisfying $\|\eta_T\|_{L^p_2} \leq c_p e^{\lambda T}$ and $\|\eta_T\|_{C^1} \leq c_p e^{\lambda T}$ for some constant $c_p > 0$, such that $\varphi_T + d\eta_T \in C^\infty(\Lambda^3_+ T^* M_T)$ and the metric $g_{\varphi_T + d\eta_T}$ on M_T has holonomy G_2 .*

The compact set I_T is the interpolation region between Z_1 and Z_2 .

Note. Of course, the M_T are topologically the same for all T , so we can view $(\varphi_T, g_{\varphi_T})$ as a one-parameter family of closed G_2 structures on a 7-manifold M .

We now have the following important result.

Proposition 7.6. *Use the notation of Theorem 7.5. Outside I_T , M_T is fibred by $K3$ surfaces which are coassociative with respect to φ_T . Moreover, for generic Q_1 and Q_2 , there exist coassociative $K3$ surfaces in $(M_T, \varphi_T, g_{\varphi_T})$ whose singularities are isolated and are ordinary double points.*

Proof. After Definition 7.3 we noted that we have a fibration $\varpi : \tilde{X}(P, Q) \rightarrow \mathbb{C}P^1$ whose fibres are $K3$ surfaces. Necessarily some of these fibres will be singular and the generic singularity is an ordinary double point. Recall that we have the freedom to choose any smooth $K3$ surface Q in the anticanonical linear system of X which meets P in a non-singular curve. Therefore, through generic choice of Q we can be assured that there are fibres other than the exceptional divisor whose only singularities are ordinary double points. Thus, for generic Q_1 and Q_2 , we have a fibration

$$\varpi_T : M_T \setminus I_T \cong (Y_1 \times \mathcal{S}^1) \sqcup (Y_2 \times \mathcal{S}^1) \rightarrow (\mathbb{C}P^1 \sqcup \mathbb{C}P^1) \times \mathcal{S}^1$$

with fibres that are $K3$ surfaces in $Y_i \times \{x\}$ for some i and some $x \in \mathcal{S}^1$. Moreover, there are some fibres of ϖ_T which only have ordinary double point singularities. Since Y_1 and Y_2 are Calabi–Yau manifolds and the almost G_2 structure on $M_T \setminus I_T$ agrees with the product G_2 structure on $Y_1 \times \mathcal{S}^1$ and $Y_2 \times \mathcal{S}^1$, a simple generalization of Corollary 2.11 leads us to deduce that the fibres of ϖ_T are coassociative with respect to φ_T . □

Remark. By studying the G_2 structure on I_T , it is shown in [21] that one can extend the coassociative fibration ϖ_T through I_T .

For convenience we introduce the following notation.

Definition 7.7. Use the notation of Theorem 7.5 and let Q_1 and Q_2 be generic so that Proposition 7.6 applies. Let Γ_T denote the set of coassociative $K3$ surfaces in $(M_T, \varphi_T, g_{\varphi_T})$, which have isolated ordinary double point singularities. By Proposition 7.6, there exist $N \in \Gamma_T$ such that $N \subseteq M_T \setminus I_T$.

Our aim now is to show that some of the singular coassociative 4-folds in Γ_T are “stable” under the deformation from the closed G_2 structure φ_T to the torsion-free G_2 structure $\varphi_T + d\eta_T$ given in Theorem 7.5(b).

7.2. Stable coassociative conical singularities

We begin with a crucial result that allows us to implement our stability theory.

Proposition 7.8. *Use the notation of Theorem 7.5 and Definition 7.7. If $N \in \Gamma_T$, then N is a CS coassociative 4-fold in $(M_T, \varphi_T, g_{\varphi_T})$ in the sense of Definition 3.2. Moreover, the singularities of N have cones in the family \mathcal{C}_\circ given in Corollary 5.12.*

Proof. First observe that N is clearly a connected coassociative integral current with $\partial N = \emptyset$. Second, if z is a singular point of N then it is an ordinary double point of a complex surface, so the tangent cone at z has multiplicity one and is modelled on a cone in \mathcal{C}_\circ by definition. Since cones in \mathcal{C}_\circ are Jacobi integrable by Corollary 6.11, we may apply Corollary 3.20 to deduce the result. □

Proposition 7.9. *Let N be a CS coassociative 4-fold in an almost G_2 manifold. Suppose that the singularities of N are z_1, \dots, z_s with rate μ and cones C_1, \dots, C_s such that:*

- (i) C_i is in the family \mathcal{C}_\circ given in Corollary 5.12 for all i ; and
- (ii) $(1, \mu] \cap \mathcal{D} = \emptyset$, where \mathcal{D} is given in Definition 3.12.

If $\mathfrak{C} = \mathcal{C}_\circ^s$, then $\mathcal{O}(N, \mu, \mathfrak{C}) = \{0\}$, in the notation of Theorem 3.14.

Proof. This follows immediately from Definition 4.12 and Corollary 6.11. □

Theorem 1.3 now follows from our final result.

Theorem 7.10. *Use the notation of Theorem 7.5 and Definition 7.7. Let $T > T_0$ and let $N \in \Gamma_T$ be such that $N \subseteq M_T \setminus I_T$. Making T_0 larger if necessary, there exists a CS deformation N' of N which is coassociative with respect to $\varphi_T + d\eta_T$.*

Proof. Note that there are $N \in \Gamma_T$ which do not lie in I_T and that we are free to make the rate μ at the singularities of N lower if necessary to satisfy $(1, \mu] \cap \mathcal{D} = \emptyset$. Hence, Propositions 7.8 and 7.9 imply N is a CS coassociative 4-fold with respect to φ_T and that $\mathcal{O}(N, \mu, \mathfrak{C}) = \{0\}$.

Recall that $M_{T'}$ is diffeomorphic to some 7-manifold M for all $T' \geq T_0$. Let $\mathcal{F} = \{(\varphi_t + \text{sd}\eta_t, g_{\varphi_t + \text{sd}\eta_t}) : s \in \mathbb{R}, t > T_0\}$. Then \mathcal{F} is a smooth

2-dimensional family of closed G_2 structures on M . Moreover, for each $T > T_0$ we may parameterize \mathcal{F} by $(u, v) \in B(0; 1)$ via

$$(u, v) \in B(0; 1) \subseteq \mathbb{R}^2 \mapsto \varphi^{(u,v)} = \varphi_{t_T(u,v)} + s_T(u, v) \, d\eta_{t_T(u,v)},$$

where

$$s_T(u, v) = \frac{(T - T_0)v}{(1 - u)^2 + v^2}$$

and

$$t_T(u, v) = \frac{T_0(1 - u)^2 + T_0v^2 + (T - T_0)(1 - u)}{(1 - u)^2 + v^2}.$$

With this parameterization of \mathcal{F} we see that $\varphi^{(0,0)} = \varphi_T$.

Notice that, since $N \cap I_T = \emptyset$ there exists $\tau > 0$ such that $T > T_0 + \tau$ and $N \cap I_{T'} = \emptyset$ for all $T' \in (T - \tau, T + \tau)$. As the G_2 structure outside $I_{T'}$ is the same for all $T' \in (T - \tau, T + \tau)$ by Theorem 7.5(a), N is coassociative with respect to $\varphi_{T'}$ for all such T' . We therefore see that $[\varphi^{(u,v)}] = 0$ in $H_{cs}^3(\hat{N})$ for all (u, v) sufficiently near $(0, 0)$. Hence, by Theorem 3.17, there exists some $\delta_N > 0$ such that for all $(u, v) \in B(0; \delta_N)$ there exists a CS deformation $N^{(u,v)}$ of N which is coassociative with respect to $\varphi^{(u,v)}$.

To complete the proof, we need to show that $\varphi^{(u,v)} = \varphi_T + d\eta_T$ for some $(u, v) \in B(0; \delta_N)$. As discussed after Theorem 3.17, this occurs if $\|d\eta_T\|_{C^1} < \epsilon$ and $\|d\eta_T\|_{L^p_2} < \epsilon$ for some $p > 4$, where $\epsilon > 0$ is determined by the geometry near N with respect to g_{φ_T} . By Theorem 7.5(b), $\|d\eta_T\|_{C^1}$ and $\|d\eta_T\|_{L^p_2}$ are of order $O(e^{\lambda T})$ for some $\lambda < 0$. Moreover, by Theorem 7.5(a), the geometry near N is not changing as T varies, so the constant ϵ can be chosen to be independent of T . Thus, we can ensure that the relevant norms of $d\eta_T$ are sufficiently small by making T_0 larger. □

Acknowledgements

The author particularly thanks Robert Bryant, Dominic Joyce and Alexei Kovalev for useful conversations, and University College, Oxford, for hospitality during the course of this project. The author also thanks the referee for helpful comments and suggestions. The author is currently supported by an EPSRC Career Acceleration Fellowship.

References

- [1] D. Adams and L. Simon, *Rates of asymptotic convergence near isolated singularities of geometric extrema*, Indiana Univ. Math. J. **37** (1988), 225–254.
- [2] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), 661–693.
- [3] L. Bérard Bergery and J.-P. Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibres*, Illinois J. Math. **26** (1982), 181–200.
- [4] R.L. Bryant, *Submanifolds and special structures on the octonions*, J. Diff. Geom. **17** (1982), 185–232.
- [5] R.L. Bryant, *Some remarks on G_2 -structures*, Proceedings of Gökova Geometry-Topology Conference 2005, eds. S. Akbulut, T. Önder and R.J. Stern, International Press, 2006.
- [6] A. Corti, M. Haskins, J. Nordström and T. Pacini, *G_2 manifolds and associative submanifolds via semi-Fano 3-folds*, Preprint, arXiv:1207.4470.
- [7] F. Dillen, L. Verstraelen and L. Vrancken, *Classification of totally real 3-dimensional submanifolds of $S^6(1)$ with $K \geq 1/16$* , J. Math. Soc. Japan **42** (1990), 565–584.
- [8] J. Etnyre and R. Ghrist, *Contact topology and hydrodynamics. I. Beltrami fields and the Seifert conjecture*, Nonlinearity **13** (2000), 441–458.
- [9] D. Fox, *Coassociative cones that are ruled by 2-planes*, Asian J. Math. **11** (2007), 535–554.
- [10] R. Harvey and H.B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–152.
- [11] M. Haskins, *The geometric complexity of special Lagrangian T^2 -cones*, Invent. Math. **157** (2004), 11–70.
- [12] A. Ikeda, *On the spectrum of homogeneous spherical space forms*, Kodai Math. J. **18** (1995), 57–67.
- [13] D.D. Joyce, *Compact manifolds with special holonomy*, OUP, Oxford, 2000.

- [14] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. V. Survey and applications*, J. Diff. Geom. **63** (2003), 299–347.
- [15] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. I. Regularity*, Ann. Global Ann. Geom. **25** (2004), 201–251.
- [16] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. II. Moduli spaces*, Ann. Global Ann. Geom. **25** (2004), 301–352.
- [17] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. III. Desingularization, the unobstructed case*, Ann. Global Ann. Geom. **26** (2004), 1–58.
- [18] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. IV. Desingularization, obstructions and families*, Ann. Global Ann. Geom. **26** (2004), 117–174.
- [19] D.D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics **12**, OUP, Oxford, 2007.
- [20] A.G. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. **565** (2003), 125–160.
- [21] A.G. Kovalev, *Coassociative $K3$ fibrations of compact G_2 -manifolds*, Preprint, [arXiv:math/0511150](https://arxiv.org/abs/math/0511150).
- [22] H.B. Lawson and R. Osserman, *Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system*, Acta Math. **139** (1977), 1–17.
- [23] R.B. Lockhart and R.C. McOwen, *Elliptic differential operators on non-compact manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **12** (1985), 409–447.
- [24] J.D. Lotay, *2-Ruled calibrated 4-folds in \mathbb{R}^7 and \mathbb{R}^8* , J. London Math. Soc. **74** (2006), 219–243.
- [25] J.D. Lotay, *Coassociative 4-folds with conical singularities*, Comm. Anal. Geom. **15** (2007), 891–946.
- [26] J.D. Lotay, *Desingularization of coassociative 4-folds with conical singularities*, Geom. Funct. Anal. **18** (2008), 2055–2100.
- [27] J.D. Lotay, *Deformation theory of asymptotically conical coassociative 4-folds*, Proc. London Math. Soc. **99** (2009), 386–424.

- [28] J.D. Lotay, *Ruled Lagrangian submanifolds of the 6-sphere*, Trans. Amer. Math. Soc. **363** (2011), 2305–2339.
- [29] K. Mashimo, *Homogeneous totally real submanifolds of S^6* , Tsukuba J. Math. **9** (1985), 185–202.
- [30] V.G. Maz'ya and B. Plamenevskij, *Elliptic boundary value problems*, Amer. Math. Soc. Transl. **123** (1984), 1–56.
- [31] R.C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), 705–747.
- [32] F. Morgan, *Geometric measure theory: a beginner's guide*, 4th edition, Academic Press, San Diego, 2009.
- [33] P.-A. Nagy, *Un principe de séparation des variables pour le spectre du Laplacien des formes différentielles et applications*, PhD thesis, Université de Savoie, 2001.
- [34] Y. Ohnita, *Stability and rigidity of special Lagrangian cones over certain minimal Legendrian orbits*, Osaka J. Math. **44** (2007), 305–334.
- [35] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Math. **201**, Longman, Harlow, 1989.
- [36] V.V. Shokurov, *Smoothness of a general anticanonical divisor on a Fano variety*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 430–441; English transl.: Math. USSR Izv **14** (1980), 395–405.
- [37] L. Simon, *Isolated singularities of extrema of geometric variational problems*, in Harmonic mappings and minimal immersions, ed. E. Giusti, Lect. Notes Math. **1161**, Springer-Verlag, Berlin, 1985, 206–277.
- [38] A. Strominger, S.T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nucl. Phys. B **479** (1996), 243–259.
- [39] S. Tanno, *The first eigenvalue of the Laplacian on spheres*, Tôhoku Math. J. **31** (1979), 179–185.
- [40] H. Urakawa, *On the least eigenvalue of the Laplacian for compact group manifolds*, J. Math. Soc. Japan **31** (1979), 209–226.
- [41] L. Vrancken, *Killing vector fields and Lagrangian submanifolds of the nearly Kähler S^6* , J. Math. Pures Appl. **77** (1998), 631–645.

- [42] B. Watson, *Manifold maps commuting with the Laplacian*, J. Diff. Geom. **8** (1973), 85–94.

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE LONDON
GOWER STREET
LONDON WC1E 6BT
ENGLAND, UK
E-mail address: j.lotay@ucl.ac.uk

RECEIVED JANUARY 6, 2011

