COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 20, Number 4, 781–802, 2012

Parabolic $(3, 5, 6)$-distributions and $GL(2)$-structures

Wojciech Kryński

We consider rank-three distributions with growth vector $(3, 5, 6)$. The class of such distributions splits into three subclasses: parabolic, hyperbolic and elliptic. In the present paper, we deal with the parabolic case. We provide a classification of such distributions and exhibit connections between them and $GL(2)$-structures. We prove that any $GL(2)$-structure on three- and four-dimensional manifold can be described as a parabolic $(3, 5, 6)$-distribution.

1. Introduction

In the present paper, we deal with rank-three distributions on manifolds of dimension 6. A generic distribution within this class is one-step bracket generating, i.e., its growth vector is $(3, 6)$. The equivalence problem for this generic class of distributions was studied by Bryant [2] who also showed that there is a canonical conformal structure associated to any distribution of this type. Our aim is to locally classify non-generic rank-three distributions on manifolds of dimension 6. Namely, distributions which have growth vector $(3, 5, 6)$ and so-called parabolic symbol. Our results extend unpublished results of Doubrov who considered elliptic and hyperbolic symbol [6]. We also exhibit links between distributions and $GL(2)$-structures.

In the recent years, $GL(2)$-structures have attracted much attention due to their links to ordinary differential equations (ODEs). The first result in this direction goes back to the paper of Chern [5] who showed that if an ODE of third order satisfies Wünschmann condition then it defines a conformal Lorentz metric on its solutions space. A similar observation for ODEs of fourth order was made by Bryant in his paper on exotic holonomies [1]. The general case was treated by Dunajski and Tod [11], whereas a more detailed analysis of equations of order 5 was given in [13]. $GL(2)$-structures were also recently studied in [10, 12, 16, 23].

Simultaneously, serious progress has been made in understanding geometry of non-holonomic distributions, and wide classes of distributions have been classified. In particular, the problem of equivalence was solved for generic distributions of rank 2 [8], of rank 3 [2, 9] and of corank 2 on odd-dimensional manifolds [14, 17]. Besides, there is a number of results, inspired by the discovery of Nurowski conformal structure, showing that certain types of distributions define G-structures on manifolds [2, 22]. In the present paper, we associate $GL(2)$-structures to non-generic distributions of rank 3.

To be more precise, we consider a rank-three distribution D on a six-dimensional manifold M and assume that D has growth vector $(3, 5, 6)$ (i.e., $\mathrm{rk}\,[D, D] = 5$ and $[D, [D, D]] = TM$; we say that D is (3,5,6)-distribution). Doubrov [6] showed that the class splits into three subclasses distinguished by the signature of a certain bilinear form associated to a distribution, i.e., D can be either parabolic, elliptic or hyperbolic. A construction of the bilinear form is given in Section 2. Doubrov concentrated on the elliptic and hyperbolic cases, which can be described in terms of Cartan geometry modelled on $SL(4)$. The distributions of elliptic and hyperbolic type correspond to systems of two partial differential equations (PDEs), elliptic or hyperbolic, respectively, for one function in two independent variables. The results of Doubrov give clear and complete picture of the geometry of such PDEs.

The geometry of parabolic $(3, 5, 6)$-distributions is more complicated. In fact, there are three inequivalent distributions which can be considered as flat (or homogeneous) models and the geometry of each of them is different. One of them is the canonical Cartan distribution on the mixed jet space $J^{2,1}(\mathbb{R}, \mathbb{R}^2)$ defined as the annihilator of the following one-forms

$$du - u_1 dt, \qquad du_1 - u_2 dt, \qquad dv - v_1 dt,$$

where (t, u, v, u_1, u_2, v_1) are natural coordinates on $J^{2,1}(\mathbb{R}, \mathbb{R}^2)$. The two other flat models can be extracted from [18] where the classification of low-dimensional, nilpotent, graded Lie algebras is given. The Lie algebras define in a natural way canonical distributions on the corresponding Lie groups (see [17] for detailed explanation of the construction) and these distributions are called flat. The two algebras corresponding to parabolic $(3, 5, 6)$-distributions are denoted m6_3_3 and m6_3_4 in the notation of [18]. It is computed in [18] that the two algebras have infinite Tanaka prolongation. It follows that the algebras of infinitesimal symmetries of the corresponding flat models are infinite-dimensional. The same holds for the Cartan distribution on $J^{2,1}(\mathbb{R}, \mathbb{R}^2)$. Therefore the picture is completely different if compared to generic $(3, 6)$-distribution, or to $(3, 5, 6)$-distribution with elliptic or hyperbolic symbol, where the dimension of the group of symmetries is always finite and the most symmetric distribution is unique up to a local equivalence. The reason for infinite dimensionality in the parabolic case is the existence of non-trivial Cauchy characteristics of the Lie square $[D,D]$ of a parabolic $(3,5,6)$-distribution D. In Section 3, we use the Cauchy characteristic to define a reduction of D. The reduction allows to interpret the original problem of equivalence of distributions as the problem of equivalence of certain structures on lower-dimensional manifolds. The process of reduction and the converse construction is analogous to the process of reduction of Goursat distributions [3, 20].

Our solution to the problem of equivalence is given in Sections 4, 5 and 6. It splits into several branches which correspond to different flat models. Moreover, we discover that all $GL(2)$-structures on three- and four-dimensional manifolds can be interpreted as certain parabolic $(3,5,6)$-distributions. In dimension 4, the result does not depend on whether a $GL(2)$-structure is defined by an ODE (in the case of dimension 3 all $GL(2)$-structures are of equation type, see [12, 16]). Therefore, as a by-product, we get a unified model for all $GL(2)$-structures on four-dimensional manifolds.

There is no clear interpretation of parabolic $(3,5,6)$-distributions as systems of PDEs. One can observe that, unlike in the elliptic and hyperbolic cases, parabolic systems of two PDEs in two variables give rise not to $(3,5,6)$- but $(3,4,6)$-distributions, which can be reduced to $(2,3,5)$-distributions considered by Cartan in his famous paper [4]. In Section 7, we provide PDE models for one branch in our classification.

2. Preliminaries

Let D be a $(3,5,6)$-distribution on a manifold M. Then, the Lie bracket of vector fields gives rise, at each point $x\in M$, to the map $D(x)\wedge D(x)\to [D,D](x)/D(x)$

$$(v_1,v_2)\mapsto [V_1,V_2](x) \mod D(x),$$

where in order to compute the Lie bracket on the right-hand side we extend vectors $v_1,v_2\in D(x)$ to local sections V_1 and V_2 of D in an arbitrary way, and the result does not depend on the extensions. Since $\dim D(x)\wedge D(x)=3$ and $\dim [D,D](x)/D(x)=2$ the mapping has one-dimensional kernel. Any element of $D(x)\wedge D(x)$ is decomposable and thus there is the unique subdistribution

$$D_2\subset D$$

of rank 2 such that $[D_2,D_2]\subset D$. (As a matter of fact D_2 is defined by the image of the *singular exponential map* associated to D [15, 19].)

There are two cases: D_2 can be integrable or $[D_2, D_2] = D$ (in a neighbourhood of a generic point). In the second case D is uniquely determined by D_2, which has growth vector $(2, 3, 5, 6)$. All distributions of type $(2, 3, 5, 6)$ were classified in [8]. Therefore, in what follows, we will assume that D_2 is integrable. We will denote

$$D_5 = [D, D].$$

Now, assume that (X_1, X_2) is a local frame of D_2, and let Y be a vector field complementing X_1 and X_2 to a local frame of D. Define

$$Y_i = [Y, X_i].$$

Then (X_1, X_2, Y, Y_1, Y_2) is a local frame of $[D, D]$ and we can complement this tuple to the full local frame on M by choosing a vector field Z. Following Doubrov [6] we define a 2×2 matrix-valued function (a_{ij}) by the formula:

$$[X_i, Y_j](x) = a_{ij}(x) Z(x) \mod [D, D](x).$$

The matrix (a_{ij}) has the following properties

1. (a_{ij}) is symmetric. Indeed

$$[X_i, Y_j] = [X_i, [Y, X_j]] = [X_j, [Y, X_i]] + [Y, [X_i, X_j]] = [X_j, Y_i] \mod [D, D]$$

   since $[X_i, X_j]$ is a section of $D_2 \subset D$.

2. If we choose different vector fields Y and Z then (a_{ij}) is multiplied by a function. Indeed, if

$$Y \mapsto \alpha Y \mod D_2, \qquad Z \mapsto \beta Z \mod D_5$$

   then

$$(a_{ij}) \mapsto \frac{\alpha}{\beta}(a_{ij}).$$

3. If we choose a different frame of D_2, $X_i \mapsto b_{i1} X_1 + b_{i2} X_2$ then (a_{ij}) transforms as a bilinear form, i.e.

$$(a_{ij}) \mapsto (b_{ij})^{\mathrm{T}} (a_{ij}) (b_{ij}).$$

Therefore, at each point $x \in M$, the matrix (a_{ij}) defines a bilinear symmetric form on $D_2(x)$ given up to multiplication by a number. There are three cases depending on the signature of $(a_{ij}(x))$: if $(a_{ij}(x))$ is definite then we say that D is *elliptic at* x, if $(a_{ij}(x))$ is indefinite then we say that D is *hyperbolic at*

x, or if $(a_{ij}(x))$ is not of a full rank then we say that D is *parabolic at* x. The parabolic case splits to the two subsequent cases: if $(a_{ij}(x))$ has rank 1 then we say that D is *non-degenerate parabolic at* x or if $(a_{ij}(x))$ has rank 0 then we say that D is *degenerate parabolic at* x.

Definition. A $(3,5,6)$-distribution D is *regular at* $x \in M$ if there exists a neighbourhood of x such that the signature of (a_{ij}) is constant in this neighbourhood (D is either elliptic or hyperbolic or non-degenerate parabolic or degenerate parabolic). Otherwise we say that D is *irregular at* x.

Clearly if D is elliptic or hyperbolic at x then it is also elliptic or hyperbolic in a small neighbourhood of x, because $(a_{ij}(x))$ depends smoothly on a point $x \in M$. Thus all elliptic and hyperbolic points are regular. On the other hand there are irregular parabolic points, but we will not consider them in the present paper. We will consider a problem of local equivalence of $(3,5,6)$-distributions at regular parabolic points and we will just say that D is *parabolic* (*degenerate* or *non-degenerate*).

In the whole paper we say that two structures on a manifold are *(locally) equivalent* if there exists a (local) diffeomorphism transforming one structure onto the other.

3. Reduction

Assume first that D is degenerate parabolic. It follows that $[X_i, Y_j] = 0 \mod D_5$. However, $[D, D_5] = TM$, so for each choice of Y as in Section 2 we have a surjection $D_5(x) \to T_xM/D_5(x)$

$$v \mapsto [Y, V](x) \mod D_5(x),$$

where V is an extension of $v \in D_5(x)$ to a local section of D_5. The mapping has a four-dimensional kernel, which does not depend on the choice of Y. Therefore there is a well-defined subdistribution $D_4 \subset D_5$. Clearly $D \subset D_4$. Now we can consider the mapping $D_2(x) \to D_5(x)/D_4(x)$

$$v \mapsto [Y, V](x) \mod D_4(x),$$

where V is an extension of $v \in D_2(x)$ to a local section of D_2. This mapping has a one-dimensional kernel $D_1 \subset D_2$, which is again invariantly assigned to a distribution.

If D is non-degenerate parabolic then the situation looks similar. Namely the matrix $(a_{ij}(x))$, which is a bilinear form on D_2, has a one-dimensional kernel for any x. Thus we have a rank-one distribution $D_1 \subset D_2$. Then we can define $D_4 = [D_1, D]$.

Denoting $D_3 = D$, in both cases we get the flag

$$D_1 \subset D_2 \subset D_3 \subset D_4 \subset D_5 \subset TM.$$

Lemma 3.1. *If D is a regular parabolic $(3,5,6)$-distribution on a manifold M, then the associated flag $(D_i)_{i=1,\dots,5}$ satisfies*

$$[D_1, D_2] = D_2, \quad [D_1, D_3] = D_4, \quad [D_1, D_4] = D_4. \tag{3.1}$$

Proof. The first equality follows from the fact that D_2 is always integrable in our context. The second follows from the definition of D_1 and D_4. In order to prove $[D_1, D_4] = D_4$, let us assume that $(X_1, X_2, Y, Y_1, Y_2, Z)$ is a local frame on M as in Section 2. Moreover assume that X_1 spans D_1. Then $Y_1 = [Y, X_1]$ complements (X_1, X_2, Y) to a local frame of D_4. We shall show that $[X_1, Y_1] = 0 \mod D_4$. In general, we have

$$[X_1, Y_1] = fY_2 \mod D_4,$$

for some f. The proof splits into two cases.

In the degenerate case we may assume that $Z = [Y, Y_2]$. Then we consider $[Y, [X_1, Y_1]]$ and apply Jacobi identity. On the one hand, we get

$$[Y, [X_1, Y_1]] = fZ \mod D_5.$$

On the other hand, we get

$$[Y, [X_1, Y_1]] = [Y_1, Y_1] + [X_1, [Y, Y_1]] = 0 + [X_1, \tilde{Y}] \mod D_5,$$

for some section $\tilde{Y}$ of D_5. But, since $(a_{ij}) = 0$, we get $[X_1, \tilde{Y}] = 0 \mod D_5$ and consequently $f = 0$.

In the non-degenerate case, we may assume that $Z = [X_2, Y_2]$. Then we consider $[X_2, [X_1, Y_1]]$ and apply the Jacobi identity. On the one hand, we get

$$[X_2, [X_1, Y_1]] = fZ \mod D_5.$$

On the other hand, we get

$$[X_2, [X_1, Y_1]] = [\tilde{X}, Y_1] + [X_1, \tilde{Y}] \mod D_5,$$

for some section $\tilde{X}$ of D_2 and some section $\tilde{Y}$ of D_5. But, by the definition of our frame, the only non-zero entry of (a_{ij}) is a_{22}, so we get $[\tilde{X}, Y_1] = [X_1, \tilde{Y}] = 0 \mod D_5$. As a result we get $f = 0$ as desired. □

Now we can define the fundamental *reduction* of parabolic $(3, 5, 6)$-distribution. Namely, from (3.1) it follows that D_1 is contained in Cauchy characteristic of both D_2 and D_4. Thus we can consider (at least locally) the quotient manifold $N = M/D_1$ with the quotient mapping

$$q\colon M \to N,$$

and with two well-defined distributions

$$B_1 = q_*(D_2), \quad B_3 = q_*(D_4),$$

such that $\operatorname{rk} B_1 = 1$, $\operatorname{rk} B_3 = 3$ and $B_1 \subset B_3$. A pair (B_1, B_3) on N will be called the *reduced pair of D*.

Remark. We do not consider global properties of D_1. Therefore, we do not know if the quotient M/D_1 is a well-defined manifold. However, as we are interested only in the problem of local classification of D near $x \in M$, we can always restrict to a sufficiently small neighbourhood $U \subset M$ of x such that the quotient $N_U = U/(D_1|_U)$ is well defined. Writing N we will always have in mind N_U even if it is not written explicitly.

There exists also the converse construction and it appears that the reduced pair (B_1, B_3) contains almost all information about the original distribution D. In fact the construction is very similar to the construction of Cartan prolongation for Goursat distributions (cf.[20, 21]). Having a pair (B_1, B_3) on a manifold N, we consider the quotient vector bundle $B_3/B_1 \to N$, and we define a manifold

$$\tilde{M} = P(B_3/B_1),$$

where $P(B_3/B_1)$ is the total space of the projectivisation of the bundle $B_3/B_1 \to N$. $\tilde{M}$ is a manifold of dimension $\dim N + 1$, and we have the projection $\pi\colon \tilde{M} \to N$.

Let $x \in \tilde{M}$. Then x is an element of $P(B_3(\pi(x))/B_1(\pi(x)))$, and it will be convenient to denote

$$\begin{aligned} L(x) = {} & \text{two-dimensional subspace of } B_3(\pi(x)) \\ & \quad \text{defining } x \text{ as an element of } P(B_3(\pi(x))/B_1(\pi(x))). \end{aligned}$$

We define a canonical rank-three distribution on $\tilde{M}$ by the following formula

$$\tilde{D}_3(x) = \{v \in T_x\tilde{M} \mid \pi_*(v) \in L(x)\}, \qquad x \in \tilde{M}.$$

By definition $\tilde{D}_3$ contains the vertical rank-one distribution

$$\tilde{D}_1 = \ker \pi_*,$$

which is tangent to the fibres of π. There is also a well-defined rank-two distribution

$$\tilde{D}_2 = \pi_*^{-1}(B_1).$$

Moreover, it can be easily seen that

$$\tilde{D}_4 = [\tilde{D}_1, \tilde{D}_3]$$

is a rank-four distribution that coincides with $\pi_*^{-1}(B_3)$. It follows that the flag $\tilde{D}_1 \subset \tilde{D}_2 \subset \tilde{D}_3 \subset \tilde{D}_4$ satisfies relations (3.1).

Lemma 3.2. *Assume that $D_1 \subset D_2 \subset D_3 \subset D_4$ satisfies* (3.1)*. Then the natural mapping $\Phi \colon M \to \tilde{M}$*

$$\Phi(x) = q_*(D_3(x))/B_1(q(x)) \in P(B_3(q(x))/B_1(q(x)))$$

is a local diffeomorphism that establishes an equivalence of flags $(D_i)_{i=1,\dots,4}$ and $(\tilde{D}_i)_{i=1,\dots,4}$. In particular, D_3 in a neighbourhood of x is locally equivalent to $\tilde{D}_3$ in a neighbourhood of $\Phi(x)$.

Proof. The first and the third relation of (3.1) allow us to define the reduction and the mapping Φ. The second relation of (3.1) will be used to prove that Φ establishes a local equivalence of flags $(D_i)_{i=1,\dots,4}$ and $(\tilde{D}_i)_{i=1,\dots,4}$. By construction $q_*(D_i(x)) = \pi_*(\Phi_*(D_i(x))) = \pi_*(\tilde{D}_i(\Phi(x)))$ for any $x \in M$ and $i = 1, 2, 3, 4$. Therefore, in order to finish the proof, it is sufficient to prove that Φ is a local diffeomorphism. Since π and q are projections to the same manifold, it is sufficient to prove that Φ_* is non-degenerate on the fibres of q. But the second relation of (3.1) implies that there exists a section X of D_1 and a section Y of D_3, transversal to $D_2 \subset D_3$ such that $[X, Y]$ is transversal to $D_3 \subset D_4$. It follows that $\Phi_*(X) \neq 0$. $\square$

Assume that D is a parabolic $(3, 5, 6)$-distribution. Lemma 3.2 does not say that $\tilde{D}_3$ is a parabolic $(3, 5, 6)$-distribution in a neighbourhood of an arbitrary $\tilde{x} \in \tilde{M}$. We can only conclude that $\tilde{D}_3$ is a parabolic

$(3,5,6)$-distribution in a neighbourhood of a point $\tilde{x}\in \operatorname{Im}\Phi$. In fact $\tilde{D}_3$ can have the growth vector $(3,4,\dots)$ at some points and this phenomenon is similar to the fact that there exist singular Goursat distributions [20, 21]. Indeed, we will prove later (Lemmas 4.1 and 5.2) the following fact:

Fact 3.1. *Let D be a parabolic $(3,5,6)$-distribution and let (B_1,B_3) be the associated reduced pair. There is a unique rank-two subdistribution B_2 of B_3 such that $[B_2,B_2]\subset B_3$.*

Taking this fact into account we can choose a local frame (Z_1,Z_2,Z_3) of B_3 such that Z_1 spans B_1 and (Z_1,Z_2) is a local frame of B_2. Then it follows that $[Z_1,Z_2]=0 \mod B_3$. Now, at a fixed point $z\in N$ the vectors $Z_2(z),Z_3(z)\in T_zN$ define a basis of $B_3(z)/B_1(z)$ and we can introduce the following parametrization of the projective space $P(B_3(z)/B_1(z))$

$$(t:s)\mapsto \operatorname{span}\{tZ_2(z)+sZ_3(z)\}.$$

We get two affine parameters t and s on the bundle $\pi\colon P(B_3/B_2)\to N$. Then

$$\tilde{D}_3(t)=\operatorname{span}\{\partial_t,Z_1,Z_3+tZ_2\}, \tag{3.2}$$

for points $(t:s)$ such that $s\neq 0$ and

$$\tilde{D}_3(s)=\operatorname{span}\{\partial_s,Z_1,sZ_3+Z_2\}, \tag{3.3}$$

for points $(t:s)$ such that $t\neq 0$. If we compute the Lie square $[\tilde{D}_3,\tilde{D}_3]$ we see that it is a rank-five distribution provided that $s\neq 0$. On the other hand, if $s=0$, then $[\tilde{D}_3,\tilde{D}_3]$ is of rank 4 (it is spanned by ∂_s,Z_1,Z_2,Z_3 and $[Z_1,Z_2]$, however $[Z_1,Z_2]$ is contained in B_3 at $s=0$). Thus, if we start with a parabolic $(3,5,6)$-distribution D then $q_*(D(x))$ does not coincide with $B_2(q(x))\subset B_3(q(x))$ for any $x\in M$. Otherwise it would contradict Lemma 3.2. To conclude we state the following.

Corollary 3.1. *A germ of a regular parabolic $(3,5,6)$-distributions D at $x\in M$ is uniquely determined by a germ of the corresponding reduced pair (B_1,B_3) at $q(x)\in N$ together with the point $\Phi(x)\in P(B_3(q(x))/B_1(q(x)))$ which corresponds to $L(\Phi(x))=q_*(D_3(x))\subset B_3(q(x))$. Moreover, $q_*(D_3(x))$ does not coincide with $B_2(q(x))$.*

4. The degenerate case

In this section, we consider degenerate parabolic $(3,5,6)$-distributions.

Lemma 4.1. *Let D be a degenerate parabolic $(3,5,6)$-distribution on M and let (B_1, B_3) be the associated reduced pair on N. Then*

1. $B_4 = [B_3, B_3] = [B_1, B_3]$ *is a rank-four distribution,*
2. $[B_1, B_4] = B_4$, $[B_4, B_4] = TN$,
3. B_3 *has a rank-one Cauchy characteristic C, and $B_2 = C \oplus B_1$ is Cauchy characteristic of B_4. In particular B_2 is integrable.*

Proof. The first two statements follow immediately from the definition of the flag $(D_i)_{i=1,\dots,5}$ and the definition of B_i. Namely, in the degenerate case, $[D_4, D_4] = [D_2, D_4] = D_5$, $[D_2, D_5] = D_5$ and $[D_5, D_5] = TM$.

It follows from statements 1 and 2 that B_3 has growth vector $(3,4,5)$, so it has Cauchy characteristic C which is of rank one. The Cauchy characteristic C does not coincide with B_1 since $[B_1, B_3] = B_4$. Therefore $B_2 = C \oplus B_1$ is a distribution of rank 2. To prove that B_2 is Cauchy characteristic of B_4, let us choose a vector field V that spans B_1 and a vector field W that spans C. Moreover, let U be a vector field complementing (V, W) to a local frame of B_3. Then $(V, W, U, [V, U])$ is a local frame of B_4. Additionally $[W, V] = [W, U] = 0 \mod B_3$, since W is a section of Cauchy characteristic of B_3. Hence

$$[W, [V, U]] = [[W, V], U] + [V, [W, U]] = 0 \mod B_4.$$

$\square$

Now we are ready to prove our first main result.

Theorem 4.1. *All degenerate parabolic $(3,5,6)$-distributions are locally equivalent to the canonical Cartan distribution on the mixed jet space $J^{2,1}(\mathbb{R}, \mathbb{R}^2)$. In natural coordinates (t, u, v, u_1, u_2, v_1) on $J^{2,1}(\mathbb{R}, \mathbb{R}^2)$ the distribution is annihilated by the following one-forms*

$$du - u_1 dt, \qquad du_1 - u_2 dt, \qquad dv - v_1 dt.$$

Proof. Lemma 4.1 implies that $B_3 \subset B_4 \subset TN$ is Goursat flag of length 2 and therefore one can choose local coordinates (t, u, u_1, u_2, v) such that

$$B_3 = \operatorname{span}\{\partial_t + u_1 \partial_u + u_2 \partial_{u_1}, \partial_{u_2}, \partial_v\} \tag{4.1}$$

(see [20], p. 462). The Cauchy characteristic of B_3 is spanned by ∂_v and the Cauchy characteristic of B_4 is spanned by ∂_{u_2} and ∂_v. Thus we have (see statement 3 of Lemma 4.1): $C = \operatorname{span}\{\partial_v\}$ and $B_1 = \operatorname{span}\{\partial_{u_2} + \alpha\partial_v\}$ for some function $\alpha\colon N \to \mathbb{R}$. However, on each leaf of the foliation tangent to the integrable distribution $B_2 = C \oplus B_1$ we can make a local change of coordinates $(v, u_2) \mapsto (\phi(v, u_2, t, u, u_1), u_2)$ in such a way that

$$C = \operatorname{span}\{\partial_v\}, \qquad B_1 = \operatorname{span}\{\partial_{u_2}\}. \tag{4.2}$$

(One just solves the ODE: $(\partial_{u_2} + \alpha\partial_v)(\phi) = 0$ with the condition $\partial_v\phi \neq 0$). In new coordinates $\partial_t + u_1\partial_u + u_2\partial_{u_1}$ is unchanged modulo ∂_v. Therefore, formula (4.1) is satisfied also in new coordinates.

Now, by Corollary 3.1 and remarks following Lemma 3.2 we can conclude that any germ of a degenerate parabolic $(3,5,6)$-distribution has the following form

$$D = \operatorname{span}\{\partial_{v_1}, \partial_{u_2}, \partial_t + u_1\partial_u + u_2\partial_{u_1} + v_1\partial v\},$$

where v_1 plays the role of t in the formula (3.2). This is just the Cartan distribution on the mixed jet space $J^{2,1}(\mathbb{R}, \mathbb{R}^2)$. $\square$

5. The non-degenerate case: symbol algebras and the reduced pair

In this section, we consider non-degenerate parabolic $(3,5,6)$-distributions and provide their basic properties. In this case, besides relations (3.1), we also have the following relations

$$[D_1, D_5] = D_5, \quad [D_2, D_3] = D_5, \quad [D_2, D_4] = D_5, \quad [D_2, D_5] = TM. \tag{5.1}$$

All of them follow directly form the definitions. Relations (3.1) and (5.1) implies that it is reasonable to introduce the following graded Lie algebra at each point $x \in M$:

$$\mathfrak{g}(x) = \bigoplus_{i=1}^{7} \mathfrak{g}_{-i}(x),$$

where

$$\begin{aligned}
&\mathfrak{g}_{-1}(x) = D_1(x), \quad \mathfrak{g}_{-2}(x) = D_2(x)/D_1(x), \quad \mathfrak{g}_{-3}(x) = D_3(x)/D_2(x),\\
&\mathfrak{g}_{-4}(x) = D_4(x)/D_3(x), \quad \mathfrak{g}_{-5}(x) = D_5(x)/D_4(x), \quad \mathfrak{g}_{-6}(x) = 0,\\
&\mathfrak{g}_{-7}(x) = T_xM/D_5(x).
\end{aligned}$$

and the bracket in $\mathfrak{g}(x)$ is defined in a standard way using the Lie bracket of vector fields. The Lie algebra $\mathfrak{g}(x)$ is assigned to a distribution D at point x in an invariant way, and it will be called the *symbol algebra*. We use negative grading in order to be consistent with Tanaka theory.

Our first aim is to classify all possible graded Lie algebras $\mathfrak{g} = \bigoplus_{i=1}^{7} \mathfrak{g}_{-i}$, that can appear as the symbol of a non-degenerate parabolic $(3,5,6)$-distribution.

Lemma 5.1. *Let $\mathfrak{g} = \bigoplus_{i=1}^{7} \mathfrak{g}_{-i}$ be a symbol algebra of a non-degenerate parabolic $(3,5,6)$-distribution D at some point $x \in M$. Then there exists a basis $e_1, \dots, e_5, e_7$ of $\mathfrak{g}$ such that e_i spans $\mathfrak{g}_{-i}$,*

1. $[e_1, e_2] = 0$,
2. $[e_1, e_3] = e_4$,
3. $[e_1, e_4] = 0$,
4. $[e_2, e_3] = e_5$,
5. $[e_2, e_5] = e_7$,
6. $[e_3, e_4] = \varepsilon e_7$,

where $\varepsilon = 0$ or $\varepsilon = 1$ and all other brackets $[e_i, e_j]$ vanish. Thus, there are exactly two inequivalent symbols.

Proof. First of all, Lie brackets $[e_i, e_j]$ not listed in the lemma necessarily vanish due to the definition of a graded Lie algebra.

If we choose a local frame $(X_1, X_2, Y, Y_1, Y_2, Z)$ as in Section 2 in such a way that X_1 spans D_1 then we can take $e_1 = X_1(x)$, $e_2 = X_2(x)$, $e_3 = Y(x)$, $e_4 = Y_1(x)$, $e_5 = Y_2(x)$ and $e_7 = Z(x)$ (e_i is defined modulo e_j, $j < i$). Then we have $[e_1, e_3] = e_4$ and $[e_2, e_3] = e_5$. Moreover, we can assume that $Z = [X_2, Y_2]$ and then $[e_2, e_5] = e_7$. The Lie bracket $[e_1, e_2]$ vanishes because D_2 is integrable. The Lie bracket $[e_1, e_4]$ vanishes due to the third relation of Lemma 3.1. Then we have $[e_3, e_4] = \varepsilon e_7$ for some $\varepsilon \in \mathbb{R}$. However, if we substitute $Y \mapsto \alpha Y$ for some $\alpha \neq 0$ then a simple calculation proves that ε becomes $\alpha\varepsilon$. Hence, if $\varepsilon \neq 0$ we can assume $\varepsilon = 1$. $\square$

Remark. The two symbols appear in the paper [18] and are denoted: m6_3_3 (for $\varepsilon = 0$) and m6_3_4 (for $\varepsilon = 1$). Flat distributions on Lie groups corresponding to the two graded Lie algebras have infinite-dimensional algebras of infinitesimal symmetries. For simplicity, m6_3_3 we will denote $\mathfrak{g}^0$ and m6_3_4 we will denote $\mathfrak{g}^1$.

In the next two lemmas, we provide basic properties of the reduced pair (B_1, B_3) associated to a non-degenerate parabolic distribution.

Lemma 5.2. *Let D be a non-degenerate parabolic $(3,5,6)$-distribution on M, and let (B_1, B_3) be the associated reduced pair on N. Then*

1. $[B_1, B_3] = B_4$ *is a rank-four distribution,*
2. $[B_1, B_4] = TN$,
3. *there exists a unique rank-two subdistribution $B_2 \subset B_3$ such that $[B_2, B_2] \subset B_3$ and $B_1 \subset B_2$.*

Conversely, if a pair (B_1, B_3) satisfies conditions 1 and 2 above, then $\tilde{D}$, as defined in Section 3 on the manifold $\tilde{M}$, is a non-degenerate parabolic $(3,5,6)$-distribution in a neighbourhood of any point $x \in \tilde{M}$ such that the corresponding two-dimensional subspace $L(x) \subset B_3$ defining x as a point in $P(B_3(\pi(x))/B_1(\pi(x)))$ does not coincide with $B_2(x)$.

Proof. The first two statements immediately follow from the definition of the flag $(D_i)_{i=1,\ldots,5}$ and the definition of B_i. Namely, in the non-degenerate case, $[D_2, D_4] = D_5$ and $[D_2, D_5] = TM$ by formula (5.1).

To prove statement 3 let us choose a vector field X that spans B_1 and consider a mapping $B_3(x) \to B_4(x)/B_3(x)$ defined by the formula $v \mapsto [X, V](x) \mod B_3(x)$, where V is an extension of $v \in B_3(x)$ to a local section of B_3. It follows from statement 1 that this mapping has a two-dimensional kernel, denoted $B_2(x)$.

Note that if B_3 has growth vector $(3,4,5)$, then it has Cauchy characteristic C, which is a distribution of rank 1. Then $B_2 = B_1 \oplus C$. If B_3 has growth vector $(3,5)$ then it is well known that the square root of B_3 exists. This square root is exactly B_2 defined above.

To prove that any pair (B_1, B_3) satisfying conditions 1 and 2 defines a non-degenerate parabolic $(3,5,6)$-distribution $\tilde{D}$ on $\tilde{M}$ in a neighbourhood of a point which does not correspond to B_2 we first have to show that $\operatorname{rk}[\tilde{D}, \tilde{D}] = 5$. This follows from formula (3.2) and the relation $[B_1, B_3] = B_4$. As a conclusion we can define $\tilde{D}_5 = [\tilde{D}, \tilde{D}] = \pi_*^{-1}(B_4)$. Then the condition $[B_1, B_4] = TN$ reads that $[\tilde{D}_2, \tilde{D}_5] = T\tilde{M}$ and, by construction, $[\tilde{D}_1, \tilde{D}_5] = \tilde{D}_5$, since $\tilde{D}_1$ is tangent to the fibres of $\pi\colon \tilde{M} \to N$ and $\tilde{D}_5$ is a pull-back of B_4, i.e., $\tilde{D}_5 = \pi_*^{-1}(B_4)$. Thus the bilinear form (a_{ij}) has rank 1 and it completes the proof of the lemma. $\square$

Lemma 5.3. *Let D be a non-degenerate parabolic $(3,5,6)$-distribution on M, let (B_1,B_3) be the associated reduced pair, and let $\mathfrak{g}(x)$ be a symbol algebra of D at $x\in M$. Then, $\mathfrak{g}(x)$ is isomorphic to $\mathfrak{g}^0$ iff B_3 has growth vector $(3,4,5)$ at x and $\mathfrak{g}(x)$ is isomorphic to $\mathfrak{g}^1$ iff B_3 at x has growth vector $(3,5)$ at x.*

Proof. By Lemma 5.2 there are only two possible growth vectors of B_3: $(3,4,5)$ or $(3,5)$, because $\operatorname{rk}[B_3,B_3]\geq\operatorname{rk}[B_1,B_3]=4$ and $[B_3,[B_3,B_3]]\supset[B_1,B_4]=TN$. Let us choose a local frame (X_1,X_2,Y,Y_1,Y_2,Z) as in the proof of Lemma 5.1. If we take into account that $B_3=q_*(D_4)=q_*(\operatorname{span}\{X_1,X_2,Y,Y_1\})$ then this characterization of $\mathfrak{g}(x)$ in terms of B_3 becomes obvious. ☐

In order to exclude irregular points, we will need one more regularity condition.

Definition. A non-degenerate parabolic $(3,5,6)$-distribution D is called *completely non-degenerate* if the associated distribution B_2 has locally constant growth vector.

It follows that if D is completely non-degenerate then either B_2 is integrable or $[B_2,B_2]=B_3$. In the second case, B_3 is determined by B_2, so we can consider the pair (B_1,B_2) instead of (B_1,B_3).

In view of Lemmas 5.3 and 5.2 there are four possibilities at a point $x\in M$. A non-degenerate parabolic $(3,5,6)$-distribution can have a symbol algebra $\mathfrak{g}^0$ or $\mathfrak{g}^1$, and B_2 can be integrable or not. Note that if D has symbol $\mathfrak{g}^1$ at x, then it has symbol $\mathfrak{g}^1$ in a neighbourhood of x.

6. The non-degenerate case: $GL(2)$-structures and canonical frames

A $GL(2)$-structure on a manifold M is a vector bundle isomorphism

$$TM=\underbrace{S\odot\cdots\odot S}_{\dim M-1},$$

where $\odot$ is a symmetric tensor product and $S\to M$ is a rank-two vector bundle over M. The natural action of the group $GL(2)$ on S extends to an irreducible action of $GL(2)$ on TM and any tangent space T_xM can be identified with the space of homogeneous polynomials in two variables of order $\dim M-1$.

$GL(2)$-structures generalize Lorentz conformal metrics in dimension 3. Indeed, the splitting $TM = S \odot \cdots \odot S$ defines the following cones

$$C(x) = \Big\{ \underbrace{v \odot \cdots \odot v}_{\dim M - 1} \mid v \in S(x) \Big\} \subset T_x M.$$

It follows that $C(x)$ is a rational curve of order $\dim M-1$ in T_xM. In the case $\dim M = 3$ it defines a cone of null-directions of a conformal Lorentz metric on M. Moreover, one can observe that the field $x \mapsto C(x)$ encodes a $GL(2)$-structure uniquely. We refer to [11] for details on $GL(2)$-structures (called paraconformal structures in this paper).

The importance of $GL(2)$-structures is based on the fact that they appear as basic geometric structures on solutions spaces of ODEs [1, 5, 11, 12]. In the simplest case the structures are described by the profound result of Chern [5]: if a third-order ODE given in the form

$$x''' = F(t, x, x', x'')$$

satisfies the Wünschmann condition

$$X_F^2(\partial_{x_2}F) - 2\partial_{x_2}F X_F(\partial_{x_2}F) - 3X_F(\partial_{x_1}F) + 6\partial_{x_0}F + \frac{4}{9}(\partial_{x_2}F)^3 \\ + 2\partial_{x_2}F\partial_{x_1}F = 0,$$

where $X_F = \partial_t + x_1\partial_{x_0} + x_2\partial_{x_1} + F(t, x_0, x_1, x_2)\partial_{x_2}$ is the total derivative, then it defines a conformal Lorentz metric on the solutions space. Moreover, an arbitrary conformal Lorentz metric can be obtained in this way. The Wünschmann condition is invariant under contact transformations of variables.

In the case of equations of order $k > 3$ the Wünschmann condition is replaced by the set of $k-2$ conditions [1, 11]. However, if $k > 3$ then not every $GL(2)$-structure comes from an ODE [12, 16]. In order to characterize $GL(2)$-structures that are defined by ODEs we have introduced in [16] the following definition:

Definition. A pair (E_1, E_2) of two distributions on a manifold M of dimension n is *regular* if

1. $\operatorname{rk} E_1 = 1$, $\operatorname{rk} E_2 = 2$, $E_1 \subset E_2$.
2. $\operatorname{rk} \operatorname{ad}^i_{E_1} E_2 = i + 2$ for $i = 1, \dots, n-2$, where $\operatorname{ad}^i_{E_1} E_2$ are distributions defined by induction: $\operatorname{ad}_{E_1} E_2 = [E_1, E_2]$ and $\operatorname{ad}^{i+1}_{E_1} E_2 = [E_1, \operatorname{ad}^i_{E_1} E_2]$.

The notion of regular pairs generalizes the notion of ODEs. Namely, for a given equation of order k

$$x^{(k)} = F(t, x, x', \ldots, x^{(k-1)}),$$

we define $E_1 = \mathrm{span}\{X_F\}$ and $E_2 = \{X_F, \partial_{x_{k-1}}\}$ where

$$X_F = \partial_t + x_1\partial_{x_0} + \cdots + x_{k-1}\partial_{x_{k-2}} + F\partial_{x_{k-1}}$$

is the total derivative. Then E_2 is the canonical Cartan distribution on the space of jets and we get that (E_1, E_2) is regular. Regular pairs locally equivalent to pairs defined by ODEs are called *of equation type*. An intrinsic characterization of such pairs is given in [16].

There is also a notion of *Wünschmann condition* for regular pairs. The notion generalizes the notion of Wünschmann condition in the case of ODEs. Roughly speaking, the evolution of E_2 along an integral curve of E_1 defines an unparameterized curve in a projective space. One can compute Wilczyński invariants [24] for the curve and the Wünschmann condition is defined as the vanishing of all Wilczyński invariants of the curve. The geometric meaning of the condition is that (E_1, E_2) satisfies the Wünschmann condition if and only if there exists a local frame (X_1, X_2) of E_2 such that X_1 spans E_1 and $\mathrm{ad}_{X_1}^n X_2 = 0 \mod E_1$. It is an easy observation [16] that there is one-to-one correspondence between germs of $GL(2)$-structures and germs of regular pairs satisfying Wünschmann condition.

Now we are in position to state our main results.

Theorem 6.1. *Let D be a completely non-degenerate parabolic $(3,5,6)$-distribution on M such that the associated distribution B_2 on N is non-integrable. Then:*

1. *The pair (B_1, B_2) is regular in the sense of [16].*
2. *There exists a canonical frame on a $T(2)$-bundle over N, where $T(2) \subset GL(2)$ is the subgroup of upper-triangular matrices. Two pairs are equivalent if and only if the corresponding frames are diffeomorphic.*
3. *The pair (B_1, B_2) is of equation type if and only if D has constant symbol algebra $\mathfrak{g}^0$.*
4. *If (B_1, B_2) satisfies the Wünschmann condition, then it defines a $GL(2)$-structure on the quotient manifold N/B_1, which is of dimension 4; conversely all germs of $GL(2)$-structures on four-dimensional manifolds can be obtained in this way.*

Proof. Statement 1 follows from the assumption $[B_2, B_2] = B_3$, which implies $[B_1, B_2] = B_3$ and together with Lemma 5.2 proves that (B_1, B_2) is a regular pair. In order to prove statement 3 let us recall from [16] that a regular pair (E_1, E_2) on five-dimensional manifold is of equation type if and only if $[E_1, E_2]$ has growth vector $(3, 4, 5)$. Thus statement 3 follows from Lemma 5.3. Statement 4 is just a consequence of statement 1 and results of [16] (Theorem 1.1).

Statement 2 in the case of regular pairs of equation type follows from [7], where a proof is given that for an arbitrary equation of fourth order there is a normal Cartan connection on a $T(2)$-bundle. Besides, the case when (B_1, B_2) satisfies the Wünschmann condition follows from [1], where a canonical connection is defined for the associated $GL(2)$-structure. The construction of the canonical frame in the general case will be provided in a forthcoming paper. In the general case, we are able to construct not a Cartan connection, but just a frame on a bundle. □

Theorem 6.2. *Let D be a completely non-degenerate parabolic $(3,5,6)$-distribution on M such that the associated distribution B_2 on N is integrable. If D has constant symbol $\mathfrak{g}^0$ then:*

1. *The Cauchy characteristic C of B_3 is contained in Cauchy characteristic of B_2, so there is a well-defined reduction $p\colon N \to O$, $A_1 = p_*(B_2)$, $A_2 = p_*(B_3)$, where $O = N/C$, and the pair (B_1, B_3) is uniquely defined by the pair (A_1, A_2).*
2. *The pair (A_1, A_2) is regular in the sense of [16].*
3. *If (A_1, A_2) satisfies the Wünschmann condition, then it defines a conformal Lorentz metric on the quotient manifold O/A_1, which is of dimension 3; conversely all germs of conformal Lorentz metrics on three-dimensional manifolds can be obtained in this way.*

Proof. If D has symbol $\mathfrak{g}^0$ then B_3 has growth vector $(3, 4, 5)$ and it defines a Goursat flag. Then, as in the proof of Theorem 4.1, we can choose local coordinates on N such that

$$B_3 = \operatorname{span}\{\partial_t + u_1\partial_u + u_2\partial_{u_1}, \partial_{u_2}, \partial_v\}.$$

Then $C = \operatorname{span}\{\partial_v\}$. But, since $[B_1, B_3] = B_4$ and $[B_1, B_4] = TN$, we get that B_1 has to be of the form

$$B_1 = \operatorname{span}\{\partial_t + u_1\partial_u + u_2\partial_{u_1} + F\partial_{u_2} + \alpha\partial_v\}$$

for some functions F and α. From the proof of Lemma 5.2 it follows that $B_2 = B_1 \oplus C$ is integrable. In order to prove that (B_1, B_3) is defined by (A_1, A_2) note that the integrability of B_2 implies that F does not depend on v and, as in the proof of Theorem 4.1, we can change local coordinates in such a way that $\alpha \equiv 0$. Since $C = \operatorname{span}\{\partial_v\}$, the functions (t, u, u_1, u_2) constitutes a system of local coordinates on $O = N/C$. B_1 is a function of (t, u, u_1, u_2) only and in coordinates we can write $A_1 = B_1$. Thus, we see that N has locally a structure of a Cartesian product $N = N/C \times C$ which is compatible with the pair (B_1, B_3), i.e., the pair (A_1, A_2) on N/C defines the pair (B_1, B_3) on N by the formula: $B_1 = A_1$ and $B_3 = A_2 \oplus C$.

Statement 2 follows from the fact that B_3 has growth vector $(3, 4, 5)$ which implies that A_2 has growth vector $(2, 3, 4)$. Additionally $[A_1, A_3] = TO$ since $[B_2, B_4] = TN$ as was proved in Lemma 5.2. Statement 3 is a corollary of Theorem 1.1 [16] applied to regular pairs on four-dimensional manifolds. □

Remark. Since all regular pairs on four-dimensional manifolds are of equation type [12, 16], the problem of equivalence of parabolic $(3, 5, 6)$-distributions described in Theorem 6.2 is reduced to the problem of contact equivalence of ODEs of third order. The last problem was solved by Chern [5], who constructed a Cartan connection taking values in $\mathfrak{sp}(4, \mathbb{R})$ (see also [7]).

We can summarize Theorems 6.1 and 6.2 in the following table

	$\mathfrak{g}^0$	$\mathfrak{g}^1$
integrable B_2	third-order ODEs	?
$[B_2, B_2] = B_3$	fourth-order ODEs	regular pairs not of equation type

Open problem. Classify all non-degenerate parabolic $(3, 5, 6)$-distributions with integrable B_2 and symbol $\mathfrak{g}^1$. In this case B_3 is equivalent to the Cartan distribution on the space $J^1(\mathbb{R}, \mathbb{R}^2)$ and B_2 is the integrable subdistribution of B_3 tangent to the fibres of the projection $J^1(\mathbb{R}, \mathbb{R}^2) \to J^0(\mathbb{R}, \mathbb{R}^2)$. However the choice of $B_1 \subset B_2$ should give non-equivalent D.

7. Symmetric models and PDEs

In this section, we will provide examples of non-degenerate parabolic $(3, 5, 6)$-distributions.

We start with two flat models. Namely, for algebras $\mathfrak{g}^0$ and $\mathfrak{g}^1$ we can construct Lie groups G^0 and G^1 such that $\mathfrak{g}^i$ is the Lie algebra of G^i. Then on G^i we can define a left-invariant rank-three distribution D^i such that at the identity element $e \in G^i$ we have $D^i(e) = \mathfrak{g}^i_{-1} \oplus \mathfrak{g}^i_{-2} \oplus \mathfrak{g}^i_{-3}$. Then it is clear that G^i admits a frame $(X_1, X_2, Y, Y_1, Y_2, Z)$ of left-invariant vector fields that is adapted to D^i in a sense of the proof of Lemma 5.1 and have structure constants such as algebra $\mathfrak{g}^i$. Moreover, any distribution which has an adapted frame with structure constants such as algebra $\mathfrak{g}^i$ is locally equivalent to the distribution D^i on G^i. We call D^i the *flat distribution* of type $\mathfrak{g}^i$. Below we will present PDE models for D^0 and D^1, but first we show normal forms of distributions corresponding to ODEs from Theorems 6.1 and 6.2. The following theorem can be relatively easy obtained in a process inverse to the reduction of Section 3.

Theorem 7.1. *Any non-degenerate parabolic (3,5,6)-distribution D with constant symbol $\mathfrak{g}^0$ can be locally put in the form*

$$du_1 - u_2dx, \quad du_2 - zdx, \quad du_3 - G(x, y, z, u_1, u_2, u_3)dx + zdy, \tag{7.1}$$

for a function G satisfying $X^2(G) = 0$, where $X = \partial_y - z\partial_{u_3}$. If B_2 is integrable, then G can be taken in the form

$$G(x, y, z, u_1, u_2, u_3) = F(x, u_1, u_2, z)y, \tag{7.2}$$

for a function F in four variables. If B_2 is non-integrable, then G can be taken in the form

$$G(x, y, z, u_1, u_2, u_3) = F(x, u_1, u_2, z, u_3 + yz) - yu_3 - y^2z, \tag{7.3}$$

for a function F in five variables.

Proof. First, we check by direct computations that (7.1) defines a parabolic $(3,5,6)$-distribution and D_2 is integrable if and only if $X^2(G) = 0$. On the other hand, applying Theorems 6.1 and 6.2 to the ODEs $\varphi''' = F(t, \varphi, \varphi', \varphi'')$ and $\varphi^{(4)} = F(t, \varphi, \varphi', \varphi'', \varphi''')$, respectively, and performing computations converse to the process of reduction, we get that the corresponding distributions are defined by functions G given by (7.2) and (7.3). $\square$

The Pfaffian system (7.1) can be also written as a system of PDEs. Substituting $u = u_1$ and $v = u_3$ we get the following system

$$u_y = 0, \quad v_y = -u_{xx}, \quad v_x = G(x, y, u, u_x, u_{xx}, v), \tag{7.4}$$

and $X^2(G) = 0$ is just an integrability condition. Taking $F = 0$ in (7.2) and (7.3) we get the models for distributions corresponding to the trivial equations $\varphi''' = 0$ and $\varphi^{(4)} = 0$. Explicitly, we have

$$u_y = 0, \quad v_y = -u_{xx}, \quad v_x = 0, \tag{7.5}$$

for order 3, and

$$u_y = 0, \quad v_y = -u_{xx}, \quad v_x = -yv - y^2 u_{xx}, \tag{7.6}$$

for order 4. Note that equation (7.5) gives also a PDE model for the flat distribution with symbol algebra $\mathfrak{g}^0$. On the other hand equation (7.6) corresponds to the flat $GL(2)$-structure on four-dimensional manifold. In general, the system (7.4) defines a $GL(2)$-structure if and only if F, defined by (7.2) and (7.3), satisfies the Wünschmann condition.

A PDE system corresponding to the flat distribution with symbol algebra $\mathfrak{g}^1$ has the following form

$$u_y = \tfrac{1}{2}(u_{xx})^2, \quad v_y = u_{xx}, \quad v_x = 0.$$

Open problem. Find normal forms (and PDE models) for parabolic $(3,5,6)$-distributions with symbol algebra $\mathfrak{g}^1$. The task seems to be more complex than the case of $\mathfrak{g}^0$ since we do not have a nice description of the reduced pair in the case of $\mathfrak{g}^1$ (in the case of symbol algebra $\mathfrak{g}^0$ we have ODEs).

Acknowledgments

I would like to express my gratitude to Boris Doubrov for his comments and questions posted to me while I was working on this paper. Partially supported by Polish Ministry of Science and Higher Education, grant N201 607540.

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Institute of Mathematics
Polish Academy of Sciences
ul. Śniadeckich 8
00-956 Warszawa
Poland and
Erwin Schrödinger Institute
Boltzmanngasse 9
A-1090 Vienna
Austria
E-mail address: krynski@impan.pl

Received March 4, 2011