

On an efficient induction step with $\text{Nklt}(X, D)$ — notes to Todorov

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Applying the effective induction on $\text{Nklt}(X, D)$ developed by Hacon–McKernan and Takayama, Todorov proved that φ_5 is birational for projective 3-fold V with $\text{vol}(V) \gg 0$, which was recently improved by Biagio in loosing the volume constraint. The observation is that the least efficient induction step can be studied in an alternative way, which allows us to assert Todorov’s statement for $\text{vol}(V) > 12^3$. The four-dimensional analog is also given in this note. The idea works well for all dimensions.

1. Introduction

We work over the complex number field \mathbb{C} . One of the fundamental questions in birational geometry has been to find an optimal integer $r_n > 0$ ($n \geq 3$) such that the pluricanonical map φ_m is birational onto the image for all $m \geq r_n$ and for all n -dimensional nonsingular projective varieties of general type. It is well known that $r_1 = 3$ and, due to Bombieri, $r_2 = 5$. For 3-folds, one has $r_3 \leq 73$ by Chen–Chen [3, 4]. For all $n \geq 3$, the remarkable boundedness theorem, i.e., $r_n < +\infty$, was proved separately by Hacon–McKernan [10], Takayama [19] and Tsuji [23]. One may refer to the very nice survey article [13] for other boundedness results in birational geometry.

We refer to [10, 19, 20] for those standard notions of “volume”, “lc center” and “ $\text{Nklt}(X, D)$ ”. In this paper, we are mainly interested in the explicit boundedness of projective varieties with large volumes.

Utilizing the effective induction on non-klt centers (i.e., $\text{Nklt}(X, D)$) developed in Hacon–McKernan [10] and Takayama [19], Todorov [20, Theorem 1.2] proved that $\varphi_5 := \Phi_{|5K_V|}$ is birational for all nonsingular projective 3-folds V with $\text{vol}(V) > 4355^3$. Recently, Biagio [9, Th. 1.2] improved this by loosing the volume constraint, say $\text{vol}(V) > 2 \times 1917^3$.

In fact, the least efficient induction step of Todorov is when V is birationally fibred by a family of surfaces with small volumes over a curve. Recalling a relevant result in our paper [7] and using the “canonical restriction inequality” in Chen–Zuo [8], the mentioned induction step of Todorov

can be handled quite well in an alternative way. This allows us to present the following improved statement:

Theorem 1.1. *Let V be a nonsingular projective 3-fold with $\text{vol}(V) > 12^3$. Then φ_m is birational for all $m \geq 5$.*

It is worthwhile to realize the idea in higher dimensions. First of all we note that the Minimal Model Program (MMP) works well for projective varieties of general type due to, for instance, Birkar–Cascini–Hacon–McKernan [1], Hacon–McKernan [12] and Siu [18]. When we speak of “minimal models” or “minimal n -folds”, it should always be understood that the variety has \mathbb{Q} -factorial terminal singularities.

Definition. For any integer $n \geq 2$, define the positive integer \tilde{r}_n to be the minimum satisfying the following properties:

for all minimal projective n -folds X_0 of general type, for all birational morphisms $\nu : X' \rightarrow X_0$ with X' nonsingular and for all nef and big \mathbb{Q} -divisors R on X' , the linear system

$$|K_{X'} + \lceil (m - 2)\nu^*(K_{X_0}) + R \rceil |$$

gives a birational map onto the image for all $m \geq \tilde{r}_n$.

Remark. We can see, later on, in Lemmas 2.1 and 2.2 that $\tilde{r}_n < +\infty$ for all $n \geq 2$, which is essentially an induction on the dimension applying Kawamata–Viehweg vanishing and the boundedness theorem of Hacon–McKernan [10], Takayama [19] and Tsuji. (In fact, it is not hard to see from Chen–Chen [4] that $\tilde{r}_3 \leq 73$.)

In the second part we shall prove the following:

Theorem 1.2. *Let Y be a nonsingular projective 4-fold. Assume $\text{vol}(Y) \gg 0$, say $\text{vol}(Y) > \frac{\tilde{r}_3 - 1}{6} \times 12^4$. Then $|mK_Y|$ gives a birational map for all $m \geq \tilde{r}_3$.*

A difficulty in high dimensions is that we do not know any practical lower bound for \tilde{r}_n ($n \geq 4$). We need to set the following:

Notation. For all $i > 0$, the real number $e_i > 0$ denotes the optimal constant so that $\text{vol}(Z) > e_i^i$ for all i -folds Z of general type. Set

$$R_n := \max \left\{ 2 \prod_{i=1}^{n-2} \left(\frac{i\sqrt[3]{2}}{e_i} + 1 \right) - 1, \tilde{r}_{n-1} \right\}.$$

Our last result is the following:

Corollary 1.1. For all $n \geq 5$, C_n are computable positive constants. Let Y be a nonsingular projective n -fold with $\text{vol}(Y) > C_n$. Then $|mK_Y|$ gives a birational map for all $m \geq R_n$.

Throughout we are in favor of the following symbols:

- “ \sim ” denotes linear equivalence or \mathbb{Q} -linear equivalence;
- “ \equiv ” denotes numerical equivalence;
- “ $A \geq B$ ”, for \mathbb{Q} -divisors A and B , means that $A - B$ is \mathbb{Q} -linearly equivalent to an effective \mathbb{Q} -divisor;
- “ $|M_1| \geq |M_2|$ ” means, for linear systems $|M_1|$ and $|M_2|$, $|M_1| = |M_2| +$ (fixed effective divisor).

2. Proof of main statements

We start with the following:

Lemma 2.1. Let S be a nonsingular projective surface of general type and $\sigma : S \rightarrow S_0$ the birational contraction onto the minimal model S_0 . Let Q be any nef and big \mathbb{Q} -divisor on S . Then the linear system $|K_S + 3\sigma^*(K_{S_0}) + [Q]|$ gives a birational map.

Proof. Write $\mathcal{L} := 3\sigma^*(K_{S_0}) + Q$. Clearly \mathcal{L} is nef and $\mathcal{L}^2 > 9$.

Case 1. S is not a (1,2) surface. For any irreducible and reduced curve $\hat{C} \subset S$, passing through a very general point $P \in S$, it is sufficient to prove $(\sigma^*(K_{S_0}) \cdot \hat{C}) \geq 2$ and so that $(\mathcal{L} \cdot \hat{C}) > 6$. Then the statement follows from either [7, Lem. 2.6] or [16, Prop. 4]. In fact, $(\sigma^*(K_{S_0}) \cdot \hat{C}) = (K_{S_0} \cdot C)$ where $C := \sigma_*(\hat{C})$. We may verify this on S_0 assuming C to be a curve in S_0 passing through a very general point of S_0 . Note that C is of general type. Thus $(K_{S_0} \cdot C) + C^2 \geq 2$. Suppose to the contrary that $(K_{S_0} \cdot C) \leq 1$. Then $C^2 \geq 1$. By the Hodge index theorem, one sees $K_{S_0}^2 = C^2 = 1$ and $K_{S_0} \equiv C$.

The surface theory implies that S is either a $(1, 1)$ surface or a $(1, 0)$ surface. If $(K_{S_0}^2, p_g(S)) = (1, 0)$, then the torsion element $\theta := K_{S_0} - C$ is of order ≤ 5 and $h^0(S_0, C) = 1$. Thus there are at most finite number of such curves on S_0 since it is determined by the group $\text{Tor}(S)$ with $|\text{Tor}(S)| \leq 5$. By the choice of C , this is impossible. If $(K_{S_0}^2, p_g(S)) = (1, 1)$, then $K_{S_0} \sim C$ since $\text{Tor}(S_0) = 0$ by Bombieri [2, Th. 15] and thus C is the unique canonical curve of S_0 , which is impossible either by the choice of C . In a word, we have $(K_{S_0} \cdot C) \geq 2$.

Case 2. S is a $(1, 2)$ surface. The statement follows from [5, Lem. 1.3, Lem. 1.4]. □

Proposition 2.1. *Let Y be a nonsingular projective 3-fold of general type, B a smooth projective curve and $h : Y \rightarrow B$ be a fibration. Denote by F a general fiber of h . Assume $K_Y \sim_{\mathbb{Q}} pF + E_p$ for some rational number $p > 5$ and an effective \mathbb{Q} -divisor E_p . Then $|mK_Y|$ gives a birational map for all $m \geq 5$.*

Proof. Take a minimal model Y_0 of Y . Modulo birational modifications over Y , we may assume for simplicity that there is a birational morphism $\pi : Y \rightarrow Y_0$. Take a sufficiently large and divisible integer $m > 0$ such that

- (i) mK_{Y_0} is a Cartier divisor;
- (ii) both $|mK_{Y_0}|$ and $|pmF|$ are base point free;
- (iii) $mK_Y \sim_{\mathbb{Z}} mpF + mE_p$;
- (iv) the support of the union of E_p and all those exceptional divisors of π is simple normal crossing.

Then one has $pmF \leq m\pi^*(K_{Y_0}) \leq mK_Y$. By Chen–Zuo [8, Lem. 3.3, Lem. 3.4], giving any small rational number $\varepsilon > 0$, one has

$$(2.1) \quad \pi^*(K_{Y_0})|_F \equiv \left(\frac{p}{p+1} - \varepsilon \right) \theta^*(K_{F_0}) + H_\varepsilon,$$

for certain effective \mathbb{Q} -divisor H_ε on F where $\theta : F \rightarrow F_0$ is the contraction onto the minimal model. Write $\pi^*(K_{Y_0}) \sim_{\mathbb{Q}} pF + E'_p$ where E'_p is another effective \mathbb{Q} -divisor. By assumption the support of E'_p is simple normal crossing.

Pick two distinct general fibers F_1, F_2 of h . Both F_1 and F_2 are known to be nonsingular projective surfaces of general type. Since

$$4\pi^*(K_{Y_0}) - F_1 - F_2 - \frac{2}{p}E'_p \equiv \left(4 - \frac{2}{p}\right)\pi^*(K_{Y_0})$$

is nef and big, Kawamata–Viehweg vanishing [15, 22] gives the surjective map

(2.2)

$$\begin{aligned} & H^0(K_Y + \lceil K_{4,p} \rceil) \\ \rightarrow & H^0(F_1, K_{F_1} + \lceil K_{4,p} - F_1 \rceil|_{F_1}) \oplus H^0(F_2, K_{F_2} + \lceil K_{4,p} - F_2 \rceil|_{F_2}), \end{aligned}$$

where $K_{4,p} := 4\pi^*(K_{Y_0}) - \frac{2}{p}E'_p$. Observing that

$$\left[4\pi^*(K_{Y_0}) - \frac{2}{p}E'_p - F_i\right]|_{F_i} \geq \lceil L_{4,i} \rceil,$$

where

$$L_{4,i} := \left(4\pi^*(K_{Y_0}) - \frac{2}{p}E'_p - F_i\right)|_{F_i} \equiv \left(4 - \frac{2}{p}\right)\pi^*(K_{Y_0})|_{F_i},$$

for $i = 1, 2$, we have

$$|5K_Y||_{F_i} \succeq |K_Y + \left[4\pi^*(K_{Y_0}) - \frac{2}{p}E'_p\right]||_{F_i} \succeq |K_{F_i} + \lceil L_{4,i} \rceil|.$$

Due to the surjectivity of (2.2), it is sufficient to show that $|K_{F_i} + \lceil L_{4,i} \rceil|$ gives a birational map for $i = 1, 2$.

Note that $\frac{p}{p+1} > \frac{5}{6}$ whenever $p > 5$. Take a small rational number $\varepsilon_0 > 0$ such that $\delta_0 := (4 - \frac{2}{p})(\frac{p}{p+1} - \varepsilon_0) - 3 > 0$. Then we have

$$L_{4,i} \equiv (3 + \delta_0)\theta_i^*(K_{F_{i,0}}) + \left(4 - \frac{2}{p}\right)H_{\varepsilon_0,i},$$

for an effective \mathbb{Q} -divisor $H_{\varepsilon_0,i}$ on F_i where $\theta_i : F_i \rightarrow F_{i,0}$ is the contraction onto its minimal model for $i = 1, 2$.

Now since

$$L_{4,i} - \left(4 - \frac{2}{p}\right)H_{\varepsilon_0,i} \equiv 3\theta_i^*(K_{F_{i,0}}) + \delta_0\theta_i^*(K_{F_{i,0}}),$$

Lemma 2.1 implies that

$$\left| K_{F_i} + \left[L_{4,i} - \left(4 - \frac{2}{p} \right) H_{\varepsilon_0,i} \right] \right|$$

gives a birational map and so does $|K_{F_i} + \lceil L_{4,i} \rceil|$ for $i = 1, 2$.

Since F_1 and F_2 are general fibers of h , $|5K_Y|$ gives a birational map. So we conclude the statement. □

Now we are prepared to prove Theorem 1.1. The outline of the proof follows that of Todorov [20, Sec. 4] except that the last step (i.e., Subcase 2.2) is implemented by an alternative argument.

Proof of Theorem 1.1. Assume $\text{vol}(V) > \alpha_0^3$ for some rational number $\alpha_0 > 0$. For any very small rational number $\varepsilon > 0$, take a birational modification $\mu : V' \rightarrow V$ such that there is a decomposition

$$\mu^*(K_V) \sim_{\mathbb{Q}} A + E_\varepsilon,$$

for some ample \mathbb{Q} -divisor A and some effective \mathbb{Q} -divisor E_ε and that the decomposition satisfies the properties of Takayama [19, Th. 3.1]. (In particular, $\text{vol}(A) \approx \text{vol}(V)$, but $\text{vol}(A) < \text{vol}(V)$.)

Take two different points $x_1, x_2 \in V'$ at very general positions. Then there is a divisor $D \sim_{\mathbb{Q}} \lambda A$ with $\lambda < \frac{3\sqrt[3]{2}}{(1-\varepsilon)\alpha_0}$ such that $x_i \in \text{Nklt}(V', D)$ for $i = 1, 2$.

Suppose the volume of those surfaces passing through very general points of V' has the lower bound α_1^2 . Of course, $\alpha_1^2 \geq 1$ and the volume of curves passing through very general points of V' has the lower bound $\alpha_2 \geq 2$.

Case 1. Assume $\alpha_1 \geq 10$. According to Takayama [19, Prop. 5.3], there is a constant $a_3 < s_3 + \frac{t_3}{\alpha_0}$ and an effective \mathbb{Q} -divisor $D_3 \sim_{\mathbb{Q}} a_3 A$ such that $x_1, x_2 \in \text{Nklt}(V', D_3)$ and that at least one point x_i is isolated in $\text{Nklt}(V', D_3)$. Thus Nadel vanishing implies that $|(\lfloor a_3 \rfloor + 2)K_{V'}|$ gives a birational map. Unfortunately, it is not enough to get the birationality of $|5K_V|$ whenever α_1 is smaller. In fact, according to Takayama, one may take

$$s_3 = \frac{2}{1-\varepsilon} + \left(1 + \frac{1}{1-\varepsilon} \right) \frac{4\sqrt{2}}{(1-\varepsilon)\alpha_1} + 2\varepsilon \left(1 + \frac{1}{1-\varepsilon} \right) \left(1 + \frac{\sqrt{2}}{(1-\varepsilon)\alpha_1} \right),$$

$$t_3 = \left(1 + \frac{1}{1-\varepsilon} \right) \left(1 + \frac{2\sqrt{2}}{(1-\varepsilon)\alpha_1} \right) \frac{3\sqrt[3]{2}}{1-\varepsilon}.$$

By taking a very small ε , while under the premise of $\alpha_0 > 12$ and $\alpha_1 \geq 10$, one can easily verify $a_3 < 4$, which means φ_5 is birational.

Case 2. Contrary to Case 1, we may assume $\text{Nklt}(V', D)$ contains an irreducible component which, passing through a very general point of V' , is a surface of volume $< 10^2$, i.e., $\alpha_1 < 10$. As already realized by McKernan [17] and in Todorov [20, p. 1328], there are two dominant morphisms $\nu : V'' \rightarrow V'$ and $f : V'' \rightarrow B$ where V'' is a normal projective 3-fold and B is a normal projective curve. For a very general point $x \in V'$, there is a surface V_x which is a pure center of a *lc* pair (V', D_x) with $D_x \sim_{\mathbb{Q}} \lambda' A$ and $\lambda' < \frac{3\sqrt[3]{2}}{(1-\varepsilon)\alpha_0}$ and, furthermore, V_x is dominated by at least one general fiber of f .

Subcase 2.1. Suppose ν is non-birational, which means that passing through a very general point $x \in V'$ there are at least two *lc* centers of the pairs (V', D) and (V', \tilde{D}) . Pick two different very general points $x_1 (= x), x_2 \in V'$. Using Todorov [20, Lem. 3.3], one may find a divisor $D'' \sim_{\mathbb{Q}} \lambda'' A$ with $\lambda'' < \frac{9\sqrt[3]{2}}{(1-\varepsilon)\alpha_0} + \varepsilon$ such that $\text{Nklt}(V', D'')$ has at worst a one-dimensional center at $x_1 = x$. This situation also fits into Takayama's induction [19, Prop. 5.3]. In fact, we may set $s_2 = 0$ and $t_2 = \frac{9\sqrt[3]{2}}{1-\varepsilon} + \varepsilon\alpha_0$ in Takayama's notation [19, 5.2(3)] and begin the induction onwards. Then one gets a divisor $\hat{D}_3 \sim_{\mathbb{Q}} \hat{a}_3 A$ with $\hat{a}_3 < s_3 + \frac{t_3}{\alpha_0}$ where

$$s_3 = \left(1 + \frac{1}{1-\varepsilon}\right)\varepsilon + \frac{2}{1-\varepsilon},$$

$$t_3 = \left(1 + \frac{1}{1-\varepsilon}\right)\left(\varepsilon\alpha_0 + \frac{9\sqrt[3]{2}}{1-\varepsilon}\right),$$

such that $\text{codimNklt}(V', \hat{D}_3) = 3$ at x_1 . Clearly one has

$$\hat{a}_3 < \varepsilon \left(1 + \frac{1}{1-\varepsilon}\right) + \frac{2}{1-\varepsilon} + \left(1 + \frac{1}{1-\varepsilon}\right) \frac{9\sqrt[3]{2}}{(1-\varepsilon)\alpha_0}.$$

Taking a very small $\varepsilon > 0$, one easily sees $\hat{a}_3 < 4$ as long as $\alpha_0 > 12$. Thus, by Nadel vanishing again, φ_5 is birational whenever $\text{vol}(V) > 12^3$ in this subcase.

Subcase 2.2. Suppose ν is birational. Without loss of generality, V'' can be considered smooth. We study the birationality of $|5K_{V''}|$ instead. We have a fibration $f : V'' \rightarrow B$ with B a normal projective curve. By assumption a general fiber S of f has the property $\text{vol}(S) < 10^2$.

Since $\text{vol}(V) > 12^3$, we have

$$\tau_0 := \frac{\text{vol}(V)}{3\text{vol}(S)} > \frac{12^3}{3 \times 10^2} = \frac{144}{25}$$

and [7, Lem. 2.7(1)] implies that

$$K_{Y''} \geq (\tau_0 - \eta)S,$$

for any very small rational number $\eta > 0$ (note here that the proof of [7, Lem. 2.7(1)] works for any curve B). Take such a number η so that $\tau_0 - \eta > 5$. Then we are in the situation of Proposition 2.1 which says φ_5 is birational. We are done. □

Next we shall study the high-dimensional analog.

Proposition 2.2. *Let Y be a nonsingular projective n -fold ($n \geq 4$) of general type, B a smooth projective curve and $f : Y \rightarrow B$ a fibration. Denote by X a general fiber of f . Assume $K_Y \sim_{\mathbb{Q}} p_n X + E_{p_n}$ for some rational number $p_n > 2\tilde{r}_{n-1} - 2$ and for some effective \mathbb{Q} -divisor E_{p_n} . Then $|mK_Y|$ gives a birational map for all $m \geq \tilde{r}_{n-1}$.*

Proof. Take a minimal model Y_0 of Y . Modulo birational modifications over Y , we may assume for simplicity that there is a birational morphism $\pi : Y \rightarrow Y_0$. Take a sufficiently large and divisible integer $m > 0$ such that

1. mK_{Y_0} is a Cartier divisor, mE_{p_n} is an integral divisor and $|mK_{Y_0}|$ is base point free;
2. $|p_n mX|$ is base point free;
3. $|p_n mK_{X_0}|$ is base point free where X_0 is a minimal model of X ;
4. $mK_Y \sim_{\mathbb{Z}} mp_n X + mE_{p_n}$;
5. the support of the union of E_{p_n} and all those exceptional divisors of π is simple normal crossing.

Then one has $p_n mX \leq m\pi^*(K_{Y_0}) \leq mK_Y$. Write $\pi^*(K_{Y_0}) \sim_{\mathbb{Q}} p_n X + E'_{p_n}$ where E'_{p_n} is another effective \mathbb{Q} -divisor. Pick general fibers X and $X_1 \neq X_2$ of f .

Step 1. The canonical restriction inequality.

If $g(B) > 0$, one may take further modification Y' of Y (still assume $Y' = Y$) such that, for the fiber X' over X (still assume $X' = X$), there is a birational morphism $\nu : X \rightarrow X_0$. Now the proof of Chen-Zuo [8, Lem. 3.4] implies

$$\pi^*(K_{Y_0})|_X \sim_{\mathbb{Q}} \nu^*(K_{X_0}).$$

(Note that the key fact used in the proof of [8, Lem. 3.4] is the rational chain connectedness of Shokurov in dimension 3 and that was proved by Hacon–McKernan [11, Cor. 1.3] for all dimensions.)

If $g(B) = 0$, we have the inclusion $\mathcal{O}_B(p_n m) \hookrightarrow f_* \omega_Y^m$. Take a very large and divisible integer l such that both lK_{Y_0} and lK_{X_0} are Cartier divisors. Then we have the inclusion

$$f_* \omega_{Y/B}^{lp_n m} \hookrightarrow f_* \omega_Y^{lp_n m + 2lm}.$$

The semi-positivity theorem (see, for instance, Viehweg [21]) implies that $f_* \omega_{Y/B}^{lp_n m}$ is generated by global sections. Since $|\nu^*(lp_n m K_{X_0})|$ (as the moving part of $|lp_n m K_X|$) is base point free and the moving part of $|(lp_n m + 2lm)K_Y|$ is exactly $|\pi^*((lp_n m + 2lm)K_{Y_0})|$ thanks to the MMP, we have

$$\pi^*((lp_n m + 2lm)K_{Y_0})|_X \geq \nu^*(lp_n m K_{X_0}).$$

In both cases, we have the following equality

$$(2.3) \quad \pi^*(K_{Y_0})|_X \equiv \frac{p_n}{p_n + 2} \nu^*(K_{X_0}) + G_n,$$

for certain effective \mathbb{Q} -divisor G_n on X .

Step 2. The induction step by means of vanishing.

Both X_1 and X_2 are known to be nonsingular projective $(n - 1)$ -folds of general type. Since

$$(\tilde{r}_{n-1} - 1)\pi^*(K_{Y_0}) - X_1 - X_2 - \frac{2}{p_n} E'_{p_n} \equiv \left(\tilde{r}_{n-1} - 1 - \frac{2}{p_n} \right) \pi^*(K_{Y_0})$$

is nef and big by assumption, Kawamata–Viehweg vanishing gives the surjective map

$$(2.4) \quad \begin{aligned} & H^0(K_Y + \lceil K_{n,p_n} \rceil) \\ \longrightarrow & H^0(X_1, K_{X_1} + \lceil K_{n,p_n} - X_1 \rceil|_{X_1}) \oplus H^0(X_2, K_{X_2} + \lceil K_{n,p_n} - X_2 \rceil|_{X_2}), \end{aligned}$$

where $K_{n,p_n} := (\tilde{r}_{n-1} - 1)\pi^*(K_{Y_0}) - \frac{2}{p_n} E'_{p_n}$. Observing that

$$\left[(\tilde{r}_{n-1} - 1)\pi^*(K_{Y_0}) - \frac{2}{p_n} E'_{p_n} - X_i \right] |_{X_i} \geq \lceil L_{n,i} \rceil,$$

where

$$L_{n,i} := \left((\tilde{r}_{n-1} - 1)\pi^*(K_{Y_0}) - \frac{2}{p_n} E'_{p_n} - X_i \right) |_{X_i} \equiv \left(\tilde{r}_{n-1} - 1 - \frac{2}{p_n} \right) \pi^*(K_{Y_0})|_{X_i},$$

for $i = 1, 2$, we have

$$|\tilde{r}_{n-1}K_Y|_{X_i} \succeq |K_Y + \left[(\tilde{r}_{n-1} - 1)\pi^*(K_{Y_0}) - \frac{2}{p_n}E'_{p_n} \right]|_{X_i} \succeq |K_{X_i} + [L_{n,i}]|.$$

Due to the surjectivity of (2.4), it is sufficient to show that $|K_{X_i} + [L_{n,i}]|$ gives a birational map for $i = 1, 2$.

Step 3. Verification on the general fiber.

Since $p_n > 2\tilde{r}_{n-1} - 2$, we have $\frac{p_n}{p_n+2} > \frac{\tilde{r}_{n-1}-1}{\tilde{r}_{n-1}}$ and $\delta_0 := (\tilde{r}_{n-1} - 1 - \frac{2}{p_n})\frac{p_n}{p_n+2} - (\tilde{r}_{n-1} - 2) > 0$. Now we have

$$L_{n,i} \equiv (\tilde{r}_{n-1} - 2 + \delta_0)\theta_i^*(K_{X_{i,0}}) + \left(\tilde{r}_{n-1} - 1 - \frac{2}{p_n} \right) G_{n,i},$$

for an effective \mathbb{Q} -divisor $G_{n,i}$ on X_i where $\theta_i : X_i \rightarrow X_{i,0}$ is the contraction onto its minimal model $X_{i,0}$ for $i = 1, 2$. Since

$$L_{n,i} - \left(\tilde{r}_{n-1} - 1 - \frac{2}{p_n} \right) G_{n,i} \equiv (\tilde{r}_{n-1} - 2)\theta_i^*(K_{X_{i,0}}) + \delta_0\theta_i^*(K_{X_{i,0}}),$$

one knows that

$$\left| K_{X_i} + \left[L_{n,i} - \left(\tilde{r}_{n-1} - 1 - \frac{2}{p_n} \right) G_{n,i} \right] \right|$$

gives a birational map by the definition of \tilde{r}_{n-1} and so does $|K_{X_i} + [L_{n,i}]|$ for $i = 1, 2$.

Since X_1 and X_2 are general fibers of f , $|\tilde{r}_{n-1}K_Y|$ gives a birational map. So we are done. □

Lemma 2.2. *For all $n \geq 3$, $\tilde{r}_n < +\infty$.*

Proof. We have already seen $\tilde{r}_2 = 5$ in Lemma 2.1. For $n > 2$, we prove this by an induction on the dimensions. We sketch the proof here.

By the boundedness theorem of Hacon–McKernan, Takayama and Tsuji, we can take a positive integer m_0 so that $P_{m_0}(X_0) \geq 2$ for all minimal projective n -fold X_0 of general type. Take a pencil $\Lambda \subset |m_0K_{X_0}|$. Given any birational morphism $\nu : X' \rightarrow X_0$ and given any nef and big \mathbb{Q} -divisor R on X' , take a birational modification $\nu_1 : X_1 \rightarrow X_0$ such that:

1. X_1 is smooth;
2. ν_1 factors through ν , i.e., $\nu_1 = \nu \circ \nu'$ and thus X_1 is over X' ;

- 3. the moving part of $\nu_1^*(\Lambda)$ is base point free;
- 4. the support of union of exceptional divisors of ν_1 and the pull back of $\{R\}$ is simple normal crossing.

Now it is sufficient to find some positive number m (independent of X_0 , the choice of ν and the \mathbb{Q} -divisor R) so that

$$|K_{X_1} + \lceil (m - 2)\nu_1^*(K_{X_0}) + \nu'^*(R) \rceil |$$

gives a birational map.

Take two generic irreducible elements F_1 and F_2 in the moving part of $\nu_1^*(\Lambda)$. In other words, the F_i is a general fiber of the induced fibration after taking the Stein factorization of $\Phi_\Lambda \circ \nu_1$. By definition, F_1 and F_2 are irreducible elements in a free pencil. Both F_1 and F_2 are smooth by Bertini's theorem and they are of general type with dimensions $n - 1$. For any integer $m \geq 2m_0 + 3$, since $(m - 2)\nu_1^*(K_{X_0}) + \nu'^*R \geq (m - 2m_0 - 2)\nu_1^*(K_{X_0}) + \nu'^*R + F_1 + F_2$, Kawamata–Viehweg vanishing gives the surjective map:

$$(2.5) \quad \begin{aligned} & H^0(X_1, K_{X_1} + \lceil (m - 2m_0 - 2)\nu_1^*(K_{X_0}) + \nu'^*R \rceil + F_1 + F_2) \\ \longrightarrow & \bigoplus_{i=1}^2 H^0(F_i, K_{F_i} + \lceil (m - 2m_0 - 2)\nu_1^*(K_{X_0}) + \nu'^*R \rceil |_{F_i}). \end{aligned}$$

Set $F = F_i$ and we study the restricted linear system on F . Modulo further birational modifications, we may assume that there is a birational contraction morphism $\sigma : F \rightarrow F_0$ onto a minimal model F_0 . By the canonical restriction inequality in Chen [6, Lem. 2.5 and Inequality (1) at p. 225], we have

$$\nu_1^*(K_{X_0})|_F \geq \frac{1}{2m_0 + 1} \sigma^*(K_{F_0}).$$

Thus

$$K_F + \lceil q\nu_1^*(K_{X_0}) + \nu'^*R \rceil |_F \geq K_F + \lceil t\sigma^*(K_{F_0}) + Q \rceil ,$$

where $q := m - 2m_0 - 2$, $t := \lfloor \frac{q}{2m_0 + 1} \rfloor$ is an integer depending only on m and Q is a certain nef and big \mathbb{Q} -divisor on F . By the induction, the statement is true on F as long as t is large enough and such an integer t should work for all $(n - 1)$ -fold F . The surjective map 2.5 says that, for some integer m , the linear system $|K_{X_1} + \lceil (m - 2)\nu_1^*(K_{X_0}) + \nu'^*(R) \rceil |$ separates different general F_1 and F_2 , and it separates general points of F_i for $i = 1, 2$. We are done. □

With Proposition 2.2 and Lemma 2.2, we are prepared for proving Theorem 1.2.

Proof of Theorem 1.2. We sketch the proof while omitting redundant calculations. We keep a similar procedure to the proof of Theorem 1.1.

First of all we know $\tilde{r}_3 \geq 27$ due to Iano-Fletcher’s example $X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$ in [14, p. 151].

Then one uses Takayama’s effective induction in [19, Prop. 5.3] and McKernan’s dominant morphism $\nu : V'' \rightarrow V'$ in [17, Lem. 3.2] to exclusively treat three cases with regard to $\text{Nklt}(V', D)$, where V' is certain birational model of Y as in the proof of Theorem 1.1 replacing V by Y :

- (i) Only sub-varieties of dimension ≤ 2 passing through very general points of Y need to be considered. Note that the situation corresponding to Subcase 2.1 in the proof of Theorem 1.1 is included here. Meanwhile, ν is non-birational by assumption. We may use Todorov [20, Lem. 3.3] to find a divisor $D'' \sim \lambda'' A$ such that $\text{Nklt}(V', D'')$ has at worst a one-dimensional lc center passing through x – a very general point of V' . Replace (V', D) by (V', D'') and go on the procedure of applying Takayama’s induction.
- (ii) Apart from i, there is a 3-fold center, with a large volume, of the pair (V', D_x) passing through a very general point $x \in V'$.
- (iii) Due to McKernan [17], Y is birationally fibred by 3-folds of small volumes over a curve B .

For Cases (i) and (ii), it is easy to find the constraint on $\text{vol}(Y)$ (say $\text{vol}(Y) \geq \alpha_0^4 > 0$) so that $|27K_Y|$ gives a birational map by assuming that all those 3-folds passing through very general points of Y have volume $> (3\sqrt[3]{16})^3$. In fact, the worse situation in Case (i) is when ν is non-birational for which we use Todorov [20, Lem. 3.3]. So we may set $s_2 = 0, t_2 = 12\sqrt[4]{2}$, while $\varepsilon \mapsto 0$, and then Takayama’s induction gives, $s_4 < 8\sqrt{2} + 2$ and

$$t_4 < 24\sqrt[4]{2}(2\sqrt{2} + 1).$$

Thus we have $a_4 < s_4 + \frac{t_4}{\alpha_0}$ and $\lceil a_4 \rceil + 1 < 27$ as long as α_0 is large enough. Actually this is the case when $\alpha_0^4 > \frac{\tilde{r}_3 - 1}{6} \times 12^4 \geq \frac{13}{3} \times 12^4$. Thus $|27K_Y|$ gives a birational map. In Case (ii), by assuming $\alpha_1 > 3\sqrt[3]{4}$, Takayama’s induction gives

$$s_4 < 8\sqrt{2} + 12(1 + 2\sqrt{2})\sqrt[3]{2}/\alpha_1,$$

$$t_4 < 8\sqrt[4]{2}(2\sqrt{2} + 1)(1 + 3\sqrt[3]{2}/\alpha_1),$$

while letting $\varepsilon \mapsto 0$. Finally, we have $a_4 < s_4 + t_4/\alpha_0$. Still under the assumption $\alpha_0^4 > \frac{13}{3} \times 12^4$, it is easy to see $[a_4] + 1 < 27$. Thus we see the birationality of $\Phi_{|27K_Y|}$.

For Case (iii), we may use Proposition 2.2 under the condition that Y is birationally fibred by 3-folds X of small volume (say $\text{vol}(X) \leq 16 \times 27$). Now using a similar method to that of [7, Lem. 2.7(1)] by computing the ratio $\frac{\text{vol}(Y)}{4\text{vol}(X)}$, it is easy for us to find the condition on $\text{vol}(Y)$ again. In fact, $\text{vol}(Y) > \frac{\tilde{r}_3-1}{6} \times 12^4$ will do for this case to get the birationality of $|mK_Y|$ for all $m \geq \tilde{r}_3$. We are done. \square

Thanks to the elaborate calculation in Di Biagio [9, Th. 5.9], we can save several pages to present here our effective result for all dimensions. So far we have the following direct result in higher dimensions:

Corollary 2.1. (*=Corollary 1.1*) *There are computable constants C_n for all $n \geq 5$. Let Y be a nonsingular projective n -fold with $\text{vol}(Y) > C_n$. Then $|mK_Y|$ gives a birational map onto its image for all $m \geq R_n$.*

We omit the proof since it can be obtained by a similar argument to that of Theorem 1.1 and Theorem 1.2. This may serve as an interesting exercise.

Finally we note that $R_3 = \tilde{r}_2 = r_2 = 5$ by Bombieri and, in fact, $R_4 = \tilde{r}_3 \geq r_3$. It is natural to ask the following:

Open Problem X_n . *Is it always true that $R_n = \tilde{r}_{n-1} = r_{n-1}$ for all $n \geq 4$?*

A positive answer to Problem X_n should be expectable.

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